Towards coordinate independent estimates for electro-magnetic GW detectors

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Work in progress

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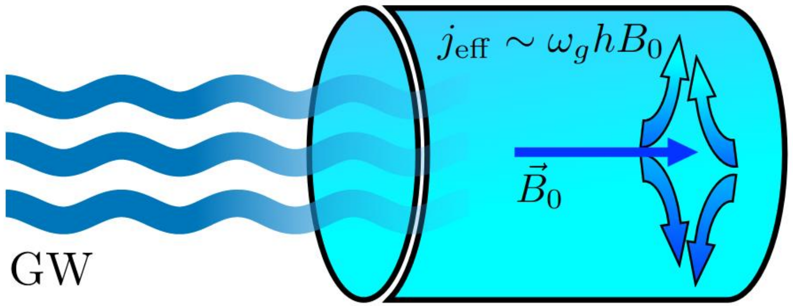
The Problem

Consider GW to EM wave conversion in background field:



from A. Berlin et. al. '21

Description of any GW detector

1. Full theory in GR
   ->coordinate invariant
2. Perturbed theory
   -> inherits gauge invariance
3. Introduce further approximations
   e.g. choice of gauge + dropping terms

(Old) Literature:

Tension

"You have to use TT gauge / proper detector frame!"

Makes Sence!
Applicability / Errors?

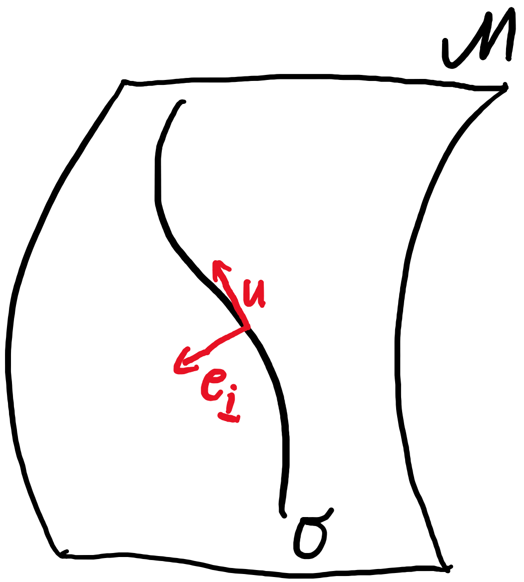
Electro Magnetism in GR

- Field strength $F_{\mu\nu}$ and 4-current j^μ satisfy:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \qquad \nabla_\nu F^{\mu\nu} = j^\mu$$

- The observed field:

$$E_{\underline{i}} = F_{\mu\nu}\, e^\mu_{\underline{i}} u^\nu \qquad B^{\underline{i}} = \frac{1}{2}\epsilon^{\underline{imn}} F_{\mu\nu}\, e^\mu_{\underline{m}} e^\nu_{\underline{n}}$$



Observers infinitesimal coord. system, tetrad:

$$g_{\mu\nu} e^\mu_{\underline{\mu}} e^\nu_{\underline{\nu}} = \eta_{\underline{\mu\nu}} \qquad e^\mu_{\underline{0}} = u^\mu$$

Electro Magnetism in GR

- Boundary conditions on conductor

Consider observer attached to surface of conductor measuring the electric field parallel to surface:

4-velocity of conductor

$$F_{\mu\nu} e^{\mu}_{\underline{1}} u^{\nu} = F_{\mu\nu} e^{\mu}_{\underline{2}} u^{\nu} = 0 \quad \Longleftrightarrow \quad \mathbf{E}_{\parallel} = 0$$

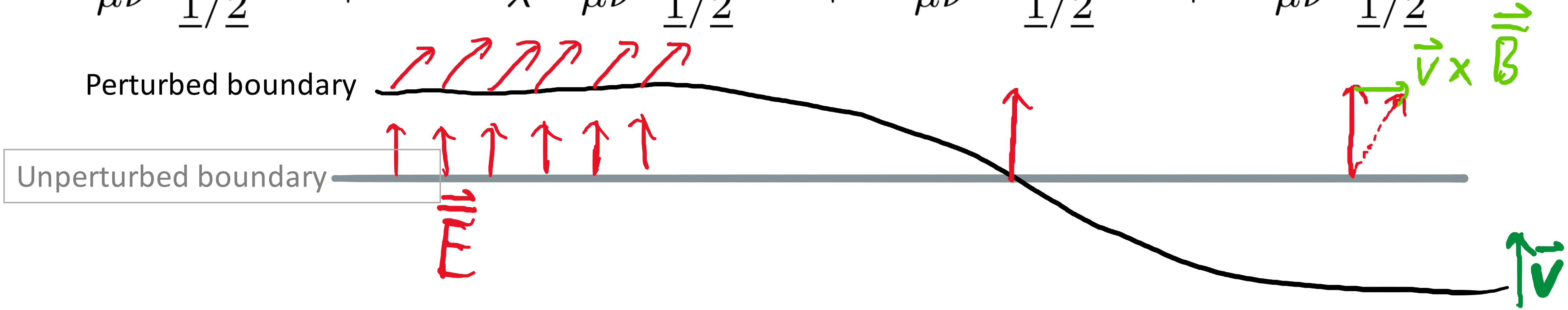
directions parallel to surface of conductor

See D. Rawson-Harris '71

Perturbations around Minkowski

- Find effective current in Maxwell equations etc.
- Perturbed boundary condition: (observed fields work similar)

$$0 = \overbrace{\delta F_{\mu\nu}\,\bar{e}^{\mu}_{\underline{1}/\underline{2}}\,\bar{u}^{\nu}}^{\text{Most Literature only considers this}} + \overbrace{\delta x^{\lambda}\partial_{\lambda}\overline{F}_{\mu\nu}\,\bar{e}^{\mu}_{\underline{1}/\underline{2}}\,\bar{u}^{\nu} + \overline{F}_{\mu\nu}\,\delta e^{\mu}_{\underline{1}/\underline{2}}\,\bar{u}^{\nu} + \overline{F}_{\mu\nu}\,\bar{e}^{\mu}_{\underline{1}/\underline{2}}\,\delta u^{\nu}}^{\text{Effects related to perceived motion of conductor surface -> Can drop in frame in which conductor is at rest!!!}}$$



Mechanical Limits

- Transverse traceless gauge
  - -freely falling masses at rest
  - -free falling limit:

$$\delta x^{TT} = h^{TT} L \left(0 \pm \mathcal{O}\left(\frac{v_s}{\omega_{GW} L}\right)\right) \text{ for } \omega_{GW} L \gg v_s$$

- Proper detector frame
  - -bodies with fixed distance at rest
  - -rigid limit:

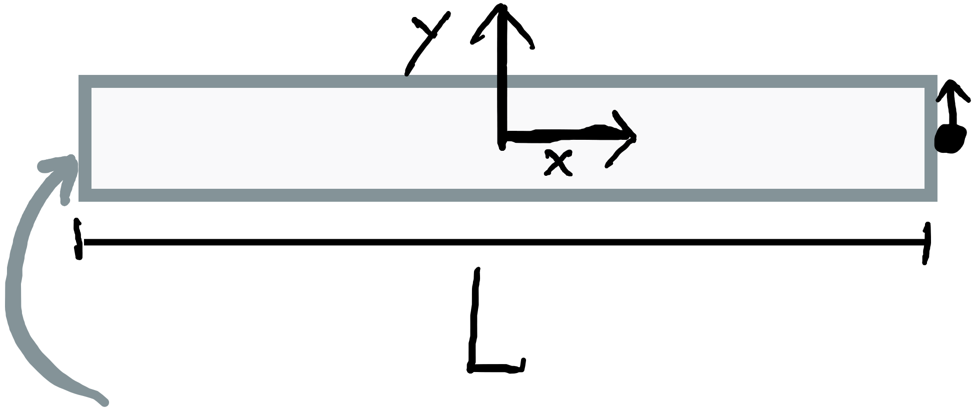
$$\delta x^{PDF} = h^{TT} L \left(0 \pm \mathcal{O}\left(\frac{\omega_{GW}^2 L^2}{v_s^2}\right)\right) \text{ for } \omega_{GW} L \ll v_s$$

  - -corrections to metric suppressed by $\omega_{GW}^2 L^2$
  - -long-wavelength limit (not mechanical):

$$h^{PDF} = h^{TT} \left(\omega_{GW}^2 L^2 \pm \mathcal{O}(\omega_{GW}^3 L^3)\right) \text{ for } \omega_{GW} L \ll 1$$

Sound velocity in solid
~10^{-5}

Toy Example modified from A. Berlin et. al. '21



Observer measuring E-field in y-direction

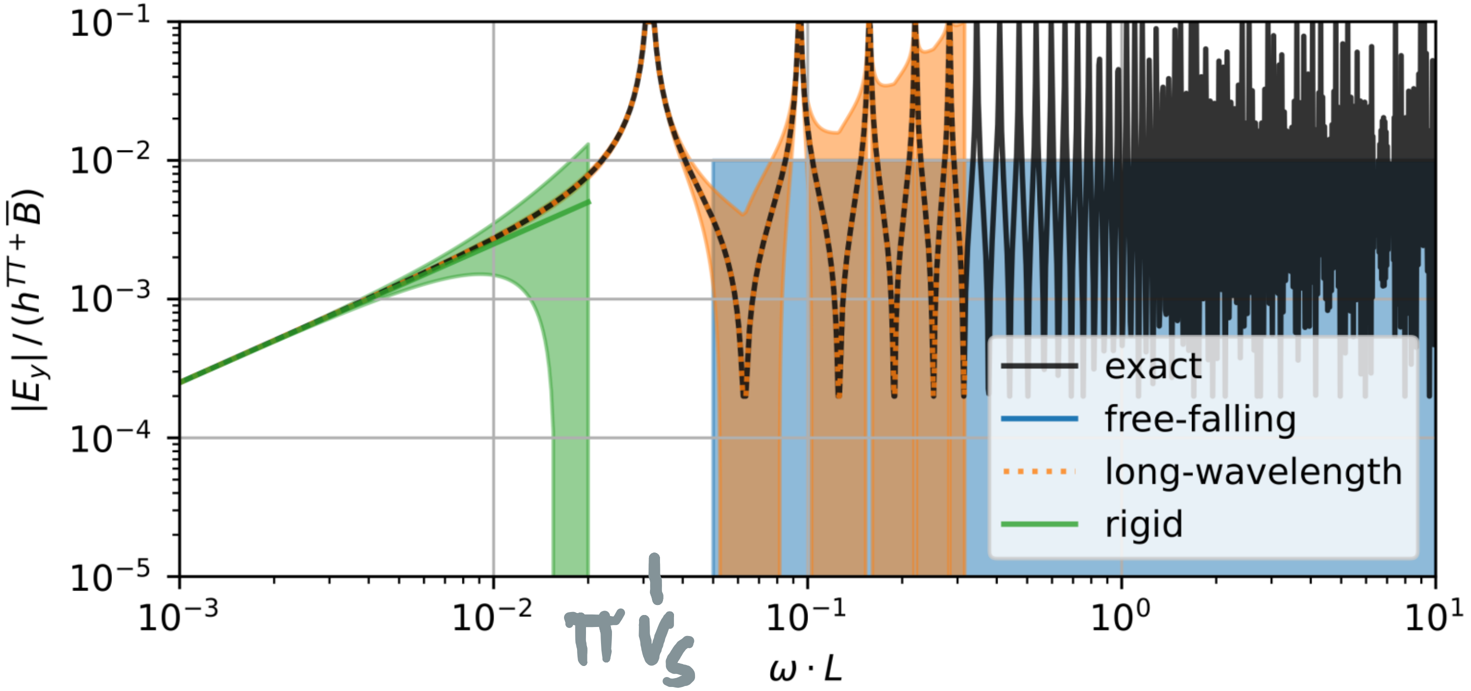
Thin Rod with $v_s = 10^{-2}$

⊙ Homogeneous B-field

$$\vec{\overline{B}} = \overline{B}\hat{e}_z$$

⊙ GW plus polarized in x-y

$$\vec{k}_{GW} = \omega\hat{e}_z$$



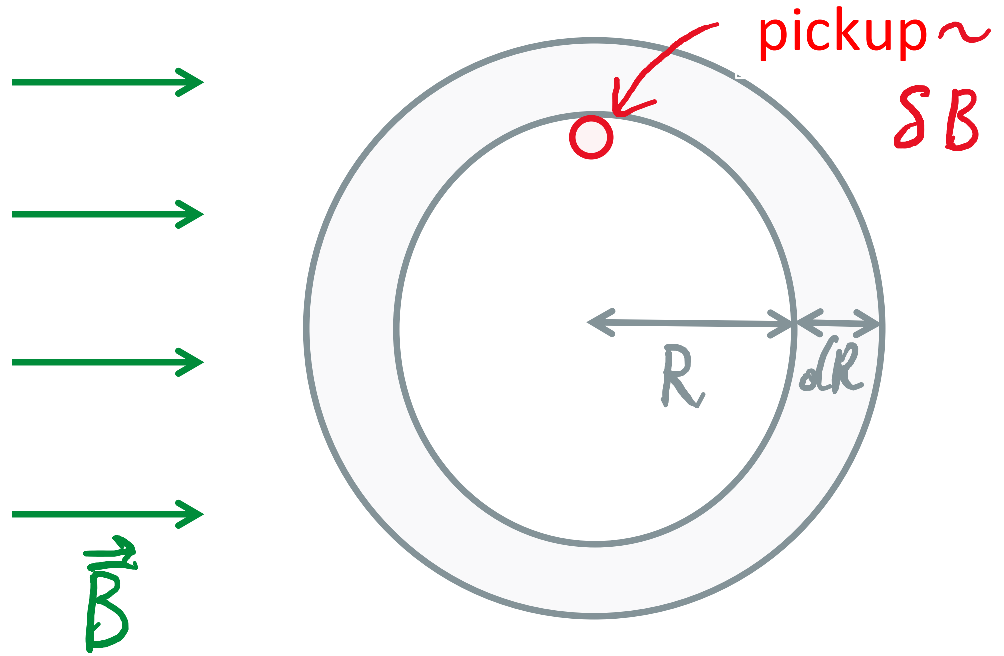
Spherical cavity in B field

- Hollow sphere with radius R

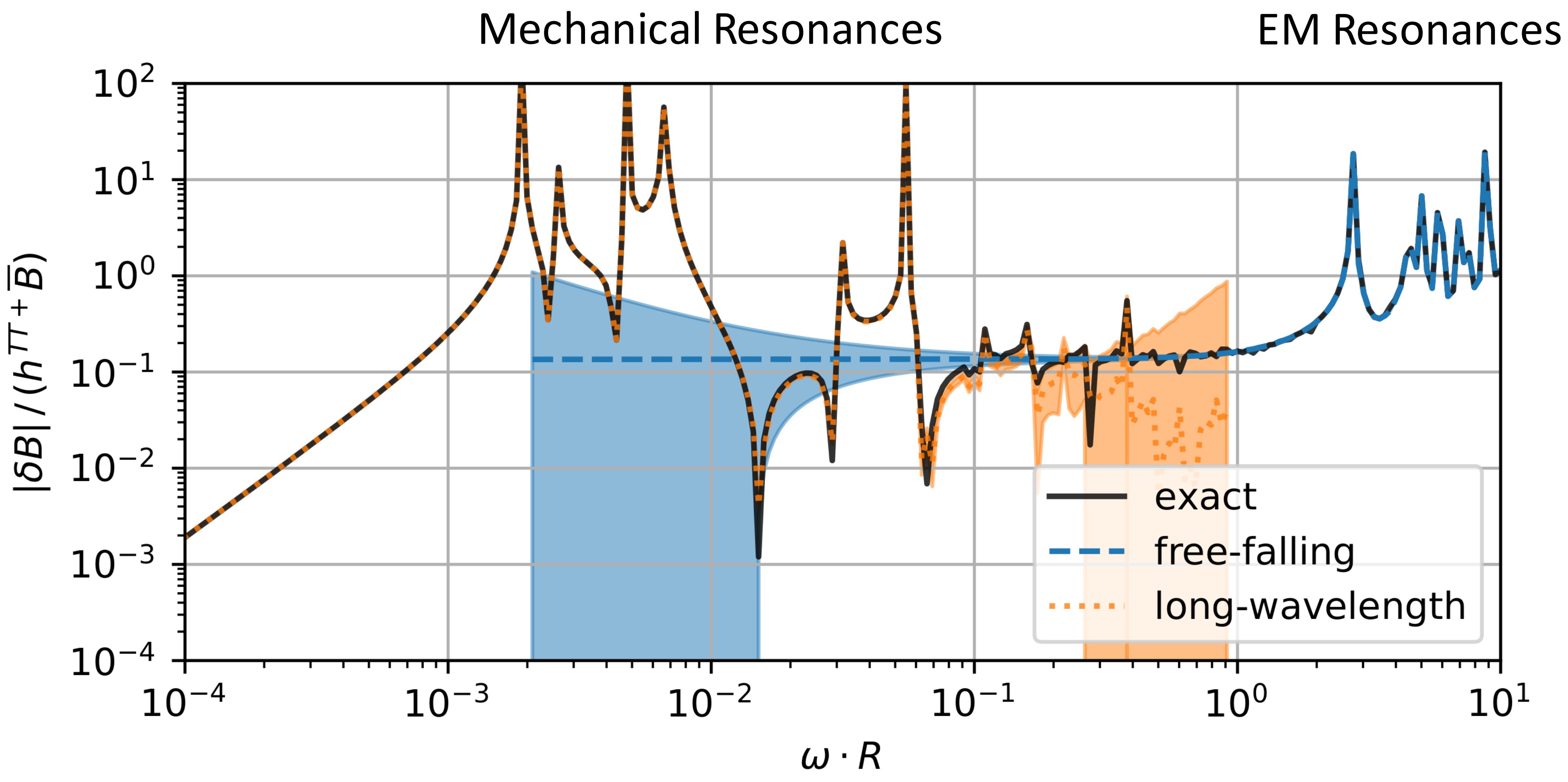
  and thickness dR=0.1 R

  -speed of sound $v_s = 10^{-3}$
- In homogeneous magnetic field
- Small pickup-loop (rigid) + freely rotating

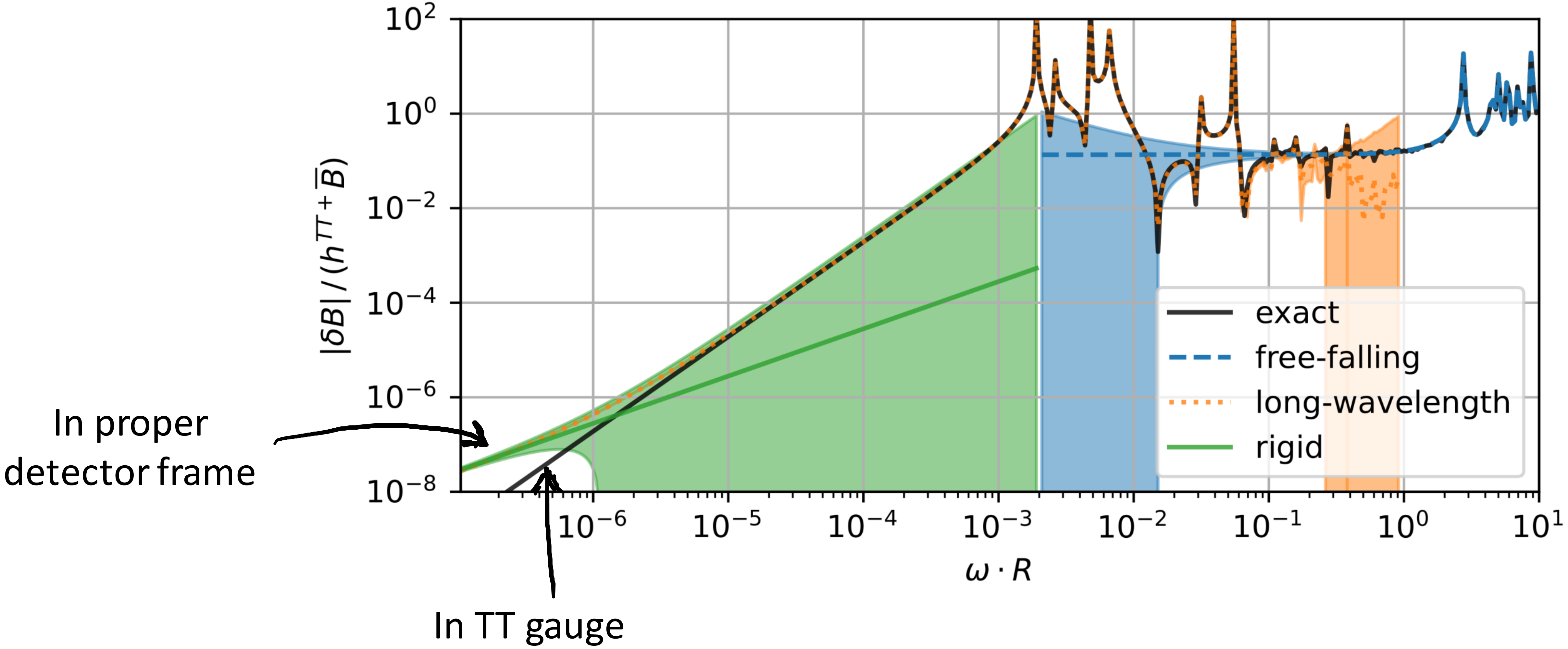
  -> Measures oscillating B field orthogonal to loop



Preliminary Result

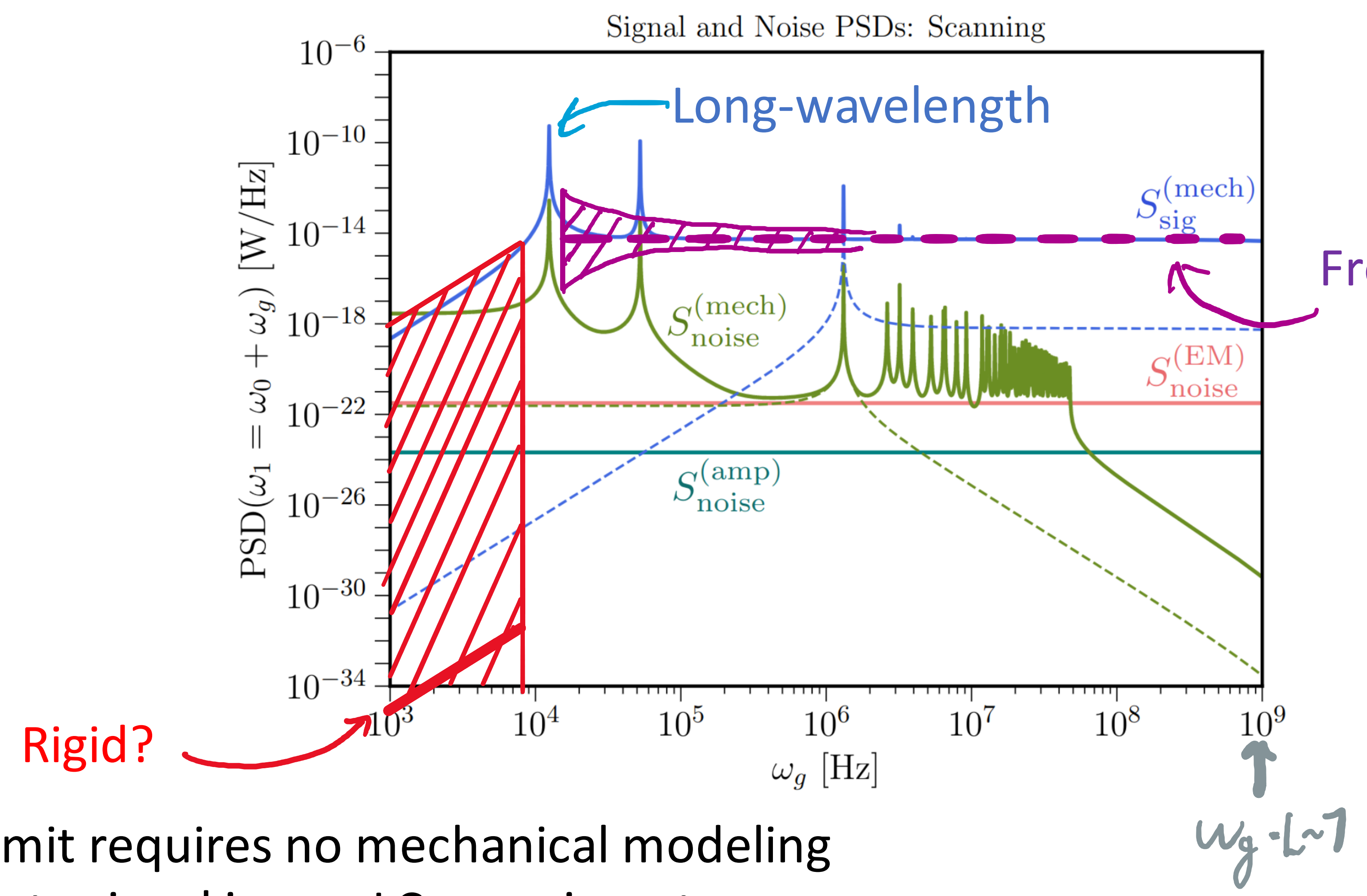


Preliminary Result



Still struggling here!

Comparison with prediction for MAGO



from A. Berlin et. Al. '23

Free-falling limit requires no mechanical modeling
=> Can estimate signal in e.g. LC experiments

Conclusion

- Bulk equations + boundary conditions + observables must be coordinate invariant

- Choice of gauge + neglecting motion, is approximation

  -> Make sure that one is in the right limit + introduce errors

Thanks

For Discussion

Perturbation Schemes 1

Thoughts:

- \+ Gauge invariance clear (hopefully)
- \- δF no measured field

Our Scheme

Transition to perturbed quantities:

$$g_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}$$
$$g^{\mu\nu} \to \eta^{\mu\nu} - h^{\mu\nu}$$
$$F_{\mu\nu} \to \overline{F}_{\mu\nu} + \delta F_{\mu\nu}$$
$$j^{\mu} \to \overline{j}^{\mu} + \delta j^{\mu}$$
$$x^{\mu}(\tau) \to \overline{x}^{\mu}(\tau) + \delta x^{\mu}(\tau)$$
$$e^{\mu}_{\underline{\mu}}(\tau) \to \overline{e}^{\mu}_{\underline{\mu}}(\tau) + \delta e^{\mu}_{\underline{\mu}}(\tau)$$

Gauge transformation:

$$x^{\mu} \to x^{\mu} + \xi^{\mu}$$

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}$$
$$\delta F_{\mu\nu} \to \delta F_{\mu\nu} - \xi^{\alpha}\partial_{\alpha}\overline{F}_{\mu\nu} - \overline{F}_{\alpha\nu}\partial_{\mu}\xi^{\alpha} - \overline{F}_{\mu\alpha}\partial_{\nu}\xi^{\alpha}$$
$$\delta j^{\mu} \to \delta j^{\mu} - \xi^{\alpha}\partial_{\alpha}\overline{j}^{\mu} + \overline{j}^{\alpha}\partial_{\alpha}\xi^{\mu}$$
$$\delta x^{\mu} \to \delta x^{\mu} + \xi^{\mu}$$
$$\delta e^{\mu}_{\underline{\mu}} \to \delta e^{\mu}_{\underline{\mu}} + \overline{e}^{\alpha}_{\underline{\mu}}\partial_{\alpha}\xi^{\mu}$$

backgrd. quantities trans. trivially

Maxwells Equations:

$$0 = \partial_{\lambda}\delta F_{\mu\nu} + \partial_{\mu}\delta F_{\nu\lambda} + \partial_{\nu}\delta F_{\lambda\mu}$$
$$\partial_{\nu}\delta F^{\mu\nu} = \delta j^{\mu} + j^{\mu}_{\text{eff}}$$
$$j^{\mu}_{\text{eff}} = -\frac{1}{2}\partial_{\alpha}h\,\overline{F}^{\mu\alpha} + \partial_{\nu}\left(h^{\mu}{}_{\alpha}\overline{F}^{\alpha\nu} + h^{\nu}{}_{\alpha}\overline{F}^{\mu\alpha}\right)$$

Implies e.g.:

$$F^{\mu\nu} = g^{\mu\alpha}F_{\alpha\beta}g^{\beta\nu} \to \overline{F}^{\mu\nu} + \delta F^{\mu\nu} - h^{\mu\alpha}\overline{F}_{\alpha}{}^{\nu} - \overline{F}^{\mu}{}_{\beta}h^{\beta\nu}$$

Another scheme, all the same except

$$F^{\mu\nu} \to \overline{F}^{\mu\nu} + \delta F^{\mu\nu} \qquad \delta F^{\mu\nu} \to \delta F^{\mu\nu} - \xi^{\alpha}\partial_{\alpha}\overline{F}^{\mu\nu} + \overline{F}^{\alpha\nu}\partial_{\alpha}\xi^{\mu} + \overline{F}^{\mu\alpha}\partial_{\alpha}\xi^{\nu}$$

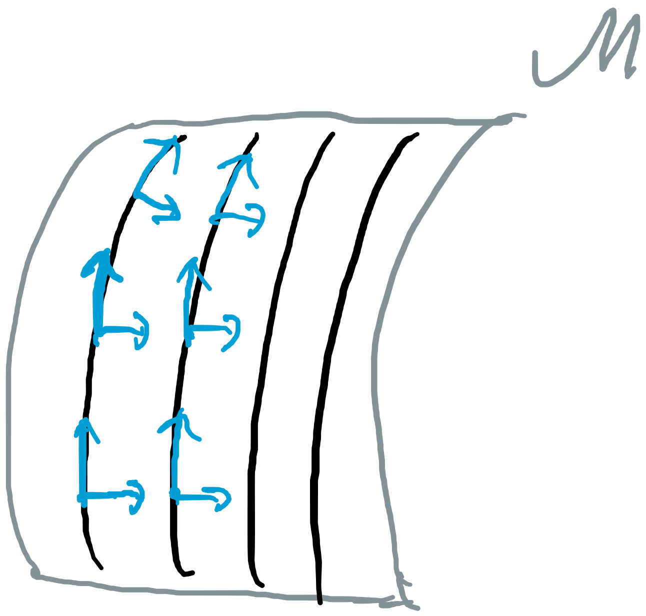
Implies e.g.:

$$F_{\mu\nu} = g_{\mu\alpha}F^{\alpha\beta}g_{\beta\nu} \to \overline{F}_{\mu\nu} + \delta F_{\mu\nu} + h_{\mu\alpha}\overline{F}^{\alpha}{}_{\nu} + \overline{F}_{\mu}{}^{\beta}h_{\beta\nu}$$

$$0 = \partial_{\lambda}(\delta F_{\mu\nu} + h_{\mu\alpha}\overline{F}^{\alpha}{}_{\nu} + \overline{F}_{\mu}{}^{\beta}h_{\beta\nu}) + \text{even perm.}(\lambda,\mu,\nu)$$
$$\partial_{\nu}\delta F^{\mu\nu} = \delta j^{\mu} - \frac{1}{2}\partial_{\alpha}h\,\overline{F}^{\mu\alpha}$$

Perturbation Schemes 2

What I think Francesco Sorge is doing (see talk + paper)



- Introduce family of observers with tetrads $e^{\mu}_{\underline{\mu}}(x)$
- Perturb in $F_{\underline{\mu\nu}} = F_{\mu\nu} e^{\mu}_{\underline{\mu}} e^{\nu}_{\underline{\nu}}$ instead of $F_{\mu\nu}$

Thoughts:

\+ δF measured field, especially usefull if all observers are part of chosen family

\- Gauge invariance not readily apparent?

\- Possible ambiguities related to choice of $e^{\mu}_{\underline{\mu}}(x)$?

See also J.-c. Hwang and H. Noh '23 for a mixed approach, perturbing in e.g. $F_{\mu\nu} e^{\nu}_{\underline{0}} = F_{\mu\nu} u^{\nu}$