HIGH-FREQUENCY GRAVITATIONAL WAVES IN ELECTROMAGNETIC WAVEGUIDES

F. Sorge, *Ann. Der Phys.*, **535** (10) 2300228 (2023).

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Ultra-High-Frequency Gravitational Waves: Where to Next?

CERN 4 - 8 December, 2023

MOTIVATION

- Recent experimental detection of GWs by LIGO-VIRGO Collaboration has strongly renewed the interest in other GW sources, both cosmological and astrophysical, in **different frequency ranges**.
- There is a growing theoretical and experimental effort in search for GWs in the High and Ultra-High Frequency band (up to GHz and beyond).
- UHFGWs will presumably require completely different detection techniques at the sub-metre scales, maybe involving other matter fields.
- According to some people, a promising route could be based on the interaction with **E.M. fields**, as in the celebrated **inverse Gertsenshtein effect** [see, e.g., the criticized Li & Baker proposal].

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«Wave resonance of light and gravitational waves»,
M. E. Gertsenshtein, Sov. Phys., JETP 14, 84 (1962).

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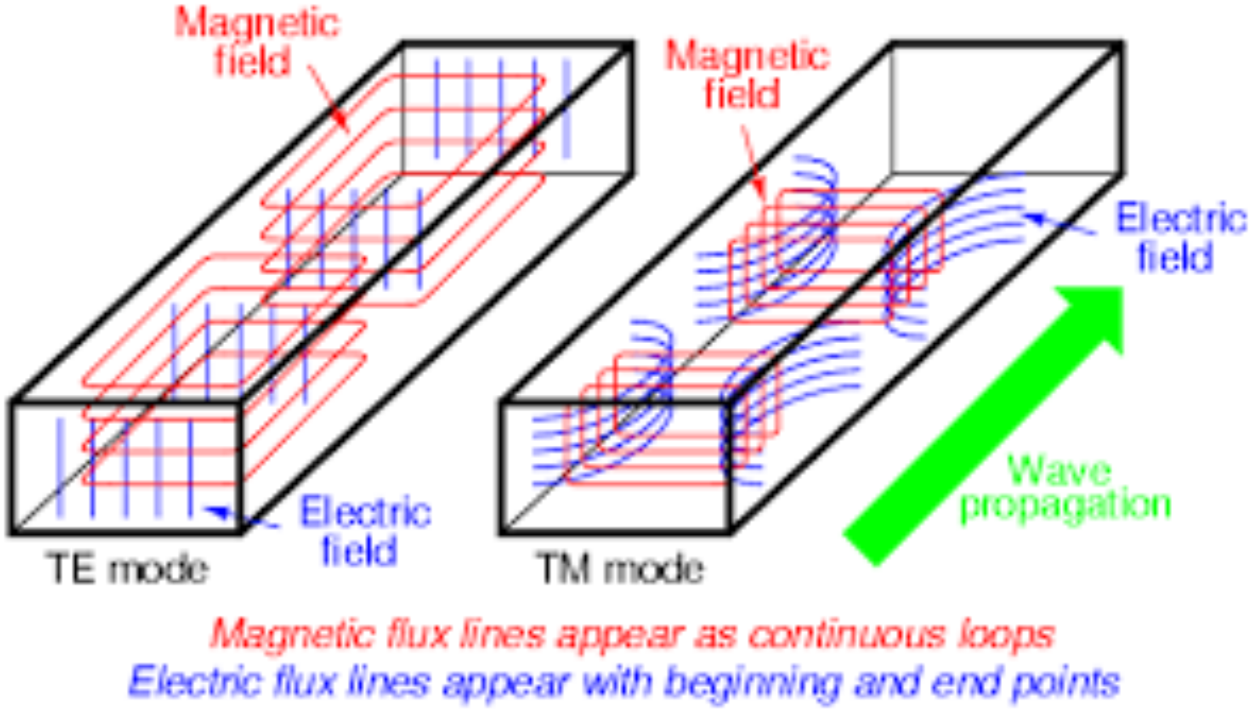
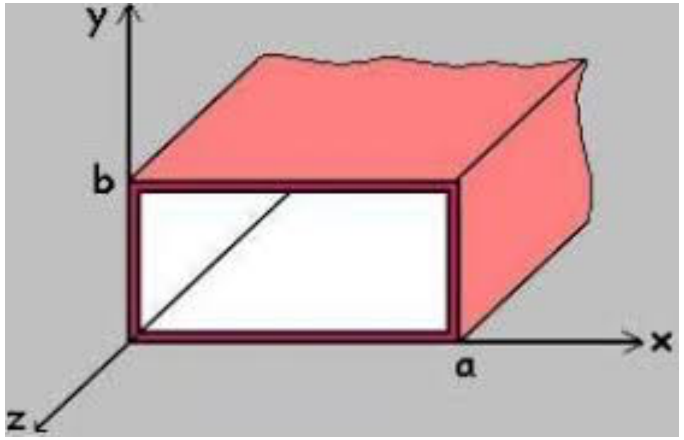
- In this talk we aim to explore possible **non-linear effects** arising from the interaction between a VHFGW and an EM wave in a waveguide.
- We will show the existence of a **Second Harmonic Generation (SHG) effect**, resembling a similar effect occurring in non-linear optics.
- As we will see, although theoretically interesting, from an experimental point of view such effect represents indeed a **distant dream**.

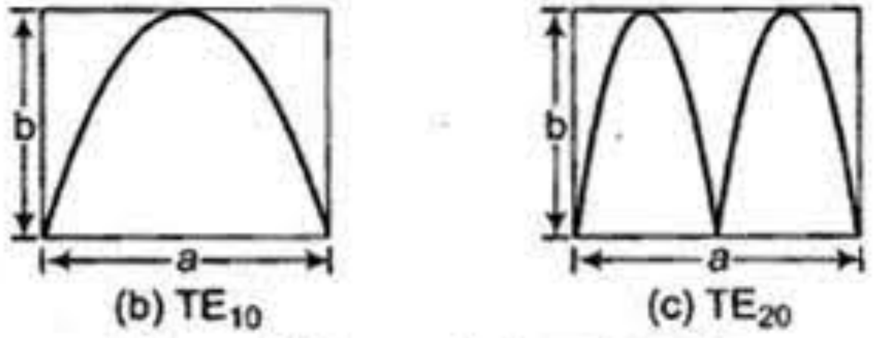
OUTLINE

- TE_{n0} MODES IN A RECTANGULAR WAVEGUIDE
- TE_{n0} MODES IN A GRAVITATIONAL WAVE BACKGROUND
- SECOND HARMONIC GENERATION
- CONCLUSIONS

TE AND TM MODES
IN A RECTANGULAR WAVEGUIDE





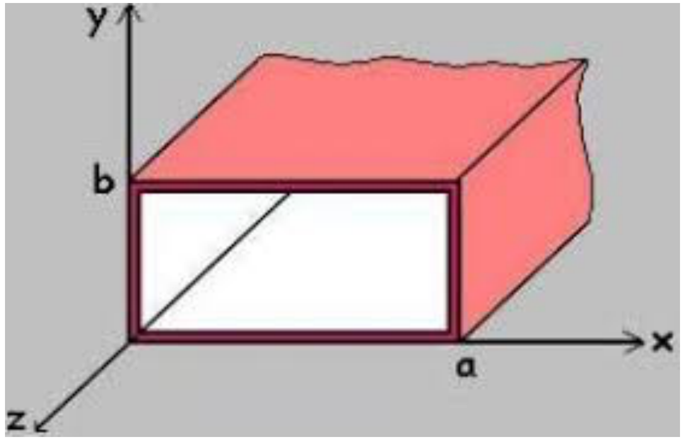


TE_{10} and TE_{20} mode in rectangular waveguide

TE_{n0} MODES
IN A RECTANGULAR WAVEGUIDE

- Solving the Maxwell equations with the required b.c. yields

$$\begin{cases} B_x^{(0)} = -\frac{ik_z a}{\pi} B_0 \sin\left(\frac{n\pi x}{a}\right) e^{ik_z z} e^{-i\omega t} \\ B_z^{(0)} = B_0 \cos\left(\frac{n\pi x}{a}\right) e^{ik_z z} e^{-i\omega t} \\ E_y^{(0)} = \frac{i\omega a}{\pi} B_0 \sin\left(\frac{n\pi x}{a}\right) e^{ik_z z} e^{-i\omega t} \end{cases}$$



$$k_z = \sqrt{\omega^2 - \omega_c^2}$$

ω_c = cutoff frequency (= π/a)

$\vec{E}^{(0)}$ and $\vec{B}^{(0)}$ = EM fields in flat background (no GW)

TE MODES
IN A GRAVITATIONAL WAVE BACKGROUND

- GWs come with two polarization states, plus (+) and cross (x).
- For sake of simplicity, let's assume a single polarization state, plus (+)
- For GW propagating along z axis, the line element reads (in the TT gauge)

$$ds^2 = -dt^2 + (1+h)dx^2 + (1-h)dy^2 + dz^2$$

- We may also assume

$$h \equiv h(t-z) = He^{ik_g z}e^{-i\omega_g t}$$

where $|H| \ll 1$ is the gravitational strain and $k_g = \omega_g$.

SOLVING MAXWELL EQUATIONS

- Let us introduce the following Orthonormal Tetrad,

$$e^{\mu}_{\ \hat{0}} = 1, \qquad e^{\mu}_{\ \hat{1}} = 1 - \frac{1}{2}h, \qquad e^{\mu}_{\ \hat{2}} = 1 + \frac{1}{2}h, \qquad e^{\mu}_{\ \hat{3}} = 1$$

- Such a tetrad is **Fermi-Walker transported**, hence representing the **best approximation to an inertial reference frame** in a curved background.
- Maxwell eqs. in a vacuum read

$$\nabla_{\nu} F^{\mu\nu} = 0,$$
$$\nabla_{\nu} \tilde{F}^{\mu\nu} = 0.$$

Hodge dual

- To the lowest order O(h):

$$\Rightarrow \quad \partial_{\nu} F^{\mu\nu} = 0,$$
$$\partial_{\nu} \tilde{F}^{\mu\nu} = 0.$$

For an antisymmetric tensor: $\nabla_{\nu} \equiv \left[\partial_{\nu} \left(\ln \sqrt{-g}\right) + \partial_{\nu}\right]$

Also: $-g = 1 + O(h^2)$

SOLVING MAXWELL EQUATIONS

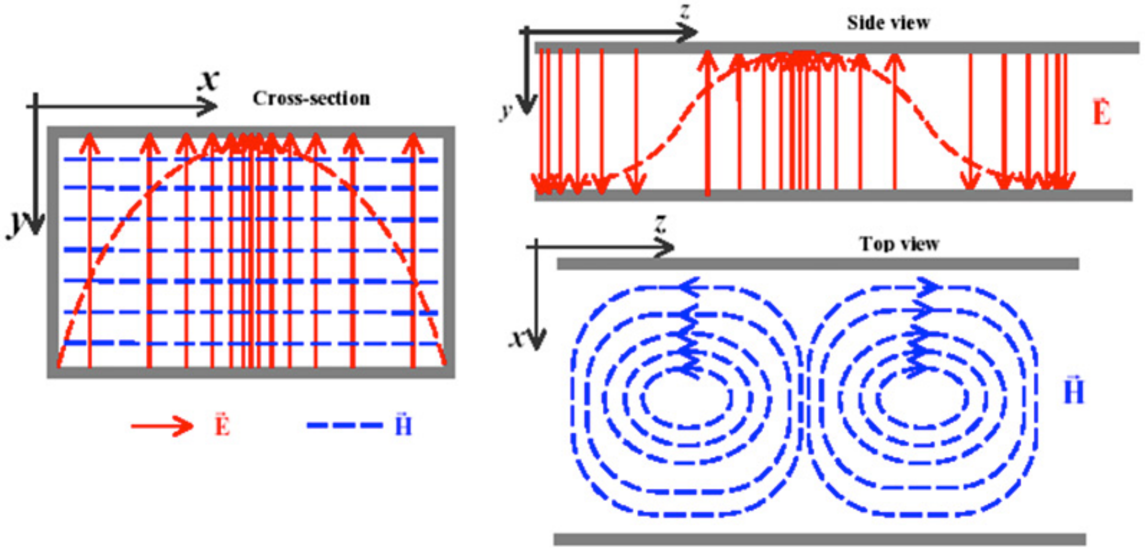
- Project the Faraday tensor onto the tetrad:

$$F^{\mu\nu} = e^{\mu}_{\ \hat{a}} F^{\hat{a}\hat{b}} e^{\nu}_{\ \hat{b}}$$

(A similar relation holds for $\tilde{F}^{\mu\nu}$)

- Taking into accout the waveguide geometry,
  we **assume a preexisting electromagnetic TE$_{10}$ mode (n=1)**

$$\begin{cases} B_x^{(0)} = -\dfrac{ik_z a}{\pi} B_0 \sin\left(\dfrac{\pi x}{a}\right) e^{ik_z z} e^{-i\omega t} \\ B_z^{(0)} = B_0 \cos\left(\dfrac{\pi x}{a}\right) e^{ik_z z} e^{-i\omega t} \\ E_y^{(0)} = \dfrac{i\omega a}{\pi} B_0 \sin\left(\dfrac{\pi x}{a}\right) e^{ik_z z} e^{-i\omega t} \end{cases}$$



SOLVING MAXWELL EQUATIONS

- We expect that (to the lowest O(h) order) the GW will excite only modes involving E_x, B_y and B_z . So let us write:

$$\begin{cases} \vec{E} = \vec{E}^{(0)} + \vec{E}^{(1)} \\ \vec{B} = \vec{B}^{(0)} + \vec{B}^{(1)} \end{cases}$$

gravitationally induced perturbations

- Using Maxwell eqs. and projecting onto the ON tetrad we find:

$$\begin{cases} \partial_t\left[\left(1+\frac{h}{2}\right)E_y\right] + \partial_x B_z - \partial_z\left[\left(1+\frac{h}{2}\right)B_x\right] = 0 \\ \partial_t\left[\left(1-\frac{h}{2}\right)B_x\right] - \partial_z\left[\left(1-\frac{h}{2}\right)E_y\right] = 0 \\ \partial_t B_z + \left(1-\frac{h}{2}\right)\partial_x E_y = 0 \\ \partial_t B_z + \left(1-\frac{h}{2}\right)\partial_x B_x = 0 \end{cases}$$

See, e.g., «Radio wave emission due to gravitational radiation», Marklund M, Brodin G, and Dunsby P K S, Ap.J. **536**, 875 (2000)

SOLVING MAXWELL EQUATIONS

- Putting all things together we get the following **inhomogeneous** wave eqs.:

$$\Box E_y^{(1)} = -\left(E_y^{(0)} + B_x^{(0)}\right)\partial_t^2 h - h\partial_x^2 E_y^{(0)}$$

$$\Box B_x^{(1)} = \left(E_y^{(0)} + B_x^{(0)}\right)\partial_t^2 h$$

$$\Box B_z^{(1)} = (\partial_t h)\partial_x\left(E_y^{(0)} + B_x^{(0)}\right)$$

Notice: no wave propagation of the EM perturbation, in the case of a freely propagating background EM wave!

- These are the **evolution equations** for the EM perturbation due to the interaction with the GW.

SECOND HARMONIC GENERATION (SHG)

- Let us specialize previous results to the case of **VHFGW**, whose frequency is **the same** as the frequency of the TE_{10} EM mode:

$$\omega_g = \omega = k_g$$

- From $\Box E_y^{(1)} = -\left(E_y^{(0)} + B_x^{(0)}\right)\partial_t^2 h - h\partial_x^2 E_y^{(0)}$ we get:

$$\Box E_y^{(1)} = \frac{i\pi H B_0 \omega}{a}\left(\frac{2\omega_g + k_z}{\omega + k_z}\right)\sin\left(\frac{\pi x}{a}\right)e^{i(k_z+k_g)z}e^{-2i\omega t}$$

SECOND HARMONIC GENERATION (SHG)

- Look for a solution like: $E_y^{(1)} = A(z)\sin\left(\frac{\pi x}{a}\right)e^{ik'_z z}e^{-2i\omega t}$

where A(z) is a **slowly varying** function of z and $k_z{}' = \sqrt{(2\omega)^2 - \omega_c^2}$.

Cutoff frequency

- Neglecting $\partial_z^2 A,$ we get:

$$A(z) = -\frac{\pi H B_0 \omega L}{4ak'_z}\left(\frac{2\omega + k_z}{\omega + k_z}\right)\frac{e^{i(k_z+k_g-k'_z)z} - 1}{iL(k_z + k_g - k'_z)}$$

- where L is the waveguide length.

SECOND HARMONIC GENERATION (SHG)

- Evaluating A(z) at the waveguide end, *z = L*, we find

$$A(L) = -\frac{\pi H B_0 \omega L}{4ak'_z}\left(\frac{2\omega + k_z}{\omega + k_z}\right)e^{iL\Delta k/2}\mathrm{sinc}\left(\frac{L\Delta k}{2}\right)$$

$$\mathrm{sinc}(x) \equiv \frac{\sin x}{x}$$

$$\Delta k = k_z + k_g - k'_z$$

wavevector mismatch

- Using $\vec{E} = \vec{E}^{(0)} + \vec{E}^{(1)}$ the **total** electric field at the waveguide end is:

$$E_y(L) = \frac{i\omega a B_0}{\pi}\sin\left(\frac{\pi x}{a}\right)e^{ik_z L}e^{-i\omega t} - \left\{\frac{\pi H B_0 \omega L}{4ak'_z}\left(\frac{2\omega + k_z}{\omega + k_z}\right)e^{iL\Delta k/2} \times \mathrm{sinc}\left(\frac{L\Delta k}{2}\right)\sin\left(\frac{\pi x}{a}\right)e^{ik'_z L}e^{-2i\omega t}\right\}$$

background TE_{10} mode

GW-induced EM perturbation

SECOND HARMONIC GENERATION (SHG)

- Following the same steps we find the B_x component of the total magnetic field at the waveguide end:

$$B_x(L) = -\frac{ik_z a}{\pi} B_0 \sin\left(\frac{\pi x}{a}\right) e^{ik_z z} e^{-i\omega t} + \left\{ \frac{HB_0 aL\omega^2}{4\pi} \frac{\omega - k_z}{k'_z} e^{iL\Delta k/2} \times \operatorname{sinc}\left(\frac{L\Delta k}{2}\right) \sin\left(\frac{\pi x}{a}\right) e^{ik'_z L} e^{-2i\omega t} \right\}$$

background TE_{10} mode

GW-induced EM perturbation

- We see that the **induced** EM perturbation propagates along the waveguide at a frequency which is **twice** the frequency of the original TE_{10} mode.
- Such result is an example of a **non-linear effect** induced by the VHFGW on the EM wave propagating inside the waveguide.

SECOND HARMONIC GENERATION (SHG)

- **CONTRIBUTION TO THE POYNTING VECTOR**

According to the general definition: $\langle \vec{S} \rangle = \frac{1}{2} \Re \left(\vec{E} \times \vec{B}^* \right)$

mean time average

In the present case, the contribution due to the induced Second Harmonic EM wave at the waveguide end, *x* = *L*, is:

$$\langle \delta S_z \rangle = \frac{H^2 B_0^2}{32} \frac{\omega^3 (\omega - k_z)(2\omega + k_z)}{k_z'^2 (\omega + k_z)} \times L^2 \mathrm{sinc}^2 \left(\frac{L \Delta k}{2} \right) \sin^2 \left(\frac{\pi x}{a} \right)$$

SECOND HARMONIC GENERATION (SHG)

- The power carried by the Second Harmonic at the waveguide end can be obtained integrating all over the waveguide section, $\mathcal{S} = ab.$

$$\delta P = \int_{\mathcal{S}} \langle \delta S_z \rangle \, dx \, dy$$

The photon flux, namely the **number of photons created with frequency 2*f***, (i.e., **twice** the frequency *f* of the initial waveguide mode) is:

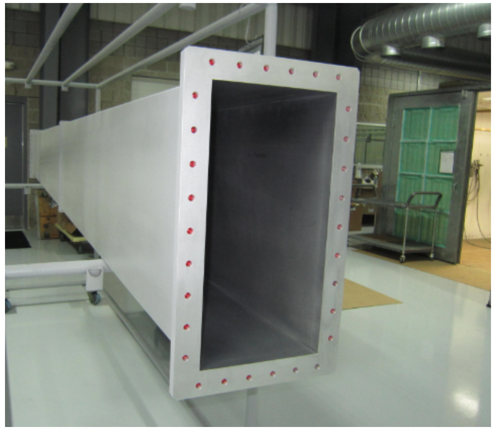
$$\frac{dN}{dt} = \frac{\delta P}{2\hbar\omega}$$

➢Notice that : $k_z = \sqrt{\omega^2 - \omega_c^2}$ and $k_z' = \sqrt{(2\omega)^2 - \omega_c^2}$,
so in general is **not possible** to make $\Delta k = 0$

Recall that the mismatch is:

$$\Delta k = k_z + k_g - k_z'$$

SHG – A NUMERICAL ESTIMATE



Assume:

- GW with $f = 10$ GHz
- waveguide WR102 (EIA) with parameters:
  - $a = 25.9$ mm, $b = 12.95$ mm, $f_c = 5.786$ GHz, $B_0 = 0.003$ T

Then (in SI units):

$$\frac{dN}{dt} \simeq 3\times10^{-4}\frac{H^2 B_0^2 c^3}{\mu_0 h f f_c^2}\left(\frac{\sin\left(\frac{\pi f L}{c}\Delta\Phi(f_c/f)\right)}{\Delta\Phi(f_c/f)}\right)^2 \quad \text{with} \quad \Delta\Phi(f_c/f) = \frac{c\Delta k}{\omega} = \sqrt{1-\left(\frac{f_c}{f}\right)^2} - 2\sqrt{1-\left(\frac{f_c}{2f}\right)^2} + 1$$

- The **Maximum dN/dt value**

$$\frac{dN}{dt} \simeq (8.6\times10^{27})H^2$$

is obtained for optimal waveguide length:

$$L_{\text{opt}} = \frac{c}{2f|\Delta\Phi|} \simeq 0.15 \text{ m}$$

SHG – A NUMERICAL ESTIMATE

- Compare with the **blackbody noise** from waveguide cavity:

➢ cryogenic temperature operation T = 4 K

➢ narrow **detection bandwidth** $\Delta f \simeq 100\ \mathrm{kHz}$

The **background** is $N_{\mathrm{bg}} = \frac{8\pi abL}{c^3} \frac{(2f)^2 \Delta f}{\exp\left(\frac{2hf}{kT}\right) - 1} \simeq 7 \times 10^{-3}$ (*photons*)

➢ The **integration time** required to detect a number of photons at least **equal to the noise background** is

$$\tau_{\mathrm{eq}} = \frac{N_{\mathrm{bg}}}{dN/dt} = \frac{8.1 \times 10^{-31}}{H^2}$$

SHG – A NUMERICAL ESTIMATE

- Assuming an **integration time = 1 hour**, the gravitational strain detection **threshold** is

$$H_{\rm th} = \frac{9 \times 10^{-16}}{\sqrt{\tau_{eq}}} \simeq 1.5 \times 10^{-17}$$

- The above sensitivity could perhaps be **improved** using a high-finesse **Fabry-Perot resonator** at the waveguide output.

- With a finesse F = 10^5, one woud reach $H_{\rm th} \simeq 4.7 \times 10^{-20}$

  @ f = 10 GHz with 1 hour integration time.

CONCLUSIONS

- An interesting feature of the effect is that the produced photons have a frequency **doubled** w.r.t. that of photons originally propagating in the waveguide. Hence such photons could be easily discriminated.
- Unfortunately, also when assuming optimistic values of the involved parameters, the number of such photons is indeed very small.
- A possible amplification could be obtained using a high-finesse Fabry-Perot cavity. But also imagining a $F = 10^9$ value, only a few photons could be obtained in any reasonable integration time.
- Nevertheless, the effect seems interesting in its own right, representing a non-trivial example of **non-linear interaction** between quantum fields (EM) and VHFGWs.
- All this also suggests that any promising route towards a quantum theory of gravity will probably need to face VHFGWs, searching for **graviton evidence**.

THANK YOU