Lévy α-stable model for the non-exponential low-|t| proton-proton differential cross section

Lévy α-stable generalization of the Bialas-Bzdak model

based on

Universe **2023**, *9*(8), 361 arXiv:2308.05000

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Preliminaries: ReBB model analysis of pp and $p\bar{p}$ data

- **the Real extended Bialas-Bzdak (ReBB) model describes elastic pp and $p\bar{p}$ $d\sigma/dt$ data in a statistically acceptable way (CL≥0.1%) in the kinematic region:**

$$546\ \mathrm{GeV} \leq \sqrt{s} \leq 8\ \mathrm{TeV}$$

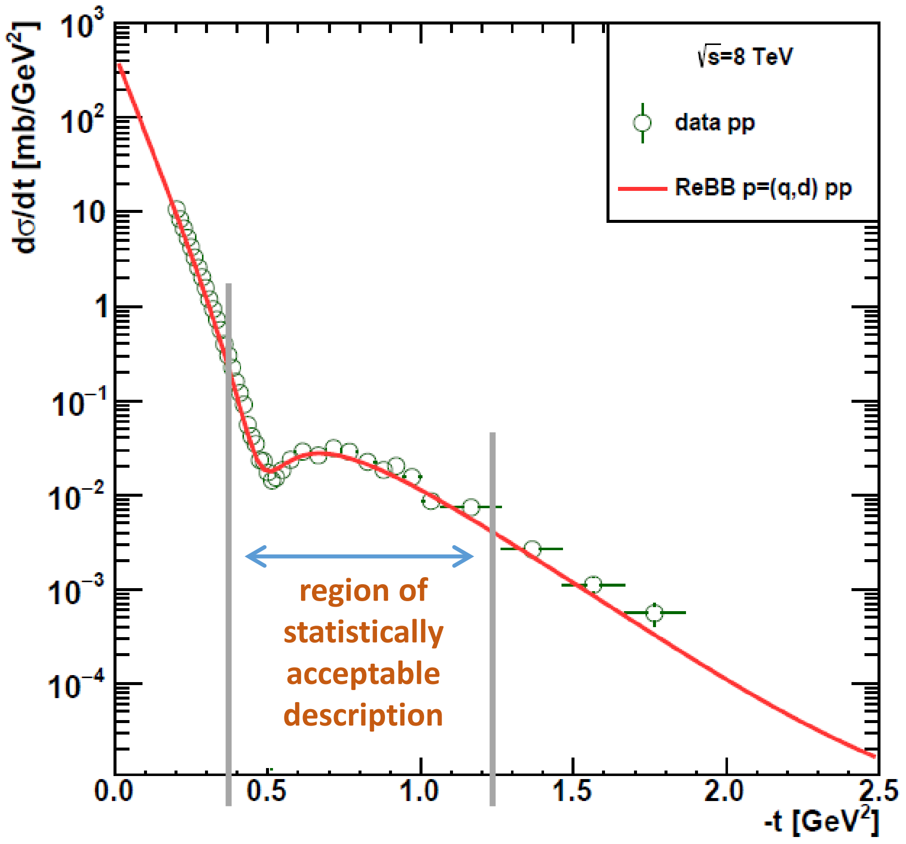
$$0.37\ \mathrm{GeV}^2 \leq -t \leq 1.2\ \mathrm{GeV}^2$$

- significant model dependent odderon signal observation

- main goal: **to improve the ReBB model to have a satistically acceptable (CL≥0.1%) description to elastic pp and $p\bar{p}$ data in a wider kinematic range**

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* ***81***, *611 (2021)*

I. Szanyi, T. Csörgő, *Eur. Phys. J. C* ***82***, *827 (2022)*



ReBB model description to the 8 TeV pp data

Unitarity and the elastic scattering amplitude

- the S-matrix is unitary expressing the conservation of probability

$$SS^{\dagger} = I$$

- **the unitaritiy constraint can be rewritten in impact parameter ($\vec{b}$) representation**

$$2\,\mathcal{Im}\,t_{el}(s,\vec{b}) = |t_{el}(s,\vec{b})|^2 + \tilde{\sigma}_{in}(s,\vec{b})$$ ($\sqrt{s}$ is the CM energy)

- **the elastic scattering amplitude $t_{el}(s,\vec{b})$ is a solution of the unitarity equation and written in terms of the inelastic cross section $\tilde{\sigma}_{in}(s,\vec{b})$**

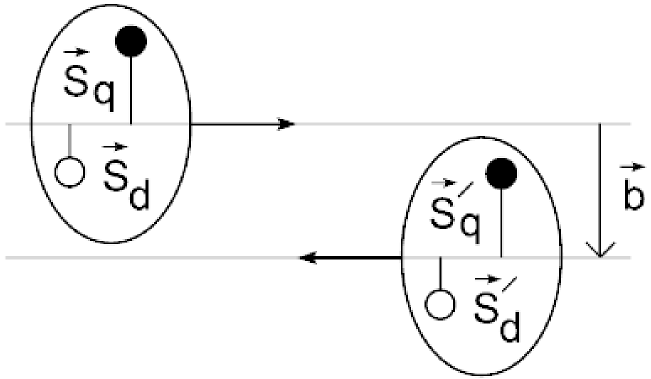
$$0 \leq \tilde{\sigma}_{in}(s,\vec{b}) \leq 1$$

- **at a given energy $\tilde{\sigma}_{in}(s,\vec{b})$ is the probability of inelastic scattering as fuction of $\vec{b}$ and can be calculated with the methods of probability calculus based on R. J. Glauber's multiple diffractive scattering theory**

The Bialas-Bzdak (BB) p=(q,d) model

A. Bialas, A. Bzdak, *Acta Phys.Polon. B 38, 159-168 (2007)*

- **in the Bialas-Bzdak (BB) p=(q,d) model the proton is a bound state of a constiuent quark and constituent a diquark**
- **the probability of inelastic scattering of protons as a function of transverse positions of quarks and diquarks inside the protons ($\vec{s}_q, \vec{s}_d, \vec{s}_q', \vec{s}_d'$) and the impact parameter ($\vec{b}$) at given energy is given by a Glauber expanasion**



Proton-proton collision in the quark-diquark model

$$\sigma(\vec{s}_q,\vec{s}_d;\vec{s}_q',\vec{s}_d';\vec{b}) = 1-\left[1-\sigma_{qq}(\vec{s}_q,\vec{s}_q';\vec{b})\right]\left[1-\sigma_{qd}\left(\vec{s}_q,\vec{s}_d';\vec{b}\right)\right]\times \times\left[1-\sigma_{dq}(\vec{s}_d,\vec{s}_q';\vec{b})\right]\left[1-\sigma_{dd}\left(\vec{s}_d,\vec{s}_d';\vec{b}\right)\right]$$

$\sigma_{ab}\left(\vec{s}_a,\vec{s}_b';\vec{b}\right)$ is the inelastic differential cross section for the collision of two constituents determined by the parton distributions of the constituents

$$\tilde{\sigma}_{in}(\vec{b}) = \int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty} d^2s_q d^2s_q' d^2s_d d^2s_d' \, D(\vec{s}_q,\vec{s}_d)\, D(\vec{s}_q',\vec{s}_d')\, \sigma(\vec{s}_q,\vec{s}_d;\vec{s}_q',\vec{s}_d';\vec{b})$$

$D\left(\vec{s}_q,\vec{s}_d\right)$ is the distribution of the quark-diquark distance inside a proton

Real extended Bialas-Bzdak (ReBB) model

F. Nemes, T. Csörgő, M. Csanád, *Int. J. Mod. Phys. A Vol. 30, 1550076 (2015)*

- **the elastic scattering amplitude was extended with a real part respecting unitarity**

$$\tilde{t}_{el}(s,\vec{b}) = i\left(1-\sqrt{1-\tilde{\sigma}_{in}(s,\vec{b})}\right) \quad \longrightarrow \quad \tilde{t}_{el}(s,\vec{b}) = i\left(1-e^{i\,\alpha_R\,\tilde{\sigma}_{in}(s,\vec{b})}\sqrt{1-\tilde{\sigma}_{in}(s,\vec{b})}\right)$$

new free parameter (α_R)

- **statistically acceptable description (CL≥0.1%) to the elastic pp and $p\bar{p}$ $d\sigma/dt$ in the kinematic region $0.546\ \mathrm{TeV} \leq \sqrt{s} \leq 8\ \mathrm{TeV}$ & $0.37\mathrm{GeV}^2 \leq -t \leq 1.2\ \mathrm{GeV}^2$**

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* ***81****, 611 (2021)*

I. Szanyi, T. Csörgő, *Eur. Phys. J. C* ***82****, 827 (2022)*

- the strong non-exponential low-|t| pp dσ/dt measured by TOTEM at LHC and earlier efficient modelling with Lévy α-stable distribution motivates the Lévy α-stable generalization of the BB model for having a statistically acceptable descriptions in a wider kinematic range

G. Antchev et al. (TOTEM Collab.), *Nucl. Phys. B, 899, 527 (2015)*

G. Antchev et al. (TOTEM Collab.), *Eur. Phys. J. C 79, 861 (2019)*

T. Csörgő, R. Pasechnik, A. Ster, *Eur. Phys. J. C 79, 62 (2019)*

Lévy α-stable generalized Bialas-Bzdak (LBB) model

- **the parton distributions of the constituent quark and diquark are Levy α-stable distributions and the inelastic differential cross section for the collision of two constiuents is:**

$$\sigma_{ab}(\vec{x}) = A_{ab}\pi S_{ab}^2 \int d^2 r_a L(\vec{r}_a|\alpha_L, R_a/2) L(\vec{x}-\vec{r}_a|\alpha_L, R_b/2) \equiv A_{ab}\pi S_{ab}^2 L(\vec{x}|\alpha_L, S_{ab}/2)$$

$$S_{ab}^{\alpha_L} = R_a^{\alpha_L} + R_b^{\alpha_L}$$

another new free parameter: α_L

- **the quark-diquark realtive distance has a Levy α-stabil distribution**:

$$D(\vec{s}_q, \vec{s}_d) = (1+\lambda)^2 L(\vec{s}_q - \vec{s}_d|R_{qd}/2)\delta^2(\vec{s}_q + \lambda\vec{s}_d)$$

$$\int d^2 s_q d^2 s_d D(\vec{s}_q, \vec{s}_d) = 1$$

$$L(\vec{x}|\alpha_L, R_L) \equiv L(\vec{x}|\beta = 0, \vec{\delta} = 0, \alpha_L, R_L) = \frac{1}{(2\pi)^2}\int d^2 q\, e^{i\vec{q}^T\vec{x}} e^{-|\vec{q}^2 R_L^2|^{\alpha_L/2}}$$

$$L(\vec{x}|\alpha_L = 2, R_L = R_G/\sqrt{2}) \equiv G(\vec{x}|R_G) = \frac{1}{2\pi R_G^2} e^{-\frac{\vec{x}^2}{2R_G^2}}$$

Difficulties with LBB model fits to the data

- difficulties with the calculation of integrals of products of Lévy α-stable distributions

- calculation is easy only if the integral can be written in a convolution form as in case of the leading order terms in $\tilde{\sigma}_{in}(s,\vec{b})$

- since multivariate Lévy α-stable distributions have forms in terms of special functions, it is hard to perform a numerical fitting procedure

- a relatively high computing capacity or improved analytic insight is needed to proceed

- **quick solution:** approximations that are valid at low –t (in the domain where the original ReBB model had difficulties to describe the strongly non-exponential features of the experimental data)

Simple Lévy α-stable model for low-|t| pp $d\sigma/dt$

- low-$|t|$ scattering corresponds to high-b scattering and at high b values $\tilde{\sigma}_{in}(s,b)$ is small
- leading order term in the Taylor expansion of the amplitude in $\tilde{\sigma}_{in}(s,b)$ dominates at low $-t$ values if α_R is small too

$$\tilde{t}_{el}(s,\vec{b}) = i\left(1 - e^{i\,\alpha_R(s)\,\tilde{\sigma}_{in}(s,\vec{b})}\sqrt{1-\tilde{\sigma}_{in}(s,\vec{b})}\right) \rightarrow \tilde{t}_{el}(s,\vec{b}) = \left(\alpha_R(s) + \frac{i}{2}\right)\tilde{\sigma}_{in}(s,\vec{b})$$

- motivated by the fact that the leading order terms in $\tilde{\sigma}_{in}(s,\vec{b})$ have Lévy α-stable shapes in the LBB model, $\tilde{\sigma}_{in}(s,\vec{b})$ is approximated with a single Lévy α-stable shape

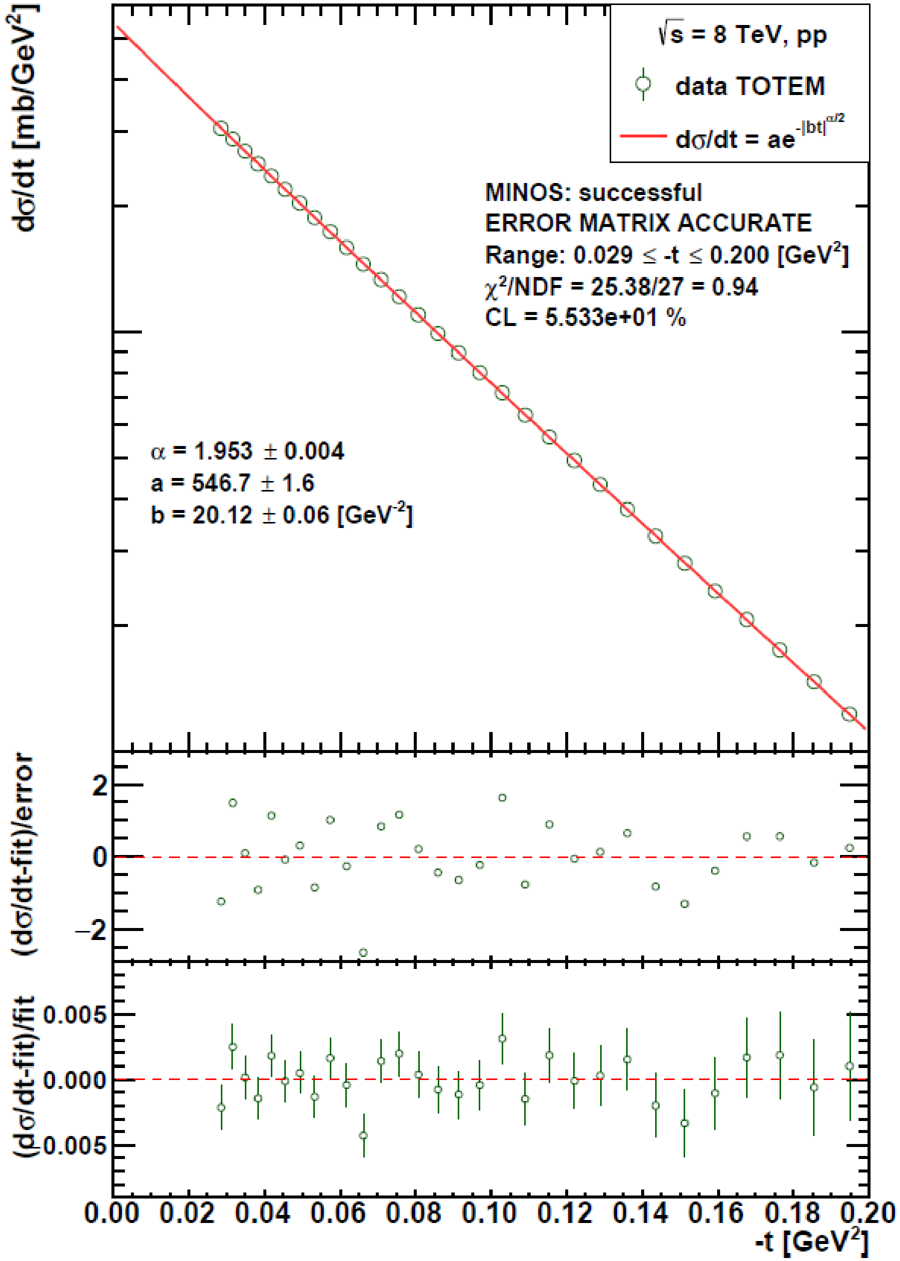
$$\tilde{\sigma}_{in}(s,\vec{b}) = \tilde{c}(s)L\left(\vec{b}|\alpha_L(s),r(s)\right)$$

- **a simple Lévy α-stable model model for low-|t| pp $d\sigma/dt$ arises**

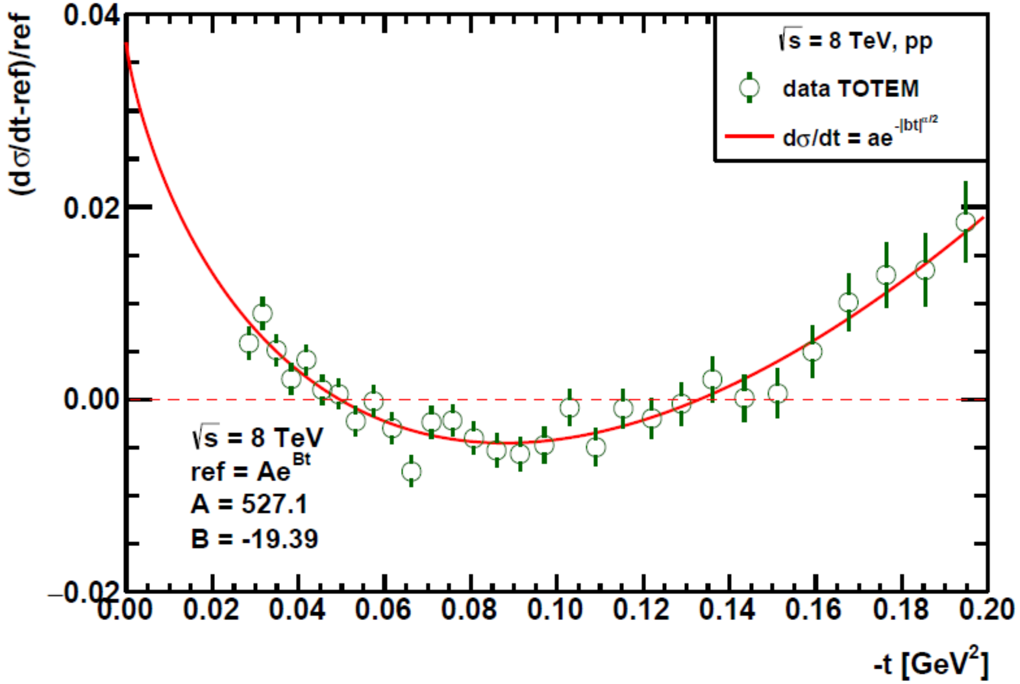
$$\frac{d\sigma}{dt}(s,t) = a(s)e^{-|tb(s)|^{\alpha_L(s)/2}}$$

- the model has three adjustable parameters, α_L, a, and b, to be determined at a given energy

Simple Lévy α-stable model and the data



T. Csörgő, S. Hegyi, I. Szanyi, *Universe 2023, 9(8), 361*



- the non-exponential Lévy α-stable model with α_L = 1.953 $\pm$ 0.004 represents the LHC TOTEM $\sqrt{s} = 8$ TeV low-$|t|$ differential cross section data with a confidence level of 55% (publieshed)
- similarly good description is obtained to all the LHC data on low-|t| pp $\boldsymbol{d\sigma/dt}$ (yet unpublished)

Simple Lévy α-stable & LBB model parameters

- parameters of the simple Levy α-stable model and the measurable quantites at $t \to 0$ can be appriximately expressed in terms of the parameters of the LBB model
- only leading order terms in $\tilde{\sigma}_{in}(s,\vec{b})$ are considered; $A_{qq} = 1$ and $\lambda = 1/2$ are fixed

$$a(s) = \frac{81}{16}\pi\left(2R_q^{\alpha_L(s)}(s)\right)^{4/\alpha_L}\left(1+\rho_0^2(s)\right)$$

$$b(s) = \frac{1}{36}\left(\frac{4}{3}\right)^{2/\alpha_L(s)}\left(\left(2+2^{\alpha_L(s)}\right)R_{qd}^{\alpha_L(s)}(s) + 3^{\alpha_L(s)}\left(2R_d^{\alpha_L(s)}(s) + R_q^{\alpha_L(s)}(s)\right)\right)^{2/\alpha_L(s)}$$

(obtained after a Taylor expansion in $t^{\alpha_L/2}$)

$$\sigma_{tot}(s) = 9\pi\left(2R_q^{\alpha_L(s)}(s)\right)^{2/\alpha_L(s)}$$

$$\rho_0(s) = 2\alpha_R(s)$$

$$\sigma_{el}(s) = \frac{a(s)}{b(s)}\Gamma\left(\frac{2+\alpha_L(s)}{\alpha_L(s)}\right)$$

- According to the analysis of elastic pp and p$\bar{\text{p}}$ data in the energy region $0.5\ \text{TeV} \leq \sqrt{s} \leq 13\ \text{TeV}$ only α_R is different for pp and p$\bar{\text{p}}$ scattering
- difference between pp and p$\bar{\text{p}}$ scattering could be seen in the data on $d\sigma/dt$, ρ_0, a (optical point), and σ_{el})

Summary

- The formal Lévy α-stable generalization of the Bialas-Bzdak model is done, the $\alpha_L = 2$ limit corresponds to the original model

- solution of difficult and complex technical (mathamatical and computational) problems is needed to fit the experimental data with the generalized model

- based on approximations a highly simplified Levy α-stable model of the pp (and $\mathrm{p\bar{p}}$) differential cross section is deduced and successfully fitted to the data in the low-|t| region

- the results indicate promising prospect for the future utility of the Lévy α-stable generalized real extended Bialas-Bzdak (LBB) model in describing experimental data

Thank you for your attention!

SUPPORTED BY:
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Backup slides

Inelastic pp collision probability

A. Bialas, A. Bzdak, *Acta Phys.Polon. B 38, 159-168 (2007)*

- **the proton-proton inelastic collision probability as a function of $\vec{b}$ at a given energy is calculated using the rules of probability calculus:**

$$\tilde{\sigma}_{in}(\vec{b}) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} d^2 s_q d^2 s'_q d^2 s_d d^2 s'_d \; D(\vec{s}_q, \vec{s}_d) \, D(\vec{s}'_q, \vec{s}'_d) \, \sigma(\vec{s}_q, \vec{s}_d; \vec{s}'_q, \vec{s}'_d; \vec{b})$$

($D(\vec{s}_q, \vec{s}_d)$ is the distribution of the quark-diquark distance inside a proton)

- **$\tilde{\sigma}_{in}(\vec{b})$ can be written as sum of 11 different terms**

$$\tilde{\sigma}_{in}(\vec{b}) = \tilde{\sigma}_{in}^{qq}(\vec{b}) + 2\tilde{\sigma}_{in}^{qd}(\vec{b}) + \tilde{\sigma}_{in}^{dd}(\vec{b}) - [2\tilde{\sigma}_{in}^{qq,qd}(\vec{b}) + \tilde{\sigma}_{in}^{qd,dq}(\vec{b}) + \tilde{\sigma}_{in}^{qq,dd}(\vec{b}) + 2\tilde{\sigma}_{in}^{qd,dd}(\vec{b})]$$
$$+[\tilde{\sigma}_{in}^{qq,qd,dq}(\vec{b}) + 2\tilde{\sigma}_{in}^{qq,qd,dd}(\vec{b}) + \tilde{\sigma}_{in}^{dd,qd,dq}(\vec{b})] - \tilde{\sigma}_{in}^{qq,qd,dq,dd}(\vec{b})$$

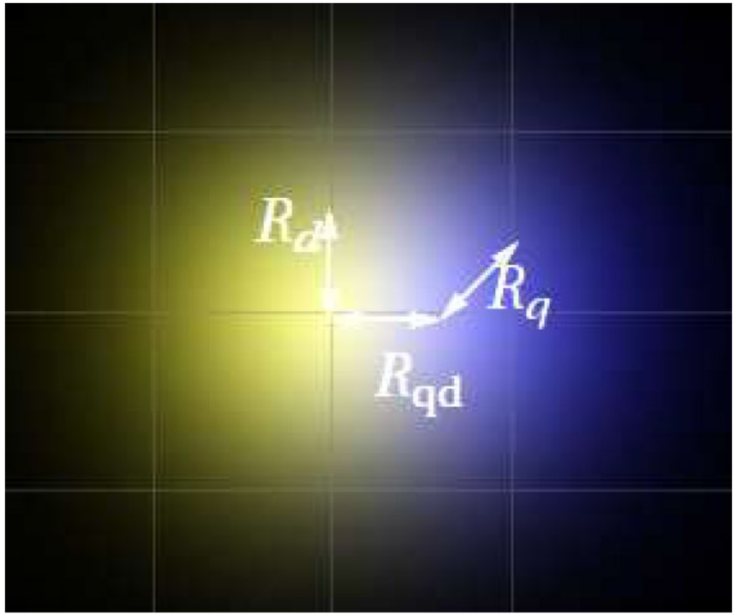
Inelastic constituent-constituent collisons

- **the inelastic differential cross section for the collision of two constituents can be written in terms of a convolution of their parton distributions**
- **in the original BB model the parton distrubutions of the constiuents are Gausian distributions**

$$G(\vec{x}|R_G) = \frac{1}{2\pi R_G^2} e^{-\frac{\vec{x}^2}{2R_G^2}}$$

$$\sigma_{ab}(\vec{x}) = A_{ab}\pi S_{ab}^2 \int d^2 r_a G(\vec{r}_a|R_a/\sqrt{2}) G(\vec{x}-\vec{r}_a|R_b/\sqrt{2}) \equiv A_{ab}\pi S_{ab}^2 G(\vec{x}|S_{ab}/\sqrt{2})$$

$$\vec{x} = \vec{b} + \vec{s}_b' - \vec{s}_a \qquad S_{ab}^2 = R_a^2 + R_b^2 \qquad a, b \in \{q,d\}$$



The picture of the proton in the quark-diquark model

- if the integrated cross sections $\sigma_{ab}^{int} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sigma_{ab}(\vec{x})\, d^2x$ have the ratio $\sigma_{qq}^{int} : \sigma_{qd}^{int} : \sigma_{dd}^{int} = 1 : 2 : 4$ expressing the idea that the diquark contains twice as many partons than the quark and the colliding constituents do not shadow each other the number of free paramters can be reduced by two

Free parameters by now:
R_q, R_d, A_{qq}

The quark-diquark distance

- **in the original BB model the the distribution of the quark-diquark distance follows Gaussian distribution**

$$G(\vec{x}|R_G) = \frac{1}{2\pi R_G^2} e^{-\frac{\vec{x}^2}{2R_G^2}}$$

- **considering the relative distence between** the quark and diquark ($\vec{s}_q - \vec{s}_d$) one can write $D(\vec{s}_q, \vec{s}_d)$ in terms of a single Gaussian distribution:

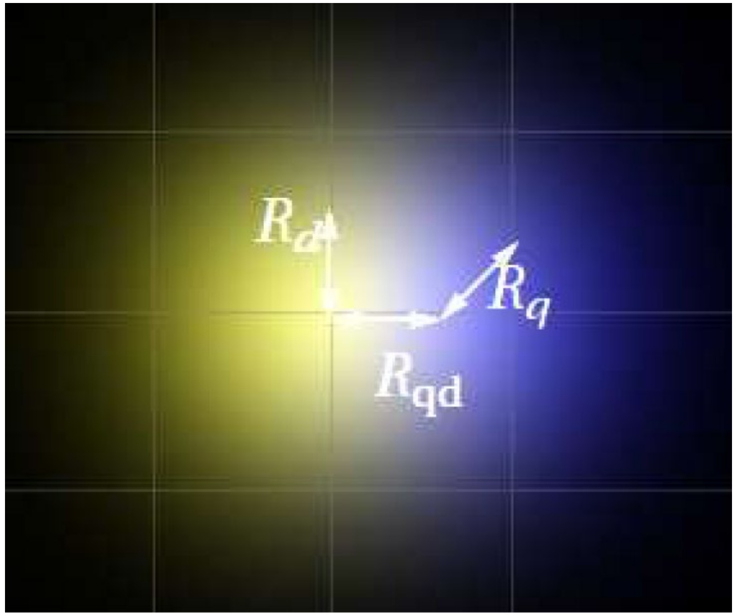
$$D(\vec{s}_q, \vec{s}_d) = (1+\lambda)^2 G(\vec{s}_q - \vec{s}_d | R_{qd}/\sqrt{2}) \delta^2(\vec{s}_q + \lambda \vec{s}_d)$$

$$\lambda = m_q/m_d$$

- the Dirac δ fixes the center of the mass of the proton making the calculations easier

- $D(\vec{s}_q, \vec{s}_d)$ is normalized as $\int d^2 s_q d^2 s_d D(\vec{s}_q, \vec{s}_d) = 1$

$(A_{qq} = 1 \; and \; \lambda = 1/2 \; can \; be \; fixed)$



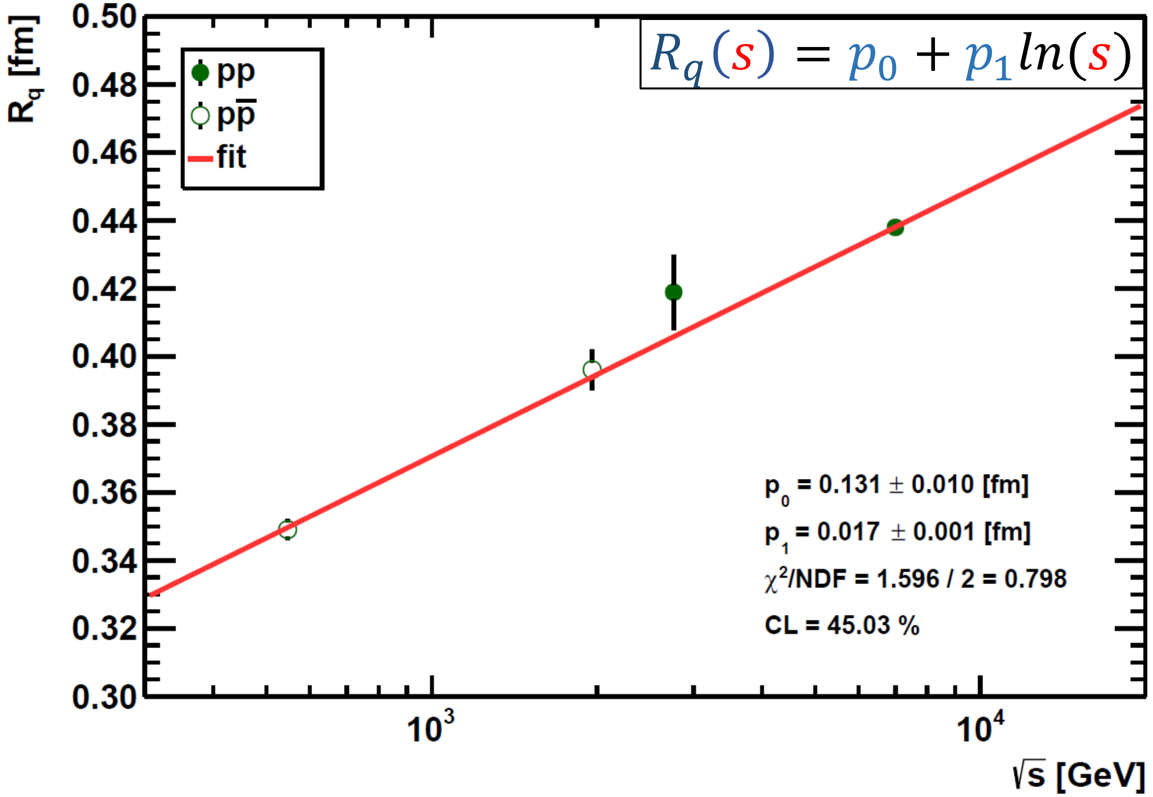
The picture of the proton in the quark-diquark model

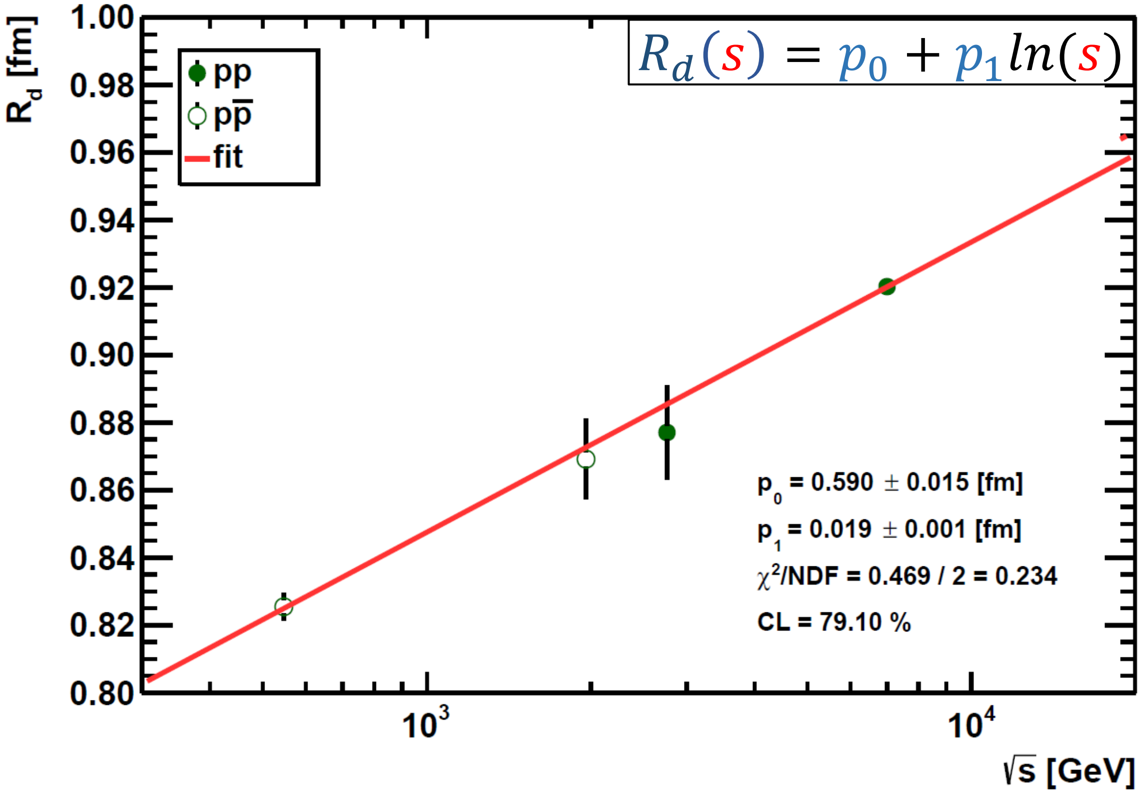
Free parameters by now: $R_q, R_d, A_{qq}, R_{qd}, \lambda$

F. Nemes, T. Csörgő, M. Csanád, Int. J. Mod. Phys. A Vol. 30, 1550076 (2015)

Energy dependences of the ReBB model parameters

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* 81, 611 (2021)



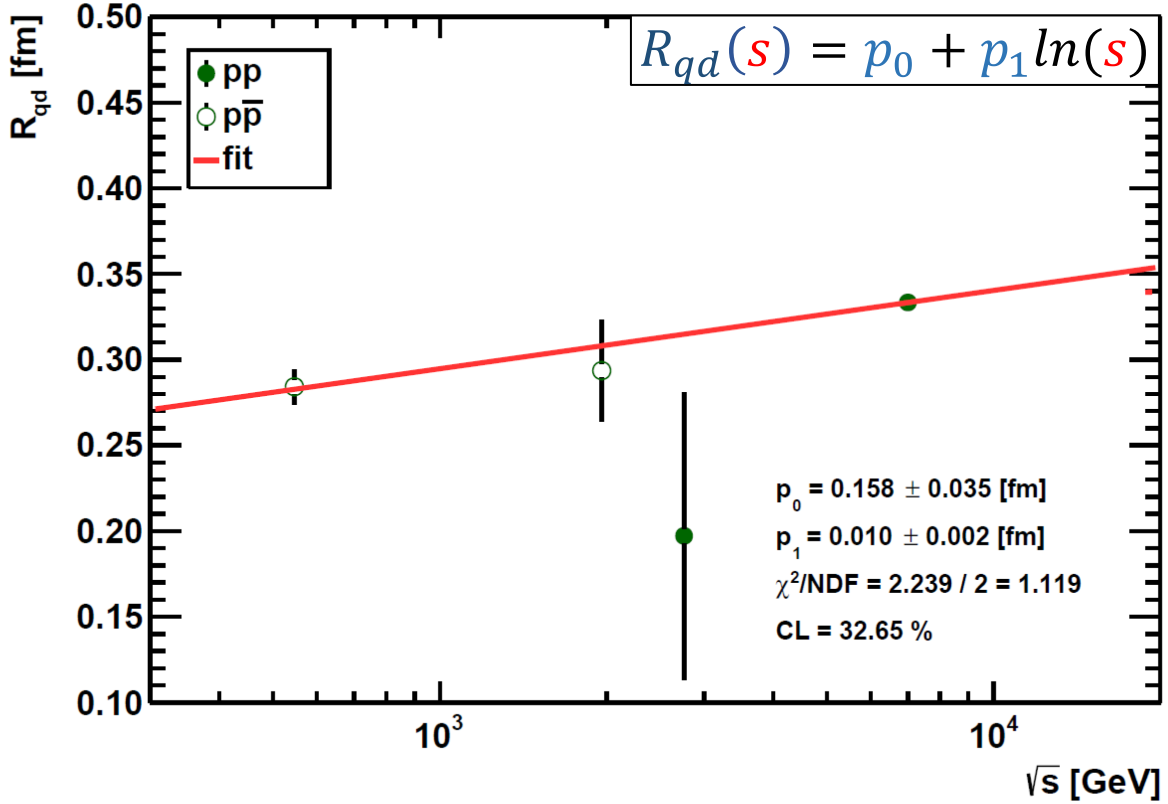


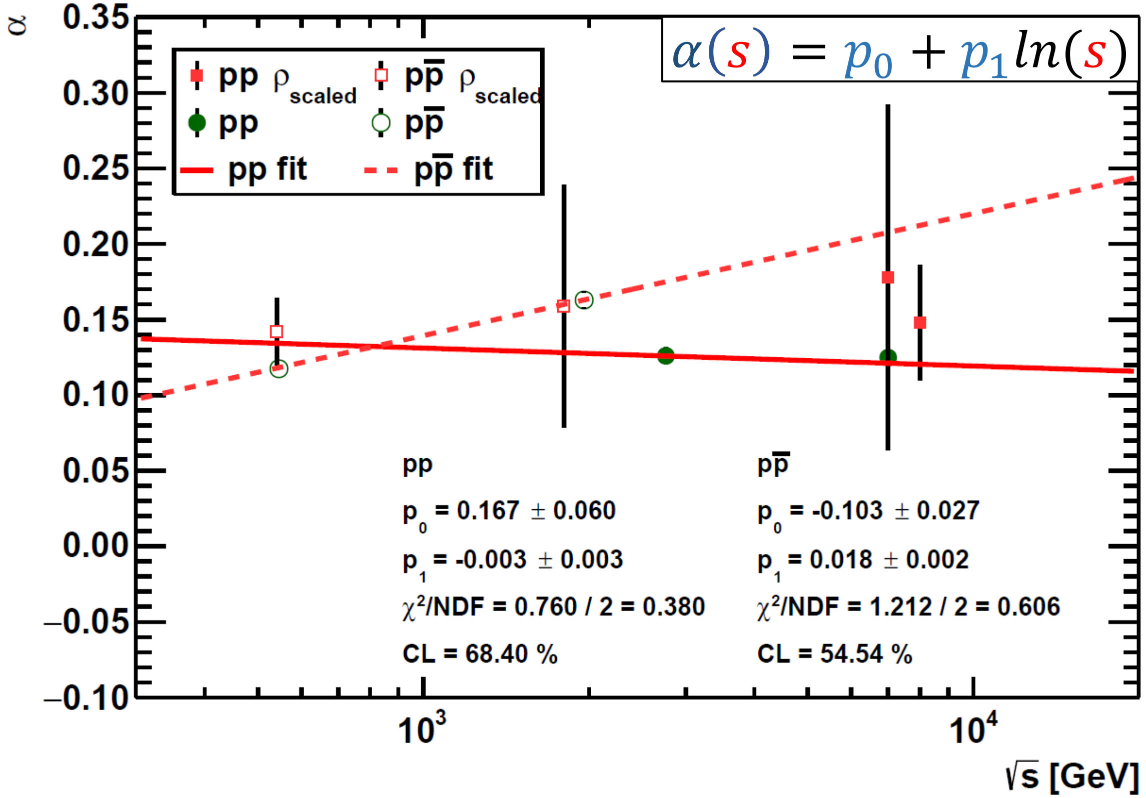
The energy dependences of the scale parameters, $R_q(s)$, $R_d(s)$, and $R_{qd}(s)$ are linear logarithmic and the same for pp and pp̄ processes!

The energy dependence of the α parameter, $\alpha(s)$ is linear logarithmic too, but not the same for pp and pp̄ processes!

Energy dependences of the ReBB model parameters

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* 81, 611 (2021)





The energy dependences of the scale parameters, $R_q(s)$, $R_d(s)$, and $R_{qd}(s)$ are linear logarithmic and the same for pp and pp̄ processes!

The energy dependence of the α parameter, $\alpha(s)$ is linear logarithmic too, but not the same for pp and pp̄ processes!

Odderon observation within the ReBB model analysis

I. Szanyi, T. Csörgő, Eur. Phys. J. C 82, 827 (2022)

$> 6\,\sigma$ combined Odderon signal from the data-model comparison at the two lowest energies, 1.96 & 2.76 TeV

$> 30\,\sigma$ combined Odderon signal from the data-model comparison at all the four energies, 1.96, 2.76, 7 & 8 TeV

$\sqrt{s}$ of combined data (TeV)	χ^2	NDF	CL	combined significance (σ) χ^2/NDF method	combined significance (σ) Stouffner method
1.96 & 2.76	124.63	36	1.0688×10^{-11}	6.79	6.3
1.96 & 2.76 & 7	2936.09	94	$< 9.1328\times 10^{-312}$	>37.7	>26.9
1.96 & 2.76 & 8	551.183	61	4.6307×10^{-80}	>18.9	>15.7
1.96 & 2.76 & 7 & 8	3362.64	119	$<8.0654\times 10^{-312}$	>37.7	>32.4

- **combination of significances by summing the individual χ^2 and NDF values:**

$$\chi^2 = \sum_i \chi_i^2$$

$$NDF = \sum_i NDF_i$$

- **combination of significances s_i by Stouffner method:**

$$s = \frac{\sum_{i=1}^{N} s_i}{\sqrt{N}}$$

Analytical approximation for significance calculation

I. Szanyi, T. Csörgő, Eur. Phys. J. C 82, 827 (2022)

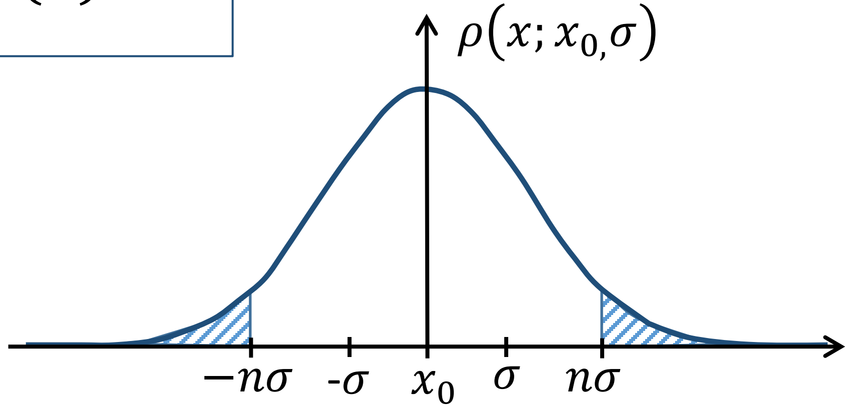
- **the Gaussian probability density function with mean x_0 and and variance σ^2 :**

$$\rho(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}$$

$$\int_{-\infty}^{\infty} dx\, \rho(x) = 1$$

- **the confidence level corresponding to $n\sigma$ significance:**

$$CL = 2\int_{x_0+n\sigma}^{\infty} dx\, \rho(x)$$



- **applying a variable change, $x \to x' = x - (x_0 + n\sigma)$:**

$$CL = \sqrt{\frac{2}{\pi\sigma^2}} \int_0^{\infty} dx'\, e^{-\frac{(x'+n\sigma)^2}{2\sigma^2}} = \sqrt{\frac{2}{\pi\sigma^2}}\, e^{-\frac{n^2}{2}} \int_0^{\infty} dx'\, e^{-\frac{x'^2+2x'n\sigma}{2\sigma^2}} \le \sqrt{\frac{2}{\pi\sigma^2}}\, e^{-\frac{n^2}{2}} \int_0^{\infty} dx'\, e^{-\frac{x'n}{\sigma}}$$

$$CL \le \sqrt{\frac{2}{\pi n^2}}\, e^{-\frac{n^2}{2}}$$

this formula gives the lower limit for the significance n in σ-s corresponding to a CL value

Proportionality between $\rho_0(s)$ and $\alpha(s)$

$$t_{el}(s,b) = i\left(1 - e^{i\,\alpha\,\tilde{\sigma}_{in}(s,b)}\sqrt{1-\tilde{\sigma}_{in}(s,b)}\right)$$

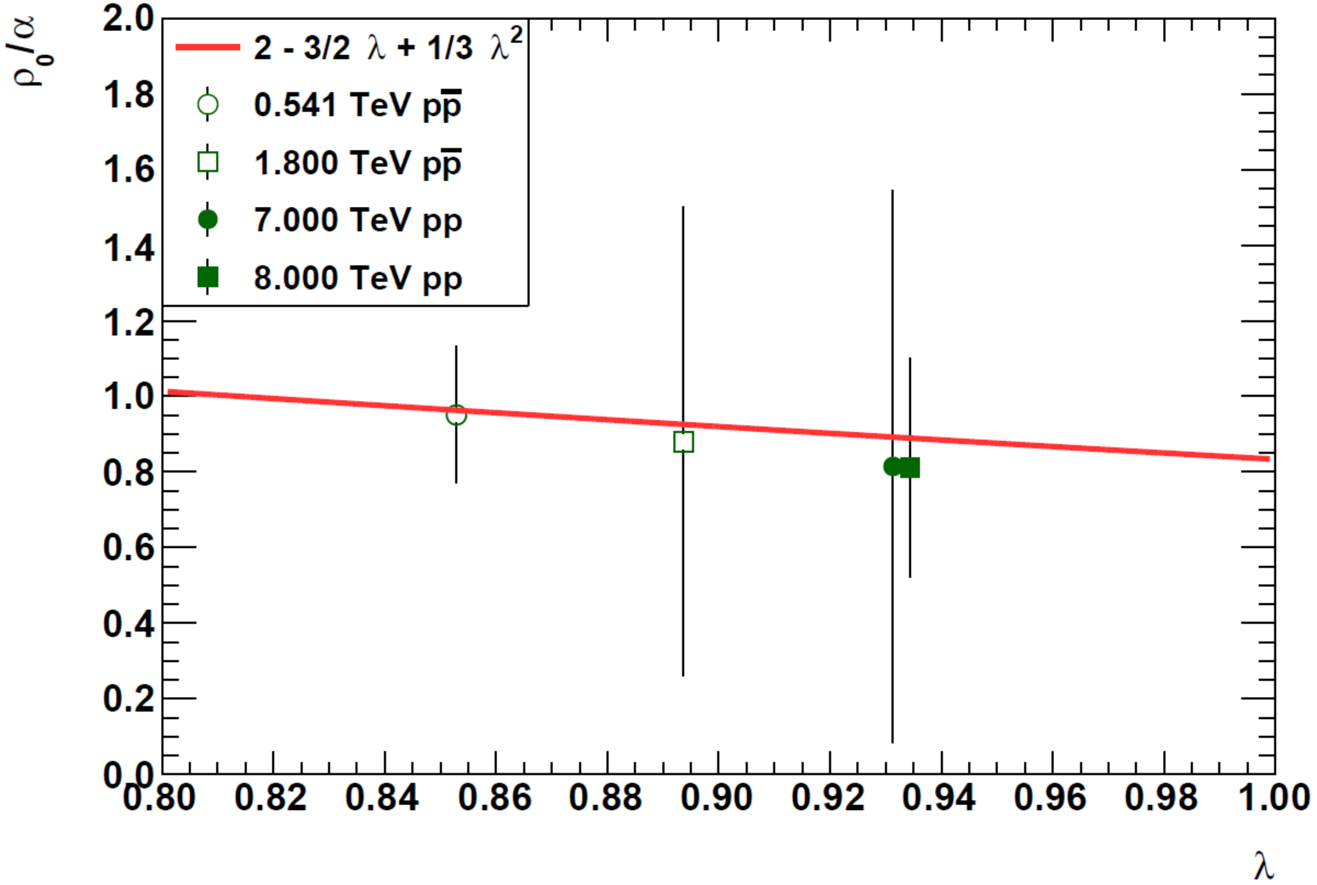
$$\alpha\tilde{\sigma}_{in} \ll 1$$

$$\mathrm{Im}\; t_{el}(s,b) \simeq \lambda(s)\exp\left(-\frac{b^2}{2R^2(s)}\right)$$

$$\rho_0(s) = \alpha(s)\left(2 - \frac{3}{2}\lambda(s) + \frac{1}{3}\lambda^2(s)\right)$$

$$\lambda(s) = \mathrm{Im}\; t_{el}(s, b=0)$$

→ by rescaling one can get additional α parameter values at energies where ρ_0 is measured (and vice versa)



The dependence of ρ_0/α on $\lambda = \mathrm{Im}\; t_{el}(s, b=0)$ in the TeV energy range. The data points are generated numerically by using the trends of the ReBB model scale parameters and the experimentally measured ρ-parameter values.

Measurable quantities

- **differential cross section:**

$$\frac{d\sigma}{dt}(s,t) = \frac{1}{4\pi}|T(s,t)|^2$$

- **total, elastic and inelastic cross sections:**

$$\sigma_{tot}(s) = 2ImT(s,t=0)$$

$$\sigma_{el}(s) = \int_{-\infty}^{0} \frac{d\sigma(s,t)}{dt} dt$$

$$\sigma_{in}(s) = \sigma_{tot}(s) - \sigma_{el}(s)$$

- **ratio ρ_0:**

$$\rho_0(s) = \lim_{t\to 0} \rho(s,t) \equiv \frac{ReT(s,t\to 0)}{ImT(s,t\to 0)}$$

- **slope of dσ/dt:**

$$\mathrm{B}(\mathrm{s},\mathrm{t}) = \frac{\mathrm{d}}{\mathrm{dt}}\left(\ln \frac{d\sigma}{\mathrm{dt}}(\mathrm{s},\mathrm{t})\right)$$

$$B_0(s) = \lim_{t\to 0} \mathrm{B}(\mathrm{s},\mathrm{t})$$

Extrapolations: ODDERON

T. Csörgő, I. Szanyi, *Eur. Phys. J. C* 81, 611 (2021)