A quantum generalization of the Cooper-Frye formula

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Outline

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Notion of freeze-out hypersurface

Usually, the sharp freeze-out hypersurface is defined by the parameter $P(t, \boldsymbol{r})$ taking the critical value P_c on the hypersurface:

$$P(t, \boldsymbol{r}) = P_\mathrm{c}$$

The parameter for defining the hypersurface can be selected as:

- The density of particles: $n(t, \boldsymbol{r}) = n_\mathrm{c}$

  D.Adamova (CERES Collaboration), Phys. Rev. Lett. **90**, 022301 (2003).

- The energy density: $\epsilon(t, \boldsymbol{r}) = \epsilon_\mathrm{c}$

  J. Sollfrank, P. Huovinen, and P.V. Ruuskanen, Eur. Phys. J. C **6**, 525 (1999).

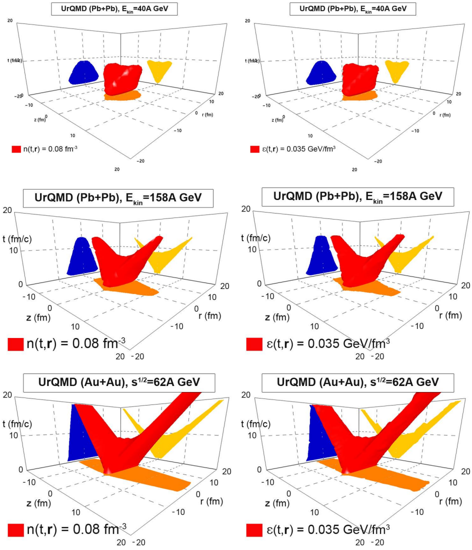
  V.N. Russkikh and Y.B. Ivanov, Phys. Rev. C **76**, 054907 (2007).

- The temperature: $T(t, \boldsymbol{r}) = T_\mathrm{c}$

  H. von Gersdorff, L. McLerran, M. Kataja, and P.V. Ruuskanen, Phys. Rev. D**34**, 794 (1986).

  P. Huovinen, Eur. Phys. J. A **37**, 121 (2008).

The freeze-out hypersurfaces in quasi-four-dimensional form: $(x, y) \to r_{\mathrm{T}} = \sqrt{x^2 + y^2}$



[1] D. Anchishkin, V. Vovchenko, and L.P. Csernai, Phys. Rev. C **87**, 014906 (2013).

Radiation of quantum fields from space-like hypersurface: the Cauchy problem

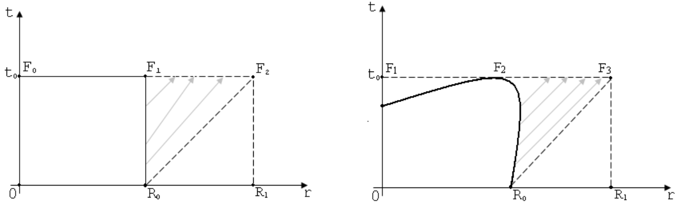


Figure: Sketch of a freeze-out hypersurface for a spherically symmetric fireball expansion. *Left panel:* Constant initial time $t = t_0$ and constant spatial boundary $R = R_0$. *Right panel:* Dependence of the initial radiation time on the radius and the dependence of the spatial boundary on time (solid curve).

Initial value problem

$$(\partial_\mu \partial^\mu + m^2)\,\hat{\varphi}(x) = 0\,,$$

where $\partial_\mu \partial^\mu = \partial_t^2 - \vec{\nabla}^2$, $\hbar = 1, \quad c = 1$.

The Cauchy problem or the initial conditions for this equation are specified on a space-like hyper-surface

$$\hat{\varphi}(x^0, \mathbf{x})\Big|_{x^0 = t_0} = \hat{\Phi}_0(\mathbf{x})\,, \qquad \frac{\partial \hat{\varphi}(x^0, \mathbf{x})}{\partial x^0}\Big|_{x^0 = t_0} = \hat{\Phi}_1(\mathbf{x})$$

Equation for the evolution of $\hat{\varphi}(x)$ together with the initial conditions can be written as:

$$(\partial_\mu \partial^\mu + m^2)\,\hat{\varphi}(x) = \delta(x^0 - t_0)\hat{\Phi}_1(\mathbf{x}) + \delta'(x^0 - t_0)\hat{\Phi}_0(\mathbf{x})$$

The Green's function:

$$G_R(x - y) = -\int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{(k_0 + i\delta)^2 - \mathbf{k}^2 - m^2}$$

Solution of the Klein-Gordon equation

Solution:

$$\hat{\varphi}(x) = \int d^4y\, \delta(y^0 - t_0) \left[G_R(x-y) \frac{\overleftrightarrow{\partial}}{\partial y^0} \hat{\Phi}(y^0, \mathbf{y}) \right]$$

where we update notations of the initial conditions:

$$\hat{\Phi}_0(\mathbf{y}) = \hat{\Phi}(y^0, \mathbf{y})\big|_{y^0=t_0}, \quad \hat{\Phi}_1(\mathbf{y}) = \partial \hat{\Phi}(y^0, \mathbf{y})/\partial y^0\big|_{y^0=t_0}$$

For the arbitrary space-like hyper-surface, $\sigma(y)$, solution looks like

$$\hat{\varphi}(x) = \int_\sigma d\sigma^\mu(y) \left[G_R(x-y) \frac{\overleftrightarrow{\partial}}{\partial y^\mu} \hat{\Phi}(y) \right] .$$

Measurement of the momentum spectrum

$$\hat{k}\psi_{\mathbf{k}}(\mathbf{r}) = \mathbf{k}\psi_{\mathbf{k}}(\mathbf{r}) \quad \to \quad \phi(\mathbf{r}) = \sum_{\mathbf{k}} \langle\psi_{\mathbf{k}}|\phi\rangle\, \psi_{\mathbf{k}}(\mathbf{r})$$

$$P(\mathbf{k}) = \Big|\langle\psi_{\mathbf{k}}|\phi\rangle\Big|^2$$

We assume that the detector measures asymptotic momentum eigenstates, i.e. that it acts by projecting the emitted single-particle state onto

$$\phi^{\text{out}}_{\mathbf{k}}(t,\mathbf{r}) = e^{-i\,\omega_k t + i\,\mathbf{k}\cdot\mathbf{r}}, \qquad \text{with} \qquad \omega_k = \sqrt{m^2+\mathbf{k}^2}$$

$$\hat{\varphi}(t,\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3\,2\omega_k}\,\left[b(\mathbf{k})\,e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{r}} + b^+(\mathbf{k})\,e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{r}}\right]$$

The single-particle spectrum and two-particle spectrum

$$P_1(\mathbf{p}) = \left\langle\, b^+(\mathbf{p})\, b(\mathbf{p})\,\right\rangle, \quad P_2(\mathbf{p}_1,\mathbf{p}_2) = \left\langle\, b^+(\mathbf{p}_1)\, b^+(\mathbf{p}_2)\, b(\mathbf{p}_2)\, b(\mathbf{p}_1)\,\right\rangle$$

Connection of the operators b and b^+ with the field of the fireball $\hat{\Phi}(x)$

Calculation of operators $b(\mathbf{k})$ and $b^+(\mathbf{k})$:

$$b(\mathbf{k}) = \int d^3r\, e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{r})}\, i\, \overleftrightarrow{\partial}_t\, \hat{\varphi}(t,\mathbf{r}) = -\int d^4y\, \delta(y^0 - t_0) \left[e^{-i(\omega_k y^0 - \mathbf{k}\cdot\mathbf{y})}\, i \frac{\overleftrightarrow{\partial}}{\partial y^0} \hat{\Phi}^+(y) \right]$$

For the arbitrary space-like hyper-surface $\sigma(x)$

$$b(\mathbf{k}) = i \int_\sigma d\sigma^\mu(x) \left[f^*_{\mathbf{k}}(x) \frac{\overleftrightarrow{\partial}}{\partial x^\mu} \hat{\Phi}(x) \right]$$

$$b^+(\mathbf{k}) = -i \int_\sigma d\sigma^\mu(x) \left[f_{\mathbf{k}}(x) \frac{\overleftrightarrow{\partial}}{\partial x^\mu} \hat{\Phi}^+(x) \right]$$

,

where $f_{\mathbf{k}}(x) = e^{-i(\omega_k x^0 - \mathbf{k}\cdot\mathbf{x})}$.

Single-particle spectrum [2]

$$2E_p \frac{dN}{d^3p} = \langle\, b^+(\mathbf{p})\, b(\mathbf{p})\, \rangle$$

$$2E_p \frac{dN}{d^3p} = i \int d^4x_1\, d^4x_2\, \delta(x_1^0 - t_0)\, \delta(x_2^0 - t_0) \left[f_{\mathbf{p}}(x_1)\, f^*_{\mathbf{p}}(x_2)\, \frac{\overleftrightarrow{\partial}}{\partial x_1^0} \frac{\overleftrightarrow{\partial}}{\partial x_2^0}\, G^<(x_2, x_1) \right]$$

Covariant form of the single-particle spectrum

$$2E_p \frac{dN}{d^3p} = i \int d\sigma^\mu(x_1)\, d\sigma^\nu(x_2) \left[f_{\mathbf{p}}(x_1)\, f^*_{\mathbf{p}}(x_2)\, \frac{\overleftrightarrow{\partial}}{\partial x_1^\mu} \frac{\overleftrightarrow{\partial}}{\partial x_2^\nu}\, G^<(x_2, x_1) \right]$$

where we have defined the correlation function or the lesser Green's function:

$$i\, G^<(x_2, x_1) = \pm \left\langle \hat{\Phi}^+(x_1)\, \hat{\Phi}(x_2) \right\rangle$$

Here the plus sign reads for bosons and the minus sign for fermions.

[2] D. Anchishkin, J.Phys.G 49, 055109 (2022)

Single-particle spectrum: local thermodynamic equilibrium

$$G^<(x_2, x_1) = \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot x} \, G^<(X;k) \, ,$$

where $x = x_1 - x_2$ and $X = (x_1 + x_2)/2$.

$$2E_p \frac{dN}{d^3p} = i \int \frac{d^4k}{(2\pi)^4} d^4X d^4x \, \delta(X^0 - t_0) \, \delta(x^0) \, e^{-i(p-k)\cdot x} \left(p^0 + k^0\right)^2 G^<(X;k) \, ,$$

where $p^0 = E_p = \sqrt{m^2 + \mathbf{p}^2}$.

Covariant form of the single-particle spectrum - 1

$$2E_p \frac{dN}{d^3p} = i \int \frac{d^4k}{(2\pi)^4} \int_\sigma d\sigma^\mu(x_1) d\sigma^\nu(x_2) e^{-i(p-k)\cdot(x_1 - x_2)}$$
$$\times (p+k)_\mu (p+k)_\nu G^<(X;k)$$

Here we use: $d^4x_1 \, \delta(x_1^0 - t_0) p^0 = d\sigma^\mu(x_1) p_\mu$.

Approximations

1. We assumed that the particles are free on the freeze-out hypersurface, this leads to the approximation:

$$G^{<}(X;k) \approx G_0^{<}(X;k)$$

In a system with slowly varying inhomogeneity, the free Green's function can be represented as:

$$i\,G_0^{<}(X;k^0,\mathbf{k}) \;=\; \frac{\pi}{\omega_k}\,\delta(k^0-\omega_k)\,f_{\mathrm{BE}}(X;k^0)$$

2. We assume that the correlation function $G^{<}(x_1,x_2)$ differs significantly from zero only if $|\mathbf{x}_1-\mathbf{x}_2| \to 0$:

$$\int_V d^4x\,\delta(x^0)\,e^{-i(p-k)\cdot x} \;=\; (2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})$$

$$2E_p\,\frac{dN}{d^3p} \;=\; \int d^4X\,\delta(X^0-t_0)\,2E_p\,f_{\mathrm{BE}}(X;E_p)$$

In a homogeneous system (no dependence on X) the formula reduces to:

$$\frac{dN}{d^3p} \;=\; V\,f_{\mathrm{BE}}(E_p)$$

$(1) + (2) \rightarrow$ Reduction to the Cooper-Frye formula

$$2E_p \frac{dN}{d^3p} = i \int \frac{d^4k}{(2\pi)^4} d^4X d^4x \, \delta(X^0 - t_0) \, \delta(x^0) \, e^{-i(p-k)\cdot x} \left(p^0 + k^0\right)^2 G^<(X;k)$$

With account for the equal-time initial conditions $dt = dX^0 = 0$ we get:

$$d^4X \, \delta\left(X^0 - t_0\right) \, p^0 \; = \; d\sigma_\mu(X) \, p^\mu$$

Reduction to the Cooper-Frye formula

$$E_p \frac{dN}{d^3p} \; = \; \int d\sigma_\mu(X) \, p^\mu \, f_{\rm BE}(X; p\cdot u) \, ,$$

where $p^0 = E_p = \sqrt{m^2 + \boldsymbol{p}^2}$, and $u(X)$ is the four-velocity at the point $X = (X^0, \mathbf{X})$ given on a space-like hypersurface.

Taking into account a finite size of the fireball

$$\int_V d^4x\, \delta(x^0)\, e^{-i(p-k)\cdot x} \neq (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{k})$$

A multiparticle system has weak inhomogeneity and is in local thermodynamic equilibrium

$$2E_p \frac{dN}{d^3p} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int_\sigma d\sigma^\mu(x_1) d\sigma^\nu(x_2) e^{-i(p-k)\cdot(x_1-x_2)} (p+k)_\mu (p+k)_\nu\, f(X;k)$$

where $p^0 = E_p = \sqrt{m^2 + \boldsymbol{p}^2}$, $k^0 = \omega_k = \sqrt{m^2 + \boldsymbol{k}^2}$.

The case of global thermodynamic equilibrium

$$2E_p \frac{dN}{d^3p} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} f(k) \left| \int_\sigma d\sigma^\mu(x)\, (p+k)_\mu\, e^{-i(p-k)\cdot x} \right|^2 .$$

where $p^0 = E_p, \mathrm{k}^0 = \omega_k$.

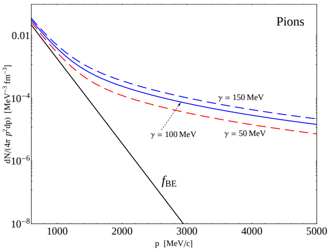
Radiation of particles from a system with a finite lifetime $\tau = 1/\gamma$

Dependence of the isotropic pion spectrum on the $|\mathbf{p}| = p$,
$d^3p = \sin\theta d\theta d\varphi p^2 dp$:

$$\frac{dN}{4\pi p^2 dp} = V \int_0^\infty \frac{dk^0}{\pi} \frac{(E_p + k_0)^2}{2E_p} \frac{m\gamma}{(k_0^2 - E_p^2)^2 + (m\gamma)^2} f_{\mathrm{BE}}(k^0)$$

Our goal is to compare this spectrum with the Cooper-Frye formula for the isotropic radiation:

$$\frac{dN}{4\pi p^2 dp} = V f_{\mathrm{BE}}(E_p)$$



Radiation of particles from a system with a finite lifetime $\tau = 1/\gamma$

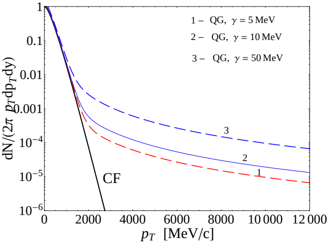
A comparison of the pion transverse spectra at midrapidity,

$(p_x, p_y, p_z) \to (\varphi, p_T, y)$, $d^3p = \sin\theta d\theta d\varphi p^2 dp$, $m_T = \sqrt{m_\pi^2 + p_T^2}$ and $y = 0$:

$$\frac{dN}{2\pi p_T dp_T dy} = V \int_0^\infty \frac{dk^0}{\pi} (m_T + k_0)^2 \frac{m\gamma}{(k_0^2 - m_T^2)^2 + (m\gamma)^2} f_{\rm BE}(k^0)$$

Our goal is to compare this spectrum with the Cooper-Frye formula for the isotropic radiation:

$$\frac{dN}{2\pi p_T dp_T} = V\, m_T(p_T)\, f_{\rm BE}\,(m_T(p_T))$$



It is interesting to compare our results with pion spectrum versus transverse momentum at midrapidity in the transverse momentum range 0.6 GeV/c $< p_T <$ 12 GeV/c measured in Pb-Pb collisions at $\sqrt{s_{NN}} = 2.76$ TeV

B. Abelev, J. Adam, D. Adamová et al. (The ALICE Collaboration), Eur. Phys. J. C **74**, 3108 (2014).

Conclusions

We obtained the quantum generalization of the Cooper-Frye for the single-particle spectrum in the framework of the following assumptions:

- The multi-particle system has a weak inhomogeneity;
- The multi-particle system is in local thermodynamic equilibrium;
- Particles are in free states on the freeze-out hypersurface;
- The correlation function $G^{<}(x_1, x_2)$ differs significantly from zero only if the differences of the arguments $\mathbf{x}_1 - \mathbf{x}_2$ are close to zero.

The latter assumption (4) cannot be applied if the system is close to a second-order phase transition, when the correlation length becomes sufficiently large. At the same time, this approximation excludes quantum effects.

Thank you for attention !

Acknowledgement

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Interference of the radiation from the freeze-out hypersurface

We can argue that formulae derived is a generalization of the Cooper-Frye formula, which takes into account quantum effects. Indeed, one can rewrite

$$2E_p \frac{dN}{d^3p} = \int_\sigma d\sigma^\mu(x_1)\, d\sigma^\nu(x_2)\, J_{\mu\nu}(x_1 - x_2, \boldsymbol{p}),$$

were we define the tensor $J_{\mu\nu}$

$$J_{\mu\nu}(x_1 - x_2, \boldsymbol{p}) = e^{-ip\cdot(x_1 - x_2)} \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{ik\cdot(x_1 - x_2)} (p+k)_\mu (p+k)_\nu f(X; k)$$

The last integral includes interference of waves with different momenta $\boldsymbol{k}$, which are projected onto the out-state with momentum $\boldsymbol{p}$. There is no such interference in the Cooper-Frye formula, where $\boldsymbol{k} = \boldsymbol{p}$ is taken.