Relativistic two-particle problem in the Lagrange formalism

Andrew Koshelkin
In-person, Moscow, Russia
52nd International Symposium on Multiparticle Dynamics (ISMD23)
21-26 August 2023, Gyöngyös, Hungary

25 August 2023
Contents

1 Introduction
2 Spinless particles
3 Spin-1/2 particles
4 Application: Para-positronium in a magnetic field
5 Conclusion
6 Acknowledgments
Introduction

Non-relativistic two-body problem - OK
Introduction

Relativistic two-body problem - ???
Introduction

NON-RELATIVISTIC DYNAMICS v.s. RELATIVISTIC DYNAMICS

Time is a variable rather than a parameter.

\[ \Downarrow \]

Relativity of simultaneity.

\[ \Downarrow \]

Non-relativistic approach is invalid.
Introduction

SOLUTION

Poincaré Invariant

\[ p_i^2 - m_i^2 = \Phi_i(p_1, m_1; p_2, m_2, q_{12}...) \]

\[ \Downarrow \]

Dirac Hamiltonian constraint dynamics

\[ \Downarrow \]

Ph. Droz-Vincent, I. Todorov, M. Kalb, P. van Alstine, A. Komar,
H. Sazdjian, H. W. Crater, C.-Y. Wong, ...

\[ \Downarrow \]

Lagrange Formalism - the current talk
Both interaction and external fields

\[ p^\mu \rightarrow p^\mu + A^\mu, \quad m \rightarrow m + S \]
Introduction

Laboratory v.s. Center Mass Frames

\[ X = \mu_1 x_1 + \mu_2 x_2, \quad x = x_2 - x_1, \quad x_1 = X - \mu_2 x, \quad x_2 = X + \mu_1 x \]

\[ P = p_1 + p_2, \quad p = \mu_1 p_2 - \mu_2 p_1, \quad p_1 = \mu_1 P - p, \quad p_2 = \mu_2 P + p \]

\[ \mu_1 = \frac{1 + \frac{m_1^2 - m_2^2}{M^2}}{2}, \quad \mu_2 = \frac{1 - \frac{m_1^2 - m_2^2}{M^2}}{2} \]

\[ \mu_1 + \mu_2 = 1, \quad M = |p_1 + p_2| = |P| \]

(Todorov)

3-rd Newton law

\[ Pp = 0, \quad (Pp)\phi = 0 \]
Spinless particles

The principles of the least action

Spinless particles

\[ A_{1+2} = \frac{\lambda_1}{2} A_1 (\text{ext. f.} + \text{int.}) + \frac{\lambda_2}{2} A_2 (\text{ext. f.} + \text{int.}) \]

1) Multiply by

\[ \int dx \phi_{1,2}^*(x) \phi_{1,2}(x) = C_{1,2} < +\infty, \]

2) Go to the momentum representation

\[ \lambda_{1,2} C_{1,2} = 1 \]
Spinless particles

Breit frame

\[ p_1 + p_2 = 0, \quad p_1^0 - p_2^0 = 0, \quad \text{assures} \quad Pp = 0 \]

\[
\left( - (p + A)^2 + (E_w + A^0)^2 - (m_w + S)^2 \right) \varphi(r) = 0
\]

\[ E_w = \frac{1}{2M} (M^2 - (m_1^2 + m_2^2)), \quad m_w = \frac{m_1 m_2}{M}, \quad M = |p_1 + p_2| \]

Covariant in \( R^{3+1} \)
Spin - $\frac{1}{2}$ particles

The principles of the least action

Spin - $\frac{1}{2}$ particles

$$\mathcal{A}_{1+2} = \frac{\lambda_1}{2} \mathcal{A}_1 (\text{ext.f.} + \text{int.}) + \frac{\lambda_2}{2} \mathcal{A}_2 (\text{ext.f.} + \text{int.})$$

1) Multiply by

$$\frac{\lambda_1}{(2 \int dx_1 \bar{\psi}_1(x_1)(m_1 + S(x_1))\psi_1(x_1))(\int dx_2 \bar{\psi}_2(x_2)\psi_2(x_2))} = 1,$$

$$\frac{\lambda_2}{(2 \int dx_2 \bar{\psi}_2(x_2)(m_2 + S(x_2))\psi_2(x_2))(\int dx_1 \bar{\psi}_1(x_1)\psi_1(x_1))} = 1$$

2) Go to the momentum representation
Spin - $\frac{1}{2}$ particles

Breit frame

$p_1 + p_2 = 0, \quad p_1^0 - p_2^0 = 0$

$$\left( - (p + A)^2 + \left( E_w + A^0 \right)^2 - (m_w + S)^2 \right) \varphi_i(r) = 0,$$

$i = 1, 2, ..., 16$$
Spin - $\frac{1}{2}$ particles

We wish to be !!!

$$
[(p_\nu + A_\nu)g^{\nu\mu}(p_\mu + A_\mu) - (m_w + S)^2] \varphi_i(r) = 0.
$$

$$
\hat{\Gamma}^0 = \mu_1 \gamma^0 \otimes I + \mu_2 \gamma^0 \otimes I,
$$

$$
\hat{\gamma} = \gamma^0 \otimes I - \gamma^0 \otimes I.
$$
Spin - $\frac{1}{2}$ particles

\[ \hat{\Gamma}^0 \hat{\Gamma}^0 + \hat{\Gamma}_0^0 = 2^{(1+2)} \left( \begin{array}{c} I \end{array} \right) ; \]

\[ \hat{\Gamma}^0 \gamma^a + \gamma^a \hat{\Gamma}^0 = 0, \quad a = 1, 2, 3; \]

\[ \gamma^a \gamma^b + \gamma^b \gamma^a = -4 \delta^{ab} \left( \begin{array}{c} I \end{array} \right) + 4 (s_1^a s_2^b + s_2^b s_1^a + s_2^a s_1^b + s_1^b s_2^a) \left( \begin{array}{c} I \end{array} \right). \]
Spin - $\frac{1}{2}$ particles

\[
\left( \Gamma^0 (E_w + A^0) - \Gamma (\hat{p} + A) + (s_1 + s_2)(\hat{p} + A) - (m_w + S) \right) \psi(r) = 0
\]

where $\psi(r)$ is the 16 component spinor.

*Covariant in $R^{3+1} \cup S^2\{R^3\}$*
**Application: Para-positronium in a magnetic field**

\[ a_B \gg a = (|e|B)^{-1/2} \]

\[
\left( \gamma^0 (E_w + A^0) - \gamma (p + A) - (m_w + S) \right) \psi(r) = 0
\]

\[
(\Delta + \frac{2(4\pi\alpha)E_w}{r} + \frac{(4\pi\alpha)^2}{r^2} + \frac{i}{2} \nabla(B \times \rho) - \frac{1}{4} e^2 B^2 \rho^2) \psi(r) = (m_w^2 - E_w^2) \psi(r)
\]

\[ A^0 = -\frac{4\pi\alpha}{r}, \quad A = \frac{1}{2} (B \times \rho), \quad r = \rho + e_z z \]
Application: Para-positronium in a magnetic field

Ground state
\[ \psi_{m=0,n=0}(\rho) = \frac{1}{a\sqrt{2\pi}} e^{-\frac{\rho^2}{4a^2}} \]

Let us look for a solution

\[ \Psi(r) = \psi(z) \psi_{m=0,n=0}(\rho) \]

\[ \psi(z) = \sqrt{\frac{k}{a}} \exp(-k|z|/a) \]

where \( k \) is a positive constant, so that \( k^2 = \varepsilon \).
Application: Para-positronium in a magnetic field

\[ k = 2^{3/2}(4\pi\alpha)m_w\varepsilon_w a\ln(a_B/a) + (4\pi\alpha)^2 \frac{\pi^{3/2}}{\sqrt{2}} \]

1) 
\[ m_e a \gg \alpha, \quad k = 2^{3/2}(4\pi\alpha)m_e a \ln(a_B/a) \]
\[ E_w = \frac{m_e}{2}(1 + (2/(m_e a)^2 - 2(4\pi\alpha)^2 \ln^2(a_B/a))) \]

( Elliott, Loudon)

2) 
\[ m_e a \ll \alpha, \quad k = (2\pi)^{7/2}\alpha \]
\[ E_w \approx \frac{1}{a^2}(1 + (m_e a)^4/2 - (2\pi)^7\alpha^4/2) \approx \frac{1}{a^2}(1 - (2\pi)^7\alpha^4/2). \]
Conclusion

1. The two-body relativistic problem is studied in terms of the principle of the least action beyond the constraint Hamiltonian dynamics.

2. In the developed approach the motion equation for a spinless particle and the Dirac-like equation are derived.

3. The developed approach is applied for study a para-positronium in a strong magnetic field.
Acknowledgments

THANK YOU FOR YOUR ATTENTION!!!