Relativistic two-particle problem in the Lagrange formalism

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Introduction

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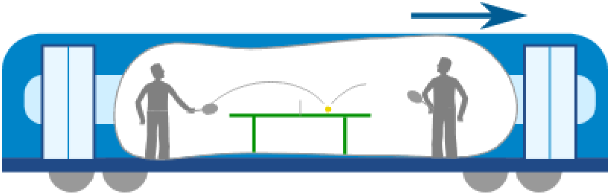

Introduction

NON-RELATIVISTIC DYNAMICS v.s. RELATIVISTIC DYNAMICS

Time is a variable rather than a parameter.

$\Downarrow$

Relativity of simultaneity.

$\Downarrow$

Non-relativistic approach is invalid

Introduction

SOLUTION

Poincare Invariant

$$p_i^2 - m_i^2 = \Phi_i(p_1, m_1; p_2, m_2, q_{12} \ldots)$$

$\Downarrow$

Dirac Hamiltonian constraint dynamics

$\Downarrow$

Ph. Droz-Vincent, I.T.Todorov, M. Kalb, P. van Alstine, A.Komar, H. Sazdjian, H.W.Crater, C.-Y.Wong, ...

$\Downarrow$

Lagrange Formalism - the current talk

Introduction

Both interaction and external fields

$$p^\mu \to p^\mu + A^\mu, \qquad m \to m + S$$

Introduction

Laboratory v.s. Center Mass Frames

$$X = \mu_1 x_1 + \mu_2 x_2, \quad x = x_2 - x_1, \quad x_1 = X - \mu_2 x, \quad x_2 = X + \mu_1 x$$

$$P = p_1 + p_2, \quad p = \mu_1 p_2 - \mu_2 p_1, \quad p_1 = \mu_1 P - p, \quad p_2 = \mu_2 P + p$$

$$\mu_1 = \frac{1 + \frac{m_1^2 - m_2^2}{M^2}}{2}, \quad \mu_2 = \frac{1 - \frac{m_1^2 - m_2^2}{M^2}}{2}$$

$$\mu_1 + \mu_2 = 1, \qquad M = |p_1 + p_2| = |P|$$

(Todorov)

3-rd Newton law

$$\boxed{Pp = 0, \qquad (Pp)\phi = 0}$$

Spinless particles

The principles of the least action
Spinless particles

$$\mathfrak{A}_{1+2} = \frac{\lambda_1}{2}\mathfrak{A}_1(ext.f. + int.) + \frac{\lambda_2}{2}\mathfrak{A}_2(ext.f. + int.)$$

1) Multiply by

$$\int dx \phi^*_{1,2}(x)\phi_{1,2}(x) = C_{1,2} < +\infty,$$

$$\boxed{\lambda_{1,2} C_{1,2} = 1}$$

2) Go to the momentum representation

Spinless particles

Breit frame

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0, \quad p_1^0 - p_2^0 = 0, \quad \text{assures} \qquad Pp = 0$$

$$\boxed{\Big(-(\boldsymbol{p} + \boldsymbol{\mathcal{A}})^2 + (E_w + \mathcal{A}^0)^2 - (m_w + \mathcal{S})^2 \Big) \varphi(\boldsymbol{r}) = 0}$$

$$E_w = \frac{1}{2M}(M^2 - (m_1^2 + m_2^2)), \quad m_w = \frac{m_1 m_2}{M}, \quad M = |p_1 + p_2|$$

Covariant in $\mathcal{R}^{3+1}$

Spin - $\frac{1}{2}$ particles

The principles of the least action
Spin - $\frac{1}{2}$ particles

$$\mathfrak{A}_{1+2} = \frac{\lambda_1}{2}\mathfrak{A}_1(ext.f. + int.) + \frac{\lambda_2}{2}\mathfrak{A}_2(ext.f. + int.)$$

1) Multiply by

$$\frac{\lambda_1}{(2\int dx_1 \bar{\Psi}_1(x_1)(m_1 + S(x_1))\Psi_1(x_1))(\int dx_2 \bar{\Psi}_2(x_2)\Psi_2(x_2))} = 1,$$

$$\frac{\lambda_2}{(2\int dx_2 \bar{\Psi}_2(x_2)(m_2 + S(x_2))\Psi_2(x_2))(\int dx_1 \bar{\Psi}_1(x_1)\Psi_1(x_1))} = 1$$

2) Go to the momentum representation

Spin - $\frac{1}{2}$ particles

Breit frame

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0, \quad p_1^0 - p_2^0 = 0$$

$$\begin{aligned} \Big(-(\boldsymbol{p} + \boldsymbol{\mathcal{A}})^2 + \quad (\quad & E_w + \mathcal{A}^0)^2 - (m_w + \mathcal{S})^2 \Big) \varphi_i(\boldsymbol{r}) = 0, \\ i \quad & = 1, 2, ..., 16 \end{aligned}$$

Spin - $\frac{1}{2}$ particles

We wish to be !!!

$$[(p_\nu + \mathcal{A}_\nu)g^{\nu\mu}(p_\mu + \mathcal{A}_\mu) - (m_w + \mathcal{S})^2]\varphi_i(\boldsymbol{r}) = 0.$$

.

$$\hat{\Gamma}^0 = \mu_1 \overset{(1)}{\gamma^0} \otimes \overset{(2)}{I} + \mu_2 \overset{(2)}{\gamma^0} \otimes \overset{(1)}{I},$$
$$\hat{\gamma} = \overset{(2)}{\gamma} \otimes \overset{(1)}{I} - \overset{(1)}{\gamma} \otimes \overset{(2)}{I}$$

Spin - $\frac{1}{2}$ particles

$$\hat{\Gamma}^0\hat{\Gamma}^0 + \hat{\Gamma}^0\hat{\Gamma}^0 = 2 \overset{(1+2)}{I};$$
$$\hat{\Gamma}^0\hat{\gamma}^a + \hat{\gamma}^a\hat{\Gamma}^0 = 0, \quad a = 1, 2, 3;$$
$$\hat{\gamma}^a\hat{\gamma}^b + \hat{\gamma}^b\hat{\gamma}^a = -4\delta^{ab} \overset{(1+2)}{I} + 4(s_1^a s_2^b + s_2^b s_1^a + s_2^a s_1^b + s_1^b s_2^a) \overset{(1+2)}{I}.$$

Spin - $\frac{1}{2}$ particles

$$\boxed{\Big(\Gamma^0(E_w + \mathcal{A}^0) - \boldsymbol{\Gamma}(\hat{\boldsymbol{p}} + \boldsymbol{\mathcal{A}}) + (\boldsymbol{s}_1 + \boldsymbol{s}_2)(\hat{\boldsymbol{p}} + \boldsymbol{\mathcal{A}}) - (m_w + \mathcal{S})\Big)\Psi(\boldsymbol{r}) = 0}$$

where $\Psi(\boldsymbol{r})$ is the 16 component spinor.

Covariant in $\mathcal{R}^{3+1} \cup \mathcal{S}^2\{\mathcal{R}^3\}$

Application: Para-positronium in a magnetic field

$$a_B \gg a = (|e|B)^{-1/2}$$

$$\Big(\gamma^0(E_w + \mathcal{A}^0) - \gamma(\boldsymbol{p} + \boldsymbol{\mathcal{A}}) - (m_w + \mathcal{S})\Big)\Psi(\boldsymbol{r}) = 0$$

$$\Big(\triangle + \frac{2(4\pi\alpha)E_w}{r} + \frac{(4\pi\alpha)^2}{r^2} + \frac{i}{2}\nabla(\boldsymbol{B}\times\boldsymbol{\rho}) - \frac{1}{4}e^2B^2\rho^2\Big)\Psi(\boldsymbol{r})$$
$$= (m_w^2 - E_w^2)\Psi(\boldsymbol{r})$$

$$A^0 = -\frac{4\pi\alpha}{r}, \quad \mathcal{A} = \frac{1}{2}(\boldsymbol{B}\times\boldsymbol{\rho}), \quad \boldsymbol{r} = \boldsymbol{\rho} + \boldsymbol{e}_z z$$

Application: Para-positronium in a magnetic field

Ground state

$$\psi_{m=0,n=0}(\boldsymbol{\rho}) = \frac{1}{a\sqrt{2\pi}} e^{-\frac{\rho^2}{4a^2}}$$

Let us look for a solution

$$\Psi(\boldsymbol{r}) = \psi(z)\ \psi_{m=0,n=0}(\boldsymbol{\rho})$$

$$\psi(z) = \sqrt{\frac{k}{a}} \exp(-k|z|/a)$$

where k is a positive constant, so that $k^2 = \varepsilon$.

Application: Para-positronium in a magnetic field

$$k = 2^{3/2}(4\pi\alpha)m_w\varepsilon_w a\ln(a_B/a) + (4\pi\alpha)^2\frac{\pi^{3/2}}{\sqrt{2}}$$

1)

$$m_e a \gg \alpha, \quad k = 2^{3/2}(4\pi\alpha)m_e a\ln(a_B/a)$$

$$E_w = \frac{m_e}{2}(1 + (2/(m_e a)^2 - 2(4\pi\alpha)^2\ln^2(a_B/a))$$

(Elliott, Loudon)

2)

$$m_e a \ll \alpha, \quad k = (2\pi)^{7/2}\alpha$$

. $$E_w = \frac{1}{a^2}(1 + (m_e a)^4/2 - (2\pi)^7\alpha^4/2) \simeq \frac{1}{a^2}(1 - (2\pi)^7\alpha^4/2).$$

Conclusion

1. The two-body relativistic problem is studied in terms of the principle of the least action beyond the constraint Hamiltonian dynamics
2. In the developed approach the motion equation for a spinless particle and the Dirac-like equation are derived.
3. The developed approach is applied for study a para-positronium in a strong magnetic field.

Acknowledgments

THANK YOU FOR YOUR ATTENTION!!!