Relativistic spin-(magneto)hydrodynamics

The Sun’s atmosphere is a superhot plasma governed by magnetohydrodynamic forces...

Ah, yes, of course.

Whenever I hear the word "magnetohydrodynamic" my brain just replaces it with "magic."

Magnetohydrodynamics combines the intuitive nature of Maxwell's equations with the easy solvability of the Navier-Stokes equations. It's so straightforward physicists add "relativistic" or "quantum" just to keep it from getting boring.
Decay of scalar particles

No anisotropy in the rest frame: isotropic decay products.
Decay of particles with spin

Preferred direction due to spin: anisotropic decay products

Basis for polarization observables.
Several random decays

Averaging over random decays should lead to isotropic decay products.
Decay of spin polarized particles

Averaging over decay of spin-polarized particles should lead to anisotropic decay products.
First evidence of a quantum effect in (relativistic) hydrodynamics

Adapted from F. Becattini
‘Subatomic Vortices’
Other effects related to spin polarization.
Einstein-de Haas effect

Electron spins get aligned in external magnetic field which is compensated by rotation of the ferromagnetic material.
Spontaneous magnetization when spun around. Transformation of orbital angular momentum into spin alignment. Angular velocity decreases with appearance of magnetic field. Explanation appeals to spin-orbit coupling.
Generation of magnetic field in heavy ion collisions

[Adapted from D. Kharzeev @ CPOD 2013.]
Magnetic field time evolution

\[ \frac{eH}{m_{\pi}^2} \]

Global angular momentum in heavy ion collisions

[B. Mohanty, ICTS News 6, 18-20 (2020).]
Angular momentum generation in non-central collisions

Au Au, $\sqrt{s_{NN}} = 200$ GeV

First evidence of a quantum effect in (relativistic) hydrodynamics

Adapted from F. Becattini
‘Subatomic Vortices’
Relativistic kinetic theory

- Kinetic theory: calculation of macroscopic quantities by means of statistical description in terms of distribution function.

- Let us consider a system of relativistic particles of rest mass $m$ with momenta $p$ and energy $p^0$

$$p^0 = \sqrt{p^2 + m^2}$$

- For large no. of particles, $f(x, p)$ gives a distribution of the four-momenta $p = p^\mu = (p^0, \mathbf{p})$ at each space-time point.

- $f(x, p) \Delta^3 x \Delta^3 p$ gives average no. of particles in the volume element $\Delta^3 x$ at point $x$ with momenta in the range $(p, p + \Delta p)$.

- Statistical assumptions:
  - No. of particles contained in $\Delta^3 x$ is large ($N \gg 1$).
  - $\Delta^3 x$ is small compared to macroscopic volume ($\Delta^3 x/V \ll 1$).

- The equilibrium distribution: $f_{eq}(x, p, s) = [\exp (\beta \cdot p - \xi) \pm 1]^{-1}$
Extended phase-space for spin degrees of freedom

- The phase-space for single particle distribution function gets extended \( f(x, p, s) \).

- The equilibrium distribution for Fermions is given by

\[
    f_{eq}(x, p, s) = \frac{1}{\exp \left[ \beta \cdot p - \xi - \frac{1}{2} \omega : s \right] + 1}
\]

\[
\begin{cases} 
    \beta \cdot p \equiv \beta_\mu p^\mu \\
    \omega : s \equiv \omega_{\mu\nu} s^{\mu\nu}
\end{cases}
\]

- Quantities \( \beta^\mu = u^\mu / T \), \( \xi = \mu / T \), \( \omega_{\mu\nu} \) are functions of \( x \).

- \( \xi, \beta^\mu, \omega^{\mu\nu} \): Lagrange multipliers for conserved quantities.

- \( s^{\mu\nu} \): Particle spin, similar to particle momenta \( p^\mu \).

- Hydrodynamics: average over particle momenta and spin.

- Classical treatment of spin.

Bhadury et. al., PLB 814, 136096 (2021); PRD 103, 01430 (2021).
Conserved currents and spin-hydrodynamics

- Express hydrodynamic quantities in terms of $f(x, p, s)$.

\[ T^\mu_\nu(x) = \int dP dS \ p^\mu p^\nu \left[ f(x, p, s) + \bar{f}(x, p, s) \right] \]

\[ N^\mu(x) = \int dP dS \ p^\mu \left[ f(x, p, s) - \bar{f}(x, p, s) \right] \]

\[ S^{\lambda,\mu_\nu}(x) = \int dP dS \ p^\lambda s^{\mu\nu} \left[ f(x, p, s) + \bar{f}(x, p, s) \right] \]

\[ dP \equiv \frac{d^3p}{E_p(2\pi)^3}, \quad dS \equiv m \frac{d^4s}{\pi \mathbf{s}} \delta(s \cdot s + \mathbf{s}^2) \delta(p \cdot s) \]

\[ \int dS = 2; \quad \mathbf{s}^2 = \frac{1}{2} \left( \frac{1}{2} + 1 \right) = \frac{3}{4}; \quad s^\mu \equiv \frac{1}{2m} \epsilon^{\mu\nu\alpha\beta} p_\nu s_{\alpha\beta} \]

- Classical treatment of spin: internal angular momentum.

- Equations of motion: $\partial_\mu T^\mu_\nu = 0$, $\partial_\mu N^\mu = 0$, $\partial_\lambda S^{\lambda,\mu_\nu} = 0$.

- Non-dissipative spin hydrodynamics: $f(x, p, s) = f_{eq}(x, p, s)$.

- Dissipative spin-hydrodynamics: Boltzmann equation for $f(x, p, s)$. 

Amaresh Jaiswal (NISER)  Spin-(magneto)hydrodynamics, ISMD 2023  17
Dissipative effects

- Shear viscosity: fluid’s resistance to shear forces

- Bulk viscosity: fluid’s resistance to compression

- Charge/heat conductivity: fluid’s resistance to flow of charge/heat.

- Dissipation to spin current new:
  
  PLB 814, 136096 (2021); PRD 103, 01430 (2021).
The particle four-current and its conservation is given by

\[ N^\mu = n u^\mu + n^\mu, \quad \partial_\mu N^\mu = 0 \]

The total stress-energy tensor of the system:

\[ T^{\mu\nu} = T_f^{\mu\nu} + T_{\text{int}}^{\mu\nu} + T_{\text{em}}^{\mu\nu} \]

\[ T_f^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu}, \]

\[ T_{\text{int}}^{\mu\nu} = -\Pi^\mu u^\nu - F_\alpha^\mu M^{\nu\alpha} \]

\[ T_{\text{em}}^{\mu\nu} = -F^{\mu\alpha} F_\alpha^\nu + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \]

Maxwell’s equation:

\[ \partial_\mu H^{\mu\nu} = J^\nu \quad \text{and} \quad H^{\mu\nu} = F^{\mu\nu} + M^{\mu\nu}, \]

\[ \partial_\mu T_{\text{em}}^{\mu\nu} = F_\alpha^\nu J^\alpha \]

Current generating external field,

\[ J^\mu = J_f^\mu + J_{\text{ext}}^\mu \quad \text{where} \quad J_f^\mu = q N^\mu, \]

\[ \partial_\mu T^{\mu\nu} = - f_\text{ext}^\nu, \quad f_\text{ext}^\nu = F_\alpha^\nu J_\text{ext}^\alpha \]
Equations of motion

- Divergence of matter part of energy-momentum tensor,
  \[ \partial_{\nu} T_{\mu \nu}^f = F_{\mu}^{\alpha} J_{\alpha}^{\alpha} + \frac{1}{2} (\partial_{\mu} F_{\nu}^{\alpha}) M_{\nu \alpha} \]

- Next, consider total angular momentum conservation:
  \[ J^{\lambda, \mu \nu} = L^{\lambda, \mu \mu} + S^{\lambda, \mu \nu} \]

- In presence of external torque its divergence leads to,
  \[ \partial_{\lambda} J^{\lambda, \mu \nu} = -\tau_{\mu \nu}^{\mu \nu}, \quad \tau_{\mu \nu}^{\mu \nu} = x^{\mu} J_{\mu \nu}^{\nu} - x^{\nu} J_{\mu \nu}^{\nu} \]

  Torque due to moment of external force; “pure” torque ignored.

- The orbital part of angular momentum and its divergence is
  \[ L^{\lambda, \mu \nu} = x^{\mu} T^{\lambda \nu} - x^{\nu} T^{\lambda \mu}, \quad \partial_{\lambda} L^{\lambda, \mu \nu} = -\tau_{\mu \nu}^{\mu \nu} \]

- Spin part of the total angular momentum is conserved
  \[ \partial_{\lambda} S^{\lambda, \mu \nu} = 0 \]

- Along with particle four-current conservation, \( \partial_{\mu} N^{\mu} = 0 \).
Boltzmann equation

- Boltzmann equation (BE) in relaxation-time approximation (RTA)
  \[
  \left( p^\alpha \frac{\partial}{\partial x^\alpha} + m F^\alpha \frac{\partial}{\partial p^\alpha} + m S^{\alpha\beta} \frac{\partial}{\partial s^{\alpha\beta}} \right) f = C[f] = - (u \cdot p) \frac{f - f_{eq}}{\tau_{eq}}
  \]

- The force term is:
  \[
  F^\alpha = \frac{q}{m} F^{\alpha\beta} p_\beta + \frac{1}{2} \left( \partial^\alpha F^{\beta\gamma} \right) m_\beta \gamma, \quad m^{\alpha\beta} = \chi s^{\alpha\beta}
  \]

- There is a “pure” torque term:
  \[
  S^{\alpha\beta} = 2 F^{\gamma[\alpha} m_{\beta]} \gamma - \frac{2}{m^2} \left( \chi - \frac{q}{m} \right) F_{\phi\gamma} s^{\phi[\alpha} p^{\beta]} p_\gamma
  \]

- We ignore this “pure” torque term for now.

- Employ the Boltzmann equation to obtain \( \delta f = \delta f_1 \).

- Evolution equations for dissipative spin-magnetohydrodynamics.
Einstein-de Haas and Barnett effects

- One can define the polarization-magnetization tensor as
  \[ M^{\mu \nu} = m \int dPdS m^{\mu \nu} (f - \bar{f}) \]

- The equilibrium polarization-magnetization tensor is
  \[ M_{eq}^{\mu \nu} = m \int dPdS m^{\mu \nu} (f_{eq} - \bar{f}_{eq}) \]

- Magnetic dipole moment \( m^{\mu \nu} = \chi s^{\mu \nu} \).
- \( \chi \): resembles the gyromagnetic ratio.

- Integrating over the momentum and spin degrees of freedom,
  \[ M_{eq}^{\mu \nu} = a_1 \omega^{\mu \nu} + a_2 u^{[\mu} u^{\gamma} \omega^{\nu]} \gamma \]

- In global equilibrium, \( \omega^{\mu \nu} \) corresponds to rotation of the fluid.

- Rotation produces magnetization (Barnett effect) and vice versa (Einstein-de Hass effect).
Other relevant works in spin hydrodynamics

- **Other parallel approaches from Wigner function** [N. Weickgenannt, X.-l. Sheng, E. Speranza, Q. Wang and D. Rischke, PRD 100 (2019) 056018].

- **Approach based on chiral kinetic theory** [S. Shi, C. Gale and S. Jeon, PRC 103 (2021) 044906].

- **Approach based on Lagrangian method** [D. Montenegro and G. Torrieri, PRD 100 (2019) 056011].


- **Relativistic spin-magnetohydrodynamics: unexplored area.**

- **Much work needed in this direction.**
Thank you!