Tensions in Standard Cosmology

And How to Avoid Them

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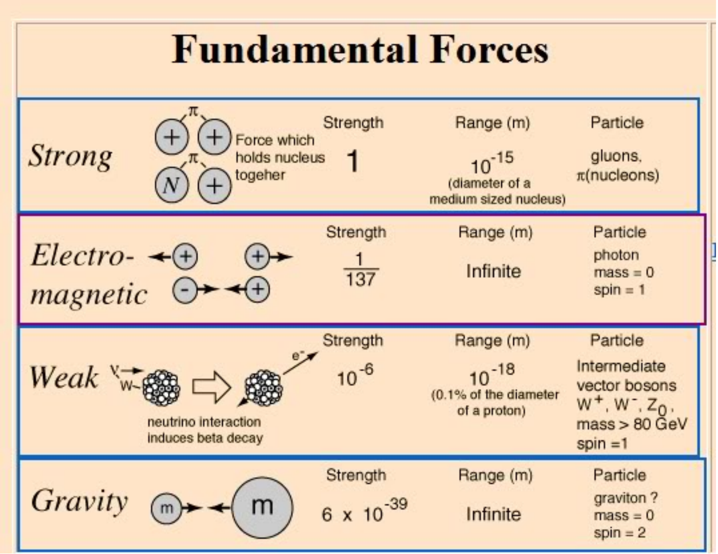
- Modern cosmology based on the cosmological principle: the universe admits a maximally symmetric (FLRW) spacetime
  - ✓ Homogeneous: all regions of space look alike, no preferred positions
  - ✓ Isotropic: no preferred directions
- Recent cosmological observations have shown that the universe is undergoing a recent epoch of accelerated expansion
- Whereas it is not conclusively known what caused this recent cosmic acceleration, the prevailing argument is that dark energy caused it
- Among the most widely considered candidates of dark energy is the vacuum energy of the cosmological constant Λ
- Some serious problems associated with the cosmological constant, among them the eponymous *cosmological constant problem* [1] and the *coincidence problem* [2]
- Several alternatives proposed, such as:
  - ✓ Inhomogeneous and/or anisotropic models
  - ✓ Interacting dark fluids (dark matter and dark energy)
  - ✓ Modifications to gravity

[1] Weinberg, S. The cosmological constant problem. Rev. Mod. Phys. 1989, 61 (1), 1

[2] Velten, H. E. et al. Aspects of the cosmological "coincidence problem". Eur. Phys. J. C 2014, 74 (11), 1

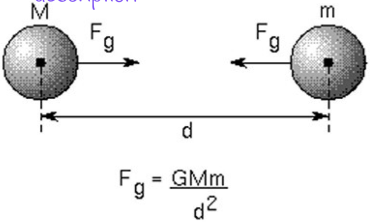
- Four fundamental forces in nature; why care only about gravity?



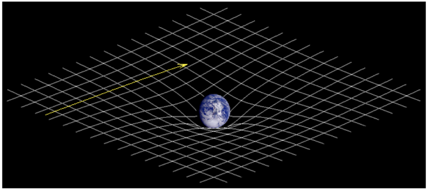
The fundamental forces of nature. [Credit: Socratic.org]

What exactly is gravity?

Newton's force-at-a distance description



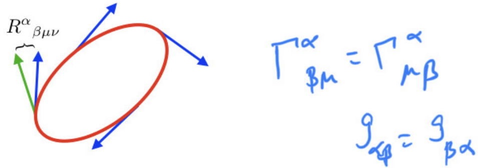
Einstein's gravity as curvature of spacetime: matter tells spacetime how to curve, spacetime tells matter how to move



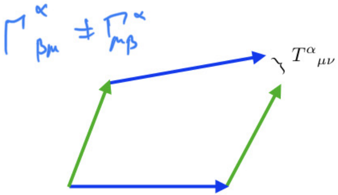
Gravitation à la Newton and Einstein

The geometrical “trinity" of gravity

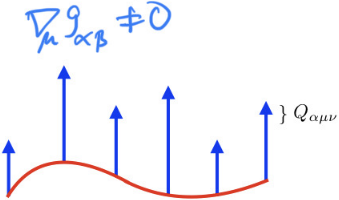
- Three different geometrical representations of spacetime curvature possible



The rotation of a vector transported along a closed curve is given by the curvature: General Relativity.



The non-closure of parallelograms formed when two vectors are transported along each other is given by the torsion: Teleparallel Equivalent of General Relativity.

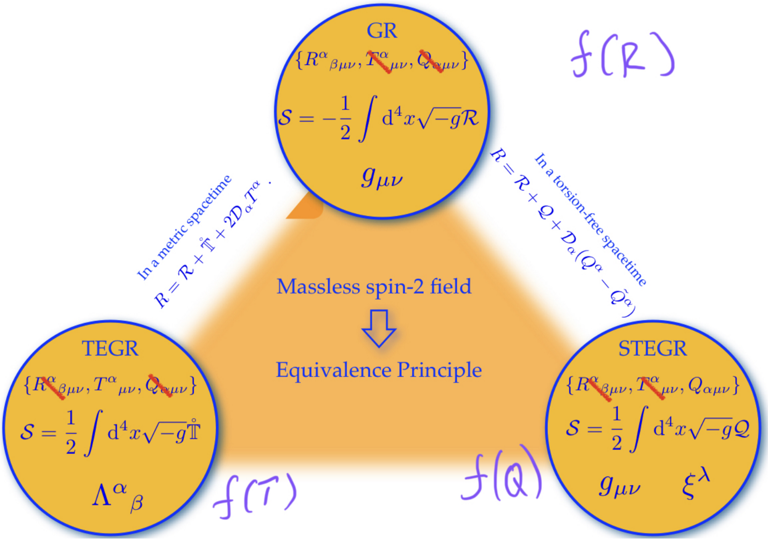


The variation of the length of a vector as it is transported is given by the non-metricity: Symmetric Teleparallel Equivalent of General Relativity.

The geometrical meaning of curvature, torsion and non-metricity. [Credit: Jimenez et al, arXiv 1903.06830]

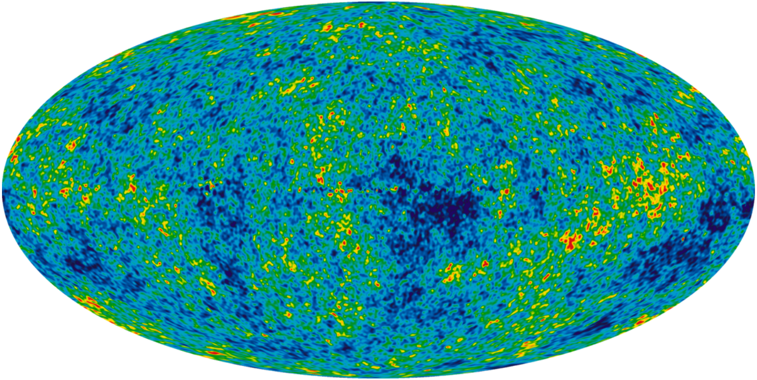
The geometrical "trinity" of gravity...

- Three possible gravitational interpretations



Three alternative gravitational descriptions. [Credit: Jimenez et al, arXiv 1903.06830]

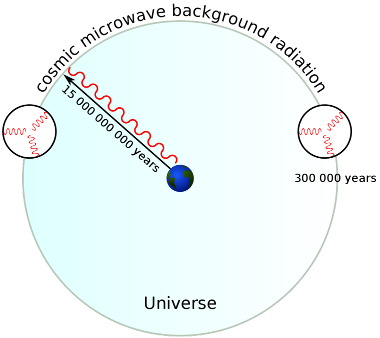
- Based on the current cosmological paradigm, the universe is a 4-dimensional homogeneous and isotropic spacetime that started off as a Big Bang



The baby universe @ $\sim$ 380,000 years old. Today, $T \sim 2.726K$, $\frac{\delta T}{T} \sim 10^{-5}$.

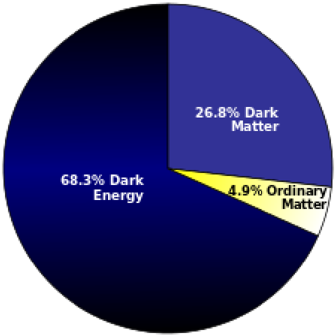
Broadly speaking, two serious puzzles remain unanswered in standard cosmology:

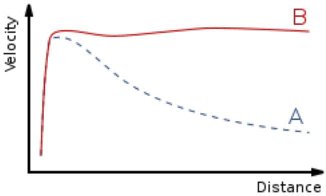
- Early-universe problems
  - ✓ Horizon problem
  - ✓ Flatness problem
  - ✓ Structure/smoothness/homogeneity problem
  - ✓ Magnetic-monopole problem



The horizon problem

- Late-universe problems
  - ✓ Rotational curves of galaxies
    - Exact nature unknown
    - No direct prediction, nor detection
  - ✓ Cosmic acceleration: the cosmological constant as dark energy?
    - Exact nature unknown
    - The *cosmological constant problem*: 120 orders of magnitude discrepancy between two predictions





The current cosmic acceleration is attributed to dark energy, whereas the discrepancy between the predicted (A) and observed (B) rotation curves of galaxies is attributed to dark matter

- Cosmological Constant Problem (vacuum catastrophe): measured energy density of the vacuum over 120 orders of magnitude less than the theoretical prediction
  - ✓ Worst prediction in the history of physics (and of science in general)
  - ✓ Casts doubt on dark energy being a cosmological constant
- Cosmic Coincidence Problem: dark matter and dark energy densities have the same order of magnitude at the present moment of cosmic history, while differing with many orders of magnitude in the past and predicted future
- Latest tensions vis-á-vis precise theoretical predictions and observational measurements:
- H_0 CMB vs local measurements, more than 3σ discrepancy
  - ✓ Planck2018, ΛCDM model

$$H_0 = 67.27 \pm 0.60 \; km/s/Mpc$$

  - ✓ Estimate using SNIa measurements (2016)

$$H_0 = 73.24 \pm 1.74 \; km/s/Mpc$$

  - ✓ Parallax measurements of Milky Way Cepheids (2018)

$$H_0 = 73.48 \pm 1.66 \; km/s/Mpc$$

- S_8 vs cosmic shear data, more than 2.5σ discrepancy between Planck data and local measurements of

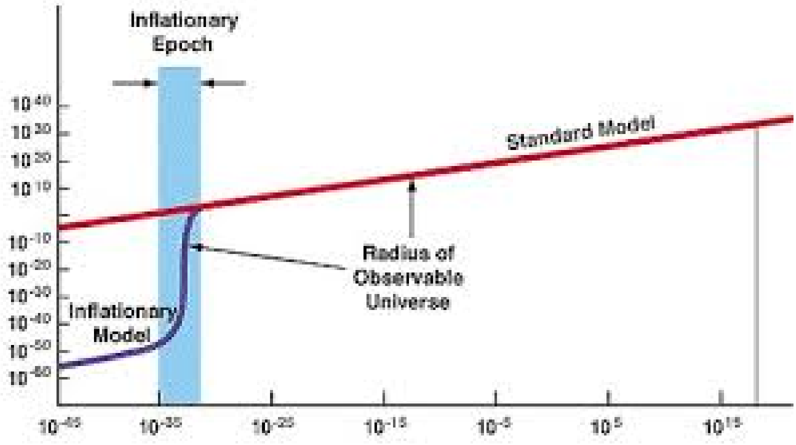
$$S_8 = \sigma_8 \sqrt{\Omega_m/0.3}$$

  σ_8 measures the amplitude of the linear power spectrum on the $8h^{-1}$Mpc scale
- Ω_K, zero or not zero? ΛCDM assumes flat universe, but Planck temperature and polarisation power spectra give an above 3σ deviation:

$$\Omega_K \approx -0.044^{+0.018}_{-0.015}$$

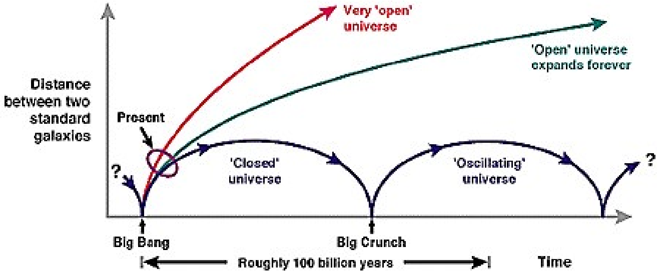
New frontiers: some suggested solutions

- Inflation: solves early-universe problems. Or does it?
  - ✓ Exact mechanisms (of start and end of inflation) still debatable
  - ✓ Over a 100 different models of inflation!



The horizon problem resolved

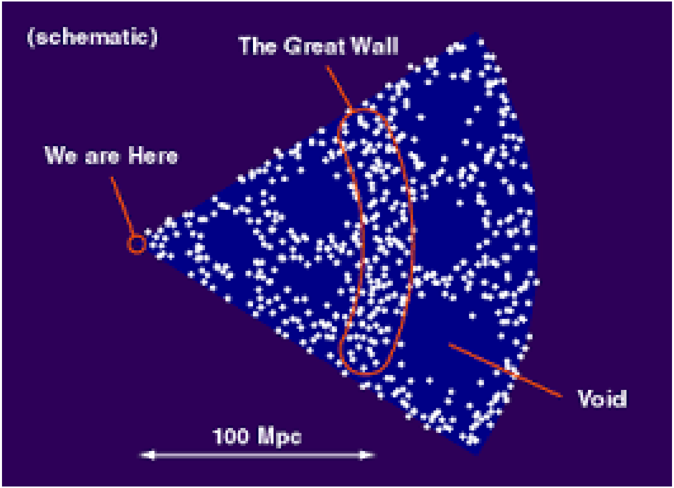
Cyclic universe models?



Could this universe be just one realization of an infinite cycle of big bangs and big crunches?

But what banged?

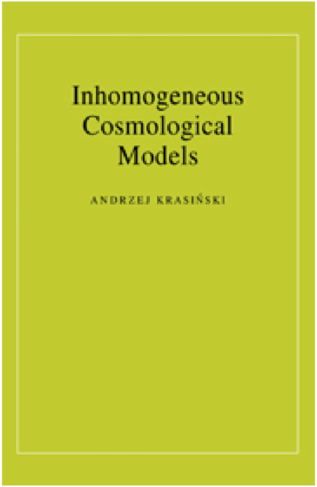
Inhomogeneous cosmological models?



Walls, voids and other inhomogeneities exist in the universe

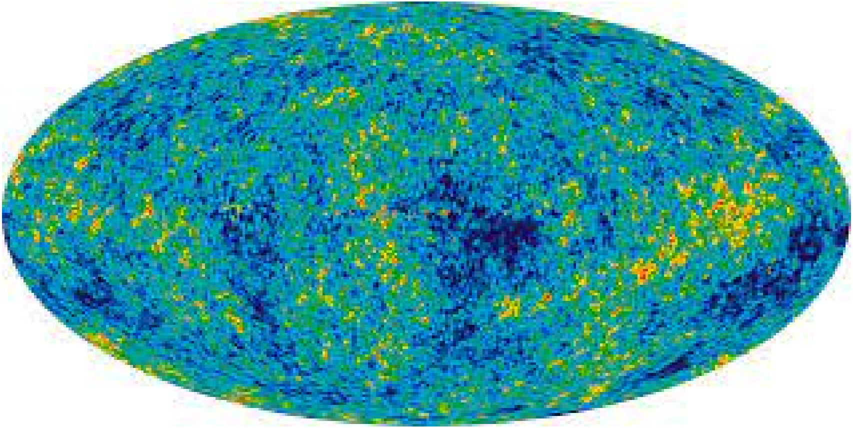
Inhomogeneous cosmological models?...

- Lemaître-Tolman-Bondi (LTB) models- isotropic expanding (contracting) but not homogenous solutions
- Szekeres models - LTB with no symmetry



A large class of inhomogeneous solutions

- Anisotropic cosmological models?



Temperature fluctuations of the CMB according to WMAP

- Since the isotropy assumption is only an approximation on large scales, there is the possibility that relaxing these assumptions may lead to solutions explaining the evolution history of the universe
- At early times, the universe was full of anisotropies with a highly irregular mechanism that isotropized later
- In fact, there are several claims regarding some degree of anisotropy in the observed universe that necessitates the consideration of a non-FLRW geometry
- There is a need for scrutiny of cosmological models that describe an early-time anisotropy with a proper mechanism to produce [near] isotropy at late times on the one hand, and an accelerated expansion at the present epoch on the other
- Bianchi models to the rescue: homogenous but anisotropic cosmological models
  - ✓ 9 possible cosmological solutions
  - ✓ Bianchi-I and Bianchi-V are the simplest, and probably the most widely explored

Group Class	Group Type	n_1	n_2	n_3
A $(a_i = 0)$	I	0	0	0
	II	+	0	0
	VI_0	0	+	−
	VII_0	0	+	+
	IX	+	+	+
B $(a_i \neq 0)$	V	0	0	0
	IV	0	0	+
	VI_h	0	+	−
	VII_h	0	+	+

Bianchi classifications

Changing [fundamental] 'constants'?

- Dirac's hypothesis [3] that the gravitational constant decreases with time has been a matter of scrutiny for some time, but recent attempts to consider both Λ and the universal gravitational constant G as dynamical quantities, and therefore not as constants, has gained more attention due to the aforementioned not-so-well-explained cosmic acceleration.
- Different forms of changing Λ and G assumptions exist in the literature, such as:

$$\Lambda = \frac{\alpha}{a^2} + \beta H^2 , \quad G = G_0 a^{\delta}$$

  - ✓ Constants α , β , δ etc to be determined from both theoretical and observational considerations

[3] $G \propto \frac{1}{t}$; physical constants depend on the age of the universe t.

- The Bianchi type-I is identified by the metric of the form [4]

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)\, dy^2 + C^2(t)\, dz^2$$

where $A(t), B(t)$ and $C(t)$ are the scale factors along $x,\ y$ and z direction

- Perfect-fluid cosmic matter distribution is given by the following energy-momentum tensor:

$$T_{ij} = (p + \rho)u_i u_j + pg_{ij}$$

where ρ is matter density, $u^i = \delta^i_t = (-1, 0, 0, 0)$ is the normalized fluid four-velocity, which is a time-like quantity such that $u^i u_i = -1$, and p is the fluid's isotropic pressure that is related to mater density through the barotropic equation of state (EoS) $p = w\rho$, with

$$w = \begin{cases} 0 & \text{for dust} \\ 1/3 & \text{for radiation} \\ -1 & \text{for dark energy} \end{cases}$$

[4] Alfedeel, A. H., & **AA**. (2022), The evolution of time-dependent Λ and G in multi-fluid Bianchi-I models, Open Astronomy 31 198 (2022)

- The Einstein Field Equations with time-dependent $\Lambda = \frac{\alpha}{a^2} + \beta H^2$, where $a = (ABC)^{1/3}$ is average scale factor, $H \equiv 1/3\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)$ is the average Hubble parameter, and $G = G(t)$ and the conservation of T^{ij} (*i.e.*, $\nabla_j T^{ij} = 0$) are reduced to the energy density evolution

$$\dot{\rho} + (\rho + p)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0$$

giving a solution for the energy density as

$$\rho = \frac{\rho_0}{a^{3(1+w)}}$$

where ρ_0 corresponds to the current value of the energy density

- The generalized Friedmann equations read:

$$8\pi G p - \Lambda = (2q - 1)H^2 - \sigma^2$$
$$8\pi G \rho + \Lambda = 3H^2 - \sigma^2$$

where σ is the shear modulus, q is deceleration parameter. Or alternatively

$$\frac{\ddot{a}}{a} + (2 - \beta)\frac{\dot{a}^2}{a^2} - \frac{\alpha}{a^2} = 4\pi G(t)(\rho - p)$$

- The time evolution equation connecting G and Λ can be given by

$$8\pi \rho \dot{G} + \dot{\Lambda} = 0$$

- Let's introduce

$$\Omega_{i0} \equiv \frac{8\pi G_0}{3H_0^2}\rho_{i0} \implies \rho_i = \frac{3H_0^2}{8\pi G_0}\,\Omega_{i0}(1+z)^{(1+w)}$$

- Background evolution

$$\frac{da}{dt} = Z \tag{2.1}$$

$$\frac{dZ}{dt} = -\frac{(2-\beta)\,Z^2}{a} + \frac{\alpha}{a} + c_1 SG \tag{2.2}$$

$$\frac{dG}{dt} = c_2\frac{Z}{a^3} + c_3\frac{Z^3}{a^3} - c_4\frac{Z}{a^2}\,G \tag{2.3}$$

where we have used the following short-hands:

$$c_1 \equiv \frac{3H_0^2}{2G_0}\left(\Omega_{m0}(1+z)^3 + \frac{2\Omega_{r0}}{3}(1+z)^4 + 2\Omega_{\Lambda 0}\right)$$

$$c_2 \equiv \frac{2\alpha(1-\beta)G_0}{3H_0^2}$$

$$c_3 \equiv \frac{2\beta(3-\beta)G_0}{3H_0^2}$$

$$c_{\equiv}\beta\left(\Omega_{m0}(1+z)^3 + \frac{2\Omega_{r0}}{3}(1+z)^4 + 2\Omega_{\Lambda 0}\right)$$

- $\Omega_{m0} + \Omega_{r0} + \Omega_{\Lambda 0} = 1$ by definition

- Rewrite Eqs. (2.2) and (2.3) in redshift space

$$\frac{dH}{dz} = \frac{(3-\beta)}{(1+z)}\, H - \frac{\alpha(1+z)}{H} - \frac{c_1 G}{(1+z)H} \tag{2.4}$$

$$\frac{dG}{dz} = -c_2(1+z) - \frac{c_3 H^2}{(1+z)} + c_4 G \tag{2.5}$$

- Define the dimensionless parameters :

$$h \equiv \frac{H}{H_0}\ , \quad \lambda \equiv \frac{\Lambda}{\Lambda_0}\ , \quad g \equiv \frac{G}{G_0}\ , \quad \gamma \equiv \frac{\alpha}{H_0^2}$$

- Equations in dimensionless parameters

$$\frac{dh}{dz} = \frac{(3-\beta)}{(1+z)} h - \frac{\gamma(1+z)}{h} - \frac{3}{2}\frac{g}{(1+z)h} \times \left(\Omega_{\mathrm{m0}}(1+z)^3 + \frac{2\Omega_{\mathrm{r0}}}{3}(1+z)^4 + 2\Omega_{\Lambda 0}\right) \tag{2.6}$$

$$\frac{dg}{dz} = \beta\left(\Omega_{\mathrm{mo}}(1+z)^3 + \frac{2\Omega_{\mathrm{r0}}}{3}(1+z)^4 + 2\Omega_{\Lambda 0}\right) g - \frac{2\beta}{3}(3-\beta)\frac{h^2}{1+z} - \frac{2\gamma}{3}(1-\beta)(1+z) \tag{2.7}$$

- The deceleration parameter is then

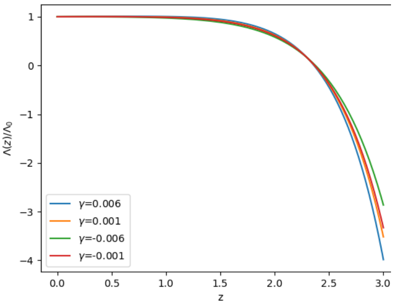
$$q = (2-\beta) - \frac{\gamma(1+z)^2}{h^2} - \frac{3}{2}\left(\Omega_{\mathrm{m0}}(1+z)^3 + \frac{2\Omega_{\mathrm{r0}}}{3}(1+z)^4 + 2\Omega_{\Lambda 0}\right)\frac{g}{h^2}$$

- Observational values used from Planck-2018 [5]

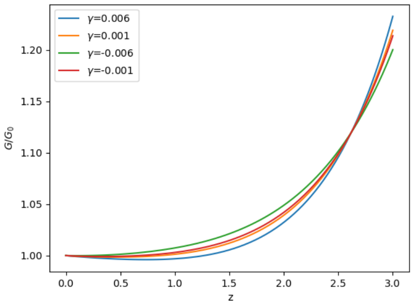
$$\Omega_{\rm m0} = 0.3111\ ,\quad \Omega_{\Lambda 0} = 0.6889\ ,\quad \Omega_{\rm r0} = 1 - \Omega_{\rm m0} - \Omega_{\Lambda 0}\ ,\quad H_0 = 67.37 \text{km/s/Mpc}$$

- Initial conditions used

$$h(0) = g(0) = 1\ ,\quad \beta = 0.02$$



Λ vs z



g vs z.

Variation of Λ and G with redshift

[5] Planck Collaboration: Planck 2018 results. VI. Cosmological parameters, Astron. Astrophys **641**, A6 (2020)

- Here we consider the Bianchi type$-V$ with spacetime metric of the form[6]

$$ds^2 = dt^2 - A^2dx^2 - e^{2mx}[B^2dy^2 + C^2dz^2]$$

where m is constant

- We assume that the universe is filled by a viscous fluid whose distribution in space is represented by the following energy momentum tensor:

$$T_{ij} = (\rho + \bar{p})u_iu_j + \bar{p}g_{ij} - 2\eta\sigma_{ij} \ , \quad \bar{p} = p - (3\xi - 2\eta) H$$

where η and ξ are coefficients of shear and bulk viscosity respectively, σ_{ij} is the shear and $\bar{p}$ is the effective pressure

- Assume a linear equation of state

$$p = w\rho \ , \qquad -1 \leq w \leq 1$$

- The shear tensor is given by

$$\sigma_{ij} = \left(u_{i;k}h^k_j + \dot{u}_{j;k}h^k_i\right) - \frac{1}{3}\theta h_{ij} \text{ ,where } h_{ij} \equiv g_{ij} + u_iu_j$$

[6] Tiwari, R.K., Alfedeel, A. H., Sofuoğlu, D., AA, Eltagani, I.H., & Shukla, B. H. (2022), A cosmological model with time-dependent Λ, G, and viscous fluid in General Relativity, *Front. Astron. Space Sci.* **9** 965652 (2022)

- The $\nabla^j T_{ij} = 0$ equation leads to the fluid continuity equation:

$$\kappa G \left[\dot{\rho} + (\overline{p} + \rho) \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \right] + \kappa \rho \dot{G} + \dot{\Lambda} - 4\kappa G \eta \sigma^2 = 0$$

- Split this into the following two equations (total matter content conserved):

$$\dot{\rho} + 3H\left[p + \rho - (3\xi - 2\eta)H\right] - 4\eta\sigma^2 = 0$$
$$\kappa \rho \dot{G} + \dot{\Lambda} = 0$$

- Meanwhile, the EFEs can be written in terms of H, σ and q as

$$\kappa G \overline{p} - \Lambda = H^2(2q - 1) - \sigma^2 + \frac{m^2}{A^2}$$
$$\kappa G \rho + \Lambda = 3H^2 - \sigma^2 - \frac{3m^2}{A^2}$$

with

$$a \equiv (ABC)^{1/3} \ , \quad H \equiv \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \ , \quad q \equiv -\frac{a\ddot{a}}{\dot{a}^2}$$

$$\sigma^2 \equiv \frac{1}{6}\left[\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right)^2 + \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right)^2 + \left(\frac{\dot{C}}{C} - \frac{\dot{A}}{A} \right)^2 \right]$$

- The generalized Raychaudhuri equation reads:

$$\dot{H} + 3H^2 - \frac{2m^2}{a^2} - \Lambda + \frac{\kappa G}{2}(p - \rho) - \kappa G \left(\frac{3\xi}{2} - \eta \right) H = 0$$

- To find the solution by introducing extra information in the form of initial conditions and a constraint, we consider the following form of the Friedmann Equation:

$$1 = \Omega_m + \Omega_\Lambda + \Omega_\sigma + \Omega_\chi$$

where the density parameters are defined as:

$$\Omega_m \equiv \frac{\kappa G \rho_m}{3H^2} \ , \qquad \Omega_\Lambda \equiv \frac{\kappa G \rho_\Lambda}{3H^2} \ , \qquad \Omega_\sigma \equiv \frac{\sigma^2}{3H^2} \ , \qquad \Omega_\chi \equiv \frac{K^2}{H^2 a^2}$$

- The current values of these dimensionless density parameters are given in terms of the current values of the quantities that describe them, as

$$\Omega_{m_0} = \frac{\kappa G_0 \rho_{m_0}}{3H_0^2} \ , \qquad \Omega_{\Lambda_0} = \frac{\kappa G_0 \rho_{\Lambda_0}}{3H_0^2} \ , \qquad \Omega_{\sigma_0} = \frac{\sigma_0^2}{3H_0^2} \ , \qquad \Omega_{\chi 0} = \frac{K^2}{H_0^2 a_0^2}$$

- We also define the following dimensionless parameters:

$$h \equiv \frac{H}{H_0} \ , \qquad a = \frac{1}{(1+z)} \ , \qquad \xi = \alpha H_0 (\rho_m/\rho_{m0})^n \ , \qquad \eta = \beta H$$

with α, β and $0 \leq n \leq \frac{1}{2}$ introduced as dimensionless constants

- We assume the ansatz, in accordance with Dirac's hypothesis

$$G(t) = G_0 a^{\delta}$$

- Here, $\delta = -1/60$ is a constant obtained from observational constraints[7] [8]
- The conservation equations in terms of the dimensionless density parameters:

$$h' = \frac{h}{(1+z)}\left[3 - 2\Omega_\chi - 3\Omega_\Lambda - \frac{3}{2}(1-w_m)\Omega_m\right] - \frac{\kappa G_0}{(1+z)^{1+\delta}}\left[\frac{3\alpha}{2}\left(\frac{h^2\Omega_m(1+z)^\delta}{\Omega_{m0}}\right)^n - \beta h\right]$$

$$\Omega_m' = -\frac{2h'}{h}\Omega_m + \frac{1}{1+z}\left(-\delta + 3 + 3w_m\right)\Omega_m - \frac{\kappa G_0}{(1+z)^{1+\delta}}\left[\frac{3\alpha}{h}\left(\frac{h^2\Omega_m(1+z)^\delta}{\Omega_{m0}}\right)^n - 2\beta + 4\beta\Omega_\sigma\right]$$

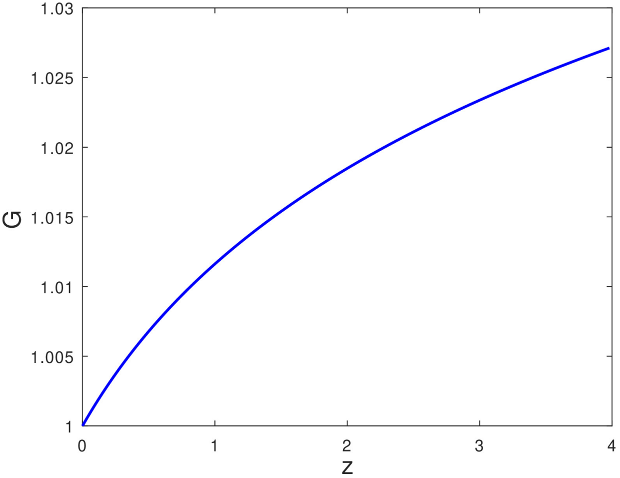
$$\Omega_\Lambda' = -\frac{2h'}{h}\Omega_\Lambda - \frac{\delta}{1+z}\Omega_m$$

$$\Omega_\chi' = -\frac{2h'}{h}\Omega_\chi + \frac{2\Omega_\chi}{1+z}\ ,$$

$$\Omega_\sigma' = -\frac{2h'}{h}\Omega_\sigma + \frac{6\Omega_\sigma}{1+z}$$

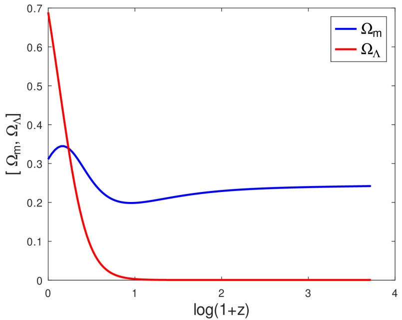
[7] Williams, J.G.; Turyshev, S.G.; Boggs, D.H. Lunar laser ranging tests of the equivalence principle with the earth and moon. Int. J. Mod. Phys. D 2009, 18, 1129–1175

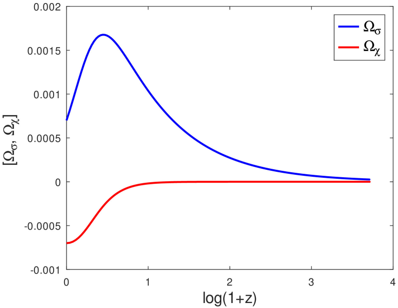
[8] Copi, C.J.; Davis, A.N.; Krauss, L.M. New nucleosynthesis constraint on the variation of G. Phys. Rev. Lett. 2004, 92, 171301



The variation of G for viscous Bianchi type-V cosmological model vs redshift

$\sim$ 2.5% change in G in about 12 billion years!





The variation of the density parameters for viscous Bianchi type-V cosmological model vs redshift.

Evolution of the matter perturbations [9]:

$$\Delta' + \frac{3}{(1+z)}\Bigg\{w - \frac{\kappa G_0}{\Omega_m h(1+z)^\delta}\left[\alpha\left(\frac{h^2\Omega_m(1+z)^\delta}{\Omega_{m0}}\right)^n - \frac{2\beta}{3}h\right] - \frac{4\beta\kappa G_0}{3(1+z)^\delta}\frac{\Omega_\sigma}{\Omega_m}$$
$$+\frac{4\beta\Omega_\sigma}{3\Omega_m}\frac{\kappa G_0}{(1+z)^\delta}\left[\frac{\alpha n\left(\frac{h^2\Omega_m(1+z)^\delta}{\Omega_{m0}}\right)^n - \frac{wh\Omega_m}{\kappa G_0(1+z)^{-\delta}}}{\frac{(1+w)h\Omega_m}{\kappa G_0(1+z)^{-\delta}} - \left[\alpha\left(\frac{h^2\Omega_m(1+z)^\delta}{\Omega_{m0}}\right)^n - \frac{2\beta}{3}h\right]}\right]\Bigg\}\Delta$$
$$-\frac{1}{h(1+z)}\Bigg\{1 + w - \frac{\kappa G_0}{\Omega_m h(1+z)^\delta}\left[\alpha\left(\frac{h^2\Omega_m(1+z)^\delta}{\Omega_{m0}}\right)^n - \frac{2\beta}{3}h\right] - \frac{4\beta\kappa G_0}{3(1+z)^\delta}\frac{\Omega_\sigma}{\Omega_m}$$
$$+\frac{4\beta\Omega_\sigma}{3\Omega_m}\frac{\kappa G_0}{h(1+z)^\delta}\left[\frac{\alpha\left(\frac{h^2\Omega_m(1+z)^\delta}{\Omega_{m0}}\right)^n - \frac{4\beta}{3}h}{\frac{(1+w)h\Omega_m}{\kappa G_0(1+z)^{-\delta}} - \left[\alpha\left(\frac{h^2\Omega_m(1+z)^\delta}{\Omega_{m0}}\right)^n - \frac{2\beta}{3}h\right]}\right]\Bigg\}\mathcal{Z}$$
$$-\frac{8\beta\kappa G_0}{3h(1+z)^{\delta+1}}\frac{\sqrt{3\Omega_\sigma}}{\Omega_m}\mathcal{S} = 0 \qquad (2.8)$$

[9] AA, Alfedeel, A. H., Sofuoğlu, D., Eltagani, I.H., & Tiwari, R.K.. (2023), Perturbations in Bianchi - V spacetimes with varying Λ, *G* and viscous fluids, Universe **9**, 61

Evolution of the perturbations in the expansion:

$$\mathcal{Z}' - \frac{1}{(1+z)\ h}\Bigg\{2h - \frac{3}{2}\frac{\kappa G_0}{(1+z)^{\delta}}\left[\alpha\left(\frac{h^2\Omega_m(1+z)^{\delta}}{\Omega_{m0}}\right)^n - \frac{4\beta}{3}h\right]$$

$$- \frac{\alpha\left(\frac{h^2\Omega_m(1+z)^{\delta}}{\Omega_{m0}}\right)^n - \frac{4\beta}{3}h}{\frac{(1+w)h^2\Omega_m}{\kappa G_0(1+z)^{-\delta}} - h\left[\alpha\left(\frac{h^2\Omega_m(1+z)^{\delta}}{\Omega_{m0}}\right)^n - \frac{2\beta}{3}h\right]}\left[-hh'(1+z) + \frac{2}{3}h^2\Omega_{\chi} - \frac{\gamma}{3}(1+z)^2\right]\Bigg]\Bigg\}\mathcal{Z}$$

$$+\frac{3}{(1+z)}\Bigg[\frac{\Omega_m h}{2}(1+3w) - \frac{3}{2}\frac{\alpha n\kappa G_0}{(1+z)^{\delta}}\left(\frac{h^2\Omega_m(1+z)^{\delta}}{\Omega_{m0}}\right)^n$$

$$+\frac{\alpha n\left(\frac{h^2\Omega_m(1+z)^{\delta}}{\Omega_{m0}}\right)^n - \frac{wh\Omega_m}{\kappa G_0(1+z)^{-\delta}}}{\frac{(1+w)h^2\Omega_m}{\kappa G_0(1+z)^{-\delta}} - h\left\{\alpha\left(\frac{h^2\Omega_m(1+z)^{\delta}}{\Omega_{m0}}\right)^n - \frac{2\beta}{3}h\right\}}\left[-hh'(1+z) + \frac{2}{3}h^2\Omega_{\chi} - \frac{\gamma}{3}(1+z)^2\right]\Bigg]\Delta$$

$$-\frac{4\sqrt{\Omega_{\sigma}}}{(1+z)}\mathcal{S} = 0$$

Evolution of the shear perturbations:

$$\mathcal{S}' - \frac{3}{(1+z)}\mathcal{S} - \frac{\sqrt{3\Omega_\sigma}}{(1+z)}\left[1 + \frac{\alpha\left(\frac{h^2\Omega_m(1+z)^\delta}{\Omega_{m0}}\right)^n - \frac{4\beta}{3}h}{\frac{(1+w)h\Omega_m}{\kappa G_0(1+z)^{-\delta}} - \left[\alpha\left(\frac{h^2\Omega_m(1+z)^\delta}{\Omega_{m0}}\right)^n - \frac{2\beta}{3}h\right]}\right]\mathcal{Z}$$

$$-\frac{3h\sqrt{3\Omega_\sigma}}{(1+z)}\left[\frac{\alpha n\left(\frac{h^2\Omega_m(1+z)^\delta}{\Omega_{m0}}\right)^n - \frac{wh\Omega_m}{\kappa G_0(1+z)^{-\delta}}}{\frac{(1+w)h\Omega_m}{\kappa G_0(1+z)^{-\delta}} - \left[\alpha\left(\frac{h^2\Omega_m(1+z)^\delta}{\Omega_{m0}}\right)^n - \frac{2\beta}{3}h\right]}\right]\Delta = 0$$

- Fix the background expansion history
- Set initial conditions at some redshift z_{in}, solve the system of perturbation equations for $\Delta(z)$ and compare it with that of standard GR/ΛCDM

$$\Delta(z_{in}) = 10^{-5}\,, \quad \mathcal{Z}(z_{in}) = 10^{-5}\,, \quad \Sigma(z_{in}) = 10^{-5}$$

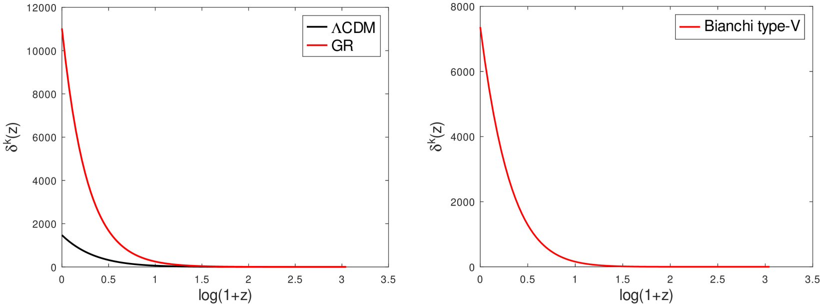
- Let's first define the normalized matter density contrast

$$\delta(z) \equiv \frac{\Delta(z)}{\Delta(z_{in})} \tag{2.9}$$

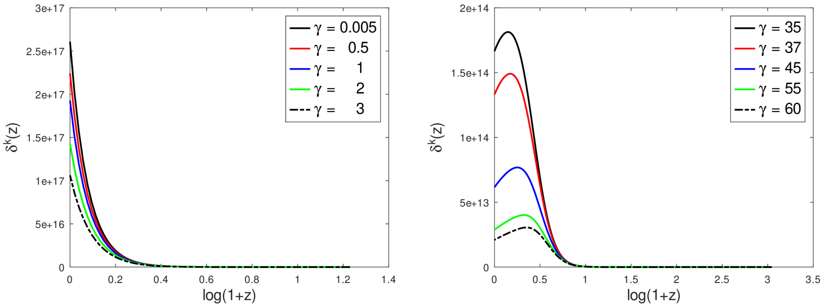
  with $z_{in} = 20$ in both GR/ΛCDM and our current models
- We have also used the following dimensionless viscosity parameters: $\alpha = 0.312, \beta = 1, n = 0.2$, as well as the current values from PLANCK2018:

$$\Omega_{m0} = 0.3111\,, \quad \Omega_{\Lambda 0} = 0.6889\,, \quad \Omega_{\chi 0} = -0.0007\,, \quad \Omega_{\sigma 0} = 1 - \Omega_{m0} - \Omega_{\Lambda 0} - \Omega_{\chi 0}$$

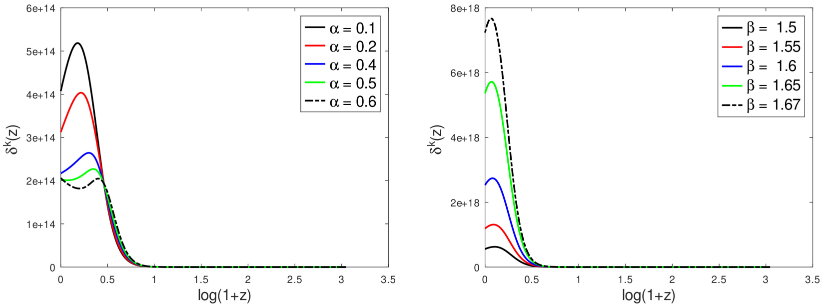
- The following are some of the highlights of our observations:
  - ✓ Increasing α decreases the late-time perturbation amplitude in the short-wavelength regime, but this effect is reversed for $z \gtrsim 0.65$
  - ✓ Increasing α increases the perturbation amplitude in the long-wavelength regime
  - ✓ Increasing β increases the perturbation amplitudes in both the short- and long-wavelength regimes
  - ✓ Increasing n increases the perturbation amplitudes in both the short- and long-wavelength regimes



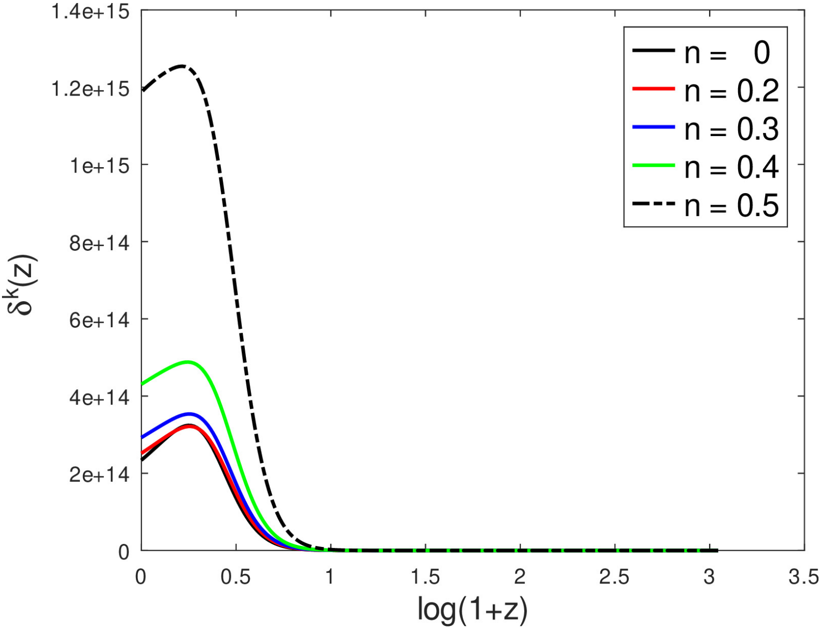
Growth of matter density perturbations $\delta(z)$ vs z. Left: for ΛCDM and GR without Λ ($\Omega_m = 1$, $\Omega_\Lambda = 0$); Right: for the Bianchi type-V model for non-viscous ($\alpha = 0 = \beta$) fluid, but with changing G and Λ



Growth of matter density perturbations $\delta(z)$ vs z for the viscous Bianchi type-V cosmological model. Left: for long-wavelength regimes; Right: for short-wavelength regimes



Growth of matter density perturbations $\delta(z)$ vs z for the viscous Bianchi type-V cosmological model. Left: for varying values of α; Right: for varying values of β



The variation of the matter density perturbations $\delta(z)$ for a viscous Bianchi type-V cosmological model vs. redshift for $\gamma = 50$, $\alpha = 0.3$, $\beta = 1$ and different values of n.

- The EMT for perfect-fluid models is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$$

- The divergence-free EMT $T^{\mu\nu}{}_{;\mu} = 0$ leads to the fluid conservation equation

$$T^{\mu\nu}{}_{;\mu} = 0 \implies \dot{\rho} + 3\frac{\dot{a}}{a}(1 + w)\rho = 0$$

- In a multi-component fluid system, it is usually assumed that the energy density of each perfect-fluid component is assumed to evolve independently of the other fluids of the system:

$$\dot{\rho}_i + 3H(1 + w)\rho_i = 0$$

  and in this case the EMT is the algebraic sum of the EMTS of each fluid, so are the total energy density and total pressure terms the algebraic sums of the individual components

- However, if we relax this assumption due to the presence of diffusion, the individual components do not obey the matter conservation equation, but the total fluid still does. For the the ith component fluid, the new conservation equation reads:

$$T_i^{\mu\nu}{}_{;\mu} = N_i^\nu$$

  where N_i^ν corresponds to the current of diffusion term for that fluid

- One can write the non-conservation equation for the fluid as [10]:

$$\dot{\rho}_i + 3\frac{\dot{a}}{a}(1+w)\rho_i = \frac{\gamma_i}{a^3}$$

where γ_i is a constant for that fluid such that $\sum_i \gamma_i = 0$

- Integrating the above equation gives

$$\rho_i = a^{-3(1+w_i)}\left[\rho_{i0} + \gamma_i \int_{t_0}^{t} a^{3w_i}\, dt'\right]$$

where ρ_{i0} is the present-day $(t = t_0)$ value of the energy density of the ith fluid

- Using a late-time, i.e., $t - t_0 \ll t_0$, expansion and expressing $a(t) = a_0\left[1-(t_0-t)H_0) + \ldots\right]$, we can write [11] the last term of the above integrand as

$$\int_{t_0}^{t} a^{3w_i}\, dt = \int_{t_0}^{t} a^{3w_i}\left[1-(t_0-t)H_0) + \ldots\right]^{3w_i} dt' \qquad (3.1)$$

[10] Maity, S., Bhandari, P., & Chakraborty, S. (2019). Universe consisting of diffusive dark fluids: thermodynamics and stability analysis. The European Physical Journal C, 79(1), 1-8.

[11] RR Mekuria, AA (2023), Observational constraints of diffusive dark-fluid cosmology, preprint arXiv:2301.02913

- Evaluating the previous integral and applying Taylor expansion around t_0 yields

$$
\begin{aligned}
\int_{t_0}^{t} a^{3w_i}\,dt &= -\frac{1}{1+3w_i}\left[(1+(t_0-t)H_0)^{1+3w_i} - (1+(t_0-t)H_0)^{1+3w_i} + \dots\right] \\
&\approx \frac{1}{1+3w_i}\left[1-(1+(t_0-t)H_0)^{1+3w_i}\right] \\
&= \frac{1}{(1+3w_i)H_0}\left[1-(2-a)^{1+3w_i}\right]
\end{aligned}
$$

where in the last step, we have normalised the scale factor to unity today: $a_0 = 1$

- The energy density of each diffusive fluid component is given according to the below relation:

$$
\rho_i = a^{-3(1+w_i)}\left\{\rho_{i0} + \frac{\gamma_i}{(1+3w_i)H_0}\left[1-(2-a)^{1+3w_i}\right]\right\}
$$

- Assuming the well-known component of radiation, dust-like matter (baryons and dark matter) and vacuum energy, the above diffusive solution leads to:

$$
\begin{aligned}
\rho_{\rm r} &= a^{-4}\left\{\rho_{\rm r0} + \frac{\gamma_{\rm r}}{2H_0}\left[1-(2-a)^2\right]\right\} \\
\rho_{\rm m} &= a^{-3}\left\{\rho_{\rm m0} + \frac{\gamma_{\rm m}}{H_0}\left[1-(2-a)\right]\right\} \\
\rho_\Lambda &= \rho_{\Lambda 0} - \frac{\gamma_\Lambda}{2H_0}\left[1-(2-a)^{-2}\right]
\end{aligned}
$$

- Let us now consider the Friedmann equation for the ΛCDM model for $k = 0$:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\left\{\rho_{r0}a^{-4} + \rho_{m0}a^{-3} + \frac{\gamma_m}{H_0}\left[1-(2-a)\right]a^{-3} + \rho_{\Lambda 0} - \frac{\gamma_\Lambda}{2H_0}\left[1-(2-a)^{-2}\right]\right\}$$

- We assume the diffusive interaction is limited between dark matter and dark energy for now, i.e., $\gamma_r = 0$, and introduce the following dimensionless quantities:

$$\Omega_i \equiv \frac{8\pi G}{3H_0^2}\rho_i\ , \qquad \Delta_i \equiv \frac{8\pi G}{3H_0^3}\gamma_i\ , \qquad 1+z \equiv a^{-1}\ , \qquad h \equiv \frac{H}{H_0}$$

- We can then show that the Friedmann equation can be recast as

$$h^2 = \Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0} - \Delta_m z(1+z)^2 - \Delta_\Lambda\left[\frac{1}{2} - \frac{1}{2}\left(\frac{1+2z}{1+z}\right)^{-2}\right]$$

- Moreover, defining the deceleration parameter as

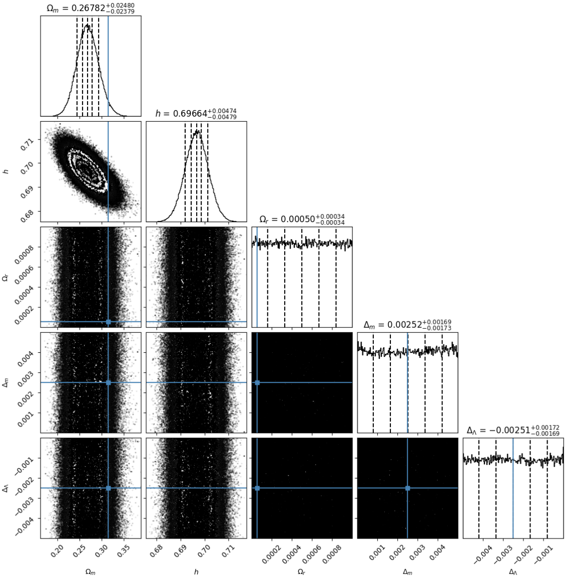
$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = \frac{4\pi G}{3H^2}\sum_i \rho_i(1+3w_i)$$

we can show that for our current model, we have

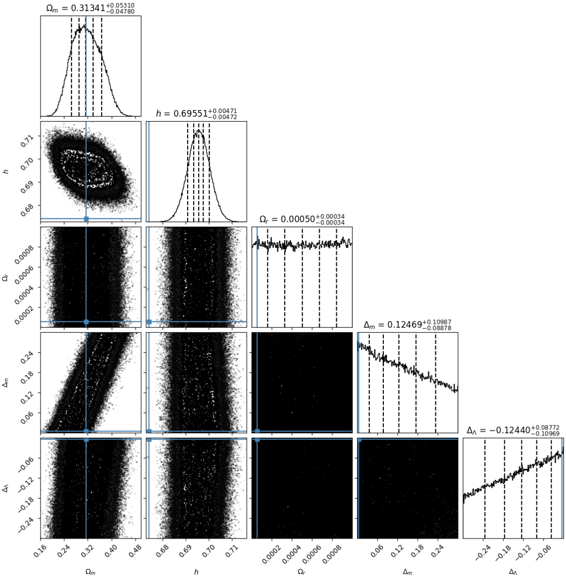
$$q = \frac{1}{2}\left\{\frac{2\Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 - 2\Omega_{\Lambda 0} - \Delta_m z(1+z)^2 + \Delta_\Lambda\left[1-\left(\frac{1+2z}{1+z}\right)^{-2}\right]}{\Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0} - \Delta_m z(1+z)^2 - \Delta_\Lambda\left[\frac{1}{2}-\frac{1}{2}\left(\frac{1+2z}{1+z}\right)^{-2}\right]}\right\}$$

Four diffusive cases of the model and their best-fitting parameter values.

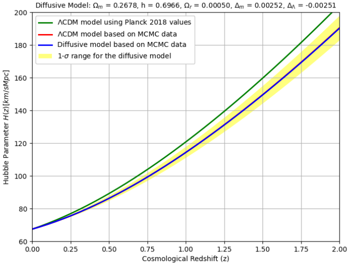
Models	h	$\Omega_{\mathrm{m}0}$	$\Omega_{\mathrm{r}0}$	Δ_{m}	Δ_{Λ}
Diffusive Case I	0.6966	0.2678	0.00050	0.00252	-0.00251
Diffusive Case II	0.6955	0.3134	0.00050	0.1246	-0.1244
ΛCDM	**0.674**	**0.315**	**0.00050**	**0**	**0**
Diffusive Case III	0.6967	0.2655	0.00050	-0.00251	0.00246
Diffusive Case IV	0.6976	0.2283	0.00050	-0.10747	0.10426



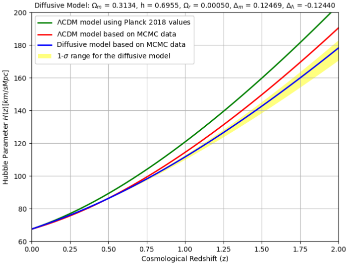
The MCMC simulation results for Case I, with the "true" values for $\Omega_{m0} = 0.315$, $\bar{h} = 0.674$, and $\Omega_{r0} = 2.47 \times 10^{-5} / \bar{h}^2$ provided by the Planck2018 data. 100 random walkers and 10000 iterations



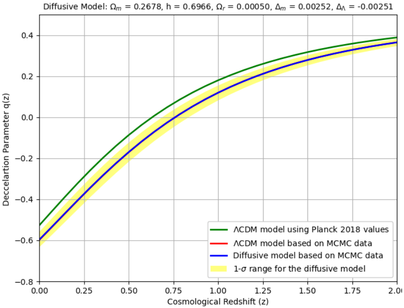
The MCMC simulation results for Case II, with the "true" values for $\Omega_{m0} = 0.315$, $\bar{h} = 0.674$, and $\Omega_{r0} = 2.47 \times 10^{-5} / \bar{h}^2$ provided by the Planck2018 data. 100 random walkers and 10000 iterations.



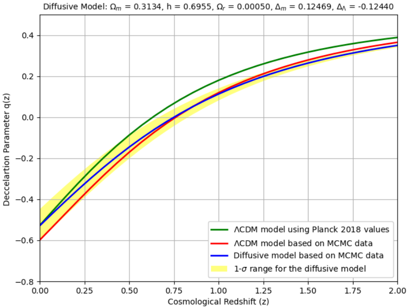
The Hubble parameter vs redshift for Case I. The blue curve represent the result obtained by considering diffusive fluid and employing MCMC simulation, with 1-σ deviation result displayed in yellowish shaded region. The red curve represents ΛCDM cosmology result using MCMC simulation where as the green curve represent one obtained directly by using the Planck 2018 data for the purpose of comparison.



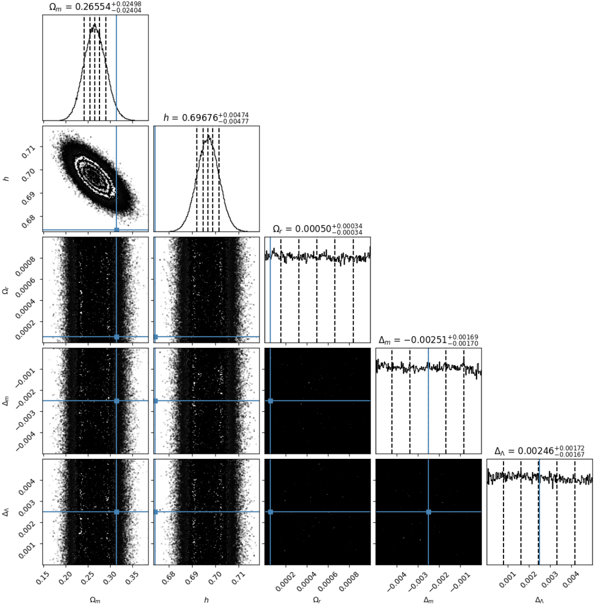
The Hubble parameter vs redshift for Case II. The blue curve represent the result obtained by considering diffusive fluid and employing MCMC simulation, with 1-σ deviation result displayed in yellowish shaded region. The red curve represent ΛCDM cosmology result using MCMC simulation where as the green curve represent one obtained directly by using the Planck 2018 data for the purpose of comparison.



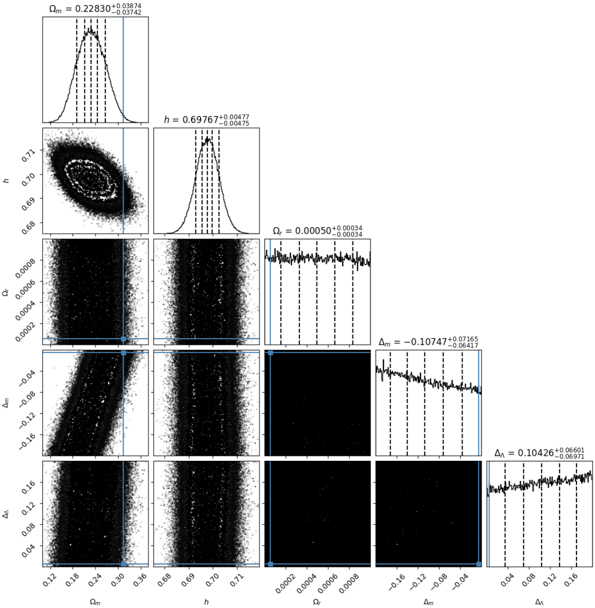
The graph of deceleration parameter vs redshift for Case I.



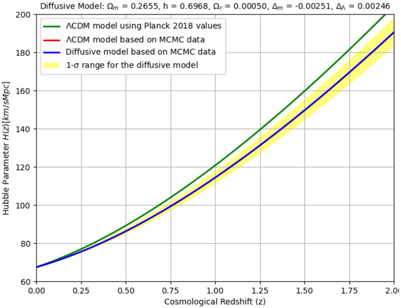
The graph of deceleration parameter vs redshift for Case II.



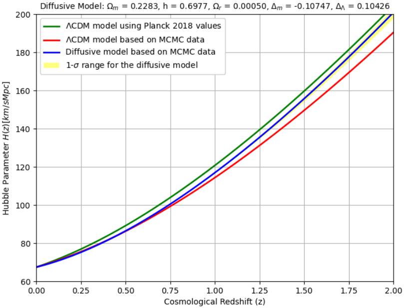
The MCMC simulation results for Case III, with the "true" values for $\Omega_{m0} = 0.315$, $\bar{h} = 0.674$, and $\Omega_{r0} = 2.47 \times 10^{-5} \ /\bar{h}^2$ provided by the Planck2018 data. 100 random walkers and 10000 iterations.



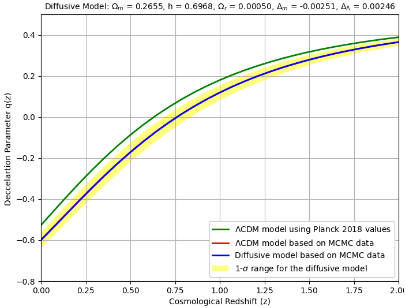
The MCMC simulation results for Case IV, with the "true" values for $\Omega_{m0} = 0.315$, $\bar{h} = 0.674$, and $\Omega_{r0} = 2.47 \times 10^{-5}\ /\bar{h}^2$ provided by the Planck2018 data. 100 random walkers and 10000 iterations.



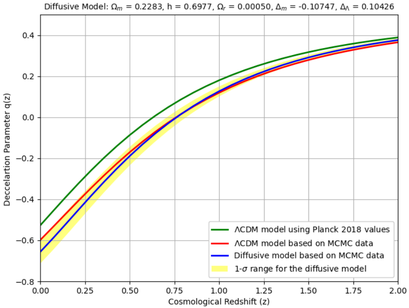
The Hubble parameter vs redshift for Case III. The blue curve represent the result obtained by considering diffusive fluid and employing MCMC simulation, with 1-σ deviation result displayed in yellowish shaded region. The red curve represents ΛCDM cosmology result using MCMC simulation where as the green curve represents one obtained directly by using the Planck 2018 data for the purpose of comparison.



The Hubble parameter vs redshift for Case IV. The blue curve represents the result obtained by considering diffusive fluid and employing MCMC simulation, with 1-σ deviation result displayed in yellowish shaded region. The red curve represents ΛCDM cosmology result using MCMC simulation where as the green curve represent one obtained directly by using the Planck 2018 data for the purpose of comparison.



The graph of deceleration parameter vs redshift for Case III.



The graph of deceleration parameter vs redshift for Case IV.

Models	$\Delta_{\rm m}$	Δ_Λ	χ^2	$\mathrm{Red.}\chi^2$	*AIC*	$\lvert\Delta AIC\rvert$	*BIC*	$\lvert\Delta BIC\rvert$
Diffusive Case II	+ve	-ve	242.3355	0.6845	252.3355	4.9405	271.7521	12.7072
Diffusive Case I	+ve	-ve	241.4118	0.6819	251.4118	4.0168	270.8285	11.7835
ΛCDM	**0**	**0**	**241.3950**	**0.6780**	**247.3950**	**0**	**259.0449**	**0**
Diffusive Case III	-ve	+ve	241.3781	0.6818	251.3781	3.9831	270.7947	11.7497
Diffusive Case IV	-ve	+ve	240.7872	0.6801	250.7872	3.3922	270.2039	11.1589

- They may be potential models to alleviate the cosmic coincidence problem by stabilising the ratio of dark matter to dark energy in both the past and future
- These models also predict a wide range of the values for H_0, thereby showing potential as a candidate for relieving the Hubble tension
- Cases having positive values of $\Delta_{\rm m}$ were showing the largest values of likelihood function. Based on the analysis of likelihood, goodness of fit, AIC and BIC criteria, one can conclude that overall Case I is the most likely to be an alternative to the ΛCDM model.
- Current work is to provide a viability test of the different cases considered, but to reject or accept any of them more work is needed
- Future directions: putting more stringent constraints on the values of the defining parameters of the model:
  - ✓ With more rigorous data and statistical analysis – using existing and upcoming cosmological data
  - ✓ Studying large-scale structure power spectrum, ISW effects, and other methods

More general interaction models

- For more general interactions

$$\dot{\rho}_{\rm dm} + 3H\rho_{\rm dm} = Q \qquad ; \qquad \dot{\rho}_{\rm de} + 3H\rho_{\rm de}(1+\omega) = -Q$$

  where Q is the rate of energy exchange, which defines the direction of energy flow between the dark sectors such that:

$$Q = \begin{cases} > 0 & \text{Dark Energy} \rightarrow \text{Dark Matter} \\ < 0 & \text{Dark Matter} \rightarrow \text{Dark Energy} \\ = 0 & \text{No interaction } (\Lambda\text{CDM case}) \end{cases}$$

- Model 1: $Q_1 = \delta H \rho_{\rm dm}$

$$\rho_{\rm dm} = \rho_{({\rm dm},0)}\, a^{(\delta-3)}$$

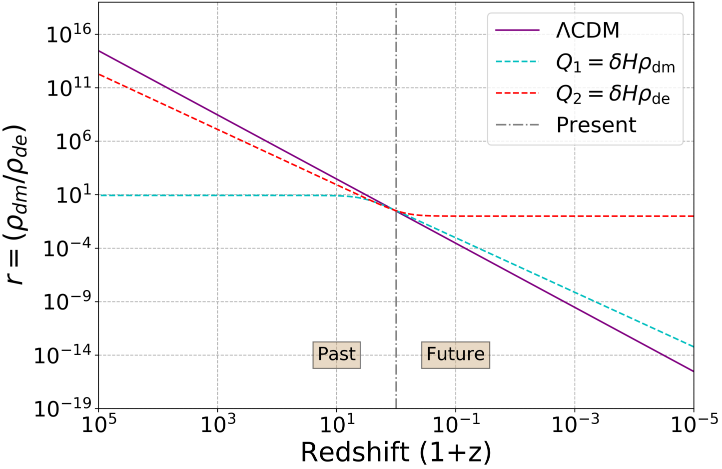
$$\rho_{\rm de} = \rho_{({\rm de},0)}\, a^{-3(1+\omega_{\rm de})} + \rho_{({\rm dm},0)} \frac{\delta}{\delta + 3\omega} \left[a^{-3\omega} - a^{\delta}\right] a^{-3}$$

- Model 2: $Q_2 = \delta H \rho_{\rm de}$

$$\rho_{\rm dm} = \rho_{({\rm dm},0)}\, a^{-3} + \rho_{({\rm de},0)} \frac{\delta}{\delta + 3\omega} \left[1 - a^{-(\delta+3\omega)}\right] a^{-3}$$

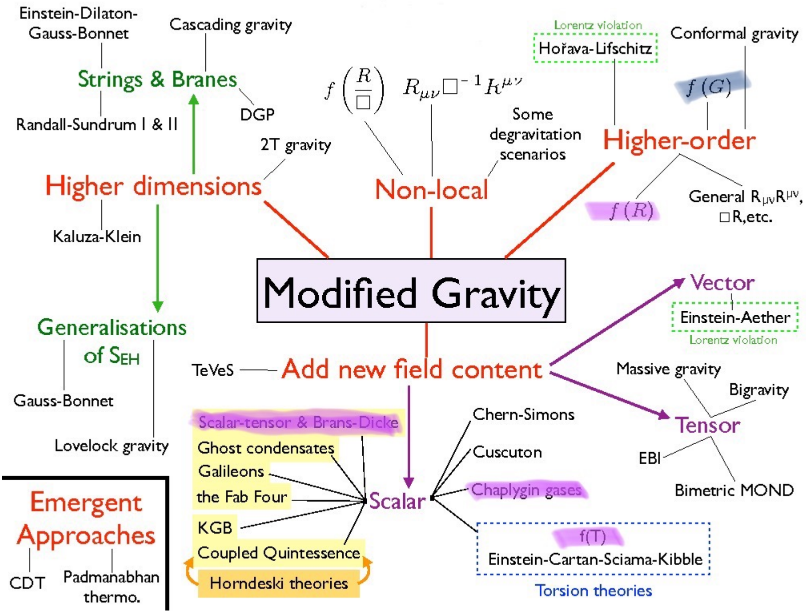
$$\rho_{\rm de} = \rho_{({\rm de},0)}\, a^{-(\delta+3\omega+3)}$$

The cosmic coincidence problem may now be addressed by considering how the ratio of dark matter to dark energy $r = (\rho_{\rm dm}/\rho_{\rm de})$ evolves with redshift z. Here it can clearly be seen that for the ΛCDM case, the current value of $r_0 \approx \left(\frac{3}{7}\right)$ seems fine tuned and coincidental in comparison to Q_1 and Q_2, where r converges and becomes constant in the past and the future respectively. Thus, alleviating the cosmic coincidence problem



Cosmic Coincidence Problem

- GR incomplete? Einstein generalized Newton's theory, and Newton modified earlier theories (such as Aristotle's), maybe it's time to rethink GR's unique status...
  - ✓ Theories of Gravity with Extra Fields
    - Scalar-Tensor theories, *e.g.*, Brans-Dicke theories
    - Einstein-Æther theories, *e.g.*, MOND
    - Bimetric theories
    - Tensor-Vector-Scalar theories (TeVeS)
  - ✓ Higher-dimensional Theories of Gravity
    - Kaluza-Klein (KK) theories
    - Braneworld models
    - Randall-Sundrum (RS) models
    - Dvali-Gabadadze-Porrati (DGP) gravity
    - (Einstein)Gauß-Bonnet (GB) gravity
  - ✓ Higher-derivative Theories of Gravity
    - Theories with Ricci and Riemann curvatures in the action
    - Hořava-Lifshitz gravity
    - Galileons
    - $f(R)$ theories: fourth-order theories. Come with far richer solutions than GR.



Some alternative gravity models. [Credit: Tessa Baker, arXiv 1512.05356]

- Einstein-Hilbert action for ΛCDM cosmology:

$$\mathcal{A}_{GR} = \frac{1}{2}\int d^4x\sqrt{-g}\,[R + 2\,(\mathcal{L}_m - \Lambda)]$$

  - ✓ Corresponding Einstein's field equations:

$$G_{ab} + \Lambda g_{ab} = T_{ab}$$

- $f(R)$ models are a sub-class of *fourth-order* theories of gravitation, with an action of the form

$$\mathcal{A}_{f(R)} = \frac{1}{2}\int d^4x\sqrt{-g}\,[f(R) + 2\mathcal{L}_m]$$

  - ✓ Corresponding $f(R)$-generalized Einstein field equations:

$$f'G_{ab} = T^m_{ab} + \tfrac{1}{2}(f - Rf')g_{ab} + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f'$$

  - ✓ Because of the highest order of the derivatives in these field equations, $f(R)$ is a fourth-order theory of gravity

- Simplest generalizations to GR
- An extra degree of freedom
- Cosmological viability:
  - ✓ Observational constraints
  - ✓ Theoretical constraints: integrability of the field equations
- Generic viability conditions on f:
  - ✓ To ensure gravity remains attractive

$$f' > 0 \quad \forall R$$

  - ✓ For stable matter-dominated and high-curvature cosmological regimes (nontachyonic scalaron)

$$f'' > 0 \quad \forall R \gg f''$$

  - ✓ GR-like law of gravitation in the early universe (BBN, CMB constraints)

$$\lim_{R\to\infty} \frac{f(R)}{R} = 1 \Rightarrow f' < 1$$

  - ✓ At recent epochs

$$|f' - 1| \ll 1$$

The matter-energy content of the Universe is specified by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_{(a}u_{b)} + \pi_{ab}$$

- Curvature and total fluid thermodynamics

$$\mu_R = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R\right]$$

$$p_R = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\left(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R - f'''\tilde{\nabla}^a R\tilde{\nabla}_a R\right)\right]$$

$$q^R_a = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \frac{1}{3}f''\Theta\tilde{\nabla}_a R\right]$$

$$\pi^R_{ab} = \frac{1}{f'}\left[f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R - \sigma_{ab}\dot{R}f''\right]$$

$$\mu \equiv \frac{\mu_m}{f'} + \mu_R\ , \quad p \equiv \frac{p_m}{f'} + p_R\ , \quad q_a \equiv \frac{q^m_a}{f'} + q^R_a\ , \quad \pi_{ab} \equiv \frac{\pi^m_{ab}}{f'} + \pi^R_{ab}$$

- The covariant derivative of the timelike vector $u^a \equiv \frac{dx^a}{d\tau}$ is decomposed into its irreducible parts as

$$\nabla_a u_b = -A_a u_b + \frac{1}{3} h_{ab}\Theta + \sigma_{ab} + \epsilon_{abc}\omega^c$$

$$A_a \equiv \dot{u}_a \;, \quad \Theta \equiv \tilde{\nabla}_a u^a \;, \quad \sigma_{ab} \equiv \tilde{\nabla}_{\langle a} u_{b\rangle} \;, \quad \omega^a \equiv \epsilon^{abc}\tilde{\nabla}_b u_c$$

- The trace-free part of the Riemann tensor defines the *Weyl conformal curvature tensor*

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{R}{3} g^{[a}{}_{[c}g^{b]}{}_{d]}$$

  - ✓ Split into its symmetric, trace-free "electric" and "magnetic" parts, E_{ab} and H_{ab} respectively given by

$$E_{ab} \equiv C_{agbh}u^g u^h, \qquad H_{ab} \equiv \frac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d$$

  - ✓ E_{ab} represents the free gravitational field (tidal forces)
  - ✓ H_{ab} is responsible for gravitational waves, no Newtonian analogue

- $1+3$ covariant splitting of the Bianchi and Ricci identities

$$\nabla_{[a}R_{bc]d}{}^{e}=0\ ,\quad (\nabla_a\nabla_b-\nabla_b\nabla_a)u_c=R_{abc}{}^{d}u_d$$

result in propagation and constraint equations

- The evolution equations uniquely determine the covariant variables on some initial hypersurface S_0 at t_0:

$$\dot{\mu}_m=-(\mu_m+p_m)\Theta-\tilde{\nabla}^a q^m_a-2A_a q^a_m-\sigma^a_b\pi^b_{a(m)} \tag{4.1}$$

$$\dot{\mu}_R=-(\mu_R+p_R)\Theta+\frac{\mu_m f''}{f'^2}\dot{R}-\tilde{\nabla}^a q^R_a-2A_a q^a_R-\sigma^a_b\pi^b_{a(R)} \tag{4.2}$$

$$\dot{\Theta}=-\tfrac{1}{3}\Theta^2-\tfrac{1}{2}(\mu+3p)+\tilde{\nabla}_a A^a-A_aA^a-\sigma_{ab}\sigma^{ab}+2\omega_a\omega^a$$

$$\begin{aligned}\dot{q}^m_a=&-\tfrac{4}{3}\Theta q^m_a-(\mu_m+p_m)A_a-\tilde{\nabla}_a p_m-\tilde{\nabla}^b\pi^m_{ab}\\&-\sigma^b_a q^m_b-A^b\pi^m_{ab}-\epsilon_{abc}\omega^b q^c_m\end{aligned} \tag{4.3}$$

$$\dot{q}^R_a = -\tfrac{4}{3}\Theta q^R_a + \frac{\mu_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b \pi^R_{ab} - \sigma^{\ b}_{a} q^R_b$$

$$- (\mu_R + p_R) A_a - A^b \pi^R_{ab} - \epsilon_{abc}\omega^b q^c_R$$

$$\dot{\omega}_a = -\tfrac{2}{3}\Theta\omega_a - \tfrac{1}{2}\epsilon_{abc}\tilde{\nabla}^b A^c + \sigma^{\ b}_{a}\omega_b \tag{4.4}$$

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a}A_{b\rangle} + A_{\langle a}A_{b\rangle} - \sigma^{c}_{\ \langle a}\sigma_{b\rangle c}$$

$$- \omega_{\langle a}\omega_{b\rangle} \tag{4.5}$$

$$\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} = \epsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - \Theta\left(E_{ab} + \tfrac{1}{6}\pi_{ab}\right) - \tfrac{1}{2}(\mu + p)\sigma_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q_{b\rangle}$$

$$+ 3\sigma^{\ \langle c}_{a}\left(E_{b\rangle c} - \tfrac{1}{6}\pi_{b\rangle c}\right) - A_{\langle a}q_{b\rangle} + \epsilon_{cd\langle a}\left[2A^c H^d_{b\rangle} + \omega^c(E^d_{b\rangle} + \tfrac{1}{2}\pi^d_{b\rangle})\right] \tag{4.6}$$

$$\dot{H}_{ab} = -\Theta H_{ab} - \epsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} + \tfrac{1}{2}\epsilon_{cd\langle a}\tilde{\nabla}^c \pi^d_{b\rangle} + 3\sigma^{\ \langle c}_{a} H_{b\rangle c}$$

$$+ \tfrac{3}{2}\omega_{\langle a}q_{b\rangle} - \epsilon_{cd\langle a}\left[2A^c E^d_{b\rangle} - \tfrac{1}{2}\sigma^c_{b\rangle}q^d - \omega^c H^d_{b\rangle}\right] \tag{4.7}$$

- Restrict the initial data to be specified; must remain satisfied on any hypersurface S_t for all t:

$$
\begin{aligned}
(C^1)_a &:= \tilde{\nabla}^b \sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a \Theta + \epsilon_{abc}\left(\tilde{\nabla}^b \omega^c + 2A^b \omega^c\right) + q_a = 0 \\
(C^2)_{ab} &:= \epsilon_{cd(a}\tilde{\nabla}^c \sigma_{b)}{}^d + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} - 2A_{\langle a}\omega_{b\rangle} = 0 \qquad (4.8) \\
(C^3)_a &:= \tilde{\nabla}^b H_{ab} + (\mu + p)\omega_a + 3\omega_b\left(E^{ab} - \tfrac{1}{6}\pi^{ab}\right) \\
&\quad + \epsilon_{abc}\left[\tfrac{1}{2}\tilde{\nabla}^b q^c + \sigma_{bd}\left(E^d{}_c + \tfrac{1}{2}\pi^d{}_c\right)\right] = 0 \\
(C^4)_a &:= \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b \pi_{ab} - \tfrac{1}{3}\tilde{\nabla}_a \mu + \tfrac{1}{3}\Theta q_a \\
&\quad - \tfrac{1}{2}\sigma_a^b q_b - 3\omega^b H_{ab} - \epsilon_{abc}[\sigma^{bd} H_d^c - \tfrac{3}{2}\omega^b q^c] = 0 \\
(C^5) &:= \tilde{\nabla}^a \omega_a - A_a \omega^a = 0 \qquad (4.9)
\end{aligned}
$$

Classic GR result (Gödel, Ellis): shear-free perfect-fluid cosmological models (homogeneous, inhomogeneous) cannot rotate and expand simultaneously, *i.e.,*

$$\Theta\omega^a = 0$$

- Turning off the shear from the propagation equations (4.5) results in a new constraint equation [12]

$$(C^6)_{ab} := E_{ab} - \tfrac{1}{2}\pi_{ab} - \tilde{\nabla}_{\langle a}A_{b\rangle} = 0$$

- Demanding consistent spatial (curl) and temporal (time derivative) propagations results in

$$\Theta\omega^a\Big\{\Big[\frac{(1-w)P}{3}\tilde{R} + \frac{(1+w)}{f'}\frac{(3w+5)f' + 4f''Q}{6f'}\rho_m\Big] + \frac{Z}{P}\Big[(\frac{1+w}{f'})\rho_m\Big]\Big\} = 0$$

[12] AA, Goswami, Dunsby. Phys. Rev. D 84 124027 (2011)

- Flat, vacuum solutions: if the 3-curvature $\tilde{R}$ vanishes, then the GR result can always be avoided for vacuum universes ($\mu_m = 0$), *i.e.,* a shear-free, spatially flat vacuum universe in any $f(R)$ theory can rotate and expand simultaneously in the linearized regime
- Non-vacuum case solutions: for a stiff fluid in R^3 gravity, there exists a flat Milne-universe solution which can rotate and expand simultaneously at the level of linearised perturbation theory
  - ✓ This suggests that there are situations where linearized fourth-order gravity shares properties with Newtonian theory not valid in GR

- Fluid flows with vanishing vorticity $\omega_a = 0$ will have the evolution equation (4.4) turned into a new constraint

$$(C^{6*})_a := \epsilon_{abc}\tilde{\nabla}^b A^c = 0 \implies A_a = \tilde{\nabla}_a \psi$$

for some scalar ψ

- Specializing to dust models

$$w = 0 = p_m \,, \qquad q_a^m = 0 = A_a \,, \qquad \pi_{ab}^m = 0 \,,$$

we observe that irrotational shear-free dust spacetimes governed by $f(R)$ gravitational physics evolve consistently if [13]

$$\left[\frac{3}{2}\left(\frac{f'''}{f'} - \frac{f''^2}{f'^2}\right)\dot{R} - \frac{\Theta f''}{6f'}\right]\epsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}R + \tfrac{3f''}{2f'}\epsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}\dot{R} = 0$$

which is an identity

  - ✓ This suggests that all irrotational shear-free dust spacetimes in $f(R)$-gravity are self-consistent

[13] AA, Elmardi. Int. J. Geom. Meth. Mod. Phys. 12 1550118 (2015)

Irrotational dust spacetimes with div $H_{ab} = 0$

A necessary condition for the propagation of gravitational waves is the vanishing of the divergence of a non-zero H_{ab}

- Prescribing this condition on the field equations results in a constraint:

$$q_a^R = \tilde{\nabla}_a \phi = \tfrac{2}{3}\tilde{\nabla}_a \Theta - \tilde{\nabla}^b \sigma_{ab}$$

- A subclass of such models, called "purely radiative" dust spacetimes, is a divergence-free E_{ab}. Such models in $f(R)$ gravity are constrained further as

$$\tilde{\nabla}_a \mu_m + f'\tilde{\nabla}_a \mu_R + f'\Theta q_a^R - \tfrac{3f'}{2}\tilde{\nabla}^b \pi_{ab}^R = 0$$

✓ In GR purely radiative irrotational dust spacetimes are spatially homogeneous:

$$\tilde{\nabla}_a \mu_m = 0$$

- Quasi-Newtonian universes: irrotational dust universes with purely gravito-electric Weyl tensor, characterized by:

$$p_m = 0 \,, \quad A_a = 0 \,, \quad q_a^m = \mu_m v_a \,, \quad \pi_{ab}^m = 0 \,, \quad \omega_a = 0 \,, \quad H_{ab} = 0$$

  ✓ Potential models for the description of gravitational collapse and late-time cosmic structure

- Choose a comoving 4-velocity $\tilde{u}^a$ such that

$$\tilde{u}^a = u^a + v^a \,, \quad v_a u^a = 0 \,, \quad v_a v^a << 1 \,,$$

  where v^a is the non-relativistic ("peculiar") velocity and vanishes in the background

For this class of models, it can ne shown that

$$\frac{1}{2}\epsilon^{abc}\tilde{\nabla}_b A_c = 0 \implies A_a \equiv \tilde{\nabla}_a \Phi$$

$$E_{ab} - \frac{1}{2}\pi_{ab} - \tilde{\nabla}_{\langle a}A_{b\rangle} = 0$$

For any fourth-order gravity model in which the anisotropic pressure π_{ab} can be given in terms of a scalar potential Ψ as

$$\pi_{ab} = \tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\Psi$$

- Two generally independent integrability conditions for generic fluid models exist [14]:

$$\tilde{\nabla}_{<a}\tilde{\nabla}_{b>}\left(\dot{\Phi} + \frac{1}{3}\Theta + \dot{\Psi}\right) + \left(\dot{\Phi} + \frac{1}{3}\Theta + \dot{\Psi}\right)\tilde{\nabla}_{<a}\tilde{\nabla}_{b>}\Phi = 0$$

$$6\tilde{\nabla}_a\ddot{\Phi} + 6\Theta\tilde{\nabla}_a\dot{\Phi} - \left(2\mu - \frac{2}{3}\Theta^2\right)\tilde{\nabla}_a\Phi + 6\tilde{\nabla}_a\ddot{\Psi} + 6\Theta\tilde{\nabla}_a\dot{\Psi}$$
$$- \left(2\mu - \frac{2}{3}\Theta^2\right)\tilde{\nabla}_a\Psi - 2\tilde{\nabla}_a(\tilde{\nabla}^2\Psi) - 3\tilde{\nabla}_a p = 0$$

- Identically the same in $f(R)$ models, due to the linearized form of π^R_{ab}

[14] AA, Dunsby, Solomons. Int. J. Mod. Phys. D 26 1750054 (2016)

- Modified Poisson equation

$$\tilde{\nabla}^2 \Phi = \frac{1}{2}(\mu + 3p) - \left[3\left(\ddot{\Phi} + \ddot{\Psi}\right) + \left(\dot{\Phi} + \dot{\Psi}\right)\Theta\right]$$

- Velocity perturbations are scale-independent, as in GR, but matter density fluctuations are scale-dependent
- Over regions of space-time where the Ricci curvature scalar is a slowly varying function of space and time
  - ✓ $f(R)$ (and its derivatives) are associated Laguerre polynomials
  - ✓ The peculiar velocity, 4-acceleration, total cosmic heat flux and anisotropic stress can be analytically calculated explicitly

- Irrotational dust universes with purely gravito-magnetic Weyl tensor $\longrightarrow$ anti-Newtonian universes, characterized by

$$p_m = 0 \,, \quad A_a = 0 \,, \quad q_a^m = 0 \,, \quad \pi_{ab}^m = 0 \,, \quad \omega_a = 0 \,, \quad E_{ab} = 0$$

  ✓ Farthest possible models from Newtonian universes

- In GR, anti-Newtonian universes suffer from severe integrability conditions, no known anti-Newtonian spacetimes that are linearized perturbations of Friedman-Lemaître-Robertson-Walker (FLRW) universes
- In fourth-order gravitational theories, anti-Newtonian models exist, subject to the integrability condition [15]

$$\tilde{\nabla}^2 q_a^R - \tilde{\nabla}_a(\tilde{\nabla}^b q_b^R) + \tilde{R} q_a^R + \frac{4f''}{f'^2}\mu_m \Theta \tilde{\nabla}_a R = 0$$

[15] AA. Class. Quantum Grav. 31 115011 (2014)

- For flat universes ($K = 0 = \tilde{R}$) this holds only if

$$f''\mu_m \Theta \tilde{\nabla}_a R = 0$$

✓ Impose $\mu_m \neq 0$ and $f'' \neq 0$. For a consistently evolving set of constraints in the flat, anti-Newtonian spacetimes, either one of the following conditions must hold:

$$\Theta = 0 \qquad \longrightarrow \text{static}$$
$$\tilde{\nabla}_a R = 0 \qquad \longrightarrow \text{homogeneous}$$

- Closed & open universes ($K = \pm 1$): any dust solution of

$$\left[\frac{f''\mu_m\Theta}{f'} \mp \frac{2}{a^2}\left(\dot{R}f''' - \tfrac{1}{3}\Theta f''\right)\right]\tilde{\nabla}_a R \mp \frac{2f''}{a^2}\tilde{\nabla}_a \dot{R} = 0$$

with $f'' \neq 0$ is an anti-Newtonian solution

- The Chaplygin gas is a dark-fluid model whose EoS

$$p = -\frac{A}{\mu^{\alpha}}$$

where A and α are positive constants, allows for a solution of the form:

$$\mu(a) = \left[A + \frac{B}{a^{3(1+\alpha)}}\right]^{\frac{1}{1+\alpha}}$$

  - ✓ Early universe: $\mu \sim a^{-3}$, behaves as dust (dark matter and baryonic matter)
  - ✓ Late universe: $\mu \sim A^{\frac{1}{1+\alpha}}$, behaves like dark energy

- Does the CG allow the simultaneous expansion and rotation of a shear-free universe?

- For a consistent propagation of the field equations in this model with $\alpha = 1$ (the original CG model)[16]:

$$\Big\{6\mu^5\left(\mu+4\Theta^2\right)+3A^2\left(6\mu^2-\frac{3}{2}\tilde{R}^2\right)$$
$$+A\mu^2\left[\tilde{R}\left(3\mu+\Theta^2\right)-24\mu\left(\mu+\Theta^2\right)\right]\Big\}\omega^a$$
$$-3A\left[9A\left(\mu-\Theta^2\right)-\mu^2\left(3\mu-5\Theta^2\right)\right]\tilde{\nabla}^2\omega^a=0 \tag{5.1}$$

- This equation automatically reduces to the well-known shear-free dust result for the case $A = 0$:

$$\omega^a(\mu+4\Theta^2)=0$$

- Using $\tilde{\nabla}^2\omega^a = -\lambda\omega^a$ yields

$$\Big[6\mu^5(\mu+4\Theta^2)+3A^2(6\mu^2-\frac{3}{2}\tilde{R}^2)$$
$$+A\mu^2\left(\tilde{R}(3\mu+\Theta^2)-24\mu(\mu+\Theta^2)\right)$$
$$+3A\lambda\left(9A(\mu-\Theta^2)-\mu^2(3\mu-5\Theta^2)\right)\Big]\,\omega^a=0$$

[16] AA., Al Ajmi, M., Elmardi, M., Nandan, H. & Sabah, N. Shear-free conditions of a Chaplygin-gas-dominated universe. Int. J. Geom. Methods Mod. Phys. 2150192 (2021)

Thus, any simultaneously expanding and rotating solution must satisfy (provided Θ remains real-valued):

$$\Theta = \pm\sqrt{\frac{A^2\left(9\tilde{R}^2/2 - 18\mu^2 - 27\lambda\mu\right) - 3A\mu^3\left(\tilde{R} - 8\mu - 3\lambda\right) - 6\mu^6}{A\left(\tilde{R}\mu^2 - 24\mu^3 + 15\lambda\mu^2 - 27A\lambda\right) + 24\mu^5}}$$

Some special cases for which simultaneously expanding and rotating solutions can exist.

- For flat space, $\tilde{R} = 0$, $\lambda \neq 0$

$$\Theta = \pm\sqrt{\frac{\mu\,(\mu^2 - 3A)\,[2\mu^3 - A(2\mu + 3\lambda)]}{A\,[9A\lambda + \mu^2(8\mu - 5\lambda)] - 8\mu^5}}$$

provided that the denominator is nonzero, i.e.,

$$A \neq \frac{5\lambda\mu^2 - 8\mu^3 \pm \mu^2\sqrt{25\lambda^2 + 208\lambda\mu + 64\mu^2}}{18\lambda}$$

  ✓ $\Theta^2 = 0$ for only $A = \{\mu^2/3\,, 2\mu^3/(2\mu + 3\lambda)\}$, it means that Eq. (5.1) can be satisfied for non-vanishing ω^a and Θ provided $A \neq \{\mu^2/3\,, 2\mu^3/(2\mu + 3\lambda)\}$

- For flat space, $\lambda = 0$ and $A \neq \{\mu^2, \mu^2/3\} \implies$ we can have non-vanishing Θ and ω^a provided

$$\Theta = \pm\sqrt{\frac{3A - \mu^2}{4\mu}}$$

- Cosmology has a long history of tensions
- Potential solutions to solve - or at least alleviate - these tensions might lie somewhere beyond the standard cosmological model based on:
  - ✓ General Relativity
  - ✓ the Copernican (Cosmological) Principle
  - ✓ Noninteracting cosmological medium
  - ✓ Perfect fluids
- Relaxing these comes at a cost of more complexity, but it might be worth the extra effort