Quantum Chaos & Phase Transitions

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QGASLAB THE LABORATORY FOR QUANTUM GRAVITY & STRINGS

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Motivating Quantum Chaos

...why you may not want to go to sleep immediately!

- Why study quantum chaos?
  - → understand properties of q-systems closer to the real world, eventually
  - → make better quantum technologies in future
  - → connections to other phenomena in physics (BHs, RMT, TPTs, cosmology, neutrino oscillations etc)
  - → interesting in its own right
- First hurdle : how to *define* quantum chaos?
  - → use notions of classical chaos and then invoke the correspondence principle (OTOCs, SFF etc)
  - → come up with unique, perhaps quantum-only, probes of quantum chaos (EE, ...???)
  - → best : Wigner statistics / RMT-like behavior / level repulsion, not-so-easy to compute (in many cases) + doesn't always work

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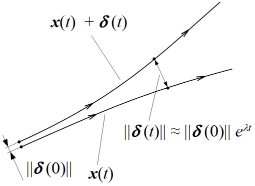
- $\exists$ a number of probes of classical chaos : phase space trajectory evolution, energy level correlation behavior, S/G-ALI etc.
- phase-space-trajectory-divergence probe (valid for intermediate times) breaks down in some cases (cue blackboard, billiards with hole etc)



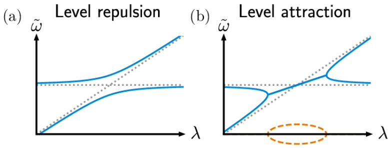


Figure: Exponential divergence of trajectories is a classical signature of onset of chaos

Quantum Chaos : Why Complexity?

- The central object in this talk is 'Quantum Complexity'.
- Historically, introduced in HEP as a means to investigate BH dynamics and study holography or AdS/CFT correspondence.
- What can you do using quantum complexity?
  - → study holography
  - → study quantum chaos (alternative to OTOCs, SFF etc.)
  - → study quantum circuits
  - → study early universe cosmology
- Recently, Jared Lichtman, used a similar notion to assist in his proof of the Erdős primitive set conjecture. The notion he used is to associate a number quantifying the size (or difficulty) of every primitive set.
- All this is to say that quantum complexity is an idea powerful & useful enough to warrant further study.

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What is Complexity?

- Roughly speaking, complexity measures the difficulty of performing a task e.g. going from home to work.
- How does one adapt this idea to quantum mechanics? In QM we work with the $\mathcal{H}$ and the operators that act on them. So an easy adaptation of complexity to QM can be through computing complexity of states and operators.
- Different notions of quantum complexity -

  $\rightarrow$ Nielsen complexity : $|\text{state}_1\rangle \rightarrow |\text{state}_2\rangle$

  $\rightarrow$ Krylov complexity : $\mathcal{O}(0) \rightarrow \mathcal{O}(t)$

  $\rightarrow$ **Spread complexity** : Krylov complexity but for states

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Nielsen Complexity

- Nielsen complexity measures, by use of unitary operations, how difficult it is to prepare a target state starting from a reference state.
- This is typically done by considering the *Fubini-Study* metric on $\mathcal{H}$.
- One constructs a quantum-circuit on the $\mathcal{H}$ from target state to reference state and measures the minimum depth of this circuit (upto some tolerance ϵ) - this defines the Nielsen geometric complexity of the circuit.
- The evolution is performed through a choice of suitable unitary operator. How this choice is made remains an open question. One works with what one is interested in or finds tractable.

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Interlude : Krylov Subspace

- Krylov complexity measures the growth of an operator through a quantum system. One may ask, what has this got to do with the “difficulty of a task” definition. Both are essentially same - they quantify a task by assigning a value to it & drawing conclusions thereby. One may tweak the explicit definition to suit specific goals but essentially the idea remains same.
- By “growth of an operator” one means how much an operator has spread through the system e.g. introducing a virus in a population like this.

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- One can see that $\mathcal{O}(t) = \mathcal{O} - it[H, \mathcal{O}] + \frac{(-it)^2}{2!}[H, [H, \mathcal{O}]] + \cdots$
- The terms in red hint at a structure if one defines a super-operator, Liouvillian : $\mathcal{L} = [H, .]$. So, $\mathcal{O}(t) = e^{-it\mathcal{L}}\mathcal{O}$.
- $\mathcal{O}(t)$ is a linear combination of operators in the subspace : Span$\{\underbrace{\mathcal{O}}_{\equiv \mathcal{L}^0\mathcal{O}}, \underbrace{[H, \mathcal{O}]}_{\equiv \mathcal{L}\mathcal{O}}, \underbrace{[H, [H, \mathcal{O}]]}_{\equiv \mathcal{L}^2\mathcal{O}}, \cdots\} \equiv \mathcal{K}$, the Krylov subspace.
- As time grows, the operator is said to become more *complex* or that it has grown because more and more commutators will become significant. Hence, the Krylov subspace method can be used to study operator growth and chaos in a quantum mechanical system.
- $K \leq D^2 - D + 1$ is the bound on the Krylov space dimension which is almost the size of the Hilbert space i.e. D^2.

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Interlude : Lanczos Algorithm

- Notice that the $\mathcal{K}$ is not orthogonal. It is much easier to work with orthonormal bases $\rightarrow$ Lanczos algorithm $\equiv$ Gram-Schmidt but for operators.
- Define an inner product for operators :
  $(\mathcal{A}, \mathcal{B}) = \frac{1}{D}\text{Tr}\left(\mathcal{A}^{\dagger}, \mathcal{B}\right) \quad ; \quad ||\mathcal{A}|| = \frac{1}{D}\text{Tr}\left(\mathcal{A}^{\dagger}, \mathcal{A}\right)$
  $\rightarrow$ the choice of this inner product happens to be pretty important
- Algorithm : (assuming Hermitian operator $\mathcal{O}$ and a Hamiltonian H)
  $\rightarrow$ set $\mathcal{O}_0 = \frac{\mathcal{O}}{||\mathcal{O}||}$
  $\rightarrow$ $\mathcal{A}_1 = [H, \mathcal{O}_0] \;\; ; \;\; b_1 = ||\mathcal{A}_1|| \;\; ; \;\; \mathcal{O}_1 = \frac{\mathcal{A}_1}{b_1}$
  $\rightarrow$ For $n > 1$ do :
    - $\mathcal{A}_n = [H, \mathcal{O}_{n-1}] - b_{n-1}\mathcal{O}_{n-2}$: Lanczos Step
    - $b_n = ||\mathcal{A}_n||$
    - if $b_n = 0$ stop ; else $\mathcal{O}_n = \frac{\mathcal{A}_n}{b_n}$

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  - → The set $\{\mathcal{O}_n\}_{n=0}^{K-1}$ is orthonormal : $(\mathcal{O}_n, \mathcal{O}_m) = \delta_{nm}$.
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- The operator's time evolution can be described by its motion along the Krylov basis : $\mathcal{O}(t) = \sum_{n=0}^{K-1} i^n \ \phi_n(t) \ \mathcal{O}_n$.
- This can be seen as expanding the time dependent operator over the Krylov basis.
- i^n makes all the terms Hermitian since operators in the Krylov basis alternate between Hermitian & anti-Hermitian.
- Heisenberg time-evolution equation $\frac{d}{dt}\mathcal{O}(t) = i[H, \mathcal{O}(t)]$ gives :

$$\dot{\phi}_n(t) = b_n \phi_{n-1}(t) - b_{n+1}\phi_{n+1}(t) \leftarrow \text{tridiagonal } \mathcal{L}.$$

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Krylov Complexity

- K-complexity is a probe of time-dependent profile of $\phi_n(t)$:

  $\rightarrow$ K-complexity = average position on the Krylov chain :

  $$C_K(t) = \sum_{n=0}^{K-1} n \left|\phi_n(t)\right|^2$$

- K-complexity is bounded, by definition, by the Krylov space dimension since the average position on the chain cannot be greater than the length of the chain itself.
- K-complexity depends only on the Hamiltonian of the system and a seed operator $\mathcal{O}$. It also depends on making a correct choice for the inner product class.
- Pro vs Nielsen Complexity : no tolerance or dependence on choice of gates/unitaries $\rightarrow$ less ambiguities.

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- One can use the Krylov subspace methods to compute complexity of a state.
- General quantum state : $|\Psi(s)\rangle = e^{iHs} |\Psi_0\rangle$, s : circuit time.
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- Why should one invest time to investigate this proposal? While this idea may seem way out there from the definition of complexity, there is some merit and motivation to such a study. The motivation I shall clarify right away and the merit, I hope to make apparent by the end of this talk.
- Motivation : TPTs, by definition, are accompanied by gap-closing i.e. level statistics play a role in determining TPTs. Since Krylov basis is constructed out of a super-operator which is a function of H - it can be expected that Krylov basis encodes information about level statistics. So, it is natural to expect spread complexity to be sensitive to TPTs.
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Kitaev Chain : The Model

- Now that we are convinced that this idea is tractable we need a "playing field". Of course, we'd prefer to find a nice & easy playing field so as to not get lost in the details and focus on the proposal.
- We take a prototypical model which shows topological phase transitions (TPTs) : the Kitaev Chain. This model has proven to be the harmonic oscillator of topological studies in the sense that this simple model elucidates ideas that go beyond itself.
- The Kitaev Chain Hamiltonian is,

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- Topological phase : $|\mu| < |J|$ - $\exists$ Majorana zero modes.
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Formalism

- Step 1 : diagonalize the H_K through BdG formalism i.e. Fourier transform $c_j, c_j^\dagger$.
- Step 2 : recast H_{BdG} as
  $$H_{\text{BdG}} = \sum_k \left[2R_3 J_0^{(k)} + iR_1 \left(J_+^{(k)} - J_-^{(k)}\right)\right].$$
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Of course, this masks some of the subtleties like dealing with BCs and odd/even number of sites, but this is roughly a formalism one can employ to calculate spread complexity quickly for the cases when the H is part of a Lie algebra.

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Results

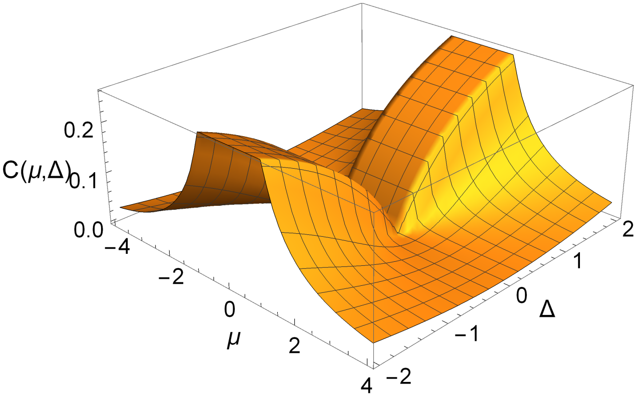

Figure: The spread complexity in the continuum limit for the circuit connecting the free fermion ground state to the Kitaev chain ground state. We have chosen $J = 1$. When $|\mu| < 1$ the system is in the topological phase and the spread complexity is a Δ-dependent constant.

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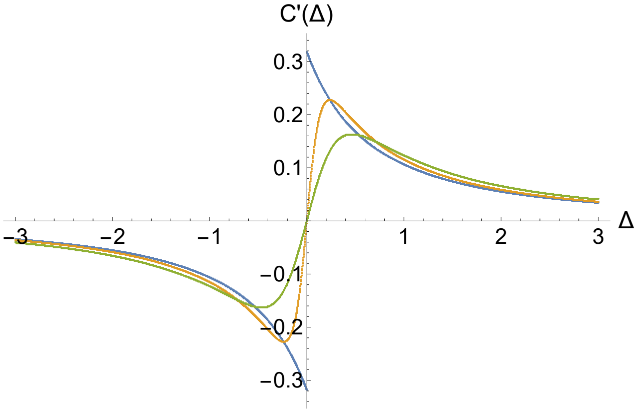

Figure: The derivative of spread complexity with respect to Δ (continuum limit) for the circuit connecting the free fermion GS and Kitaev chain GS. We have $J = 1, \mu = 0.98, \mu = 1.02, \mu = 1.1$. When crossing the TPT points at $|\mu| = 1$ the derivative develops a discontinuity.

Discussion & Outlook

- We have shown that, conservatively speaking, spread complexity is a sensitive and efficient of TPTs for the Kitaev Chain - at least.
- Furthermore, we have done so by considering three different circuits demonstrating that spread complexity is robustly sensitive.
- The formalism relies on being able to associate the H to a Lie algebra. If not, this formalism breaks down. Currently, this formalism is the only one that has been used to study sensitivity of spread complexity to TPTs.
- A simple case in which to look for alternative methods is given by the Kitaev chain itself in the form of TBC case. TBC breaks translation invariance and hence one cannot follow BdG formalism which relies on Fourier transformation to diagonalize the H.

Discussion & Outlook

- Fluctuations diverge at quantum critical points and one may expect the complexity of a quantum state, say the GS, to do so too. In our work we have made an attempt to explore this notion and add to an ever growing list of literature.
- The critical points may correspond to different classes of phase transitions like the conventional PTs, BKT PTs and deconfined critical points. It would be interesting to see if complexity is sensitive to such phase transitions too.

Thank You