$f(R,T)$ gravity and Gödel's Solution

Application of Quantum Information in Astrophysics and Cosmology

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Overview

$f(R,T)$ gravity

- $f(R,T)$ gravity is a modified theory of gravition
- In 2011, Harko et al. [1] developed the $f(R,T)$ gravity.
- $f(R,T)$ gravity generalizes the $f(R)$ gravity

The action for $f(R,T)$ gravity is given as

$$S = \frac{1}{2k}\int \sqrt{-g}\,[f(R,T) + 2kL_m]\,d^4x \quad (1)$$

Let us define the energy-momentum tensor as

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{ab}} \tag{2}$$

Further, assume that the Lagrangian density L_m depends on the metric tensor components g_{ab}, and not on its derivatives. Thus we have

$$T_{ab} = g^{ab}L_m - 2\frac{\partial L_m}{\partial g_{ab}} \tag{3}$$

Varying the action S as in (1) with respect to the metric components, the modified field equations for $f(R,T)$ gravity are expressed as

$$\begin{aligned} & f_R(R,T)R_{ab} - \frac{1}{2}f(R,T)g_{ab} + (g_{ab}\Box - \nabla_a\nabla_b)\, f_R(R,T) = \\ & T_{ab} - f_T(R,T)\,(T_{ab} + \Theta_{ab}) \end{aligned} \tag{4}$$

where

$$\Theta_{ab} = g_{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{ab}} = -2T_{ab} + g_{ab}L_m - 2g^{\alpha\beta}\frac{\partial^2 L_m}{\partial g^{ab}\partial g^{\alpha\beta}}$$

For the matter Lagrangian $L_m = -p$ we arrive at

$$\Theta_{ab} = -2T_{ab} - p g_{ab} \tag{5}$$

Therefore, the modified field equations (4) finally take the form

$$f_R(R,T)R_{ab} - \frac{1}{2}f(R,T)g_{ab} + (g_{ab}\Box - \nabla_a\nabla_b)\, f_R(R,T) = \\ (1 + f_T(R,T))\, T_{ab} + p f_T(R,T) g_{ab} \tag{6}$$

This field equations reduces to

- $f(R)$ gravity field equations when $f(R,T)$=$f(R)$
- GR field equations when $f(R,T)$=$f(R) = R$

In literature we found three forms of $f(R,T)$ as

- $f(R,T) = R + h(T)$
- $f(R,T) = f_1(R) + f_2(T)$
- $f(R,T) = f_1(R) + f_2(R) f_3(T)$

$f(R,T) = R + 2h(T)$:

$$G_{ab} = [1 + 2h_T(T)]T_{ab} + [2ph_T(T) + h(T)]g_{ab} \quad (7)$$

$f(R,T) = f_1(R) + f_2(T)$:

$$G_{ab} = G_{eff}T_{ab} + T^{eff}_{ab} \quad (8)$$

where

$$G_{eff} = \frac{1}{f_1'(R)}(1 + f_2'(T))$$

and

$$T^{eff}_{ab} = \frac{1}{f_1'(R)}\left[\frac{1}{2}(f_1(R) - Rf_1'(R) + f_2(T))g_{ab} - (g_{ab}\Box - \nabla_a\nabla_b)f_1'(R)\right]$$

$f(R,T) = f_1(R) + f_2(R) f_3(T)$:

$$[f_1'(R) + f_2'(R)f_3(T)]R_{ab} - \frac{1}{2}f_1(R)g_{ab} + (g_{ab}\Box - \nabla_a\nabla_b)[f_1'(R) + f_2'(R)f_3(T)] =$$

$$(1 + f_2(R)f_3'(T))T_{ab} + \frac{1}{2}f_2(R)f_3(T)g_{ab} \quad (9)$$

Gödel metric

- Gödel [2] had proposed a stationary cosmological model with rotation in the form (coordinates $\{t, x, y, z\}$, a = constant).

$$ds^2 = a^2 \left(dt^2 - 2e^x dt dy + \frac{1}{2} e^{2x} dy^2 - dx^2 - dz^2 \right) \quad (10)$$

- In the Gödel's model, the matter is described as dust with the energy density ρ
- Cosmological constant Λ is nontrivial and negative
- The angular velocity ω of the cosmic rotation in (10) is

$$\omega^2 = \frac{1}{2a^2} = 4\pi G\rho = -\Lambda \quad (11)$$

G as Newton's gravitational constant.

The Gödel metric (10) represents a particular case of a wider family of stationary cosmological models described by

$$ds^2 = dt^2 - 2\sqrt{\sigma}e^{mx}dtdy - (dx^2 + ke^{2mx}dy^2 + dz^2) \qquad (12)$$

here $m > 0$, $\sigma > 0$and k are constant parameters. The line element (12) is called the model with rotation of the Gödel type.

- Gödel [3] himself outlined a more physical expanding generalizations of (10), although without giving explicit solutions.
- Considerable numbers of exact and approximate models with rotation and with or without expansion were developed
- Maitra [4] provided a cylindrically symmetric inhomogeneous stationary dust-filled world with rotation and shear.
- Wright [5] presented inhomogeneous cylindrically symmetric solution for dust plus cosmological term.

- Schucking and Ozsvath [5, 6],discussed a series of paper dealing with spatially homogeneous models with expansion, rotation and shear
- however, that it is impossible to combine pure rotation and expansion in a solution of the general relativity field equations for a simple physical matter source, such as a perfect fluid
- There are two ways to overcome that difficulty: one should either take a more general energy-momentum or to add cosmic shear
- The possibility of combining cosmic rotation with expansion was the first successful step towards a realistic cosmology.

Now generalized model is obtained from (12) by introducing the time-dependent scale factor $R(t)$

$$ds^2 = dt^2 - 2\sqrt{\sigma}R(t)e^{mx}dtdy - R(t)(dx^2 + ke^{2mx}dy^2 + dz^2) \quad (13)$$

The metric (13) is usually called the Gödel type model with rotation and expansion and also known as Gödel-Obukhov line element.

Investigated Problems

Problems

- Gödel metric, known as a simplest metric allowing for the closed time-like curves (CTCs) is compatible with this theory or not.
- Smallness of the Cosmological constant Λ

Let us consider the Gödel metric in the form

$$ds^2 = a^2 \left[dt^2 - dx^2 + \frac{1}{2} e^{2x} dy^2 - dz^2 + 2e^x dt dy \right], \qquad (14)$$

where a is a positive number. The non-zero components of the Ricci tensor for the Gödel metric (14) are expressed as

$$R_{22} = e^{2x}, R_{24} = R_{42} = e^x, R_{44} = 1.$$

The Ricci scalar or scalar curvature $R = g^{ij} R_{ij}$ is $R = 1/a^2$.

We consider the energy momentum tensor in the following form

$$T_{ij} = (\rho + p)u_i u_j + (p + \Lambda)g_{ij}, \tag{15}$$

Here u is the unit vector along the t line with $u_i = (a, 0, ae^x, 0)$, ρ, p and Λ represent the energy density, pressure and cosmological constant, respectively. The non-zero components of the energy momentum tensor are

$$\begin{aligned}
T_{11} &= -(p + \Lambda)a^2 \\
T_{22} &= (\rho + \frac{3p}{2} + \frac{\Lambda}{2})a^2 e^{2x} \\
T_{33} &= -(p + \Lambda)a^2 \\
T_{44} &= (\rho + 2p + \Lambda)a^2 \\
T_{24} &= (\rho + 2p + \Lambda)a^2 e^x
\end{aligned} \tag{16}$$

The trace of the energy momentum tensor is expressed as $T = \rho + 5p + 4\Lambda$.

Solutions under various functional forms

Motivations:

- Singh et al.[7] investigated the cosmological constant in $f(R,T)$ gravity by considering $f(T) = \mu T^2$ for the class $f(R,T) = R + f(T)$.
- Pettorino et al.[8] devoted their study to class of extended quintessence cosmologies, in which scalar field act as dark energy and coupled to the Ricci scalar exponentially.
- Harko and Lobo [9] proposed the exponential dependence of Hilbert–Einstein action by replacing R with $\Lambda e^{\frac{R+2L_m}{\Lambda}}$, where L_m is the the matter Lagrangian.
- Moraes et al.[10] discussed the issue of cosmic acceleration in $f(R,T) = R + e^T$ gravity.
- Godani and Samanta [11] presented the existence of a wormhole solution in $f(R,T) = R + 2\alpha \ln T$ gravity, where α is a constant.

Solution for $h(T) = \lambda T^2$

In this case, the modified field equations (6) take the form

$$G_{ab} = [1 + 4\lambda T]T_{ab} + \lambda T[4p + T]g_{ab} \tag{17}$$

The non-zero components of equation (17) for the Gödel metric (14) along with (16) are expressed as

$$-65\,\lambda\,p^2+(-92\,\Lambda\,\lambda - 18\,\lambda\,\rho - 1)\,p-32\,\Lambda^2\lambda-12\,\Lambda\,\lambda\,\rho-\lambda\,\rho^2-\Lambda = \frac{1}{2a^2} \tag{18}$$

$$105\,\lambda\,p^2+(124\,\Lambda\,\lambda + 66\,\lambda\,\rho + 3)\,p+32\,\Lambda^2\lambda+44\,\Lambda\,\lambda\,\rho+9\,\lambda\,\rho^2+\Lambda+2\,\rho = \frac{3}{2a^2} \tag{19}$$

$$85\,\lambda\,p^2+(108\,\Lambda\,\lambda + 42\,\lambda\,\rho + 2)\,p+32\,\Lambda^2\lambda+28\,\Lambda\,\lambda\,\rho+5\,\lambda\,\rho^2+\Lambda+\rho = \frac{1}{2a^2} \tag{20}$$

Let us assume that the universe is filled with a perfect fluid and for simplicity, we choose the case in which the pressure is zero, i.e., $p = 0$. Thus the system of equations (18)-(20) reduce to

$$-32\,\Lambda^2\lambda - 12\,\Lambda\,\lambda\,\rho - \lambda\,\rho^2 - \Lambda = \frac{1}{2a^2} \tag{21}$$

$$32\,\Lambda^2\lambda + 44\,\Lambda\,\lambda\,\rho + 9\,\lambda\,\rho^2 + \Lambda + 2\,\rho = \frac{3}{2a^2} \tag{22}$$

$$32\,\Lambda^2\lambda + 28\,\Lambda\,\lambda\,\rho + 5\,\lambda\,\rho^2 + \Lambda + \rho = \frac{1}{2a^2} \tag{23}$$

For $\lambda = 0$, the system of equations (21)-(23) has a solution in the form $\rho = \frac{1}{a^2}$ and $\Lambda = -\frac{1}{2a^2}$. This solution reduces to the solution of Santos [12] and Gödel [3].

For $\lambda \neq 0$. With the removal of the Λ^2 term from the system of equations, we now have a single equation

$$4\,\lambda\,\rho^2 + (16\,\Lambda\,\lambda + 1)\,\rho = \frac{1}{a^2}. \tag{24}$$

This equation (24) has infinite solutions depending on ρ and/or Λ. Let us assume that [13]

$$\rho = \frac{1}{a^2\,(1 + 2\,\lambda)} \tag{25}$$

is a solution of the equation. Then the corresponding cosmological constant is given as

$$\Lambda = \frac{-2 + a^2\,(1 + 2\,\lambda)}{8\,(1 + 2\,\lambda)\,a^2} \tag{26}$$

ρ and Λ satisfy the field equations under the constraint $\lambda = -\frac{a^2+2}{4a^2}$. Thus, these quantity takes the form

$$\rho = \frac{2}{a^2 - 2} \tag{27}$$

and

$$\Lambda = \frac{a^2 - 6}{8(a^2 - 2)} \tag{28}$$

The relationship between the matter energy density and cosmological constant for this model is expressed as

$$\Lambda = \left(\frac{a^2 - 6}{16}\right)\rho \tag{29}$$

Solution for $h(T) = \lambda \ln(T)$

In this case the modified field equations (6) take the form

$$G_{ab} = \left[\frac{T+2\lambda}{T}\right] T_{ab} + \lambda \left[\frac{2p + T\ln(T)}{T}\right] g_{ab} \tag{30}$$

The non-zero components of equation (30) for the Gödel metric (14) along with (16) are expressed as

$$\frac{\rho + 5p + 4\Lambda}{2a^2} = -(\rho + 5p + 4\Lambda + 2\lambda)(p + \Lambda) - \lambda\,[2p + (\rho + 5p + 4\Lambda)\ln(\rho + 5p + 4\Lambda)] \tag{31}$$

$$\frac{3(\rho + 5p + 4\Lambda)}{4a^2} = (\rho + 5p + 4\Lambda + 2\lambda)\left(\rho + \frac{3p}{2} + \frac{\Lambda}{2}\right) + \frac{\lambda}{2}[2p + (\rho + 5p + 4\Lambda)\ln(\rho + 5p + 4\Lambda)] \tag{32}$$

$$\frac{\rho + 5p + 4\Lambda}{2a^2} = (\rho + 5p + 4\Lambda + 2\lambda)(\rho + 2p + \Lambda) + \lambda\,[2p + (\rho + 5p + 4\Lambda)\ln(\rho + 5p + 4\Lambda)] \tag{33}$$

For pressure is zero ($p = 0$), the system of equations (31)-(33) reduce to

$$\frac{\rho + 4\,\Lambda}{2a^2} = -\,(\rho + 4\,\Lambda + 2\,\lambda)\,\Lambda - \lambda\ (\rho + 4\,\Lambda)\ln(\rho + 4\,\Lambda) \tag{34}$$

$$\frac{3(\rho + 4\,\Lambda)}{4a^2} = (\rho + 4\,\Lambda + 2\,\lambda)\left(\rho + \frac{\Lambda}{2}\right) + \frac{\lambda}{2}\ (\rho + 4\,\Lambda)\ln(\rho + 4\,\Lambda) \tag{35}$$

$$\frac{\rho + 4\,\Lambda}{2a^2} = (\rho + 4\,\Lambda + 2\,\lambda)\,(\rho + \Lambda) + \lambda\ (\rho + 4\,\Lambda)\ln(\rho + 4\,\Lambda) \tag{36}$$

For $\lambda = 0$, the system of equations (34)-(36) reduces to

$$\frac{\rho + 4\,\Lambda}{2a^2} = -4\,\Lambda^2 - \Lambda\,\rho \tag{37}$$

$$\frac{3(\rho + 4\,\Lambda)}{4a^2} = 2\,\Lambda^2 + \frac{9}{2}\,\Lambda\,\rho + \rho^2 \tag{38}$$

$$\frac{\rho + 4\,\Lambda}{2a^2} = 4\,\Lambda^2 + 5\,\Lambda\,\rho + \rho^2 \tag{39}$$

Solution of the above system are $(\rho, \Lambda) = \left\{ \left(\frac{1}{a^2}, -\frac{1}{2a^2}\right), \left(\frac{2}{a^2}, -\frac{1}{2a^2}\right) \right\}$ and $\Lambda = -\frac{1}{4}\rho$.

For $\lambda \neq 0$ and the elimination of $\lambda\ (\rho + 4\,\Lambda)\ln(\rho + 4\,\Lambda)$ from the system of equations, (34)-(36) reduce to a single equation

$$\frac{\rho + 4\,\Lambda}{a^2} = 4\,\Lambda\,\rho + 2\,\lambda\,\rho + \rho^2 \tag{40}$$

This equation has an infinite number of solutions depending on ρ and/or Λ. As a result of using ρ in the form (25), we get $\rho < 0$ and $\Lambda > 0$, which represents unrealistic behavior of the physical parameters. Therefore, let us assume that

$$\rho = \frac{1}{a^2\,(1 + k\lambda)} \tag{41}$$

is a solution of (40) where $k \neq 2$ is a constant quantity which needs to be determined. The corresponding cosmological constant has the form

$$\Lambda = \frac{2\,a^2(k\lambda + 1) - k}{4a^2\,(k\lambda + 1)\,k} \tag{42}$$

ρ and Λ satisfy the field equations (34)-(36) under the constraint

$$\lambda = -\frac{2\,a^2 + k}{2a^2 k\left(2\,\ln\left(\frac{2}{k}\right) + 1\right)}. \tag{43}$$

Therefore, the matter energy density ρ and cosmological constant Λ will take the form

$$\rho = \frac{4\ln(\frac{2}{k}) + 2}{4a^2\ln(\frac{2}{k}) - k} \tag{44}$$

$$\Lambda = \frac{(2a^2 - k)\ln(\frac{2}{k}) - k}{(4a^2\ln(\frac{2}{k}) - k)k} \tag{45}$$

For this model, the relationship between the matter energy density and the cosmological constant is as follows:

$$\Lambda = \frac{(2a^2 - k)\ln(\frac{2}{k}) - k}{(4\ln(\frac{2}{k}) + 2)k}\rho \tag{46}$$

We have the restriction on the parameters k and a^2 as follows:
$k \in (0, 2)$ and $\frac{k}{4\ln(\frac{2}{k})} < a^2 < \frac{k}{2}\left(\frac{1}{2\ln(\frac{2}{k})} + 1\right)$ for the matter energy density $\rho > 0$ and cosmological constant $\Lambda < 0$.

Solution for $h(T) = \lambda e^T$

In this case the modified field equations (6) take the form

$$G_{ab} = \left[1 + 2\lambda e^T\right] T_{ab} + \lambda e^T \left[2p + 1\right] g_{ab} \tag{47}$$

The non-zero components of equation (47) for the Gödel metric (14) along with (16) are expressed as

$$\frac{1}{2a^2} = -\left(1 + 2\,\lambda\,\mathrm{e}^{\rho+5\,p+4\,\Lambda}\right)(p + \Lambda) - \lambda\,\mathrm{e}^{\rho+5\,p+4\,\Lambda}\,(2\,p + 1) \tag{48}$$

$$\frac{3}{4a^2} = \left(1 + 2\,\lambda\,\mathrm{e}^{\rho+5\,p+4\,\Lambda}\right)\left(\rho + \frac{3p}{2} + \frac{\Lambda}{2}\right) + \frac{\lambda}{2}\,\mathrm{e}^{\rho+5\,p+4\,\Lambda}\,(2\,p + 1) \tag{49}$$

$$\frac{1}{2a^2} = \left(1 + 2\,\lambda\,\mathrm{e}^{\rho+5\,p+4\,\Lambda}\right)(\rho + 2\,p + \Lambda) + \lambda\,\mathrm{e}^{\rho+5\,p+4\,\Lambda}\,(2\,p + 1) \tag{50}$$

For pressure is zero ($p = 0$), the system reduces to

$$\frac{1}{2a^2} = -\left(1 + 2\,\lambda\,\mathrm{e}^{\rho+4\,\Lambda}\right)\Lambda - \lambda\,\mathrm{e}^{\rho+4\,\Lambda} \tag{51}$$

$$\frac{3}{4a^2} = \left(1 + 2\,\lambda\,\mathrm{e}^{\rho+4\,\Lambda}\right)\left(\rho + \frac{\Lambda}{2}\right) + \frac{\lambda}{2}\mathrm{e}^{\rho+4\,\Lambda} \tag{52}$$

$$\frac{1}{2a^2} = \left(1 + 2\,\lambda\,\mathrm{e}^{\rho+4\,\Lambda}\right)(\rho + \Lambda) + \lambda\,\mathrm{e}^{\rho+4\,\Lambda} \tag{53}$$

For $\lambda = 0$, the system of equations (51)-(53) has solution

$$(\rho, \Lambda) = \left(\frac{1}{a^2}, -\frac{1}{2a^2}\right)$$

, which reduces to the solution of Santos [12] and Gödel [3]

For $\lambda \neq 0$, the elimination of $\lambda\, e^{\rho+4\,\Lambda}$ from the above system of equations (51)-(53) reduces to a single equation

$$\rho\,\left(1 + 2\,\lambda\, e^{\rho+4\,\Lambda}\right) = \frac{1}{a^2} \tag{54}$$

Following the above procedure we have the Solution

$$\rho = \frac{1}{a^2\,(1 + 2\,\lambda)} \tag{55}$$

$$\Lambda = -\frac{1}{4a^2\,(1 + 2\,\lambda)} \tag{56}$$

These two parameters satisfy the field equations under the constraint $\lambda = -\frac{1}{4a^2}$. Thus, ρ and Λ takes the form

$$\rho = \frac{2}{2a^2 - 1} \tag{57}$$

and

$$\Lambda = -\frac{1}{2(2a^2 - 1)} \tag{58}$$

Based on this model, the matter energy density ρ and cosmological constant Λ are related as

$$\Lambda = -\frac{1}{4}\rho \tag{59}$$

Concluding Remarks

- The modified field equations are developed for various trace dependent functions $h(T)$, including $h(T) = \lambda T^2$, $h(T) = \lambda \ln(T)$, and $h(T) = \lambda e^T$.
- For all the discussed forms of $h(T)$, the functional form $f(R,T) = R + 2h(T)$ produced the same results as Santos [12] and Gödel [3] for $\lambda = 0$.
- For $\lambda \neq 0$, the cosmological constant depends on the matter and geometry for $h(T) = \lambda T^2$ and $h(T) = \lambda \ln(T)$, but only on the matter for $h(T) = \lambda e^T$.
- The expressions in (28), (45) and (58) indicate the smallness of the cosmological constant Λ depending on a followed by the prescribed λ.
- It is noted that the λ parameter is a small negative quantity and depends on a. The parameter λ lies in the interval $-0.5 < \lambda < -0.33$ and $-\infty < \lambda < -0.5$ for $h(T) = \lambda T^2$ and $h(T) = \lambda e^T$, respectively. For the scenario $h(T) = \lambda \ln(T)$, the range of λ is determined by k. We will use $k = 1$ and $a^2 \in (0.36068, 1.2214)$ as an example, so $-1 < \lambda < -0.59$.
- In the considered form of $h(T)$, we are able to find the solutions of the field equations for Gödel metric. Thus there is a possibility of arising the CTCs in $f(R,T)$ gravity.

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THANK YOU