Towards a gravity dual for spread complexity

Jaco van Zyl

based on [A Chattopadhyay, A Mitra, HJRvZ, 2302.10489]

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Talk Layout

Motivation

- Complexity is related to the holographic description of black holes
- Growth of complexity = growth of black hole interiors
- Thermofield double is a famous example of this

[Chapman et al, 1810.05151]

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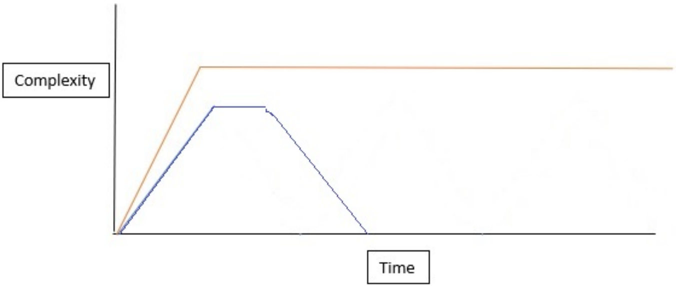
  [Chapman et al, 1810.05151]
- Complexity can be used as a diagnostic of quantum chaos

  [Chapman, Pelicastro, 2110.14672]
- Supplements diagnostics such as SFF, OTOC, Loschmidt echo...

Motivation



[Balasubramanian, DeCross, Kar, Li, Parrikar, 2101.02209]

Complexity

- Central question: How hard is it to synthesize a desired target state with the gates at your disposal?
- Need, $|\phi_r\rangle$, $|\phi_t\rangle$, $\{U_1, U_2, \cdots, U_n\}$, $g(U_1, U_2, \cdots, U_n)$

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- E.g. $U_1 U_2 U_1 U_3 (U_1)^3 U_2 |\phi_r\rangle = U_3 U_1 U_2 U_1 U_3 (U_1)^3 U_2 U_3 |\phi_r\rangle$, "complexity $= 8$"

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- Discrete notion of complexity closely related to quantum computational setups
- We will, however, be interested in a continuous notion of complexity

Nielsen Complexity

- Accessible gates are taken to be from some symmetry group
  [Nielsen, quant-ph/0502070]
- E.g. $SU(2)$: Gates $U = e^{i(s_1 J_1 + s_2 J_2 + s_3 J_3)}$
- Target states: $|\phi_t(s_1, s_2, ..., s_n)\rangle = U(s_1, \cdots, s_n)|\phi_r\rangle$

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- Target states: $|\phi_t(s_1, s_2, ..., s_n)\rangle = U(s_1, \cdots, s_n)|\phi_r\rangle$
- We have a manifold of target states on which one can define a metric
- Complexity = shortest distance connecting points
- Can introduce a circuit parameter $s_i = s_i(\sigma)$

Nielsen Complexity

- Two examples of metrics
- F_1 cost function: $\mathcal{F}_1 d\sigma = |\langle\phi_r|U^\dagger dU|\phi_r\rangle|$

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- Group symmetries are encoded as metric isometries
- $\mathcal{F}_1 : \quad F_i = \partial_i \left(\langle\phi_t(s'_1, s'_2, \cdots, s'_n)|\phi_t(s_1, s_2, ..., s_n)\rangle\right)|_{s'=s}$
- FS metric:
  $g_{ij} = \partial_i\partial'_j \log\left(\langle\phi_t(s'_1, s'_2, \cdots, s'_n)|\phi_t(s_1, s_2, ..., s_n)\rangle\right)\Big|_{s'=s}$

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- Bigger stability subgroup leads to simpler expressions (especially for FS metric)
- Manifold of states $\Leftrightarrow$ group elements of G/H

Krylov Complexity

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- Given a Hamiltonian and reference state one first builds the basis $|O_n) = H^n|\phi_r\rangle$

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- Given a Hamiltonian and reference state one first builds the basis $|O_n) = H^n|\phi_r\rangle$
- From a Gram-Schmidt process one then obtains the Krylov basis $|K_n\rangle$
- The K-complexity (or spread complexity) of a state is then given by $C_K = \sum_n n\langle\phi_t|K_n\rangle\langle K_n|\phi_t\rangle \equiv \langle\phi_t|\hat{K}|\phi_t\rangle$
- The Krylov basis provides an **ordered** basis for the Hilbert space of target states

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Krylov Complexity

- Given some basis for the Hilbert space of target space in increasing complexity $|B_n\rangle$
- We can define complexity as $C = \sum_n c_n \langle\phi_t|B_n\rangle\langle B_n|\phi_t\rangle$
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- With c_n strictly increasing
- The choice $c_n = n$ minimises the complexity of the time-evolved reference state

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K-Complexity as Volume

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- Which notion of complexity is the right one?
- Many ambiguities in the definition of complexity
- Are these different notions of complexity related?
- Krylov complexity related to the geodesic volume [Caputa, Magan, Patramanis, arXiv:2109.03824]

$SU(1,1)$ K-complexity

- $[L_0, L_{\pm 1}] = \mp L_{\pm 1} \quad [L_{-1}, L_{+1}] = 2L_0$
- Reference state $L_{+1}|h\rangle = 0 \quad L_0|h\rangle = h|h\rangle$
- Hamiltonian $H = \alpha(L_{+1} + L_{-1})$

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- Spread complexity of the time-evolved reference state $|t\rangle = e^{itH}|h\rangle$ given by $K(t) = 2h\sinh^2(\alpha t)$

$SU(1,1)$ Geodesic Volume

- States represented on target state manifold are $SU(1,1)$ coherent states $|z) = e^{zL_{-1}}|h\rangle$
- Fubini-Study metric $ds^2 = \frac{2h dz d\bar{z}}{(1-z\bar{z})^2}$

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- Useful change of coordinates $z = e^{i\phi}\tanh(\rho)$
- $\cosh(\frac{\sqrt{2}L}{\sqrt{h}}) = \cosh(\rho_f)\cosh(\rho_i) - \cos(\Delta\phi)\sinh(\rho_f)\sinh(\rho_i)$
- Note that geodesics from the center all have $L = \sqrt{\frac{h}{2}}\rho_f$

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- For this choice of Hamiltonian we have $\rho = 2\alpha t$, $\phi = \frac{\pi}{2}$

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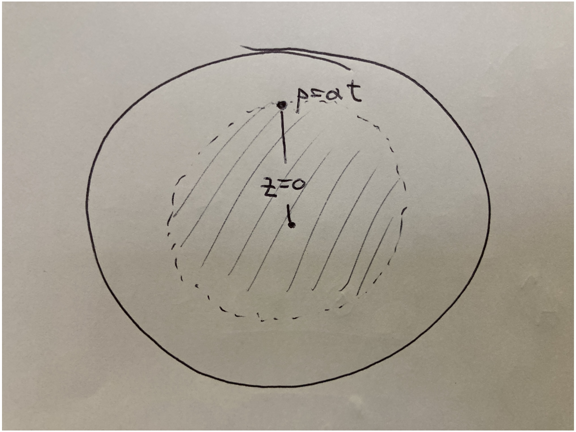
Geodesic Volume

$$V = \int_0^{2\alpha t} d\rho \int_0^{2\pi} \sqrt{g} = \pi K(t)$$



Geodesic Volume

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Geodesic Volume

- Note that a few special choices have been made - volume as measured from the center, specific choice of $su(1,1)$ Hamiltonian
- In general the isometry generator of the Fubini-Study metric close on the same algebra as the group
- The K-complexity operator corresponds to some combination of the isometry generators
- For example, for this choice of Hamiltonian and reference state, we have $\hat{K} = L_0 + h$

Geometric QM

- General relation between Killing vectors of the FS metric and generators [Ashtekar, Schilling, gr-qc/9706069]
- Quantum state space permits a Riemmanian metric, symplectic structure [Kibble, 1979], [Provost, Vallee, 1980]
- Given a family of states parametrised by a set of continuous coordinates $|s\rangle = |s_1, s_2, \cdots, s_n\rangle$
- $g_{ij}(s) = \partial_i \partial'_j \log |\langle s|s'\rangle| \Big|_{s=s'}$ and $\sigma_{ij}(s) = \frac{1}{2i} \partial_i \partial'_j \log \frac{\langle s|s'\rangle}{\langle s'|s\rangle} \Big|_{s=s'}$

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- If the states may be parametrised holomorphically $g_{a\bar{b}} = g_{\bar{b}a} = \frac{1}{2}\partial_a \partial_{\bar{b}} \log(\bar{z}|z)$ and $\sigma_{a\bar{b}} = -\sigma_{\bar{b}a} = i g_{a\bar{b}}$

Vector fields

- Consider an operator $\hat{G}$ acting as $\hat{G}|s\rangle = \phi(s)|s\rangle - iX_{\hat{G}}|s\rangle$
- One can prove that $X_{\hat{G}} = k^a\partial_a + k^{\bar{a}}\partial_{\bar{a}}$
- $k^a = -\frac{1}{2}\sigma^{a\bar{b}}\partial_{\bar{b}}\langle G\rangle$ and $k^{\bar{a}} = -\frac{1}{2}\sigma^{\bar{a}b}\partial_b\langle G\rangle$

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- On the manifold of states the symmetries of the quantum states (associated with the Killing vectors) are related to expectation values of the symmetry generators

Dilaton Gravity

- Consider the following action
- $S = \frac{1}{2\pi} \int d^2x \sqrt{g}(R + V)\eta + S_{boundary}$
- The equations of motion are $R = -V$ and $\nabla_\mu \nabla_\nu \eta - \frac{1}{2} g_{\mu\nu} V \eta = 0$

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- $\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0 \quad ; \quad k^\mu = -\frac{1}{2}\sigma^{\mu\nu}\nabla_\nu\eta$
- The dilaton must be the expectation value of a symmetry generator!

Low rank algebras

- $L_+ = L_-^\dagger \quad ; \quad [[L_-, L_+], L_\pm] = \pm 2 f L_\pm$
- Highest weight state $L_-|w\rangle = 0$, $[L_-, L_+]|w\rangle = w_0|w\rangle$
- An arbitrary group element action may be written as $e^{i(a_+ L_+ + a_+^* L_- + a_0 [L_-, L_+])}|w\rangle = N e^{z L_+}|w\rangle$
- The manifold of target states is a two-dimensional manifold and each state is in a one-to-one correspondence with elements of $G/([L_-, L_+])$

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Fubini-Study metric

- $ds^2 = \frac{2w_0}{(1-fz\bar{z})^2}$
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- We can compute the geodesic distance between any two points represented on the manifold

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- For spread complexity the most general thing we can do is generate the Krylov basis from some state $|\psi_{z_0}\rangle = U(\bar{z}_0, z_0)|w\rangle$
- $U(\bar{z}_0, z_0) = e^{z_0 L_+} e^{\frac{1}{2f} \log(1-f\bar{z}_0 z_0)[L_-, L_+]} e^{-\bar{z}_0 L_-}$

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- $\frac{(\bar{z}|\hat{K}|z)}{(\bar{z}|z)} = z'\partial_{z'} \log((\bar{z}'|z'))$
- One finds $K(|z\rangle; H, U|w\rangle) = \frac{w_0(z-z_0)(\bar{z}-\bar{z}_0)}{(1-f\bar{z}_0 z_0)(1-f\bar{z}z)}$

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Spread Complexity as Dilaton Solutions

- Dilaton action $S = \frac{1}{2\pi}\int d^2x\sqrt{g}\eta\left(R + \frac{4f}{w_0}\right)$
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- We need to supplement this with boundary conditions
- The crucial condition is that $\partial_z\eta|_{z=z_0} = 0$
- This gives $\eta = 2c_3 f\frac{2_0(z-z_0)(\bar{z}-\bar{z}_0)}{(1+fz_0\bar{z}_0)(1-fz\bar{z})} + \eta_0$
- This is the spread complexity up to an additive constant and overall factor!

Spread Complexity as Dilaton Solutions

- Physically, the boundary condition imposes that the complexity is minimised at the appropriate reference state
- The dilaton we have chosen matches exactly with cost factors of the form $\hat{K} = \sum_n (mn + c)|K_n\rangle\langle K_n|$

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- Though we have chosen the minimal complexity state we still need to specify its value. And we need to determine the scaling factor m
- In terms of JT gravity quantities the matching of the scaling is determined by setting $MR = -1$

Outlook

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- Are the fluctuations of the dilaton captured by some notion related to complexity? What about the boundary action of the dilaton?
- Dilaton models with non-linear potential terms - are these related to different choices of c_n?
- Does this picture continue to hold in higher dimensions?

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- Are the fluctuations of the dilaton captured by some notion related to complexity? What about the boundary action of the dilaton?
- Dilaton models with non-linear potential terms - are these related to different choices of c_n?
- Does this picture continue to hold in higher dimensions?
- Can the construction be altered to give a bulk /boundary interpretation?
- Dilatons have a natural interpretation of (varying) volume of compactified dimensions. Is this a way to get a handle on the complexity = volume conjecture?

Thank you for your attention!

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