New exact vacuum solutions and wormholes in $\mathcal{R}^2$ gravity

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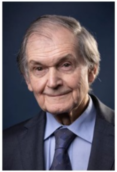
Applications of Quantum Information in Astrophysics and Cosmology Workshop
Cape Town, South Africa

In the news ...

The Nobel Prize in Physics 2020



© Nobel Prize Outreach. Photo: Fergus Kennedy

Roger Penrose

Prize share: 1/2

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Reinhard Genzel

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Andrea Ghez

Prize share: 1/4

The Nobel Prize in Physics 2020 was divided, one half awarded to Roger Penrose "for the discovery that black hole formation is a robust prediction of the general theory of relativity", the other half jointly to Reinhard Genzel and Andrea Ghez "for the discovery of a supermassive compact object at the centre of our galaxy"

Robert Roy Britt

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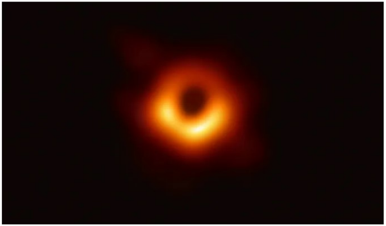

The shadow of a black hole is observed at the center of the galaxy M87. The bright ring is created by light that actually bends in the intense gravity around the object. Image: Event Horizon Telescope Collaboration

Black Hole's Shadow Seen for the First Time

The first-ever direct image of a black hole's point of no return confirms theories, thrills scientists.

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Article | Published: 30 November 2022

Traversable wormhole dynamics on a quantum processor

Daniel Jafferis, Alexander Zlokapa, Joseph D. Lykken, David K. Kolchmeyer, Samantha I. Davis, Nikolai Lauk, Hartmut Neven & Maria Spiropulu

Nature **612**, 51–55 (2022) | Cite this article

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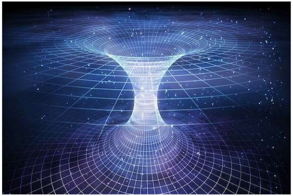
Abstract

The holographic principle, theorized to be a property of quantum gravity, postulates that the description of a volume of space can be encoded on a lower-dimensional boundary. The anti-

DECEMBER 12, 2022

Did physicists make a wormhole in the lab? Not quite, but a new experiment hints at the future of quantum simulations

by Sam Baron, The Conversation



A wormhole is a hypothetical 'shortcut' between two regions of space. Credit: Shutterstock

Scientists made headlines last week for supposedly generating a wormhole. The research, reported in *Nature*, involves the use of a quantum computer to simulate a wormhole in a simplified model of physics.

Morris-Thorne 1988 Am.J.Phys. paper ...

Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity

Michael S. Morris and Kip S. Thorne
Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125

(Received 16 March 1987; accepted for publication 17 July 1987)

Rapid interstellar travel by means of spacetime wormholes is described in a way that is useful for teaching elementary general relativity. The description touches base with Carl Sagan's novel *Contact*, which, unlike most science fiction novels, treats such travel in a manner that accords with the best 1986 knowledge of the laws of physics. Many objections are given against the use of black holes or Schwarzschild wormholes for rapid interstellar travel. A new class of solutions of the Einstein field equations is presented, which describe wormholes that, in principle, could be traversed by human beings. It is essential in these solutions that the wormhole possess a throat at which there is no horizon; and this property, together with the Einstein field equations, places an extreme constraint on the material that generates the wormhole's spacetime curvature: In the

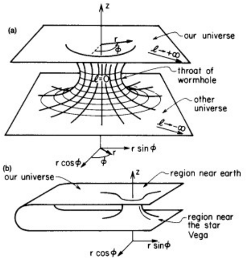


Fig. 1. (a) Embedding diagram for a wormhole that connects two different universes. (b) Embedding diagram for a wormhole that connects two distant regions of our own universe. Each diagram depicts the geometry of an equatorial ($\theta = \pi/2$) slice through space at a specific moment of time ($t =$ const). These embedding diagrams are derived quickly in item (b) of Box 2, and—in a more leisurely fashion—in Sec. III C, where they are also discussed. This figure is adapted from Ref. 1, Fig. 31.5.

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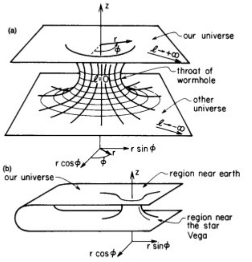

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To sustain wormholes, Weak Energy Condition needs be violated...

- In General Relativity (GR): by introducing "exotic" matter
- In modified gravity: "exotic" matter may *not* be needed...

Presentation based on:

- *Beyond Schwarzschild-de Sitter spacetimes I*, Phys. Rev. D 106, 104004 (2022)
- *Beyond Schwarzschild-de Sitter spacetimes II*, to appear in PhysRevD
- *Beyond Schwarzschild-de Sitter spacetimes III*, to appear in PhysRevD
- *Properties of Buchdahl-inspired metrics*, to be posted
- *Traversable Morris-Thorne-Buchdahl wormholes in quadratic gravity* (with M. Azreg-Aïnou)
- *A stationary axisymmetric vacuum solution for pure R2 gravity* (with M. Azreg-Aïnou)

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My thanks to:

- Richard Shurtleff
- Dieter Lüst
- Tiberiu Harko
- Mustapha Azreg-Aïnou
- Timothy Clifton

$\mathcal{R}^2$ field equation

$$\mathcal{A}_{\,\mathrm{GR}} = \int d^4x\sqrt{-g}\,\mathcal{R} \tag{1}$$

$$\mathcal{A}_{\,\mathrm{R2}} = \int d^4x\sqrt{-g}\,\mathcal{R}^2 \quad \textbf{Scale invariant !} \tag{2}$$

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$\mathcal{R}^2$ field equation in vacuo:

$$\mathcal{R}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right) + \left(g_{\mu\nu}\nabla_\lambda\nabla^\lambda - \nabla_\mu\nabla_\nu\right)\mathcal{R} = 0 \tag{3}$$

Trace of $g_{\mu\nu} = 4 \quad \Rightarrow \quad$ Trace eqn: $\nabla_\lambda\nabla^\lambda\,\mathcal{R} = 0$

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Schwarzschild-de Sitter metric ($\mathcal{R} = 4\Lambda$ constant) is a solution to $\mathcal{R}^2$ field eqn.

But there are solutions with **non-constant** Ricci scalar curvature !

The general Buchdahl-inspired metric

From Buchdahl's 1962 Nuovo Cimento paper [Buchdahl-1962]...

I derived the metric

$$ds^2 = e^{k\int\frac{dr}{r\,q(r)}}\left\{-\frac{p(r)\,q(r)}{r}dt^2 + \frac{p(r)\,r}{q(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right\} \quad (4)$$

Evolution rules

$$\frac{dp}{dr} = \frac{3k^2}{4r}\frac{p}{q^2} \quad (5)$$

$$\frac{dq}{dr} = \left(1 - \Lambda\, r^2\right) p \quad (6)$$

Ricci scalar

$$\mathcal{R}(r) = 4\Lambda\, e^{-k\int\frac{dr}{r\,q(r)}} \qquad \text{Non-constant !} \quad (7)$$

Two new higher-derivative parameters: Λ and k (the Buchdahl parameter)

The *special* Buchdahl-inspired metric: $\Lambda = 0$

Metric with $\Lambda = 0$:

$$ds^2 = \left|1 - \frac{r_s}{\rho}\right|^{\tilde{k}} \left\{ -\left(1 - \frac{r_s}{\rho}\right) dt^2 + \frac{r^4(\rho)\, d\rho^2}{\rho^4 (1 - \frac{r_s}{\rho})} + r^2(\rho) \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \right\} \quad (8)$$

$$r^2(\rho) := \frac{\zeta^2 r_s^2 \left|1 - \frac{r_s}{\rho}\right|^{\zeta - 1}}{\left(1 - \text{sgn}\left(1 - \frac{r_s}{\rho}\right) \left|1 - \frac{r_s}{\rho}\right|^{\zeta}\right)^2}; \quad \zeta := \sqrt{1 + 3\tilde{k}^2}; \quad \tilde{k} := \frac{k}{r_s} \quad (9)$$

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Compared with Schwarzschild

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (10)$$

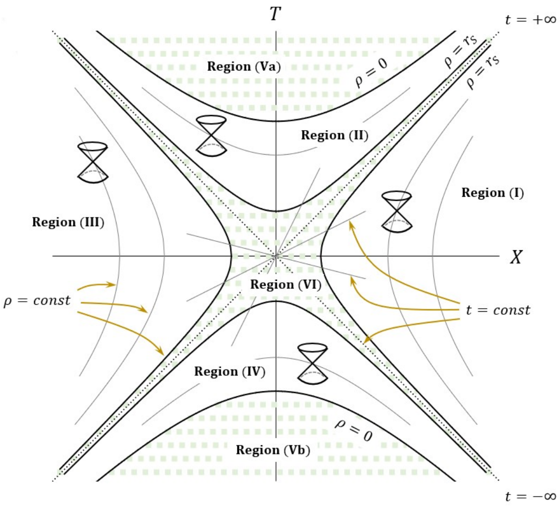
At $\tilde{k} = 0$

$$\zeta = 1 \quad \text{and} \quad r(\rho) = \rho \quad \forall \rho \in \mathbb{R} \quad (11)$$

$\Rightarrow$ Recover Schwarzschild metric !

Buchdahl parameter $\tilde{k}$ is new higher-derivative characteristic

$\zeta-$Kruskal-Szekeres diagram



"Gulf" as Region (VI) is a new feature

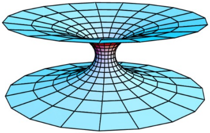
Traversable Morris-Thorne-Buchdahl WH

MT ansatz [Morris/Thorne-1988]

$$ds^2 = -e^{2\Phi(R)}dt^2 + \frac{dR^2}{1-\frac{b(R)}{R}} + R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{12}$$

Four WH conditions:

1. $\Phi(R)$ be finite (no horizon)
2. $b(R_0) = R_0$ at the "throat" $R = R_0$
3. $b(R)/R \leq 1$ for $R \geq R_0$. Equal sign at "throat". g_{RR} does not flip sign as $R \to R_0^+$.
4. Asymptotically flat: $b(R)/R \to 0$ as $R \to +\infty$



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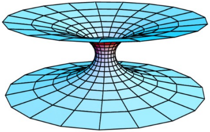

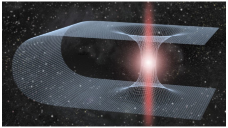

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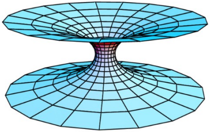
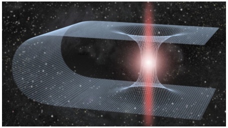
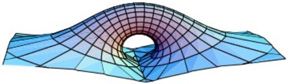

Traversable Morris-Thorne-Buchdahl WH

The special Buchdahl-inspired metric can be brought to the MT form

$$ds^2 = -e^{2\Phi(R)}dt^2 + \frac{dR^2}{1 - \frac{b(R)}{R}} + R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{13}$$

with

- $y(R)$ implicit

$$y := \left(1 - \frac{r_s}{\rho}\right)^{\zeta}; \qquad R = (\zeta r_s)\frac{y^{\frac{1}{2}\left(\frac{\tilde{k}-1}{\zeta}+1\right)}}{1-y} \tag{14}$$

- Redshift function

$$e^{2\Phi(R)} = y^{\frac{\tilde{k}+1}{\zeta}} \tag{15}$$

- Shape function

$$1 - \frac{b(R)}{R} = \frac{1}{4y}\left[\left(\frac{\tilde{k}-1}{\zeta}+1\right) - \left(\frac{\tilde{k}-1}{\zeta}-1\right)y\right]^2 \tag{16}$$

Traversable Morris-Thorne-Buchdahl WH

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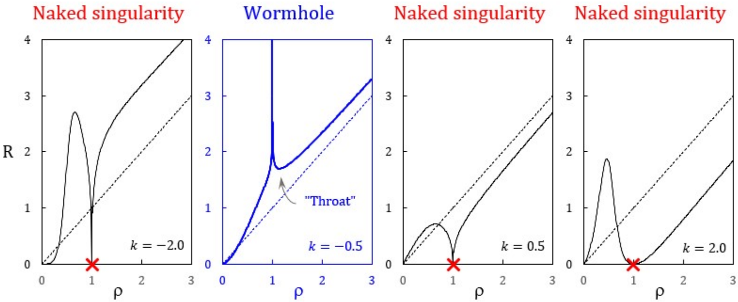

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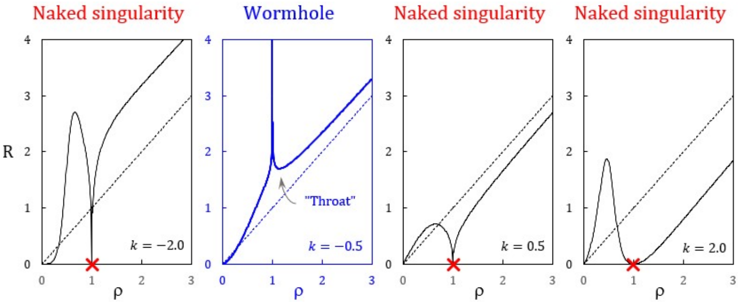
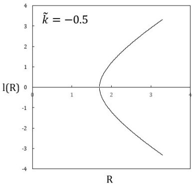


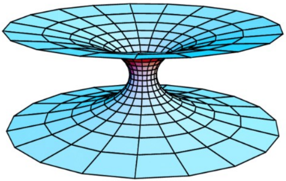
- For $\tilde{k} \in (-1, 0)$: $R(\rho)$ has a minimum in the exterior $\Rightarrow$ A traversable Morris-Thorne-Buchdahl wormhole connecting 2 asymptotically flat sheets of spacetime
- For $\tilde{k} \in (-\infty, -1) \cup (0, +\infty)$: a naked singularity at $\rho = r_s$

Traversable Morris-Thorne-Buchdahl WH

The proper radial distance $l(R)$

$$l(R) = \pm \int_{R_0}^{R} \frac{dR}{\sqrt{1 - \frac{b(R)}{R}}} = \pm \frac{\zeta r_s}{1 + \frac{\tilde{k}-1}{2\zeta}} \left[y^{1+\frac{\tilde{k}-1}{2\zeta}} {}_2F_1\left(2, 1 + \frac{\tilde{k}-1}{2\zeta}; 2 + \frac{\tilde{k}-1}{2\zeta}; y\right) - \text{const} \right] \tag{17}$$





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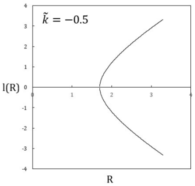
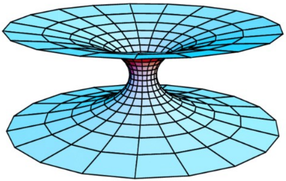


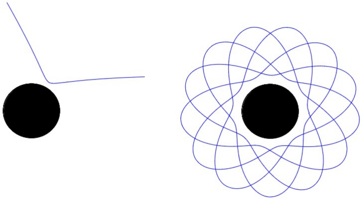


When a ***wormhole*** is formed, Weak Energy Condition is ***violated***:

$$G_{00} = \frac{\tilde{k}(\tilde{k}+1)}{2 r_s^2 \zeta^4} \frac{\left[1 \mp \left|1 - \frac{r_s}{\rho}\right|^{\zeta}\right]^4}{\left|1 - \frac{r_s}{\rho}\right|^{2(\zeta - 1)}} \quad \Rightarrow \quad G_{00} < 0 \quad \text{for } \tilde{k} \in (-1, 0) \tag{18}$$

... despite that there is no “exotic” matter !

Orbits, shadows, etc.



Perihelion and aphelion are not on opposite sides of massive object !

Model Sagittarius A* as $\mathcal{R}^2$ wormhole ? Extracting bound for $\tilde{k}$?

In [Axisymmetric-2023] Azreg-Aïnou and I extended my work to axisymmetric setup and applied for M87* shadow, and obtained:

$$-0.155 \leq \tilde{k} \leq 0.004$$

Thank you !

[Buchdahl-1962] Buchdahl H.A., *On the gravitational field equations arising from the square of the Gaussian curvature*, Nuovo Cimento, Vol 23 No 1, 141 (1962)

[Morris/Thorne-1988] Morris M.S. and K.S. Thorne, *Wormholes in space-time and their use for interstellar travel*, Am. J. Phys. 56, 395 (1988); M.S. Morris, K.S. Thorne and U. Yurtsever, *Wormholes, Time Machines, and the WEC*, Phys. Rev. Lett. 61, 1446 (1988)

[Nguyen-2022] HKN, *Beyond SdS spacetimes I*, Phys. Rev. D. 106, 104004 (2022); *Beyond SdS spacetimes II*, to appear in PRD (2211.03542 [gr-qc]); *Beyond SdS spacetimes III*, to appear in PRD (2211.07380 [gr-qc]); *Properties of Buchdahl-inspired metrics* (to be posted)

[MTB-wormhole-2023] HKN and M. Azreg-Aïnou, *Traversable Morris-Thorne-Buchdahl wormholes in quadratic gravity* (to be posted)

[Axisymmetric-2023] HKN and M. Azreg-Aïnou, A stationary axisymmetric vacuum solution for pure R2 gravity (2304.08456 [gr-qc])