‘Relativistic’ Quantum Theories

from the

Proper Symmetry Theoretical Formulation

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Lorentz Covariant Quantum Physics :-

• Schrödinger wavefunction $\phi(x^\mu)$

— basic operators x_μ and $-i\hbar\partial_\mu$

• abstract operators as Minkowski four-vectors

— $\hat{X}_i \longrightarrow \hat{X}_\mu$ and $\hat{P}_i \longrightarrow \hat{P}_\mu$

— $[\hat{X}_\mu, \hat{P}_\nu] = i\hbar\eta_{\mu\nu}$

• Heisenberg-Weyl symmetry — $[Y_\mu, E_\nu] = i\hbar\, c\, \eta_{\mu\nu}\, M$

— M is an effective Casimir element → Newtonian mass m

— $m\hat{X}_\mu \longleftarrow Y_\mu$, different m for different irr. representations

— $\hat{P}_\mu \longleftarrow \frac{1}{c}E_\mu$, constant c (. . . $c \to \infty$ limit)

Against Poincaré Symmetry :-

- $\hat{P}_\mu \hat{P}^\mu$ **cannot be Casimir operator** $(= -m_E^2 c^2)$

—$[\hat{X}_i, \hat{P}_\mu \hat{P}^\mu] = 2i\hbar \hat{P}_i$

- consider E-dynamics with **Lorentz force**

$$\hat{H}_\tau = \frac{1}{2m}\left(\hat{P}_\mu - \frac{e}{c}\hat{A}_\mu\right)\left(\hat{P}^\mu - \frac{e}{c}\hat{A}^\mu\right)$$

$$\frac{d\hat{X}^\mu}{d\tau} = \frac{1}{m}\left(\hat{P}^\mu - \frac{e}{c}\hat{A}^\mu\right) \equiv \frac{1}{m}\hat{\pi}^\mu \,, \qquad \frac{d}{d\tau}\left(\hat{P}^\mu - \frac{e}{c}\hat{A}^\mu\right) = 0$$

— $\hat{\pi}^\mu = m\frac{d\hat{X}^\mu}{d\tau}$ satisfies $\hat{\pi}_\mu \hat{\pi}^\mu = -m_E^2 c^2$ (no dynamical content)

— **but only $\hat{P}_\mu$ is the true momentum** (*cf.* Einstein 1935)

— $\hat{P}_\mu \hat{P}^\mu$ **evolves nontrivially**

Position Operator for Composite System :-

— product representation with all abstract generators G as

$$\hat{G} = \hat{G}_a \otimes \hat{I} + \hat{I} \otimes \hat{G}_b$$

- **additive mass** $\hat{M} = \hat{M}_a \otimes \hat{I} + \hat{I} \otimes \hat{M}_b$ $(m = m_a + m_b)$

- **additive momentum** $\hat{P}^\mu = \hat{P}^\mu_a \otimes \hat{I} + \hat{I} \otimes \hat{P}^\mu_b$ $(P^\mu = \frac{1}{c}E^\mu)$

- **position is not an additive notion**

— $\hat{X}^\mu \equiv \frac{1}{m}\hat{Y}^\mu = \frac{1}{m}(m_a\hat{X}^\mu_a \otimes \hat{I} + \hat{I} \otimes m_b\hat{X}^\mu_b)$ **as center of mass**

- **Heisenberg-Weyl as part of Galilean** ($\rightarrow$ **3+1 version**)

Symmetry Theoretical Formulation :-

• take **regular representation of $\boldsymbol{H(1,3)}$** (Heisenberg-Weyl)

— generators as left-invariant vector fields ($\hbar = 2$)

$$Y^L_\mu = x_\mu i\partial_\zeta + i\partial_{v^\mu} \ , \quad E^L_\mu = c\, v_\mu i\partial_\zeta - ic\,\partial_{x^\mu} \ , \quad M^L = i\partial_\zeta$$

• to **irr. components** (inverse Fourier-Plancherel transform)

$$\alpha(v^\mu, x^\mu, \zeta) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int dm\, \boldsymbol{\alpha_m(v^\mu, x^\mu)}\, e^{-im\zeta} |m|^n$$

— $\boldsymbol{\hat{X}^L_\mu = x_\mu + i\partial_{p^\mu}} \ , \quad \boldsymbol{\hat{P}^L_\mu = p_\mu - i\partial_{x^\mu}} \ , \quad \boldsymbol{\hat{M}^L = m}$

• **complete rep.** of proper extensive enveloping/group C^*-algebra

— operators as $\boldsymbol{\alpha(p_\mu, x_\mu)\star = \alpha(p_\mu\star, x_\mu\star) \equiv \alpha(\hat{P}^L_\mu, \hat{X}^L_\mu)}$

• wavefunctions $\boldsymbol{\phi(p^\mu, x^\mu) = {}_\eta\langle p^\mu, x^\mu | \phi\rangle = \langle p_\mu, x_\mu | \phi\rangle}$

Minkowski Metric Operator $\hat{\eta}$ on Krein Space :-

• **Minkowski** nature of proper invariant **inner product**

— effectively, bra as ${}_\eta\langle\cdot| = \langle\cdot|\,\hat{\eta}$

— naive $|\phi(x^\mu)|^2$ **integral** cannot avoid **divergence**

• **observables (pseudo-)Hermitian,** ${}_\eta\left\langle\cdot|\hat{A}^{\dagger^\eta}\cdot\right\rangle = {}_\eta\left\langle\hat{A}\cdot|\cdot\right\rangle$

— $\hat{X}_\mu = \hat{\eta}\hat{X}^\mu\hat{\eta}^{-1}$ and $\hat{P}_\mu = \hat{\eta}\hat{P}^\mu\hat{\eta}^{-1}$

• **noncommutative geometric picture**

— $\hat{X}^\mu$ **and** $\hat{P}^\mu$ **as coordinates**

The Symplectic Geometry — NC Vs C :-

- **Heisenberg** — $\frac{d}{ds}\alpha(\hat{P}_\mu, \hat{X}_\mu) = \frac{1}{i\hbar}[\alpha(\hat{P}_\mu, \hat{X}_\mu), \hat{H}_s]$

- **Schrödinger** — $\frac{d}{ds} f_\alpha(z^n, \bar{z}^n) = \{f_\alpha(z^n), f_{H_s}\}$

- $f_\alpha(z^n, \bar{z}^n) \equiv \frac{{}_\eta\langle\phi|\alpha(\hat{P}_\mu, \hat{X}_\mu)|\phi\rangle}{{}_\eta\langle\phi|\phi\rangle}$ $\qquad \left(|\phi\rangle = \sum_n z^n |n\rangle\right)$

— as the pull-back of $\hat{\alpha}$ under $(z^n, \bar{z}^n) \longrightarrow (\hat{P}_\mu, \hat{X}_\mu)$

- $\rightarrow$ **bijective homomorphism between NC Poisson algebras**

— **NC Kähler product** $\qquad f_\alpha \star_\kappa f_{\alpha'} = f_{\alpha\alpha'}$ *Cirelli et.al 90*

NC values of NC coordinates :-

— NC number as the new convenient fiction

- **state as evaluative homorphism**

— mapping observable algebra to algebra of their NC values

$[\hat{\alpha}]_\phi = \{ f_\alpha|_\phi, V_{\alpha_n}|_\phi \}$ $\quad (V_{\alpha_n} = \frac{\partial f_\alpha}{\partial z^n} = -f_\beta \bar{z}^n + \sum_m \bar{z}^m \langle m|\hat{\alpha}|n\rangle)$

- **Kähler product** — $\quad [\hat{\alpha}\hat{\alpha}']_\phi = [\hat{\alpha}]_\phi \star_\kappa [\hat{\alpha}']_\phi$

$f_{\alpha\alpha'} = f_\alpha f_{\alpha'} + \sum_n V_{\alpha_n} V_{\alpha'_{\bar{n}}}$, $\quad V_{\alpha\alpha'_n} = -f_{\alpha\alpha'} \bar{z}_n + \sum_{m,l} \bar{z}_m \langle m|\hat{\alpha}|l\rangle\langle l|\hat{\alpha}'|n\rangle$

Deutsch & Hayden 00 ; Kong 23

- **locality of quantum information** (**Heisenberg picture**)

Galvão & Hardy 03

- Substituting a **Qubit** for an Arbitrarily Large Number of Classical Bits'

Concept of Numbers (in history) :

- $x + 2 = 0 \quad \longrightarrow \quad$ negative numbers
- $2x - 1 = 0 \quad \longrightarrow \quad$ rational numbers
- $x^2 - 2 = 0 \quad \longrightarrow \quad$ real numbers
- $x^2 + 1 = 0 \quad \longrightarrow \quad$ complex numbers
- $xy - x - i = 0 \quad \longrightarrow \quad$ (i,2), ($\frac{1}{i-1}$,-i), . . .
- $xy - yx - 1 = 0 \quad \longrightarrow \quad$ noncommutative numbers

★ $\hat{x}\hat{p} - \hat{p}\hat{x} - i\hbar = 0$

needs NC/q-number values for the variables

$\boldsymbol{H_R(1,3)}$ symmetry :-

$$[J'_{\mu\nu}, J'_{\rho\sigma}] = i\hbar\, c\left(\eta_{\nu\sigma} J'_{\mu\rho} + \eta_{\mu\rho} J'_{\nu\sigma} - \eta_{\mu\sigma} J'_{\nu\rho} - \eta_{\nu\rho} J'_{\mu\sigma}\right) \ ,$$

$$[J'_{\mu\nu}, Y_\rho] = i\hbar\, c\,(\eta_{\mu\rho} Y_\nu - \eta_{\nu\rho} Y_\mu) \ ,$$

$$[J'_{\mu\nu}, E_\rho] = i\hbar\, c\,(\eta_{\mu\rho} E_\nu - \eta_{\nu\rho} E_\mu) \ ,$$

$$[Y_\mu, E_\nu] = i\hbar\, c\, \eta_{\mu\nu}\, M$$

- **semi-direct product of $\boldsymbol{H(1,3)}$ and Lorentz symmetry**
- **Casimir elements – $\boldsymbol{M}$, $\boldsymbol{\frac{1}{2} T_{\mu\nu} T^{\mu\nu}}$, $\boldsymbol{\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} T_{\mu\nu} T_{\rho\sigma}}$**

$$T_{\mu\nu} \equiv M J'_{\mu\nu} - (Y_\mu E_\nu - Y_\nu E_\mu)$$

- **$\boldsymbol{H_R(1,3)}$ irr. reps. as direct products of those of $\boldsymbol{H(1,3)}$ and Lorentz symmetry generated by $\boldsymbol{T_{\mu\nu}}$**

— spin operators $\hat{S}_{\mu\nu} = \frac{1}{mc}\hat{T}_{\mu\nu} = \hat{J}_{\mu\nu} - \hat{L}_{\mu\nu}$

Symmetry Contraction $c \longrightarrow \infty$:-

$$(J_{\mu\nu} = \frac{1}{c} J'_{\mu\nu}),\ K_i = \frac{1}{c} J_{i0},\ P_i = \frac{1}{c} E_i,\ T' = \frac{-1}{c} Y_0,\ H \equiv -E_0$$

- $\boldsymbol{H_R(1,3) \longrightarrow H_{GH}(3) \supset H_R(3)}$
- **dynamical theory $\longrightarrow$ 'nonrelativistic' theory**

Symmetry Contraction to Classical :-

$$Y^c_\mu = \frac{1}{k_y} Y_\mu,\ E^c_\mu = \frac{1}{k_e} E_\mu; \qquad k_y, k_e \longrightarrow \infty$$

- **M decouples and $[Y^c_\mu, E^c_\nu] = 0$**
- **quantum theory $\longrightarrow$ classical theory**
- projective Krein/Hilbert space $\longrightarrow$ classical phase space

— **Heisenberg picture direct** (NC to C)

— **Schrödinger picture to Koopman-von Neumann picture**

Covariant Hamiltonian Dynamics :-

- $\frac{d}{d\tau} = -\frac{1}{i\hbar}[\hat{H}_\tau, \cdot]$

— evolution parameter τ defined by the Hamiltonian (flow)

- Schrödinger equation $\frac{d}{d\tau}\phi(x^\mu) = \frac{1}{i\hbar}\hat{H}_\tau\phi(x^\mu)$

  — $\hat{H}_\tau = \frac{\hat{P}_\mu \hat{P}^\mu}{2m} + V(\hat{X}^\mu)$

$V(\hat{X}^\mu) = 0$, **τ-independent eq. as Klein-Gordon eq.**

- **only $V(\hat{X}^\mu)$ allows potential in 'nonrelativistic' limit**
- **Piron-Reuse (75) frame** for two-particle dynamics

$p^i = \mu \frac{dx^i}{d\tau}$, $\quad \frac{dp_i}{d\tau} = -\frac{\partial V(x_i x^i)}{\partial x^i}$, $\quad$ with $\frac{E_{c.m.}}{m\,c}\tau$ as common time

‘Quantum Field Theory’ :-

- **obtained from** $H_{R}(1,3)$ **symmetry** + ‘2nd-quantization’

- $\boldsymbol{m=0}$ **irr. representations** (? quantum)

— $[\hat{Y}_\mu, \hat{E}_\nu] = 0$, $\hat{E}_\mu \hat{E}^\mu$ as effective Casimir operator

— e.g. vector space of $|\vec{p}\rangle$ with $p_\mu p^\mu = -m_{_E}^2 c^2$ $(\hat{E}_\mu = c p_\mu)$

- $\boldsymbol{\hat{a}_{\vec{p}}\,|0\rangle = 0\ , \quad |\vec{p}_1 \ldots \vec{p}_n\rangle = \sqrt{2E_{\vec{p}_1} \ldots 2E_{\vec{p}_n}}\,\hat{a}^\dagger_{\vec{p}_1} \ldots \hat{a}^\dagger_{\vec{p}_n} \ldots |0\rangle}$

- **not ‘2nd-quantization’ of RQM**

— true $\phi(x^\mu)$ or $\tilde{\phi}(p^\mu)$, as span of $|p^\mu\rangle$, theory

- ‘2nd-quantization’ of **NRQM** $\boldsymbol{m \to 0}$ **limit + Lorentz sym**

Concluding Remarks :-

- **fully consistent theory of quantum particle dynamics**
- **$(\hat{X}_\mu, \hat{P}_\mu)$-NC geometry as spacetime**

— irr. representation, *cf.* classical Minkowski spacetime

— $(3+1)$ Gel'fand-Kirillov dimension (module of Weyl algebra)

- **quantum phase space as 'Euclidean' NC-geometry ?**
- **noncommutative reality of NC-values**
- **issues about quantum field theory**
- **metric operator for quantum gravity ?**
- **noncommutative numbers ?**

— Dirac's q-number, Takesaki's NC number theory

- **X-X and P-P noncommutating physics ?**

THANK YOU !

Fundamental (Special) Quantum Relativity:

$SO(2,4) \sim SU(2,2)$ (*cf.* deformed S.R.)

- contains noncommuting Y_μ and P_μ , M
- contains Lorentz symmetry $J_{\mu\nu}$
- stable symmetry, no deformation
- G, $\hbar$, c in structural constants

$$
\begin{array}{ccccc}
SO(2,4) \longrightarrow H_R(1,3) & \longrightarrow & H_{GH}(3) \supset \tilde{G}(3) \supset H_R(3) & \\
\downarrow & (\frac{1}{c^2} \rightarrow 0) & \downarrow & (\hbar \downarrow 0) \\
ISO(1,3) \subset S(1,3) & \longrightarrow & S_G(3) \supset G(3) \supset S(3) &
\end{array}
$$

- (Lie algebra) contractions as approximations
- deformation is 'inverse' of contraction