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Dynamical System Analysis in Teleparallel Scalar-Tensor Gravity

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Outline

Introduction:

The minimally coupled scalar field contributes to the energy density of the Universe in the form of dynamical vacuum energy.

The major success of scalar field models is their capability to offer a valid alternative explanation of the smallness of the present vacuum energy density.

The study of dynamical system analysis of dark energy scalar field models is explained by Copland et. al. [1]

[1] E. J. Copeland, M. Sami, S. Tsujikawa, *Dynamics of dark energy*, **arXiv:hep-th/0603057**.

Benifits and Limitations of Dynamical System Analysis Approach :

When we study cosmology in modified gravity theories, we usually obtain complicated systems of equations with ambiguous initial conditions.

Dynamical system analysis is a powerful mathematical and computational tool for studying the behavior of complex systems over time.

One limitation of the dynamical system approach is the dependence on the choice of variables which characterize the solution associated with critical points[2].

[2] J. Dutta, W. Khyllep, and H. Zonunmawia, The Eur. Phys. J. C 79 (2019).

Outline

Teleparallel Scalar Tensor Gravity :

One of the most important and simplest modifications to the GR was suggested by Brans and Dicke in 1961[3].

Horndeski gravity is the most general scalar tensor theory[4], its current form, in curved space-time, was given by Deffayet, Deser, and Esposito-Farese.[5]

The teleparallel analog of Horndeski gravity was first proposed in [6].

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[3] C. Brans and R. H. Dicke, *Phys. Rev.* **124**, 925 (1961).
[4] G. W. Horndeski, *Int. J. Theor. Phys.* **10**, 363 (1974).
[5] C. Deffayet, S. Deser, and G. Esposito-Farese, *Phys. Rev. D*, **80**, 064015 (2009).
[6] S. Bahamonde, K. F. Dialektopoulos, J. L. Said, *Phys. Rev. D*, **100**, 064018 (2019).

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CAN HORNDESKI THEORY BE RECAST USING TELEPARALLEL … PHYS. REV. D **100,** 064018 (2019)

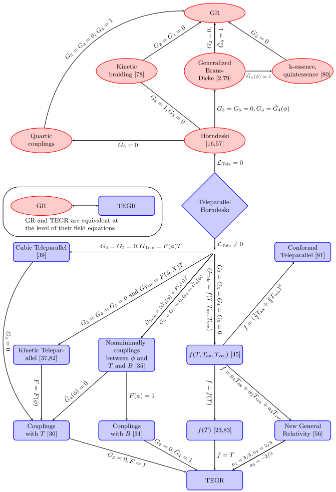


FIG. 1. Relationship between teleparallel Horndeski and various theories.

Outline

Friedmann–Lemaître–Robertson–Walker (FLRW) metric.

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\,, \tag{1}$$

Where $a(t)$ is the scale factor and the tetrad field can be described as follows,

$$e_\mu^A = (1, a(t), a(t), a(t))\,, \tag{2}$$

The tetrad e_μ^A (and its inverses E_A^μ) relate to the metric as the fundamental variable of theory through the relations,

$$g_{\mu\nu} = e_\mu^A e_\nu^B \eta_{AB}\,, \qquad \eta_{AB} = E_A^\mu E_B^\nu g_{\mu\nu}\,, \tag{3}$$

The tetrads must satisfy orthogonality conditions, which take the form,

$$e_\mu^A E_B^\mu = \delta_B^A\,, \qquad e_\mu^A E_A^\nu = \delta_\mu^\nu\,, \tag{4}$$

The Weitzenböck connection can be defined as,

$$\Gamma^{\sigma}_{\nu\mu} := E^{\sigma}_{A}\left(\partial_{\mu} e^{A}_{\nu} + \omega^{A}_{B\mu} e^{B}_{\nu}\right) , \tag{5}$$

The torsion tensor can be described as follows,

$$T^{\sigma}_{\mu\nu} := 2\Gamma^{\sigma}_{[\nu\mu]} , \tag{6}$$

The tetrad e^{A}_{μ} (and its inverses E^{μ}_{A}) relate to the metric as the fundamental variable of theory through the relations,

$$g_{\mu\nu} = e^{A}_{\mu} e^{B}_{\nu} \eta_{AB} , \qquad \eta_{AB} = E^{\mu}_{A} E^{\nu}_{B} g_{\mu\nu} , \tag{7}$$

By an appropriate combination of contractions of torsion tensors, a torsion scalar can be written as follow,

$$T := \frac{1}{4} T^{\alpha}{}_{\mu\nu} T_{\alpha}{}^{\mu\nu} + \frac{1}{2} T^{\alpha}{}_{\mu\nu} T^{\nu\mu}{}_{\alpha} - T^{\alpha}{}_{\mu\alpha} T^{\beta\mu}{}_{\beta} , \tag{8}$$

We consider the irreducible pieces of the torsion tensor as follow,

$$a_\mu := \frac{1}{6}\epsilon_{\mu\nu\lambda\rho}T^{\nu\lambda\rho}\,, \tag{9}$$

$$v_\mu := T^{\lambda}{}_{\lambda\mu}\,, \tag{10}$$

$$t_{\lambda\mu\nu} := \frac{1}{2}\left(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}\right) + \frac{1}{6}\left(g_{\nu\lambda}v_\mu + g_{\nu\mu}v_\lambda\right) - \frac{1}{3}g_{\lambda\mu}v_\nu\,, \tag{11}$$

The scalar invariant can be consider as follow,

$$T_{\text{ax}} := a_\mu a^\mu = -\frac{1}{18}\left(T_{\lambda\mu\nu}T^{\lambda\mu\nu} - 2T_{\lambda\mu\nu}T^{\mu\lambda\nu}\right)\,, \tag{12}$$

$$T_{\text{vec}} := v_\mu v^\mu = T^{\lambda}{}_{\lambda\mu}T_\rho{}^{\rho\mu}\,, \tag{13}$$

$$T_{\text{ten}} := t_{\lambda\mu\nu}t^{\lambda\mu\nu} = \frac{1}{2}\left(T_{\lambda\mu\nu}T^{\lambda\mu\nu} + T_{\lambda\mu\nu}T^{\mu\lambda\nu}\right) - \frac{1}{2}T^{\lambda}{}_{\lambda\mu}T_\rho{}^{\rho\mu}\,. \tag{14}$$

The scalar invariants which are linear in the torsion tensor can be obtained as follow,

$$I_2 = v^{\mu}\phi_{;\mu} \,, \tag{15}$$

The complete set of quadratic contractions of the torsion tensor can be written as follow,

$$\begin{aligned} J_1 &= a^{\mu}a^{\nu}\phi_{;\mu}\phi_{;\nu} \,, \\ J_3 &= v_{\sigma}t^{\sigma\mu\nu}\phi_{;\mu}\phi_{;\nu} \,, \\ J_5 &= t^{\sigma\mu\nu}t_{\sigma}^{\alpha}{}_{\nu}\phi_{;\mu}\phi_{;\alpha} \,, \\ J_6 &= t^{\sigma\mu\nu}t_{\sigma}^{\alpha\beta}\phi_{;\mu}\phi_{;\nu}\phi_{;\alpha}\phi_{;\beta} \,, \\ J_8 &= t^{\sigma\mu\nu}t_{\sigma\mu}^{\alpha}\phi_{;\nu}\phi_{;\alpha} \,, \\ J_{10} &= \epsilon^{\mu}{}_{\nu\sigma\rho}a^{\nu}t^{\alpha\rho\sigma}\phi_{;\mu}\phi_{;\alpha} \,, \end{aligned} \tag{16}$$

Naturally, the regular Horndeski terms from curvature-based gravity also appear in this framework.

$$
\begin{aligned}
\mathcal{L}_2 &:= G_2(\phi, X)\,, \\
\mathcal{L}_3 &:= G_3(\phi, X)\mathring{\Box}\phi\,, \\
\mathcal{L}_4 &:= G_4(\phi, X)\,(-T+B) + G_{4,X}(\phi, X)\left((\mathring{\Box}\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right)\,, \\
\mathcal{L}_5 &:= G_5(\phi, X)\mathring{G}_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5,X}(\phi, X)\left((\mathring{\Box}\phi)^3 + 2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\alpha}\phi_{;\alpha}{}^{\mu} - 3\phi_{;\mu\nu}\phi^{;\mu\nu}\,\mathring{\Box}\phi\right)\,,
\end{aligned}
\tag{17}
$$

Where the kinetic term is defined as $X := -\frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi$. BDLS (Bahamonde Dialektopoulos Levi Said) theory simply adds the further Lagrangian component,

$$
\mathcal{L}_{\mathrm{Tele}} := G_{\mathrm{Tele}}\left(\phi, X, T, T_{\mathrm{ax}}, T_{\mathrm{vec}}, I_2, J_1, J_3, J_5, J_6, J_8, J_{10}\right)\,. \tag{18}
$$

This results in the BDLS action given by [7]

$$\mathcal{S}_{\text{BDLS}} = \frac{1}{2\kappa^2}\int d^4x\, e\mathcal{L}_{\rm Tele} + \frac{1}{2\kappa^2}\sum_{i=2}^{5}\int d^4x\, e\mathcal{L}_i + \int d^4x\, e\mathcal{L}_{\rm m}\,, \tag{19}$$

In this work, we consider the class of models in which,

$$G_2 = X - V(\phi)\,, \quad G_3 = 0 = G_5\,, \quad G_4 = 1/2\kappa^2\,, \tag{20}$$

We have studied the models as follow,

$$G_{\rm Tele_1} = X^{\alpha} T\,, \quad G_{\rm Tele_2} = X^{\alpha} I_2\,, \tag{21}$$

The torsion scalar and torsion contraction scalar invariant can be obtained as follows,

$$T = 6H^2\,, \quad I_2 = 3H\dot{\phi}\,. \tag{22}$$

[7] S. Bahamonde, K. F. Dialektopoulos, J. L. Said, *Phys. Rev D*, **100**, 064018 (2019).

Thus, we can write the effective Friedmann equations as

$$\frac{3}{\kappa^2}H^2 = \rho_{\rm m} + \rho_{\rm r} + X + V + 6H\dot{\phi}G_{{\rm Tele},I_2} + 12H^2 G_{{\rm Tele},T} + 2XG_{{\rm Tele},X} - G_{\rm Tele}\,, \quad (23)$$

$$-\frac{2}{\kappa^2}\dot{H} = \rho_{\rm m} + \frac{4}{3}\rho_{\rm r} + 2X + 3H\dot{\phi}G_{{\rm Tele},I_2} + 2XG_{{\rm Tele},X} - \frac{d}{dt}\left(4HG_{{\rm Tele},T} + \dot{\phi}G_{{\rm Tele},I_2}\right)\,, \quad (24)$$

The scalar field equation is given by,

$$\frac{1}{a^3}\frac{d}{dt}\left[a^3\dot{\phi}\left(1 + G_{{\rm Tele},X}\right)\right] = -V'(\phi) - 9H^2 G_{{\rm Tele},I_2} + G_{{\rm Tele},\phi} - 3\frac{d}{dt}\left(HG_{{\rm Tele},I_2}\right)\,. \quad (25)$$

The action for Model-I [8] is described as below,

$$S=\int d^4xe[P(\phi)X-V(\phi)-\frac{T}{2\kappa^2}+X^\alpha T]+S_m+Sr\,, \tag{26}$$

Where $T=6H^2$ is the torsion scalar, ϕ is scalar field and $X=-\partial_\mu\phi\partial^\nu\frac{\phi}{2}$ is kinetic term.

$$\begin{aligned}
-p_r &= -V(\phi)+\frac{T}{2\kappa^2}+P(\phi)X-X^\alpha T+\frac{2\dot{H}}{\kappa^2}-4X^\alpha\dot{H}-8\alpha X^\alpha H\frac{\ddot{\phi}}{\dot{\phi}}\,, && (27)\\
\rho_m+\rho_r &= \frac{T}{2\kappa^2}-V(\phi)-p(\phi)X-X^\alpha T-2\alpha X^\alpha T\,, && (28)\\
0 &= V^{'}(\phi)+3HP(\phi)\dot{\phi}+P^{'}(\phi)X+\frac{2X^\alpha\alpha T}{\dot{\phi}}\left(3H+\frac{2\dot{H}}{H}\right)\\
&\quad+\ddot{\phi}[P(\phi)-\alpha X^{\alpha-1}T+\alpha^2X^{\alpha-2}T]\,. && (29)
\end{aligned}$$

[8] S. Bahamonde, K. F. Dialektopoulos, J. L. Said, *Phys. Rev D*, **100**, 064018 (2019).

The equations of pressure and energy density for effective DE can be defined a,

$$\begin{aligned} \rho_{de} &= X^{\alpha}T + 2\alpha X^{\alpha}T + V(\phi) + P(\phi)X \,, \\ p_{de} &= -V(\phi) + P(\phi)X - X^{\alpha}T - 4X^{\alpha}\dot{H} - \frac{8\alpha X^{\alpha}H\ddot{\phi}}{\dot{\phi}} \,. \end{aligned} \tag{30}$$

In this the set of dimensionless variables are [9],

$$\begin{aligned} x &= \frac{\kappa\dot{\phi}}{\sqrt{6}H} \,, \quad y = \frac{\kappa\sqrt{V}}{\sqrt{3}H} \,, \quad u = 2X^{\alpha}\kappa^{2} \,, \\ \rho &= \frac{\kappa\sqrt{\rho_{r}}}{\sqrt{3}H} \,, \quad \lambda = \frac{-V'(\phi)}{\kappa V(\phi)} \,, \quad \Gamma = \frac{V(\phi)V''(\phi)}{V'(\phi)^{2}} \,, \end{aligned} \tag{31}$$

These dimensionless variables satisfy the constraint equation as follow,

$$x^{2} + y^{2} + \rho^{2} + (1+2\alpha)u + \Omega_{m} = 1 \,. \tag{32}$$

[9] M. Gonzalez-Espinoza, G. Otalora, *Eur. Phys. J. C.*, **81**, 480 (2021).

The dynamical system in this case is as follows,

$$\frac{dx}{dN} = \frac{x\left(-x^2\left(\rho^2+3(\alpha(2\alpha+5)+1)u-3y^2-3\right)\right)}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)} - \frac{\alpha ux\left(2\alpha\left(\rho^2+3\right)+\rho^2+(6\alpha+3)u-3(2\alpha+1)y^2-3\right)}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)} + \frac{\sqrt{6}\lambda xy^2(2\alpha u+u-1)-3x^4}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)},$$

$$\frac{dy}{dN} = \frac{-y\left(x^2\left(\rho^2+\left(6\alpha^2+9\alpha-3\right)u-3y^2+3\right)-2\sqrt{6}\alpha\lambda uxy^2+3x^4\right)}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)} + \frac{\alpha u(-y)\left((6\alpha+3)u-(2\alpha-1)\left(-\rho^2+3y^2-3\right)\right)}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)} - y\sqrt{\frac{3}{2}}\lambda x,$$

$$\frac{du}{dN} = \frac{\alpha u\left(2\alpha u\left(\rho^2+3x^2-3y^2\right)+(u-1)x\left(6x-\sqrt{6}\lambda y^2\right)\right)}{\alpha u(2\alpha(u+1)+u-1)-(u-1)x^2},$$

$$\frac{d\rho}{dN} = \frac{\rho\left(-x^2\left(\rho^2+6\alpha^2u+9\alpha u+u-3y^2-1\right)+2\sqrt{6}\alpha\lambda uxy^2-3x^4\right)}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)} + \frac{\alpha\rho u\left(2\alpha u+u+(2\alpha-1)\left(-\rho^2+3y^2+1\right)\right)}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)},$$

Critical Points:

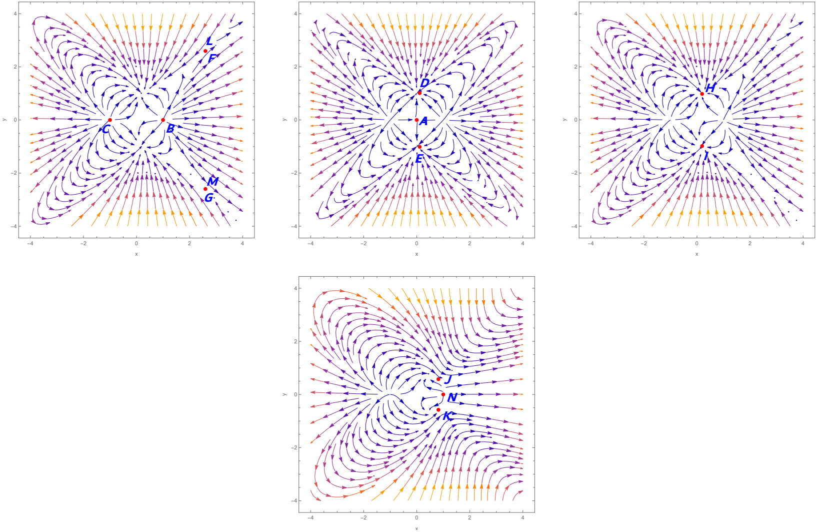
Table 1: Critical Points for Dynamical System Corresponding to Model-I, for General α.

Critical Point	x_c	y_c	u_c	ρ_c	(q)	ω_{tot}
A, $2\alpha^2\tau^2 + \alpha\tau^2 + 2\alpha^2\tau - \alpha\tau \neq 0$	0	0	τ	0	$\frac{1}{2}$	0
B	1	0	0	0	2	1
C	-1	0	0	0	2	1
D, in this case $\lambda = 0$	$\zeta, (\alpha-1)^2\alpha\zeta \neq \left(\alpha^2-1\right)\zeta^3$	$\frac{\sqrt{(\alpha+1)\zeta^2+\alpha}}{\sqrt{\alpha}}$	$-\frac{\zeta^2}{\alpha}$	0	-1	-1
E, in this case $\lambda = 0$	$\zeta, (\alpha-1)^2\alpha\zeta \neq \left(\alpha^2-1\right)\zeta^3$	- $\frac{\sqrt{(\alpha+1)\zeta^2+\alpha}}{\sqrt{\alpha}}$	$-\frac{\zeta^2}{\alpha}$	0	-1	-1
F	$\frac{\sqrt{\frac{3}{2}}}{\lambda}$	$\sqrt{\frac{3}{2}}\sqrt{\frac{1}{\lambda^2}}$	0	0	$\frac{1}{2}$	0
G	$\frac{\sqrt{\frac{3}{2}}}{\lambda}$	$-\sqrt{\frac{3}{2}}\sqrt{\frac{1}{\lambda^2}}$	0	0	$\frac{1}{2}$	0
H	$\frac{\lambda}{\sqrt{6}}$	$\sqrt{1-\frac{\lambda^2}{6}}$	0	0	$\frac{1}{2}\left(\lambda^2-2\right)$	$-1+\frac{\lambda^2}{3}$
I	$\frac{\lambda}{\sqrt{6}}$	$-\sqrt{1-\frac{\lambda^2}{6}}$	0	0	$\frac{1}{2}\left(\lambda^2-2\right)$	$-1+\frac{\lambda^2}{3}$
J, in this case $\lambda = 2$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	0	0	1	$\frac{1}{3}$
K, in this case $\lambda = 2$	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$	0	0	1	$\frac{1}{3}$
L	$\frac{\sqrt{\frac{3}{2}}}{\lambda}$	$\frac{\sqrt{\frac{3}{2}}}{\lambda}$	$\chi, \chi - 1 \neq 0$	0	$\frac{1}{2}$	0
M	$\frac{\sqrt{\frac{3}{2}}}{\lambda}$	$-\frac{\sqrt{\frac{3}{2}}}{\lambda}$	$\chi, \chi - 1 \neq 0$	0	$\frac{1}{2}$	0
$N, \alpha = 0$	$\delta, \delta \neq 0$	0	$1-\delta^2$	0	2	1

Stability of Critical Points:

Table 2: Eigenvalues and Stability of Eigenvalue at Corresponding Critical Points Corresponding to Model-I General α

Name of Critical Point	Corresponding Eigenvalues	Stability
A	$\left\{\frac{3}{2},\frac{3}{2},-\frac{1}{2},0\right\}$	Unstable
B	$\left\{3,1,-6\alpha,\frac{1}{2}\left(6-\sqrt{6}\lambda\right)\right\}$	Unstable
C	$\left\{3,1,-6\alpha,\frac{1}{2}\left(6+\sqrt{6}\lambda\right)\right\}$	Unstable
D	$\{0,-3,-3,-2\}$	Stable
E	$\{0,-3,-3,-2\}$	Stable
F	$\left\{-\frac{1}{2},-3\alpha,\frac{3\left(-\lambda^2-\sqrt{24\lambda^2-7\lambda^4}\right)}{4\lambda^2},\frac{3\left(\sqrt{24\lambda^2-7\lambda^4}-\lambda^2\right)}{4\lambda^2}\right\}$	Stable for $\alpha>0$ $\wedge\left(-2\sqrt{\frac{6}{7}}\leq\lambda<-\sqrt{3}\vee\sqrt{3}<\lambda\leq 2\sqrt{\frac{6}{7}}\right)$
G	$\left\{-\frac{1}{2},-3\alpha,\frac{3\left(-\lambda^2-\sqrt{24\lambda^2-7\lambda^4}\right)}{4\lambda^2},\frac{3\left(\sqrt{24\lambda^2-7\lambda^4}-\lambda^2\right)}{4\lambda^2}\right\}$	Stable for $\alpha>0$ $\wedge\left(-2\sqrt{\frac{6}{7}}\leq\lambda<-\sqrt{3}\vee\sqrt{3}<\lambda\leq 2\sqrt{\frac{6}{7}}\right)$
H	$\left\{-\alpha\lambda^2,\frac{1}{2}\left(\lambda^2-6\right),\frac{1}{2}\left(\lambda^2-4\right),\lambda^2-3\right\}$	Stable for $\alpha>0\wedge\left(-\sqrt{3}<\lambda<0\vee 0<\lambda<\sqrt{3}\right)$
I	$\left\{-\alpha\lambda^2,\frac{1}{2}\left(\lambda^2-6\right),\frac{1}{2}\left(\lambda^2-4\right),\lambda^2-3\right\}$	Stable for $\alpha>0\wedge\left(-\sqrt{3}<\lambda<0\vee 0<\lambda<\sqrt{3}\right)$
J	$\{-1,1,0,-4\alpha\}$	Unstable
K	$\{-1,1,0,-4\alpha\}$	Unstable
L	$\left\{0,-\frac{1}{2},\frac{3}{4}\left(\frac{\sqrt{-\lambda^2(\chi-1)\left(7\lambda^2(\chi-1)+24\right)}}{\lambda^2(\chi-1)}-1\right),-\frac{3\sqrt{-\lambda^2(\chi-1)\left(7\lambda^2(\chi-1)+24\right)}}{4\lambda^2(\chi-1)}-\frac{3}{4}\right\}$	Stable for $\lambda\in\mathbb{R}\wedge\lambda\neq 0\wedge\frac{7\lambda^2-24}{7\lambda^2}\leq\chi<\frac{\lambda^2-3}{\lambda^2}$
M	$\left\{0,-\frac{1}{2},\frac{3}{4}\left(\frac{\sqrt{-\lambda^2(\chi-1)\left(7\lambda^2(\chi-1)+24\right)}}{\lambda^2(\chi-1)}-1\right),-\frac{3\sqrt{-\lambda^2(\chi-1)\left(7\lambda^2(\chi-1)+24\right)}}{4\lambda^2(\chi-1)}-\frac{3}{4}\right\}$	Stable for $\lambda\in\mathbb{R}\wedge\lambda\neq 0\wedge\frac{7\lambda^2-24}{7\lambda^2}\leq\chi<\frac{\lambda^2-3}{\lambda^2}$
N	$\left\{0,1,3,\frac{1}{2}\left(6-\sqrt{6}\delta\lambda\right)\right\}$	Unstable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Outline

In this study, we probe possible cosmological behaviors of scalar-tensor models using a dynamical systems approach.

By taking a flat homogeneous and isotropic background solution, dynamical systems are analyzed to determine the number and nature of the critical points.

The stability of critical points of the system is used to express whether these positions in the cosmic evolution are stable or not.

The phase space diagrams are also presented to analyse phase space trajectory behaviour at these critical points.

S. A. Kadam, B.Mishra, J. L. Said, *Eur. Phys. J. C*, **82**, 680 (2022).

Thank you for patients listening