# Representation of fermions in Pati-Salam model

De-Sheng Li
lideshengjy@126.com
Hunan Institute of Engneering

[arXiv: 1412.2578, 2011.14362]

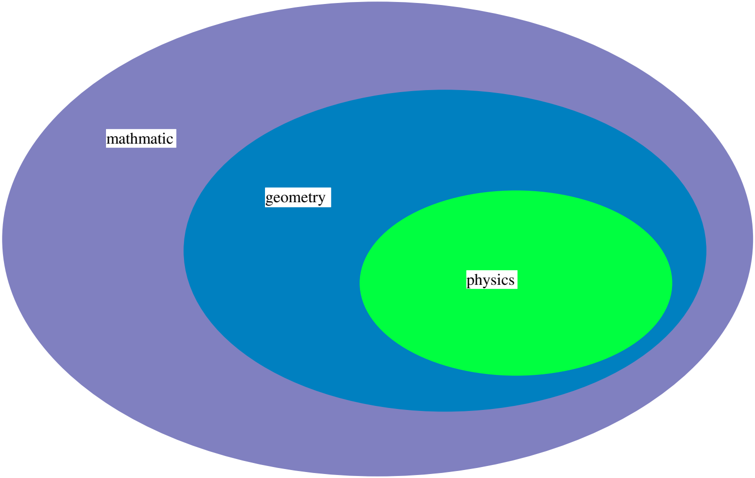
November 9, 2023

We thank Hong-Fei Zhang, Zhao li, Cai-Dian Lv, Ming Zhong, Yong-Chang Huang, Jun-Bao Wu and Chao-Guang Huang for valuable discussions and guidances.

# Contents

# Contents

# Pure geometry philosophy for fundamental physics




Geometry is a logic system.

# Geometry background of General Relativity (GR) and Standard Model of particle physics (SM)

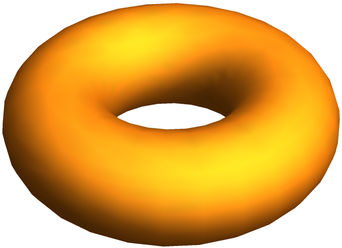


Figure 1: Geometry background of GR is curved, smooth manifold, pseudo-Riemannian manifold [M. Fecko, Differential geometry and Lie groups for physicists, (2006)] (more precisely, Lorentzian manifold). The gravitational field is determined by the metric of manifold.

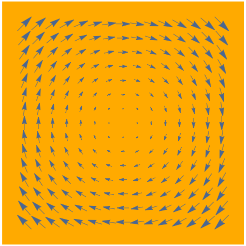


Figure 2: Geometry background of SM is very similar with the flat space-time with G-bundle [D. Husemoller, Fibre bundles, (1966)]. The electromagnetic field, weak bosons fields, gluon bosons fields are originated from the principal G-bundle connections. Leptons, quarks are originated from the sections of associated bundle.

# The curved manifold with G-bundle is a good option for geometry background of Yang-Mills theory in curved space-time

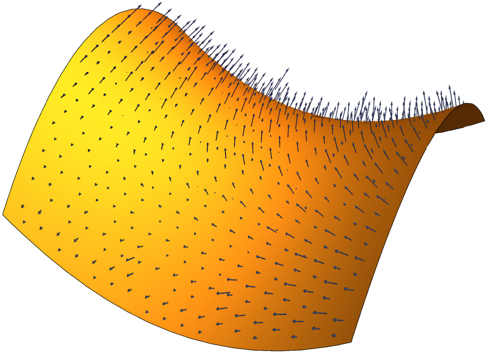


Figure 3: It can be found that "square root metric" Lorentz manifold not only with metric, but also equipped with $U(4') \times U(4)$-bundle at the same time. This geometry might give intrinsic geometrical interpretation to all the fields being observed.

# Square root something usual leads to unusual

- $$\sqrt{-1} = i$$
- $$\sqrt{Klein - Gordon\ equation} \quad \Rightarrow \quad Dirac\ equation$$
- $$\sqrt{Metric\ g} \quad \Rightarrow \quad ?$$

# Contents

# Metric

- Riemannian manifold is described by metric

$$g(x) = -g_{\mu\nu}(x)dx^{\mu} \otimes dx^{\nu}, g_{\mu\nu}(x) = g_{\nu\mu}(x), \det(g_{\mu\nu}(x)) \neq 0. \tag{2.1}$$

- The inverse metric is defined

$$g^{-1}(x) = -g^{\mu\nu}(x)\partial_{\mu}\partial_{\nu}, \langle \partial_{\nu}, dx^{\mu} \rangle = \delta^{\mu}_{\nu}. \tag{2.2}$$

- And it can be described by orthonormal frame formalism as

$$g^{-1}(x) = -\eta^{ab}\theta_a(x)\theta_b(x). \tag{2.3}$$

- Here orthonormal frame $\theta_a(x) = \theta^{\mu}_a(x)\partial_{\mu}$ describes gravitational field.

# Square root metric

- Similar with square root Klein-Gordon equation gives us Dirac equation, is there any explicit mathematic formulas for square root inverse metric?

$$\sqrt{g^{-1}(x)} \quad \Rightarrow \quad ? \tag{2.4}$$

- Similar ideas have “Kaluze-Klein theory without extra dimensions: curved Clifford space” [M. Pavsic, Phys. Lett. B **614**, 85–95 (2005)], etc.

# Square root metric

- Similar with square root Klein-Gordon equation gives us Dirac equation and Dirac fermions, after ten years researching, we write an explicit mathematic formula for square root inverse metric

$$\sqrt{g^{-1}(x)} \quad \Rightarrow \quad \tilde{l}(x), \quad l(x), \tag{2.5}$$

with

$$g^{-1}(x) = \frac{1}{4} tr[\tilde{l}(x) l(x)], \tag{2.6}$$

where

$$l(x) \quad = \quad i\gamma^0\gamma^a\theta_a(x), \tag{2.7a}$$
$$\tilde{l}(x) \quad = \quad i\gamma^a\gamma^0\theta_a(x). \tag{2.7b}$$

# Square root metric

- The definition of Dirac matrices is

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} I_{4\times 4}, \tag{2.8}$$

  where $\eta^{ab} = diag(1, -1, -1, -1)$.

- The Hermiticity conditions for Dirac matrices can be chosen

$$\gamma^a \gamma^{b\dagger} + \gamma^{b\dagger} \gamma^a = 2I^{ab} I_{4\times 4}, \tag{2.9}$$

  where $I^{ab} = diag(1, 1, 1, 1)$.

# Square root metric

- The definition of Dirac matrices has $U(4)$ rotation freedom

$$\gamma^{a\prime} = \Psi^\dagger \gamma^a \Psi, \quad \Psi \in U(4), \tag{2.10}$$

such that $\gamma^{a\prime}$ still Dirac matrices.

- Then, the entity $l$ can be rewriten as

$$\begin{aligned} l(x) &= i\gamma^0\gamma^a(x)\theta_a(x) = i\bar{\Psi}\gamma^a\Psi\theta_a(x) \\ &= i\bar{\Psi}_j\gamma^a\Psi_i\theta_a(x)e_j^\dagger \otimes e_i\theta_a(x), \end{aligned} \tag{2.11}$$

where $tr(e_j^\dagger \otimes e_i) = e_i e_j^\dagger = \delta_{ij}$. One simple choice of $e_i$ is

$$e_1 = (e^{i\theta_1}, 0, 0, 0), \qquad e_2 = (0, e^{i\theta_2}, 0, 0), \tag{2.12a}$$

$$e_3 = (0, 0, e^{i\theta_3}, 0), \qquad e_4 = (0, 0, 0, e^{i\theta_4}). \tag{2.12b}$$

- Direct calculation shows that

$$l^\dagger(x) = -l(x). \tag{2.13}$$

# Connections and gauge field

- Coefficients of affine connections on coordinates, coefficients of spin connections on orthonormal frame [S. Chern, W. Chen, and K. Lam, Lectures on differential geometry, (1999)] and gauge fields on $U(4')\times U(4)$-bundle are defined as follows

$$\begin{aligned}
\nabla_\mu \partial_\nu &= \Gamma^\rho{}_{\nu\mu}(x)\partial_\rho, && (2.14a)\\
\nabla_\mu \theta_a(x) &= \Gamma^b{}_{a\mu}(x)\theta_b(x), && (2.14b)\\
\nabla_\mu(\gamma^0\gamma^a) &= i[V_\mu(x)\gamma^0\gamma^a - \gamma^0\gamma^a V_\mu(x)], && (2.14c)\\
\nabla_\mu e_i^\dagger &= iW_{\mu ij}(x)e_j^\dagger, && (2.14d)
\end{aligned}$$

- The relation between coefficients of affine connections on coordinates and coefficients of spin connections on orthonormal frame is found

$$\Gamma^b{}_{a\mu}(x)\theta_b^\rho(x) = \partial_\mu\theta_a^\rho(x) + \theta_a^\nu(x)\Gamma^\rho{}_{\nu\mu}(x). \qquad (2.15)$$

# Connections and gauge field

- The Hermiticity conditions for gauge fields are

$$V_\mu^\dagger(x) = V_\mu(x), \qquad W_{\mu ij}^*(x) = W_{\mu ji}(x). \tag{2.16}$$

- The gauge field $V_\mu(x)$ and $W_{\mu ij}(x)$ can be decomposed by the generators of the $U(4)$ group

$$V_\mu(x) = V_\mu^\alpha(x)\mathcal{T}^\alpha, \qquad W_{\mu ij}(x) = W_\mu^\alpha(x)\mathcal{T}_{ij}^\alpha, \tag{2.17}$$

where $\alpha = 0, 1, 2, \cdots, 15$ and

$$V_\mu^{\alpha *} = V_\mu^\alpha, \qquad W_\mu^{\alpha *} = W_\mu^\alpha. \tag{2.18}$$

# Equation and Lagrangian

- A equation satisfy $U(4') \times U(4)$ gauge invariant, locally Lorentz invariant and generally covariant principles is constructed

$$\mathbf{tr}\nabla l(x) \quad = \quad 0. \tag{2.19}$$

- This equation describes a manifold parallel transporting itself.
- Eliminate index $x$, the explicit formula of equation (2.19) is

$$\Big[(i\partial_\mu\bar{\Psi}_i - \bar{\Psi}_i\tilde{V}_\mu + W_{\mu ij}\bar{\Psi}_j)\gamma^a\Psi_i + \bar{\Psi}_i\gamma^a(i\partial_\mu\Psi_i + V_\mu\Psi_i - \Psi_j W_{\mu ji}) \\ + i\bar{\Psi}_i\gamma^b\Psi_i\Gamma^a{}_{b\mu}\Big]\,\theta^\mu_a = 0.(2.20)$$

- We define a Lagrangian

$$\mathcal{L} \quad = \quad \bar{\Psi}_i\gamma^a(i\partial_\mu\Psi_i + V_\mu\Psi_i - \Psi_j W_{\mu ji})\theta^\mu_a + \bar{\Psi}_i\phi\Psi_i, \tag{2.21a}$$

$$\phi \quad = \quad \frac{i}{2}\gamma^b\Gamma^a{}_{b\mu}\theta^\mu_a. \tag{2.21b}$$

One find that Lagrangian (2.21a) have relation with (2.19)

$$\mathbf{tr}\nabla l(x) = \mathcal{L} - \mathcal{L}^\dagger. \tag{2.22}$$

# Lagrangian and equation

- If equation (2.19) being satisfied, Lagrangian (2.21a) is Hermitian

$$\mathcal{L} = \mathcal{L}^{\dagger}. \tag{2.23}$$

- Curvature tensor and gauge field strength tensors are defined as follows

$$R^{a}{}_{b\mu\nu} = \partial_{\mu}\Gamma^{a}{}_{b\nu} - \partial_{\nu}\Gamma^{a}{}_{b\mu} + \Gamma^{c}{}_{b\nu}\Gamma^{a}{}_{c\mu} - \Gamma^{c}{}_{b\mu}\Gamma^{a}{}_{c\nu}, \tag{2.24a}$$

$$H_{\mu\nu} = \partial_{\mu}V_{\nu} - \partial_{\nu}V_{\mu} - iV_{\mu}V_{\nu} + iV_{\nu}V_{\mu}, \tag{2.24b}$$

$$F_{\mu\nu ij} = \partial_{\mu}W_{\nu ij} - \partial_{\nu}W_{\mu ij} - iW_{\mu ik}W_{\nu kj} + iW_{\nu ik}W_{\mu kj}, \tag{2.24c}$$

where $R_{ab\mu\nu} = -R_{ba\mu\nu}$ if $\nabla g = 0$ and

$$H^{\dagger}_{\mu\nu} = H_{\mu\nu}, \quad F^{*}_{\mu\nu ij} = F_{\mu\nu ji}. \tag{2.25}$$

# Curvature, gauge field strength tensor and identity

- There is Yang-Mills [C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191–195 (1954)] Lagrangian for gauge bosons in this model

$$\mathcal{L}_Y = \frac{-1}{2}\mathbf{tr}\left(H^{\mu\nu}H_{\mu\nu}\right) - \frac{\zeta}{2}F^{\mu\nu}_{ij}F_{\mu\nu ji}, \tag{2.26}$$

where $\zeta \in \mathbb{R}$ is constant.

# Lagrangian of Gravity

- For gravity, Einstein-Hilbert action be showed as follows

$$S = \int R\omega, \tag{2.27}$$

where $R$ is Ricci scalar curvature, $\omega = \sqrt{-g}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ is volume form. The variation of action give us Einstein tensor.

- The Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}. \tag{2.28}$$

- Einstein say: “The reason for the formalism of left hand is to let its divergence identically zero in the meaning of covariant derivative. The right hand of equation are the sum up of all the things still problems in the meaning of field theory.”

# Lagrangian of Gravity, Einstein-Cartan gravity

- After defining $\nabla^2 = \nabla_{[\mu}\nabla_{\nu]} dx^\mu \wedge dx^\nu$, the equation of motion for this gravity theory is constructed

$$\mathbf{tr}\nabla^2[\tilde{l}(x)l(x)] = 0. \tag{2.29}$$

- This equation (2.29) is obviously $U(4') \times U(4)$ gauge invariant, locally Lorentz invariant and generally covariant. The explicit formula of equation (2.29) is

$$R\Psi_i^\dagger\Psi_i = i\left(F_{abij}\Psi_j^\dagger(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger})\Psi_i - \Psi_i^\dagger H_{ab}(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger})\Psi_i\right).$$

- We define a Hermitian Lagrangian

$$\begin{aligned}\mathcal{L}_g = R\Psi_i^\dagger\Psi_i - i\Big(&F_{abij}\Psi_j^\dagger(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger})\Psi_i \\ &-\Psi_i^\dagger H_{ab}(\gamma^a\gamma^b - \gamma^{b\dagger}\gamma^{a\dagger})\Psi_i\Big),\end{aligned} \tag{2.30}$$

where $R\Psi_i^\dagger\Psi_i$ is Einstein-Hilbert action.

# Total Lagrangian

- The total lagrangian for $U(4') \times U(4)$ Pati-Salam model in curved space-time and Einstein-Cartan gravity [W. Drechsler, Z. Phys. C **41**, 197–205 (1988); M. Tecdhiolli, Universe **5**, 206 (2019)] is

$$\mathcal{L}_T = \mathcal{L} + \tilde{g}\mathcal{L}_{YM} + g\mathcal{L}_g, \tag{2.31}$$

where $\tilde{g}$ and $g$ are parameters

$$\tilde{g}, g \in \mathbb{R}. \tag{2.32}$$

# Sheaf quantization, path integral quantization and canonical quantization

- Sheaf（层）[R. Harshorne, Algebraic geometry, (2013);M. Kashiwara and P. Masaki, Sheaves on manifolds (1990)] is a geometry structure natural than section.

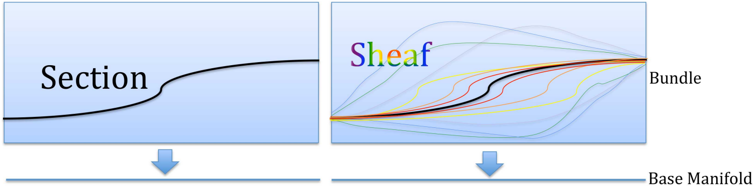



Figure 4: Fibre bundle is a map from bundle to base manifold. The sheaf is the contravariant functor of the map.

- The sheaf quantization [K, Nakayama, J.Math.Phys.**55**,102103 (2014); A. Doring and C. Isham, J. Math. Phys. **49**, 053515 (2018); C. Flori, A first course in topos quantum theory, (2013)], path integral quantization and canonical quantization of this theory are consistent with each other.

# Sheaf quantization, path integral quantization and canonical quantization

- The path integral formulation of transition amplitude can be derived from sheaf quantization

$$\alpha_\kappa(t, x^q) = \int_{t' \in (t_0, t)} D\pi_\kappa(t', x^q) e^{i \int \omega \hat{\mathcal{L}}[\phi_\kappa(t', x^q), \partial_\mu \phi_\kappa(t', x^q)]} \alpha_\kappa(t_0, x^q). \quad (2.33)$$

- The sheaf quantiztion suggest the canonial quantiztion formulation based on Lgrangian (not Lagrangian density).

# Contents

# Flat space-time limits of Pati-Salam model in curved space-time

- The flat space-time limits version of Lagrangian $\mathcal{L}_T$ is

$$\begin{aligned}\mathcal{L}_{T-L} &= \mathbf{tr}\Big[i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi + f\bar{\Psi}\gamma^{\mu}V_{\mu}\Psi - g\bar{\Psi}\gamma^{\mu}\Psi W_{\mu} + \bar{\Psi}\phi\Psi + V(\phi) \\ &\quad -\frac{1}{2}H^{\mu\nu}H_{\mu\nu} - \frac{\eta}{2}F^{\mu\nu}F_{\mu\nu} - igF_{\mu\nu}\Psi^{\dagger}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu\dagger}\gamma^{\mu\dagger})\Psi \\ &\quad +if\Psi^{\dagger}H_{\mu\nu}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu\dagger}\gamma^{\mu\dagger})\Psi\Big], \end{aligned} \tag{3.1}$$

  where each terms are Dirac, minimal coupling, Yukawa coupling , Higgs potential, Yang-Mills and magnetic moment terms.

- $\Psi$, $V_\mu$ and $W_\mu$ are $4 \times 4$ matrices. Without loss of generality, we choose minimal coupling model to analyse the interaction vetexes

$$\mathcal{L}_{Min} = \mathbf{tr}\left[i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi + f\bar{\Psi}\gamma^{\mu}V_{\mu}\Psi - g\bar{\Psi}\gamma^{\mu}\Psi W_{\mu}\right]. \tag{3.2}$$

# Flat space-time limits of Pati-Salam model in curved space-time

- We observe that the second term in Lagrangian (3.2) is difficult to decompose due to chiral symmetry, but the third term can be decomposed

$$\mathcal{L}_{Min} = \mathbf{tr}\Big[ i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi + \sum_{\alpha=1}^{15}\big( f\bar{\Psi}\gamma^{\mu}V_{\mu}^{\alpha}T^{\alpha}\Psi - g\bar{\Psi}_{L}\gamma^{\mu}\Psi_{L}W_{\mu}^{\alpha}T^{\alpha} - g\bar{\Psi}_{R}\gamma^{\mu}\Psi_{R}W_{\mu}^{\alpha}T^{\alpha}\big)\Big], \tag{3.3}$$

accordingly, the second term in Lagrangian (3.2) describes the $SU(4')$ color gauge interaction, and the third term in Lagrangian (3.2) describes the $SU(4)_L \times SU(4)_R$ chiral flavor gauge interaction.

# Flat space-time limits of Pati-Salam model in curved space-time

- Lagrangian (3.1) is invariant under local gauge transformations of color space and flavor space rotation $\tilde{U}$ and $U$, respectively,

$$\Psi' = \tilde{U}\Psi U, \tag{3.4}$$

where

$$\tilde{U} \in U(4'), \quad U \in U(4), \tag{3.5}$$

such that

$$\gamma^{\mu\prime} = \tilde{U}\gamma^\mu\tilde{U}^\dagger \Rightarrow \gamma^{0\prime}\gamma^{\mu\prime} = \tilde{U}\gamma^0\gamma^\mu\tilde{U}^\dagger, \tag{3.6a}$$

$$V'_\mu = \tilde{U}V_\mu\tilde{U}^\dagger - (\partial_\mu\tilde{U})\tilde{U}^\dagger, \tag{3.6b}$$

$$W'_\mu = U^\dagger(\partial_\mu U) - U^\dagger W_\mu U. \tag{3.6c}$$

# Representation of fermions

- Then the column of the fermion matrix $\Psi$ corresponding to color and the row corresponding to flavor, and transfer as $U(4') \times U(4)$ fundamental representation.
- So, fermions are filled into $SU(4)$ fundamental representation naturally as Table 1. In Pati-Salam model,"lepton number as the fourth color" [J. C. Pati and A. Salam, Phys.Rev. **D10**, 275 (1974)].

Table 1: Fermions are filled into $SU(4)$ fundamental representation $\mathbf{4} \otimes \mathbf{6}$.

| $SU(4)$ | | | **6** | | | | | |
|---|---|---|---|---|---|---|---|---|
| **4** | Quarks | R G B | u | c | t | d | s | b |
| | Leptons | | $e$ | $\mu$ | $\tau$ | $\nu_e$ | $\nu_\mu$ | $\nu_\tau$ |

- Anti-fermions are filled into $\bar{\mathbf{4}} \otimes \mathbf{6}$ similarly.

# Representation of fermions

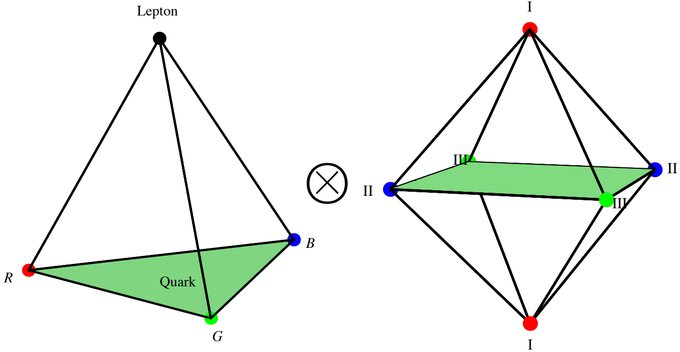



Figure 5: Weight diagram of $SU(4)$ fundamental representation $\mathbf{4} \otimes \mathbf{6}$. Representation $\mathbf{4} = \mathbf{3} + \mathbf{1}$, $\mathbf{3}$ is 3 kinds of color, red, green and blue, $\mathbf{1}$ is the lepton number. Representation $\mathbf{6}$ gives us 6 flavor of quarks and leptons. 6 flavors devided to 3 generations, I ,II and III, each generation has 2 kinds of quarks or leptons.

# Representation of fermions

- An explicit fermions representation in this model might be

$$\Psi = \begin{pmatrix} \sqrt{2}u_R & \sqrt{2}c_R & \sqrt{2}t_R & d'_R \\ \sqrt{2}u_G & \sqrt{2}c_G & \sqrt{2}t_G & d'_G \\ \sqrt{2}u_B & \sqrt{2}c_B & \sqrt{2}t_B & d'_B \\ e & \mu & \tau & \nu' \end{pmatrix}, \tag{3.7}$$

  where $u, c, t$ and $d'$ are quarks fields, $e, \mu, \tau$ and $\nu'$ are electron, mu, tau and neutrinos fields.

- The corresponding fermions electric charges of (3.7) are

$$Q_\Psi = \begin{pmatrix} 2/3 & 2/3 & 2/3 & -1/3 \\ 2/3 & 2/3 & 2/3 & -1/3 \\ 2/3 & 2/3 & 2/3 & -1/3 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

- The quarks states like $|d\rangle, |s\rangle, |b\rangle$ and neutrinos states $|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle$ are eigen states of the Lagrangian.

# Gauge bosons

- $V_\mu^\alpha$ and $W_\mu^\alpha$ $(\alpha = 0, 1, \cdots, 15)$ are gauge bosons fields.
- The interactions related with $W_\mu^\alpha$ always preserves the possibility of chiral symmetry breaking such that the gauge group can decomposed to $U(4') \times U(4)_L \times U(4)_R$, where $U(4')$ is color group and $U(4)_L \times U(4)_R$ is chiral flavor group.
- The $V_\mu^0$ is dark photon [arXiv: 1311.0029; B. Holdom, Phys. Let. B. **166** (2):196–198 (1986)] and $W_\mu^0$ is Fiona (芳) particle.
- The left over part gauge group is a Pati-Salam gauge group $SU(4') \times SU(4)_L \times SU(4)_R$ and the $SU(4')$ can be decomopsed as follow

$$SU(4') = SU(3') \oplus U(1') + U_{X^+} + U_{X^-}\,. \tag{3.8}$$

# Gauge bosons,Color $SU(4')$ processes

- The $SU(3')$ is the gauge group of quantum chramodynamics (QCD) and the corresponding gauge bosons $V_\mu^\alpha(\alpha = 1, 2 \cdots, 8)$ are gluons.
- The $U(1')$ is electro–magnetic interaction gauge group and corresponding gauge boson $V_\mu^{15}$ is photon $\gamma$.

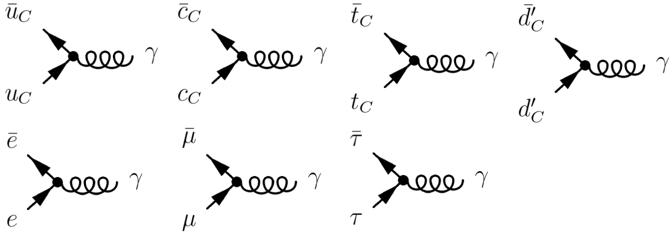



Figure 6: The fermion–anti-fermion–boson interaction vertexes of photon.

# Gauge bosons,Color $SU(4')$ processes

- The $X^{\pm C}$ particles transport semi-leptonic processes and

$$X^{\pm C} = V_\mu^{8+C} \pm i V_\mu^{9+C}. \tag{3.9}$$

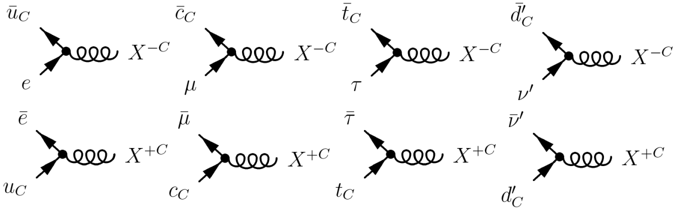



Figure 7: The fermion–anti-fermion–boson interaction vertexes of $X$ bosons. All three external legs are momentum in.

- The electric charge of $X^{+C}$ and $X^{-C}$ are $\frac{1}{3}$ and $-\frac{1}{3}$.

# Gauge bosons,Color $SU(4')$ processes

- The representation (filling scheme) of matrix $V_\mu$ might be

$$V_\mu = \begin{pmatrix} G_\mu^{RR} + V_\mu^{15} & G_\mu^{RG} & G_\mu^{RB} & X_\mu^{-R} \\ G_\mu^{GR} & G_\mu^{GG} + V_\mu^{15} & G_\mu^{GB} & X_\mu^{-G} \\ G_\mu^{BR} & G_\mu^{BG} & G_\mu^{BB} + V_\mu^{15} & X_\mu^{-B} \\ X_\mu^{+R} & X_\mu^{+G} & X_\mu^{+B} & -3V_\mu^{15} \end{pmatrix}, \quad (3.10)$$

where $G_\mu$ are gluons and $V_\mu^{15}$ are photon. The corresponding electric charge matrix of $V_\mu$ is

$$Q_V = \begin{pmatrix} 0 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & -1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix}. \quad (3.11)$$

# Gauge bosons,Color $SU(4')$ processes,flavor mixing

- The left-handed flavor eigenstates $|d'_{LC}\rangle, |s'_{LC}\rangle, |b'_{LC}\rangle$ of $d, s, b$ quark states are

$$-\frac{\sqrt{2}}{2}g\mathbf{Tr}[\bar{u}_{LC}\gamma^{\mu}d'_{LC}W^{+}_{\mu}]|d'_{LC}\rangle = \alpha_1|d'_{LC}\rangle, \tag{3.12a}$$

$$-\frac{\sqrt{2}}{2}g\mathbf{Tr}[\bar{c}_{LC}\gamma^{\mu}d'_{LC}W^{+}_{\mu}]|s'_{LC}\rangle = \alpha_2|s'_{LC}\rangle, \tag{3.12b}$$

$$-\frac{\sqrt{2}}{2}g\mathbf{Tr}[\bar{t}_{LC}\gamma^{\mu}d'_{LC}W^{+}_{\mu}]|b'_{LC}\rangle = \alpha_3|b'_{LC}\rangle. \tag{3.12c}$$

- The left-handed mass eigenstates of the $d, s$ and $b$ quarks are

$$i\mathbf{Tr}\left[\bar{d}'_{LC}\gamma^{\mu}\partial_{\mu}d'_{LC}\right]|d_{LC}\rangle = m_{dL}|d_{LC}\rangle, \tag{3.13a}$$

$$i\mathbf{Tr}\left[\bar{d}'_{LC}\gamma^{\mu}\partial_{\mu}d'_{LC}\right]|s_{LC}\rangle = m_{sL}|s_{LC}\rangle, \tag{3.13b}$$

$$i\mathbf{Tr}\left[\bar{d}'_{LC}\gamma^{\mu}\partial_{\mu}d'_{LC}\right]|b_{LC}\rangle = m_{bL}|b_{LC}\rangle. \tag{3.13c}$$

- The Cabibbo-Kobayashi-Maskawa (CKM) matrix is

$$\begin{pmatrix} |d'_{LC}\rangle \\ |s'_{LC}\rangle \\ |b'_{LC}\rangle \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} |d_{LC}\rangle \\ |s_{LC}\rangle \\ |b_{LC}\rangle \end{pmatrix}. \tag{3.14}$$

# Gauge bosons, Chiral flavor $SU(4)_L \times SU(4)_R$ processes

- The chiral gauge group $SU(4)_{L,R}$ can be decomposed as

$$SU(4)_{L,R} = SU(3)_Y \oplus U(1)_Z + U_{W^+} + U_{W^-}, \tag{3.15}$$

and related gauge bosons $W^\alpha_\mu (\alpha = 1, 2, \cdots, 15)$ contain weak bosons $W^\pm$ and $Z$

$$W^\pm_\mu = W^9_\mu \pm iW^{10}_\mu = W^{11}_\mu \pm iW^{12}_\mu = W^{13}_\mu \pm iW^{14}_\mu, \tag{3.16a}$$

$$Z_\mu = W^3_\mu = W^8_\mu = W^{15}_\mu. \tag{3.16b}$$

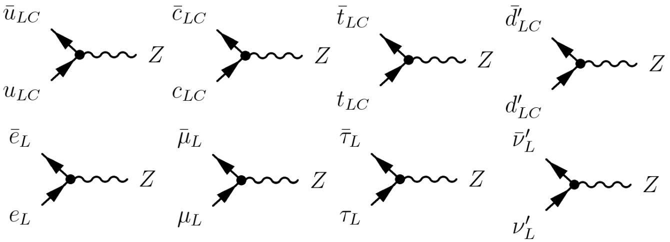



Figure 8: The fermion–anti-fermion–boson interaction vertexes of $Z$ bosons.

# Gauge bosons, Chiral flavor $SU(4)_L \times SU(4)_R$ processes

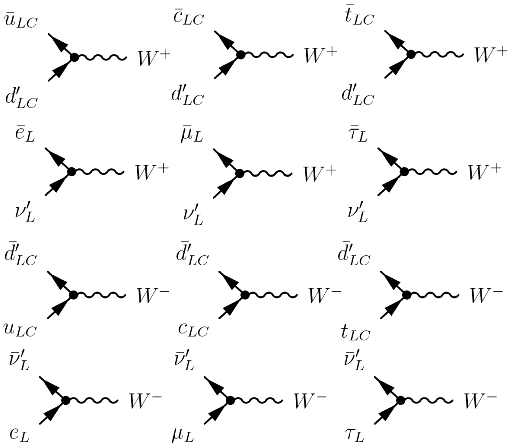



Figure 9: The fermion–anti-fermion–boson interaction vertexes of *W* bosons. All three external legs are momentum in.

# Gauge bosons, Chiral flavor $SU(4)_L \times SU(4)_R$ processes

- The left over gauge bosons are $Y^1, Y^2$ and $Y^1_*, Y^2_*$ with 0 eletric charge.

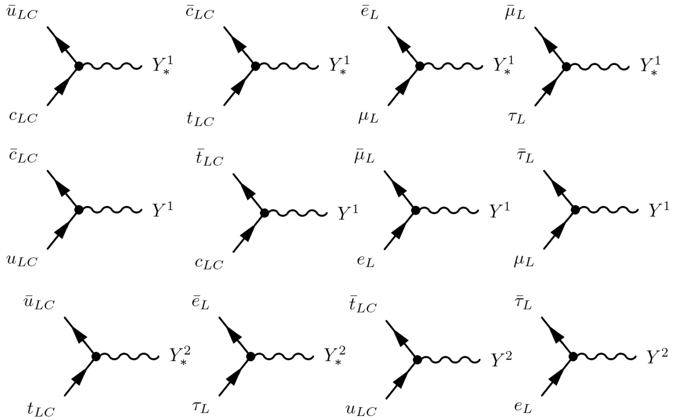



Figure 10: The gauge bosons $Y^1, Y^2, Y^1_*, Y^2_*$ transport fermion–anti-fermion–boson interaction vertexes about beyond-SM flavor changing neutral currents (FCNCs). All three external legs are momentum in.

# Gauge bosons, Chiral flavor $SU(4)_L \times SU(4)_R$ processes

- The $W_\mu$ matrix might be

$$W_\mu = \frac{1}{2}\begin{pmatrix} \zeta_1 Z_\mu & Y^1_\mu & Y^2_\mu & W^-_\mu \\ Y^1_{*\mu} & \zeta_2 Z_\mu & Y^1_\mu & W^-_\mu \\ Y^2_{*\mu} & Y^1_{*\mu} & \zeta_3 Z_\mu & W^-_\mu \\ W^+_\mu & W^+_\mu & W^+_\mu & \zeta_4 Z_\mu \end{pmatrix}. \quad (3.17)$$

The corresponding electric charge matrix of $W_\mu$ is

$$Q_W = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \quad (3.18)$$

# Gauge bosons, Chiral flavor $SU(4)_L \times SU(4)_R$ processes, flavor mixing

- The left-handed flavor eigenstates of neutrinos are

$$-\frac{1}{2}g\mathbf{Tr}[\bar{e}_L\gamma^\mu\nu'_L W^+_\mu]|\nu_{eL}\rangle = \alpha_4|\nu_{eL}\rangle, \tag{3.19a}$$

$$-\frac{1}{2}g\mathbf{Tr}[\bar{\mu}_L\gamma^\mu\nu'_L W^+_\mu]|\nu_{\mu L}\rangle = \alpha_5|\nu_{\mu L}\rangle, \tag{3.19b}$$

$$-\frac{1}{2}g\mathbf{Tr}[\bar{\tau}_L\gamma^\mu\nu'_L W^+_\mu]|\nu_{\tau L}\rangle = \alpha_6|\nu_{\tau L}\rangle. \tag{3.19c}$$

- The left-handed mass eigenstates of neutrinos are

$$i\mathbf{Tr}\left[\bar{\nu}'_L\gamma^\mu\partial_\mu\nu'_L\right]|\nu_{1L}\rangle = m_{1L}|\nu_{1L}\rangle, \tag{3.20a}$$

$$i\mathbf{Tr}\left[\bar{\nu}'_L\gamma^\mu\partial_\mu\nu'_L\right]|\nu_{2L}\rangle = m_{2L}|\nu_{2L}\rangle, \tag{3.20b}$$

$$i\mathbf{Tr}\left[\bar{\nu}'_L\gamma^\mu\partial_\mu\nu'_L\right]|\nu_{3L}\rangle = m_{3L}|\nu_{3L}\rangle. \tag{3.20c}$$

- The Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix is

$$\begin{pmatrix} |\nu_{eL}\rangle \\ |\nu_{\mu L}\rangle \\ |\nu_{\tau L}\rangle \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} |\nu_{1L}\rangle \\ |\nu_{2L}\rangle \\ |\nu_{3L}\rangle \end{pmatrix}. \tag{3.21}$$

# Gauge bosons

- The masses of $X^{\pm}$ and $Y^1$, $Y^2$, $Y^1_*$, $Y^2_*$ must be superheavy from the restrictions of experimental data.

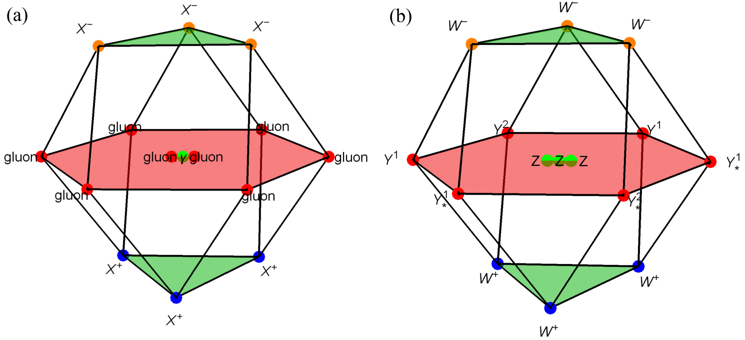



Figure 11: Weight diagram of $SU(4)$ adjoint representation and corresponding gauge bosons. The decomposition of $SU(4)$ adjoint representation is $\mathbf{15} = \mathbf{8} \oplus \mathbf{1} + \mathbf{3} + \mathbf{3}^*$. (a) The wight diagram of $V^{\alpha}_{\mu}(\alpha = 1, 2, \cdots, 15)$ related gauge bosons. (b) The wight diagram of $W^{\alpha}_{\mu}(\alpha = 1, 2, \cdots, 15)$ related gauge bosons.

# Contents

# Summary

- This theory unify fermions, gauge bosons, Higgs and gravitational fields into a pair of "entities", square root metric

$$\sqrt{g^{-1}(x)} \quad \Rightarrow \quad l(x), \quad \tilde{l}(x), \tag{4.1}$$

  and its connections.
- The interactions between fields can be derived from self-parallel transportation principle

$$\mathbf{tr}\nabla l(x) = 0, \qquad \mathbf{tr}\nabla^2[\tilde{l}(x)l(x)] = 0. \tag{4.2}$$

- The sheaf quantization, path integral quantization and canonical quantization are consistent with each other.
- Particles spectrum, representation of fermions, fermion–anti-fermion–boson interaction vetexes and flavor mixing are discussed.

**Thank you！　　谢谢！**