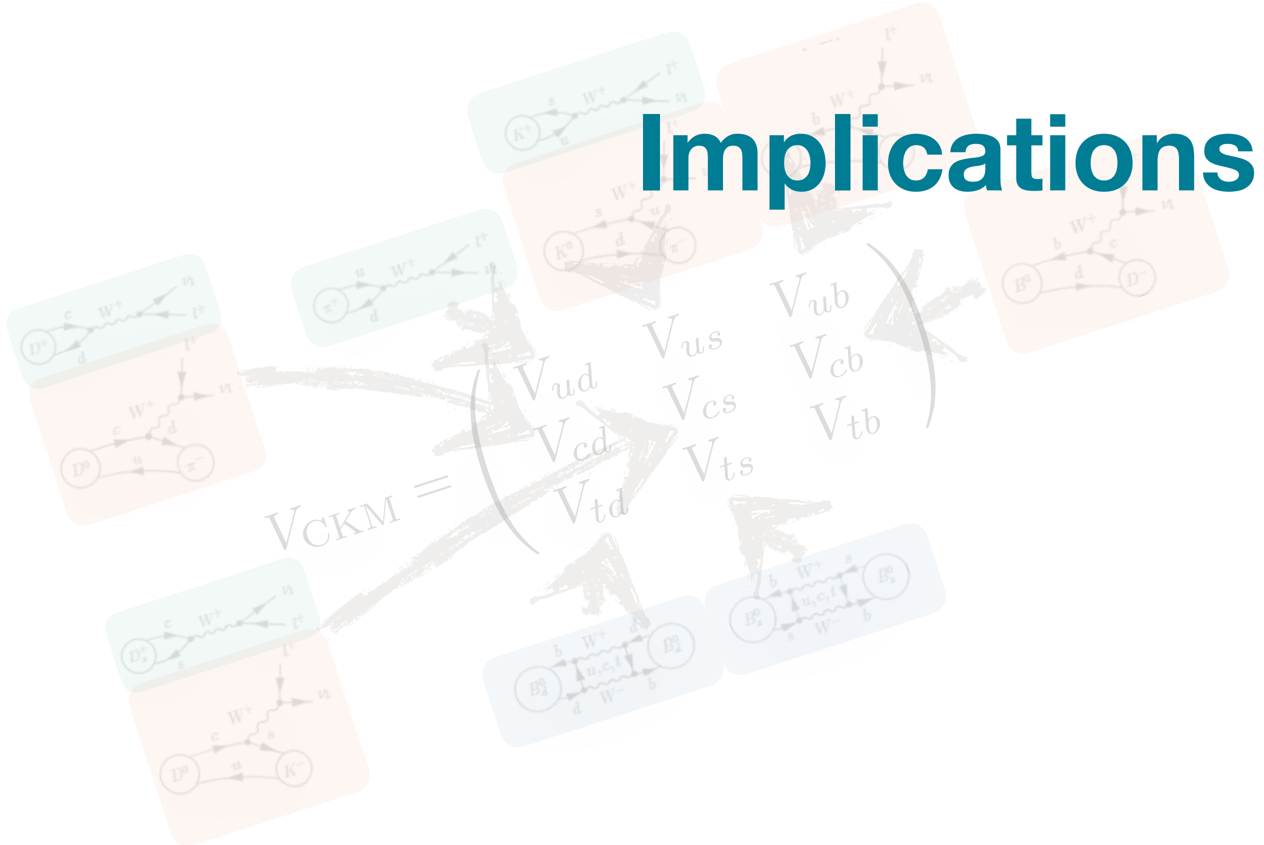
Bayesian inference for form-factor fits regulated by unitarity and analyticity



Implications of LHCb measurements and future prospects
25–27 Oct 2023
CERN

Andreas Jüttner

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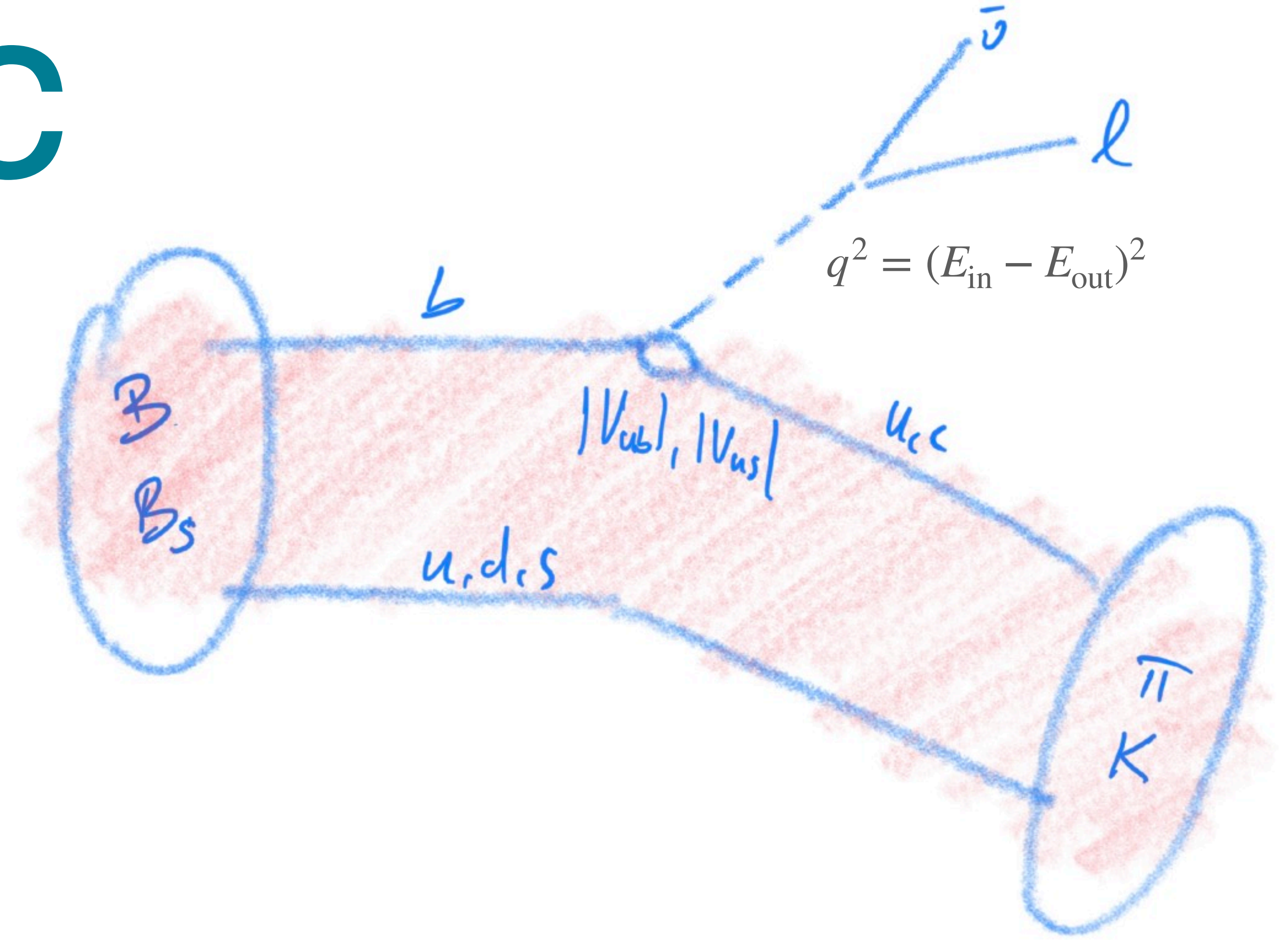
Intro: exclusive semileptonic meson decay



Objective: obtain model-independent theory prediction over entire kinematical range

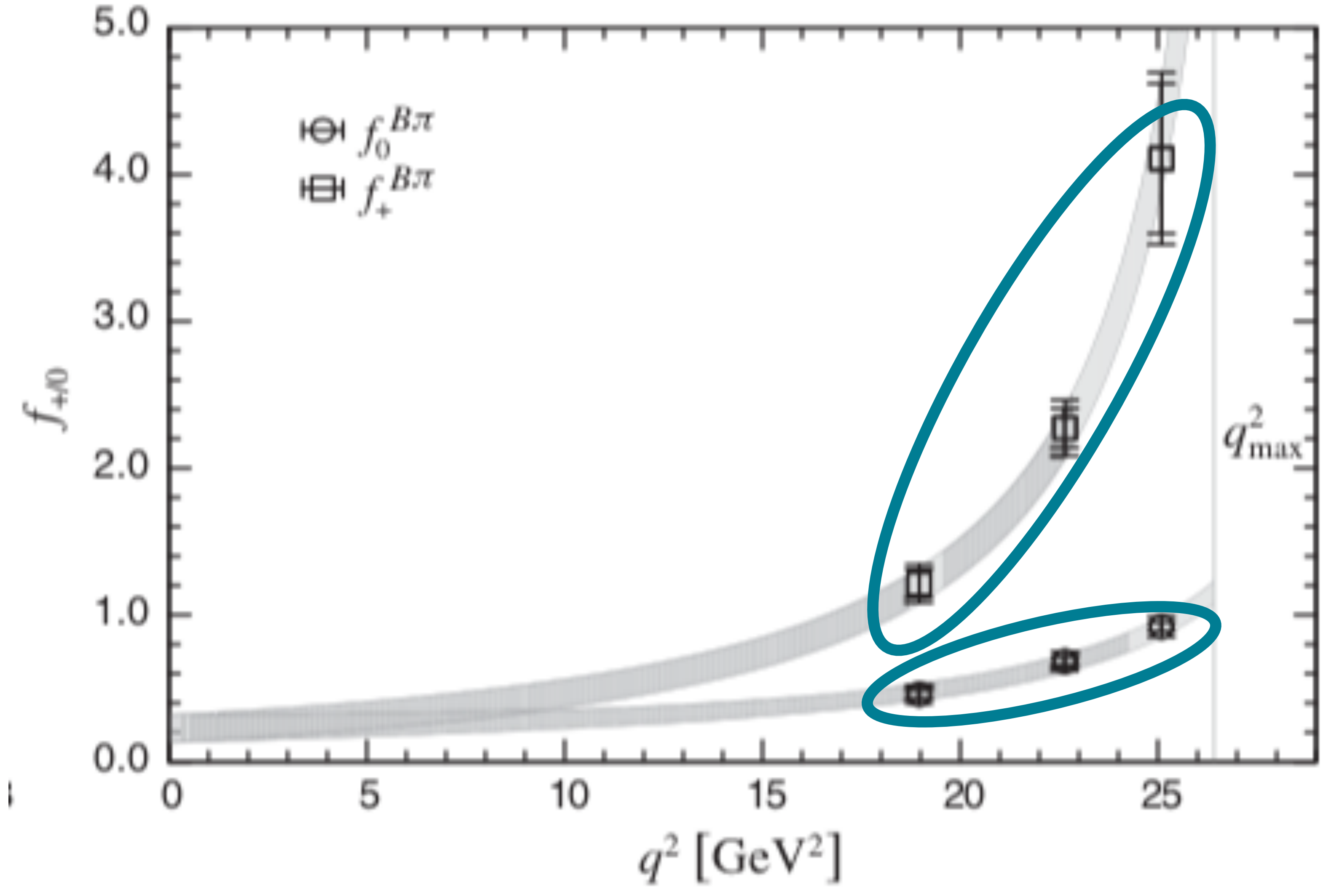
Input:

- sum rules: $q^2 \approx 0$
- lattice QCD:
  - finite lattice spacing (UV)
  - finite volume (IR)
  - worsening signal-to-noise

→ limited kinematic reach

↓

need extra/interpolation



RBC/UKQCD PRD 91, 074510 (2015)

BGL fitting strategies

$$f_X(q_i^2) = \frac{1}{B_X(q_i^2)\phi_X(q_i^2, t_0)} \sum_{n=0}^{K_X-1} a_{X,n} z(q_i^2)^n$$

unitarity constraint: $\mathbf{a}_X \;^2 \leq 1$

Boyd, Grinstein, Lebed, PRL 74 (1995)

Determine all $a_{X,n}$ from finite set of theory data

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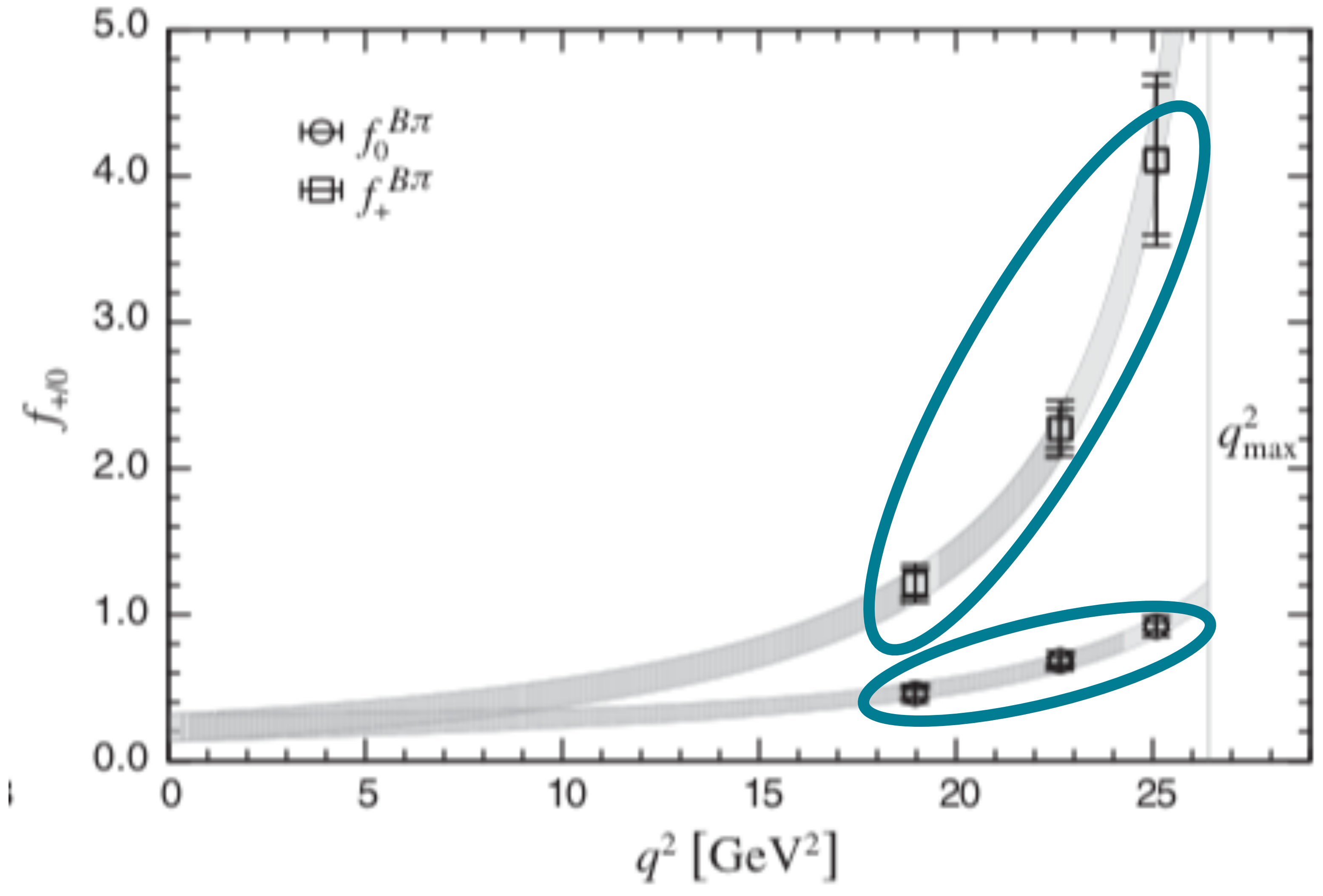
Determine all $a_{X,n}$ from finite set of theory data

Frequentist fit:

- $N_{\text{dof}} = N_{\text{data}} - K_X \geq 1$
  $\rightarrow$ in practice truncation K at low order
- induced systematic difficult to estimate
- meaning of Frequentist with unitarity constraint?



RBC/UKQCD PRD 91, 074510 (2015)

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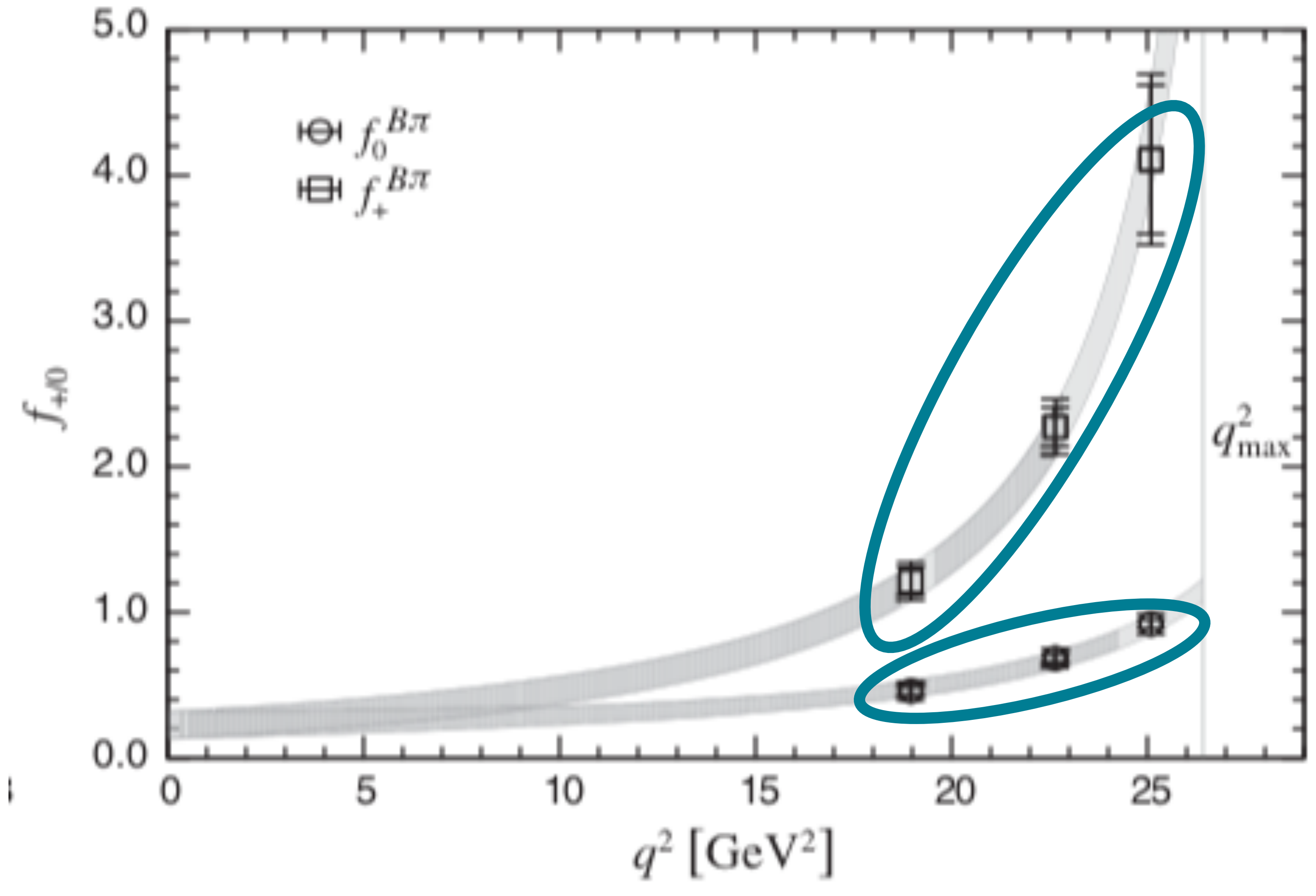
Bayesian fit:
- fit including higher order z expansion meaningful
- need *regulator* to control higher-order coefficients
- well-defined meaning of unitarity constraint



RBC/UKQCD PRD 91, 074510 (2015)

BGL fitting strategies

$$f_X(q_i^2) = \frac{1}{B_X(q_i^2)\phi_X(q_i^2, t_0)} \sum_{n=0}^{K_X-1} a_{X,n} z(q_i^2)^n$$

unitarity constraint: $\mathbf{a}_X{}^2 \le 1$

Boyd, Grinstein, Lebed, PRL 74 (1995)

Determine all $a_{X,n}$ from finite set of theory data

Frequentist fit:

- $N_{\text{dof}} = N_{\text{data}} - K_X \ge 1$
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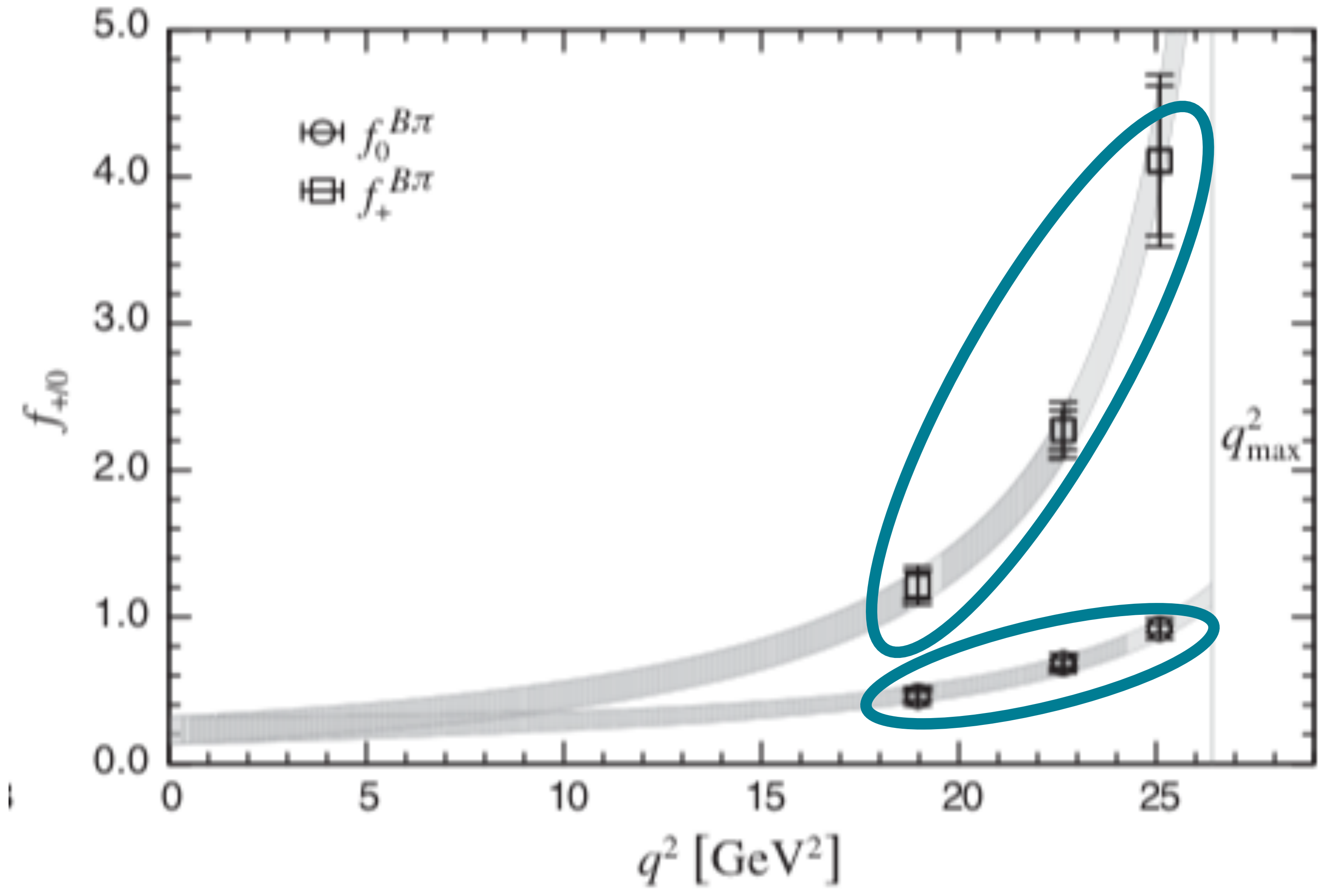

RBC/UKQCD PRD 91, 074510 (2015)

Here: Combined Frequentist + Bayesian perspective

BGL — conventions

$$f_X(q_i^2) = \frac{1}{B_X(q_i^2)\phi_X(q_i^2, t_0)} \sum_{n=0}^{K_X-1} a_{X,n} z(q_i^2)^n = Z_{XX,in} a_{X,n}$$

(Z is $N_{\text{data}} \times K$ matrix)

E.g. $P \to P$ transition:

Input (e.g. lattice ff): $\mathbf{f}^T = (\mathbf{f}_+^T, \mathbf{f}_0^T) = (f_+(q_0^2), f_+(q_1^2), \ldots, f_+(q_{N_+-1}^2), f_0(q_0^2), f_0(q_1^2), \ldots, f_0(q_{N_0-1}^2))$

Output (BGL params): $\mathbf{a}^T = (\mathbf{a}_+^T, \mathbf{a}_0^T) = (a_{+,0}, a_{+,1}, a_{+,2}, \ldots, a_{+,K_+-1}, a_{0,1}, \ldots, a_{0,K_0-1})$

BGL — frequentist fit with kinematical constraint

Flynn, AJ, Tsang, arXiv:2303.11285

Frequentist fit $\chi^2(\mathbf{a}, \mathbf{f}) = [\mathbf{f} - Z\mathbf{a}]^T C_{\mathbf{f}}^{-1} [\mathbf{f} - Z\mathbf{a}]$ $f_X(q_i^2) = Z_{XX,in} a_{X,n}$

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For combined fit over $f_+(q^2)$ and $f_0(q^2)$ with constraint $f_+(0) = f_0(0)$:

$$Z = \begin{pmatrix} Z_{++} & Z_{+0} \\ Z_{0+} & Z_{00} \end{pmatrix}$$

$$a_{0,0} = B_0(0)\phi_0(0,t_0)f_+(0) - \sum_{k=1}^{K_0-1} a_{0,k} z^k(0)$$

expressions in arXiv:2303.11285

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expressions in arXiv:2303.11285

Solution:

$$\mathbf{a} = \left(Z^T C_{\mathbf{f}}^{-1} Z\right)^{-1} Z C_{\mathbf{f}}^{-1} \mathbf{f}$$

$$C_{\mathbf{a}} = \left(Z^T C_{\mathbf{f}}^{-1} Z\right)^{-1}$$

Bayesian form-factor fit

Flynn, AJ, Tsang, arXiv:2303.11285

Compute BGL parameters as expectation values $\langle g(\mathbf{a})\rangle = \mathcal{N}\int d\mathbf{a}\, g(\mathbf{a})\, \pi(\mathbf{a}\ \ \mathbf{f}, C_{\mathbf{f}})\, \pi_{\mathbf{a}}$

where *probability for parameters given model and data (assume input Gaussian)*

$$\pi(\mathbf{a}\ \ \mathbf{f}, C_{\mathbf{f}}) \propto \exp\left(-\frac{1}{2}\chi^2(\mathbf{a},\mathbf{f})\right) \quad \text{where} \quad \chi^2(\mathbf{a},\mathbf{f}) = (\mathbf{f} - Z\mathbf{a})^T C_{\mathbf{f}}^{-1}(\mathbf{f} - Z\mathbf{a})$$

where *prior knowledge is* ***only*** *QFT unitarity constraint (flat prior for BGL params):*

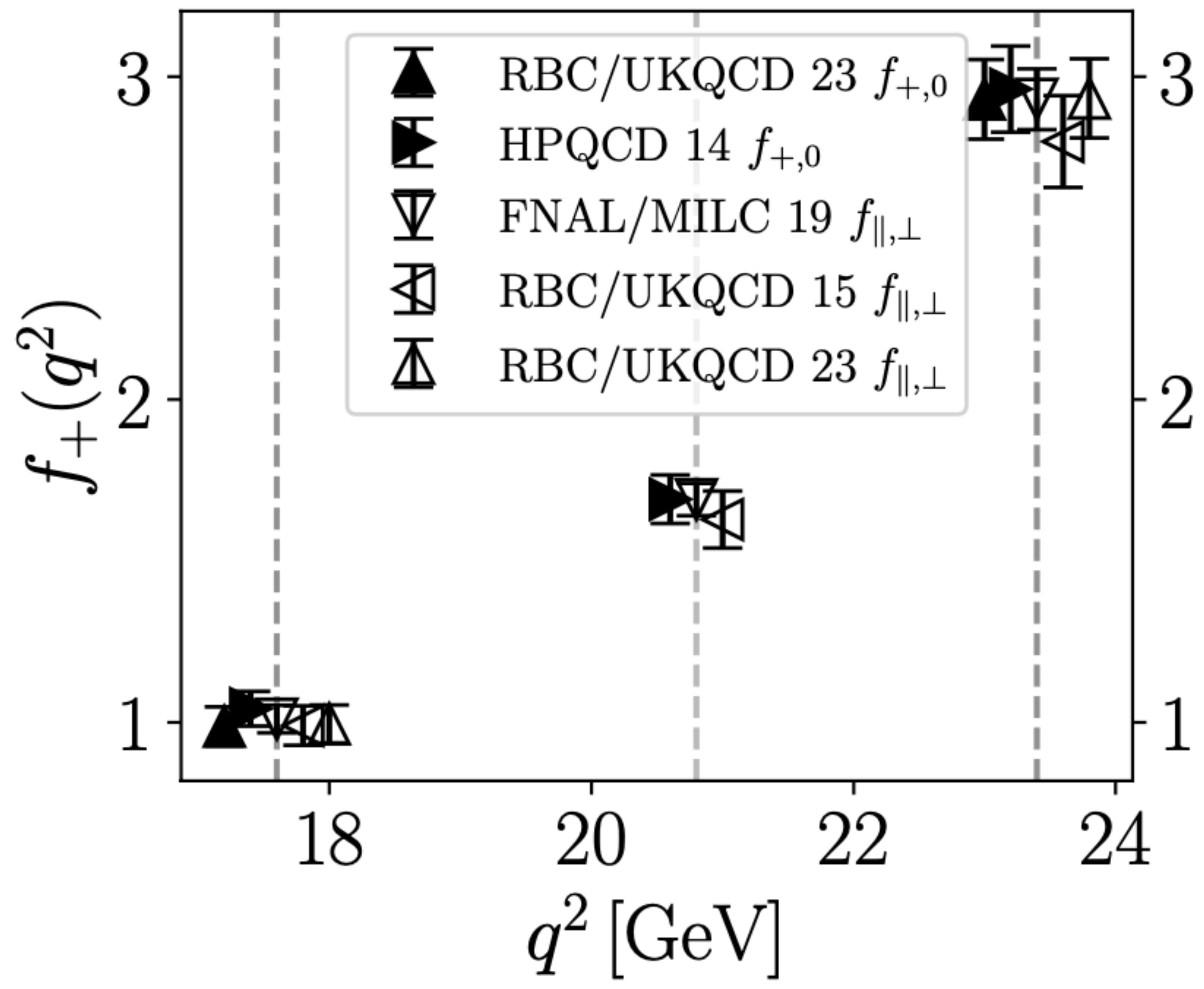
$$\pi_{\mathbf{a}} \propto \theta\left(1 - \ \ \mathbf{a}_+\ \ {}^2_{\alpha_{B_sK}}\right)\theta\left(1 - \ \ \mathbf{a}_0\ \ {}^2_{\alpha_{B_sK}}\right)$$

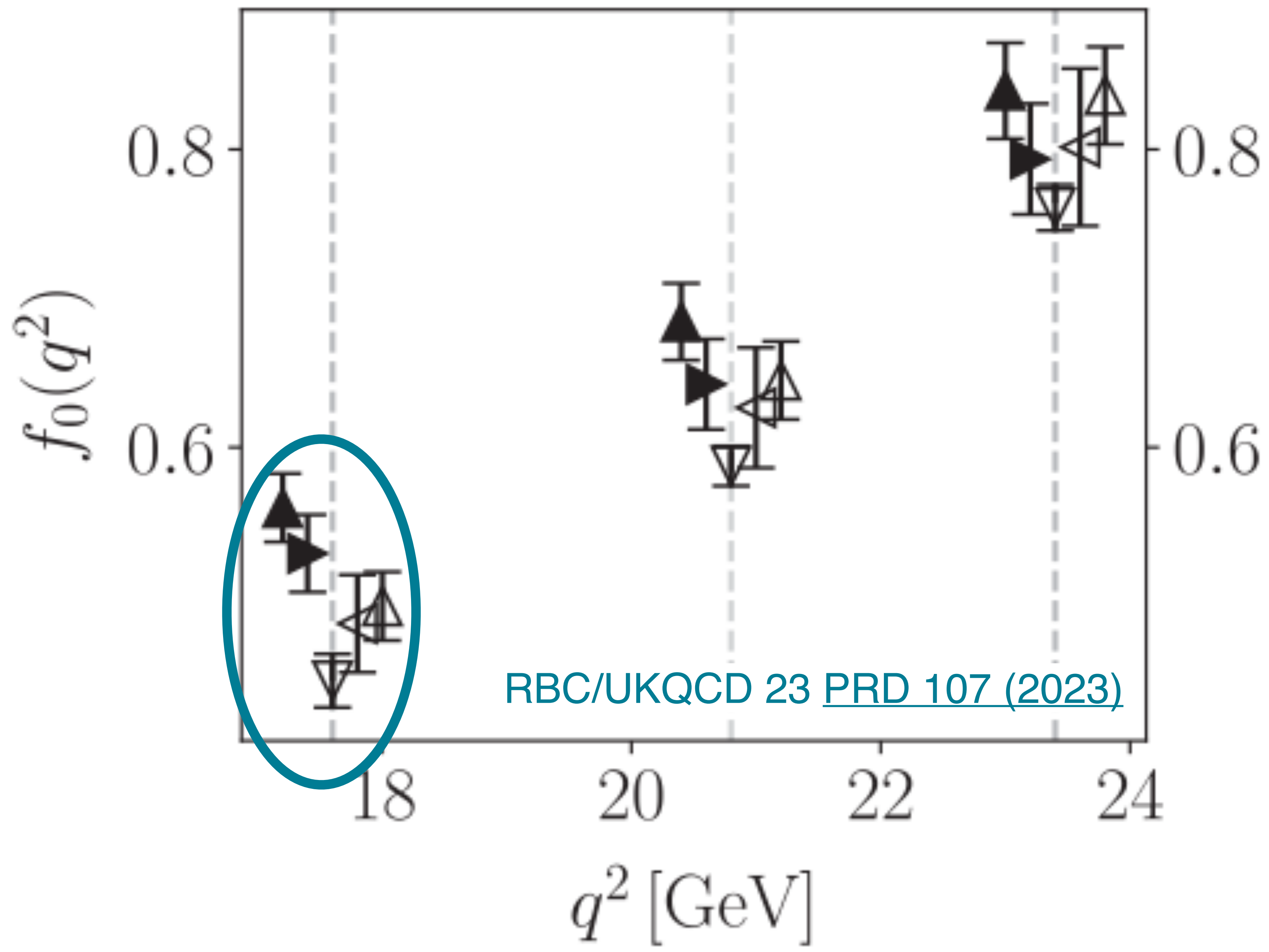
In practice MC integration: draw samples for **a** from multivariate normal distribution and drop samples not compatible with unitarity

Example 1: $B_s \to K\ell\nu$

Lattice input from HPQCD 14, FNAL/MILC 19, RBC/UKQCD 23

[HPQCD 14 PRD 90 (2014), RBC/UKQCD 23 PRD 107 (2023) FNAL/MILC 19 PRD 100 (2019)]





RBC/UKQCD 23: **Results for $f_0(q^2)$ disagree depending on strategy for continuum/chiral limit; requires independent confirmation** [RBC/UKQCD PRD 107 (2023)]

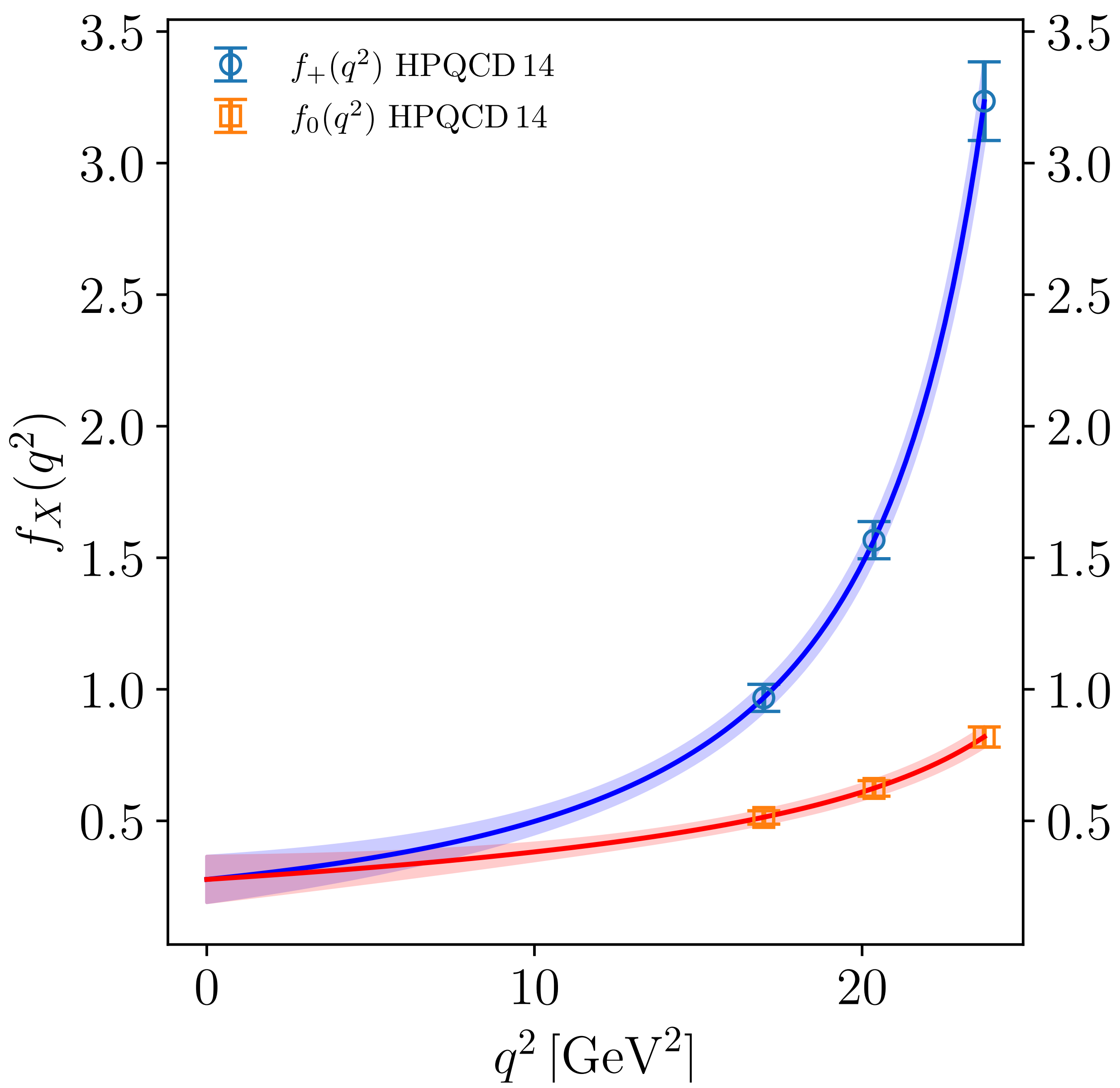
Example 1: $B_s \to K\ell\nu$ — fit to HPQCD 14

HPQCD 14
PRD 90 (2014) 054506

HPQCD 14 – $\mathbf{a}_+$

Frequentist

K_+	K_0	$a_{+,0}$	$a_{+,1}$	$a_{+,2}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
2	2	0.0270(13)	-0.0792(50)	-	0.03	2.93	3
2	3	0.0273(13)	-0.0760(63)	-	0.02	4.06	2
3	2	0.0257(14)	-0.0805(50)	0.068(31)	0.15	1.89	2
3	3	0.0262(14)	-0.0727(64)	0.096(34)	0.97	0.00	1



HPQCD 14 – $\mathbf{a}_0$

Frequentist

K_+	K_0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
2	2	0.0883(44)	-0.250(17)	-	0.03	2.93	3
2	3	0.0880(44)	-0.242(19)	0.053(65)	0.02	4.06	2
3	2	0.0906(45)	-0.240(17)	-	0.15	1.89	2
3	3	0.0908(46)	-0.215(22)	0.138(71)	0.97	0.00	1

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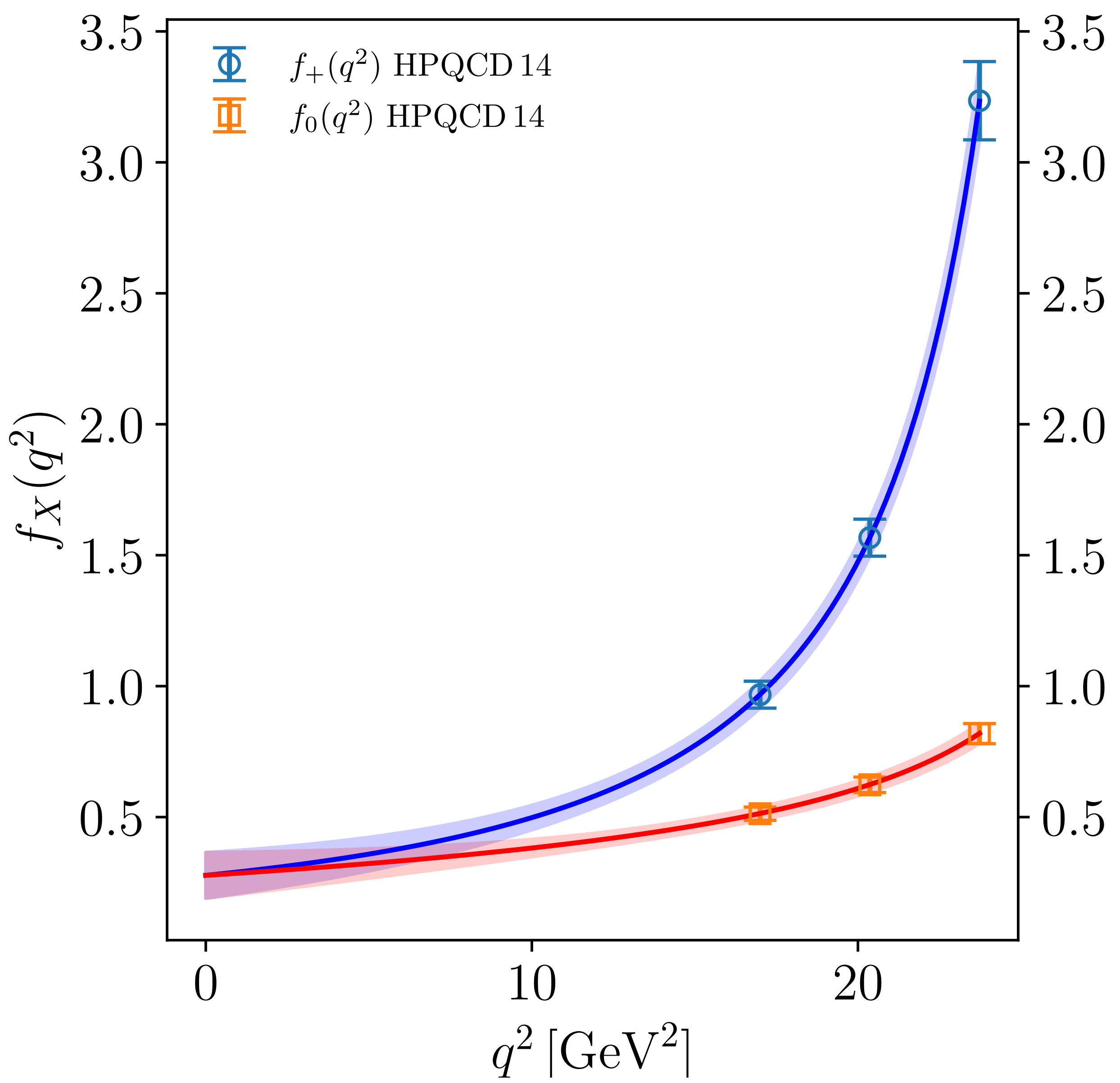

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HPQCD 14 – $\mathbf{a}_+$

Flynn, AJ, Tsang, arXiv:2303.11285

Bayesian

K_+	K_0	$a_{+,0}$	$a_{+,1}$	$a_{+,2}$	$a_{+,3}$	$a_{+,4}$	$a_{+,5}$	$a_{+,6}$	$a_{+,7}$	$a_{+,8}$	$a_{+,9}$
2	2	0.0270(12)	-0.0792(49)	-	-	-	-	-	-	-	-
2	3	0.0273(13)	-0.0761(63)	-	-	-	-	-	-	-	-
3	2	0.0257(14)	-0.0805(49)	0.069(30)	-	-	-	-	-	-	-
3	3	0.0261(14)	-0.0728(64)	0.096(34)	-	-	-	-	-	-	-
3	4	0.0261(14)	-0.0728(76)	0.096(39)	-	-	-	-	-	-	-
4	3	0.0261(14)	-0.0729(68)	0.096(35)	0.008(90)	-	-	-	-	-	-
4	4	0.0261(14)	-0.0730(77)	0.091(62)	-0.02(20)	-	-	-	-	-	-
5	5	0.0262(15)	-0.0735(79)	0.084(67)	-0.03(19)	0.03(68)	-	-	-	-	-
6	6	0.0261(14)	-0.0735(79)	0.086(69)	-0.03(19)	-0.00(64)	0.01(65)	-	-	-	-
7	7	0.0262(14)	-0.0732(84)	0.088(69)	-0.02(18)	0.01(65)	0.02(73)	-0.03(70)	-	-	-
8	8	0.0261(14)	-0.0732(80)	0.089(72)	-0.02(18)	-0.00(66)	0.03(86)	-0.04(90)	0.03(73)	-	-
9	9	0.0261(14)	-0.0729(84)	0.095(75)	-0.02(19)	-0.04(68)	0.1(1.0)	-0.1(1.2)	0.1(1.1)	-0.06(79)	-
10	10	0.0261(14)	-0.0726(89)	0.101(79)	-0.01(20)	-0.09(73)	0.2(1.3)	-0.3(1.7)	0.2(1.8)	-0.2(1.4)	0.08(87)

HPQCD 14 – $\mathbf{a}_0$

Bayesian

K_+	K_0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	$a_{0,5}$	$a_{0,6}$	$a_{0,7}$	$a_{0,8}$	$a_{0,9}$
2	2	0.0883(44)	-0.250(17)	-	-	-	-	-	-	-	-
2	3	0.0880(44)	-0.243(19)	0.052(65)	-	-	-	-	-	-	-
3	2	0.0907(46)	-0.240(17)	-	-	-	-	-	-	-	-
3	3	0.0906(44)	-0.215(22)	0.137(73)	-	-	-	-	-	-	-
3	4	0.0907(47)	-0.215(22)	0.14(11)	-0.01(31)	-	-	-	-	-	-
4	3	0.0907(45)	-0.214(22)	0.139(72)	-	-	-	-	-	-	-
4	4	0.0907(46)	-0.215(25)	0.12(19)	-0.08(60)	-	-	-	-	-	-
5	5	0.0909(46)	-0.218(25)	0.10(19)	-0.12(55)	0.04(63)	-	-	-	-	-
6	6	0.0907(45)	-0.217(25)	0.10(19)	-0.11(53)	0.06(66)	-0.02(66)	-	-	-	-
7	7	0.0907(46)	-0.217(26)	0.11(20)	-0.08(51)	0.03(73)	0.03(81)	-0.04(70)	-	-	-
8	8	0.0908(46)	-0.217(25)	0.11(20)	-0.08(50)	-0.01(84)	0.1(1.0)	-0.09(96)	0.08(74)	-	-
9	9	0.0907(46)	-0.215(25)	0.13(22)	-0.05(50)	-0.06(95)	0.2(1.4)	-0.2(1.5)	0.1(1.2)	-0.05(82)	-
10	10	0.0907(46)	-0.214(27)	0.15(24)	-0.03(49)	-0.2(1.1)	0.4(1.8)	-0.5(2.2)	0.4(2.1)	-0.3(1.6)	0.13(90)

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HPQCD 14
PRD 90 (2014) 054506

Frequentist

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2	3	0.0273(13)	-0.0760(63)	-	0.02	4.06	2
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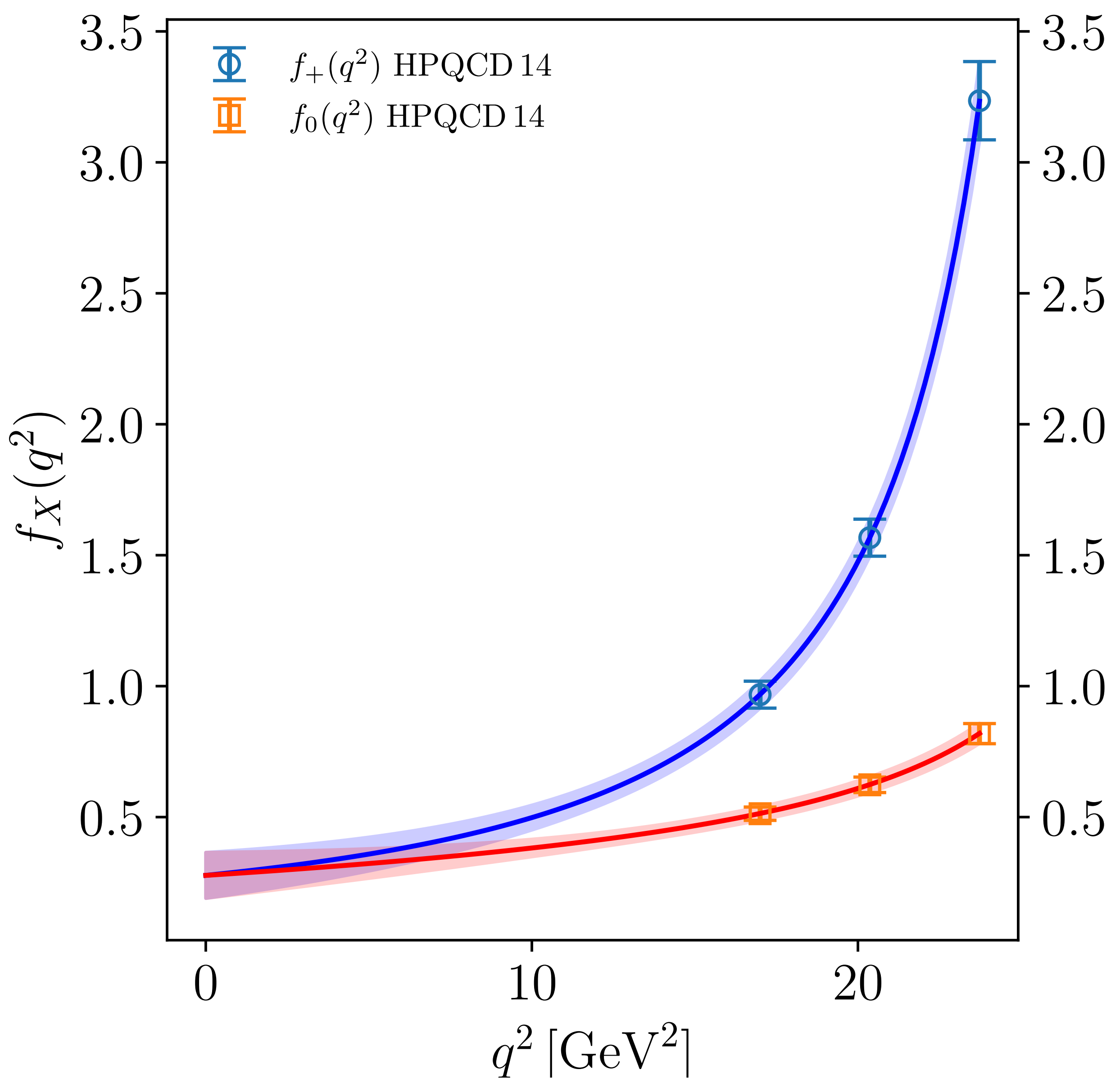

HPQCD 14 – $\mathbf{a}_0$

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2	2	0.0883(44)	-0.250(17)	-	0.03	2.93	3
2	3	0.0880(44)	-0.242(19)	0.053(65)	0.02	4.06	2
3	2	0.0906(45)	-0.240(17)	-	0.15	1.89	2
3	3	0.0908(46)	-0.215(22)	0.138(71)	0.97	0.00	1

Frequentist

HPQCD 14 – $\mathbf{a}_+$ — Flynn, AJ, Tsang, arXiv:2303.11285

Bayesian

K_+	K_0	$a_{+,0}$	$a_{+,1}$	$a_{+,2}$	$a_{+,3}$	$a_{+,4}$	$a_{+,5}$	$a_{+,6}$	$a_{+,7}$	$a_{+,8}$	$a_{+,9}$
2	2	0.0270(12)	-0.0792(49)	-	-	-	-	-	-	-	-
2	3	0.0273(13)	-0.0761(63)	-	-	-	-	-	-	-	-
3	2	0.0257(14)	-0.0805(49)	0.069(30)	-	-	-	-	-	-	-
3	3	0.0261(14)	-0.0728(64)	0.096(34)	-	-	-	-	-	-	-
3	4	0.0261(14)	-0.0728(76)	0.096(39)	-	-	-	-	-	-	-
4	3	0.0261(14)	-0.0729(68)	0.096(35)	0.008(90)	-	-	-	-	-	-
4	4	0.0261(14)	-0.0730(77)	0.091(62)	-0.02(20)	-	-	-	-	-	-
5	5	0.0262(15)	-0.0735(79)	0.084(67)	-0.03(19)	0.03(68)	-	-	-	-	-
6	6	0.0261(14)	-0.0735(79)	0.086(69)	-0.03(19)	-0.00(64)	0.01(65)	-	-	-	-
7	7	0.0262(14)	-0.0732(84)	0.088(69)	-0.02(18)	0.01(65)	0.02(73)	-0.03(70)	-	-	-
8	8	0.0261(14)	-0.0732(80)	0.089(72)	-0.02(18)	-0.00(66)	0.03(86)	-0.04(90)	0.03(73)	-	-
9	9	0.0261(14)	-0.0729(84)	0.095(75)	-0.02(19)	-0.04(68)	0.1(1.0)	-0.1(1.2)	0.1(1.1)	-0.06(79)	-
10	10	0.0261(14)	-0.0726(89)	0.101(79)	-0.01(20)	-0.09(73)	0.2(1.3)	-0.3(1.7)	0.2(1.8)	-0.2(1.4)	0.08(87)

results stable as truncation relaxed

HPQCD 14 – $\mathbf{a}_0$

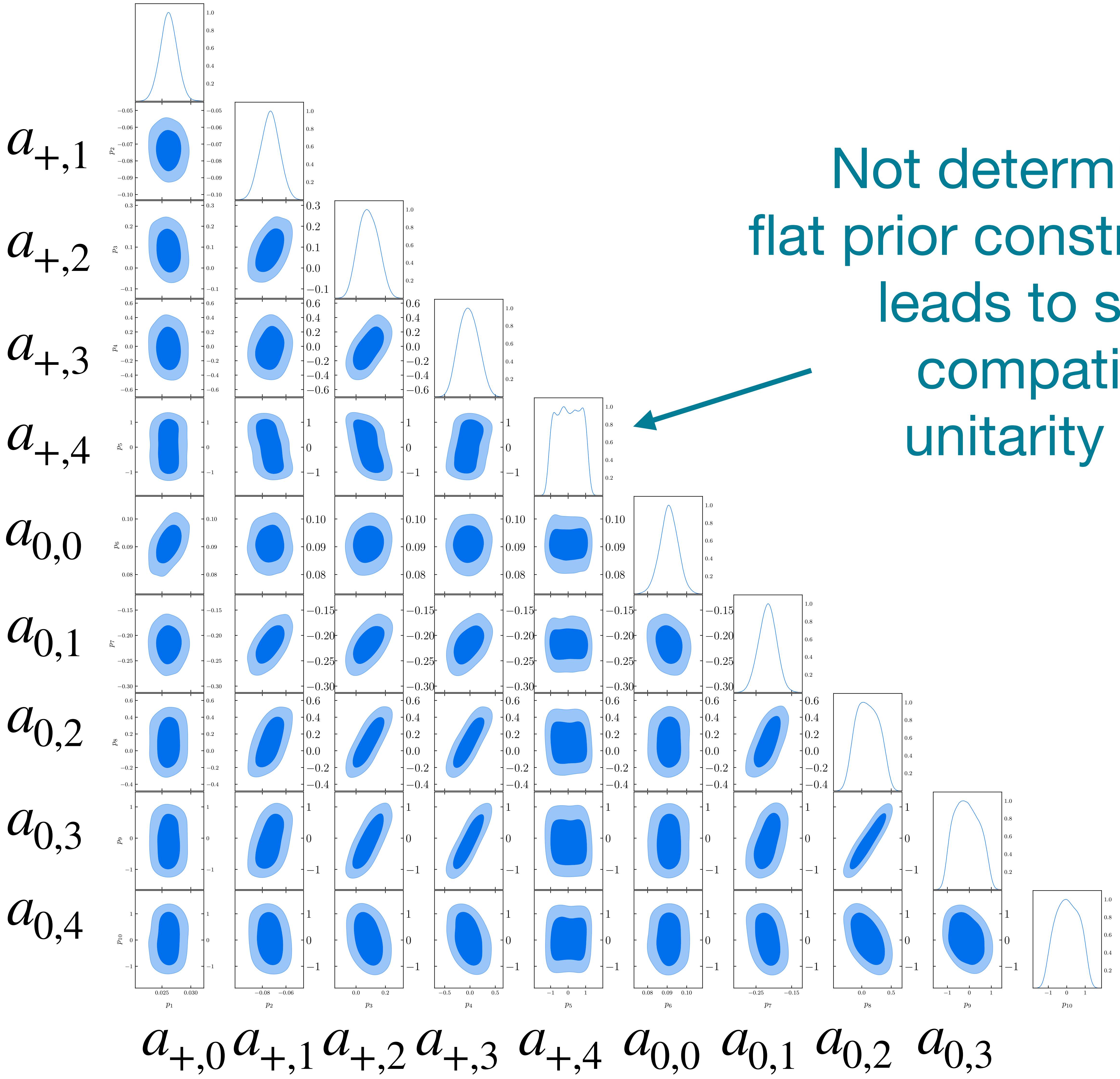
Bayesian

K_+	K_0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	$a_{0,5}$	$a_{0,6}$	$a_{0,7}$	$a_{0,8}$	$a_{0,9}$
2	2	0.0883(44)	-0.250(17)	-	-	-	-	-	-	-	-
2	3	0.0880(44)	-0.243(19)	0.052(65)	-	-	-	-	-	-	-
3	2	0.0907(46)	-0.240(17)	-	-	-	-	-	-	-	-
3	3	0.0906(44)	-0.215(22)	0.137(73)	-	-	-	-	-	-	-
3	4	0.0907(47)	-0.215(22)	0.14(11)	-0.01(31)	-	-	-	-	-	-
4	3	0.0907(45)	-0.214(22)	0.139(72)	-	-	-	-	-	-	-
4	4	0.0907(46)	-0.215(25)	0.12(19)	-0.08(60)	-	-	-	-	-	-
5	5	0.0909(46)	-0.218(25)	0.10(19)	-0.12(55)	0.04(63)	-	-	-	-	-
6	6	0.0907(45)	-0.217(25)	0.10(19)	-0.11(53)	0.06(66)	-0.02(66)	-	-	-	-
7	7	0.0907(46)	-0.217(26)	0.11(20)	-0.08(51)	0.03(73)	0.03(81)	-0.04(70)	-	-	-
8	8	0.0908(46)	-0.217(25)	0.11(20)	-0.08(50)	-0.01(84)	0.1(1.0)	-0.09(96)	0.08(74)	-	-
9	9	0.0907(46)	-0.215(25)	0.13(22)	-0.05(50)	-0.06(95)	0.2(1.4)	-0.2(1.5)	0.1(1.2)	-0.05(82)	-
10	10	0.0907(46)	-0.214(27)	0.15(24)	-0.03(49)	-0.2(1.1)	0.4(1.8)	-0.5(2.2)	0.4(2.1)	-0.3(1.6)	0.13(90)

Example 1: $B_s \to K\ell\nu$ — fit to HPQCD 14

Distribution and correlation of BGL coefficients

truncation, e.g. $(K_+, K_0) = (5,5)$



Not determined by data, flat prior constrains coefficient, leads to stability and compatibility with unitarity constraint

Example 1: $B_s \to K\ell\nu$ — V_{ub}

Lattice input from HPQCD 14, FNAL/MILC 19, RBC/UKQCD 23

[HPQCD 14 PRD 90 (2014), RBC/UKQCD 23 PRD 107 (2023) FNAL/MILC 19 PRD 100 (2019)]

Experimental input — two bins from LHCb [LHCb PRD 108 (2023)]

$$R_{\rm BF} = \frac{\mathcal{B}(B_s^0 \to K^-\mu^+\nu_\mu)}{\mathcal{B}(B_s^0 \to D_s^-\mu^+\nu_\mu)}$$

$$R_{BF}^{\rm low} = 1.66(08)(09) \times 10^{-3},$$
$$R_{BF}^{\rm high} = 3.25(21)(^{+18}_{-19}) \times 10^{-3},$$
$$R_{BF}^{\rm total} = 4.89(21)(^{+24}_{-25}) \times 10^{-3},$$

$$\mathcal{B}(B_s^0 \to D_s^-\mu^+\nu_\mu) = 2.49(12)(21) \times 10^{-2}.$$

[LHCb PRD 101 (2020)]

$$|V_{ub}| = \sqrt{\frac{R_{BF}^{\rm bin}\mathcal{B}(B_s^0 \to D_s^-\mu^+\nu_\mu)}{\tau_{B_s^0}\Gamma_0^{\rm bin}(B_s \to K\ell\nu)}},$$

Example 1: $B_s \to K\ell\nu$ — fit to HPQCD 14

Results for HPQCD 14 PRD 90 (2014) 054506

Flynn, AJ, Tsang, arXiv:2303.11285

increasing truncation

K_+	K_0	$f(q^2=0)$	$R^{\rm impr}_{B_s\to K}$	$R_{B_s\to K}$	$\frac{\Gamma^\tau}{\lvert V_{ub}\rvert^2}\,[\frac{1}{\rm ps}]$	$\frac{\Gamma^\mu}{\lvert V_{ub}\rvert^2}\,[\frac{1}{\rm ps}]$	$V^{\rm low}_{\rm CKM}$	$V^{\rm high}_{\rm CKM}$	$V^{\rm full}_{\rm CKM}$
2	2	0.208(25)	1.524(37)	0.727(25)	4.51(45)	6.23(76)	0.00383(47)	0.00352(35)	0.00363(37)
2	3	0.226(34)	1.511(41)	0.704(39)	4.67(49)	6.67(97)	0.00361(53)	0.00344(34)	0.00349(38)
3	2	0.233(27)	1.609(58)	0.733(27)	4.44(45)	6.08(77)	0.00368(45)	0.00367(37)	0.00367(38)
3	3	0.293(41)	1.592(57)	0.664(40)	4.84(51)	7.3(1.1)	0.00310(44)	0.00349(35)	0.00333(36)
3	4	0.293(56)	1.593(60)	0.667(59)	4.85(58)	7.4(1.4)	0.00313(55)	0.00349(37)	0.00338(40)
4	3	0.294(42)	1.594(60)	0.663(40)	4.85(52)	7.4(1.1)	0.00309(44)	0.00348(36)	0.00332(36)
4	4	0.285(92)	1.593(60)	0.677(88)	4.83(62)	7.3(1.7)	0.00328(86)	0.00350(38)	0.00346(42)
5	5	0.277(88)	1.595(62)	0.685(85)	4.81(62)	7.2(1.7)	0.00333(85)	0.00351(38)	0.00348(42)
6	6	0.277(88)	1.592(63)	0.685(86)	4.79(63)	7.2(1.7)	0.00335(88)	0.00350(38)	0.00348(43)
7	7	0.282(89)	1.592(60)	0.680(87)	4.82(64)	7.3(1.7)	0.00332(89)	0.00350(38)	0.00347(43)
8	8	0.283(88)	1.594(61)	0.679(85)	4.83(64)	7.3(1.7)	0.00330(85)	0.00351(37)	0.00347(41)
9	9	0.289(91)	1.594(62)	0.674(88)	4.85(64)	7.4(1.8)	0.00327(89)	0.00350(38)	0.00347(42)
10	10	0.293(95)	1.593(60)	0.670(91)	4.87(67)	7.5(1.9)	0.00325(92)	0.00349(38)	0.00346(42)

results for phenomenology in Bayesian setup independent of truncation

increasing truncation

K_+	K_0	$I[\mathcal{A}^\tau_{\rm FB}]\,[\frac{1}{\rm ps}]$	$I[\mathcal{A}^\mu_{\rm FB}]\,[\frac{1}{\rm ps}]$	$\bar{\mathcal{A}}^\tau_{\rm FB}$	$\bar{\mathcal{A}}^\mu_{\rm FB}$	$I[\mathcal{A}^\tau_{\rm pol}]\,[\frac{1}{\rm ps}]$	$I[\mathcal{A}^\mu_{\rm pol}]\,[\frac{1}{\rm ps}]$	$\bar{\mathcal{A}}^\tau_{\rm pol}$	$\bar{\mathcal{A}}^\mu_{\rm pol}$
2	2	1.22(13)	0.0278(51)	0.2708(37)	0.00443(34)	0.74(15)	6.15(75)	0.164(29)	0.98767(96)
2	3	1.26(14)	0.0314(70)	0.2709(38)	0.00465(44)	0.81(18)	6.59(96)	0.173(31)	0.9872(12)
3	2	1.23(13)	0.0319(59)	0.2780(43)	0.00524(51)	0.46(19)	5.99(76)	0.103(40)	0.9852(15)
3	3	1.36(15)	0.045(10)	0.2814(48)	0.00612(66)	0.53(20)	7.2(1.1)	0.110(40)	0.9830(18)
3	4	1.37(17)	0.046(14)	0.2814(50)	0.00611(83)	0.53(22)	7.3(1.3)	0.109(41)	0.9830(22)
4	3	1.37(15)	0.046(10)	0.2815(50)	0.00616(71)	0.53(22)	7.2(1.1)	0.109(42)	0.9829(20)
4	4	1.36(19)	0.046(21)	0.2810(69)	0.0060(15)	0.53(21)	7.2(1.7)	0.109(42)	0.9834(41)
5	5	1.35(19)	0.044(20)	0.2806(67)	0.0058(15)	0.53(22)	7.1(1.6)	0.109(44)	0.9837(39)
6	6	1.35(20)	0.044(20)	0.2803(69)	0.0058(15)	0.53(22)	7.1(1.7)	0.111(44)	0.9838(39)
7	7	1.35(20)	0.045(20)	0.2806(69)	0.0059(15)	0.53(21)	7.2(1.7)	0.111(43)	0.9835(39)
8	8	1.36(20)	0.045(20)	0.2808(69)	0.0059(15)	0.53(22)	7.2(1.7)	0.109(44)	0.9835(39)
9	9	1.36(20)	0.047(21)	0.2812(71)	0.0060(15)	0.53(22)	7.3(1.7)	0.109(44)	0.9832(40)
10	10	1.37(21)	0.048(23)	0.2815(72)	0.0061(15)	0.53(22)	7.4(1.8)	0.109(43)	0.9831(41)

Example 1: $B_s \to K\ell\nu$ — global fit

Combined fit over three lattice data sets —
let's first look at **Frequentist**:

[HPQCD 14 PRD 90 (2014), RBC/UKQCD 23 PRD 107 (2023) FNAL/MILC 19 PRD 100 (2019)]

K_+	K_0	$a_{+,0}$	$a_{+,1}$	$a_{+,2}$	$a_{+,3}$	$a_{+,4}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
2	2	0.02641(58)	-0.0824(26)	-	-	-	0.00	5.15	14
2	3	0.02668(68)	-0.0811(31)	-	-	-	0.00	5.50	13
3	2	0.02477(68)	-0.0829(26)	0.054(12)	-	-	0.00	3.95	13
3	3	0.02534(73)	-0.0792(31)	0.062(12)	-	-	0.00	3.89	12
3	4	0.02534(73)	-0.0781(34)	0.067(14)	-	-	0.00	4.19	11
4	3	0.02535(73)	-0.0776(38)	0.074(20)	0.023(30)	-	0.00	4.19	11
4	4	0.02592(97)	-0.033(50)	0.69(69)	2.1(2.3)	-	0.00	4.53	10
5	5	0.0266(10)	0.052(65)	2.21(97)	11.1(5.6)	17.2(15.1)	0.00	5.04	8

K_+	K_0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
2	2	0.0854(17)	-0.2565(75)	-	-	-	0.00	5.15	14
2	3	0.0856(18)	-0.2527(91)	0.021(27)	-	-	0.00	5.50	13
3	2	0.0858(18)	-0.2501(77)	-	-	-	0.00	3.95	13
3	3	0.0864(18)	-0.2379(95)	0.061(28)	-	-	0.00	3.89	12
3	4	0.0869(19)	-0.231(13)	0.067(29)	-0.08(10)	-	0.00	4.19	11
4	3	0.0869(19)	-0.229(15)	0.091(48)	-	-	0.00	4.19	11
4	4	0.0887(27)	-0.08(17)	2.2(2.4)	7.0(7.9)	-	0.00	4.53	10
5	5	0.0887(28)	0.07(20)	6.1(3.3)	41.5(19.0)	93.3(44.0)	0.00	5.04	8

Example 1: $B_s \to K\ell\nu$ — global fit

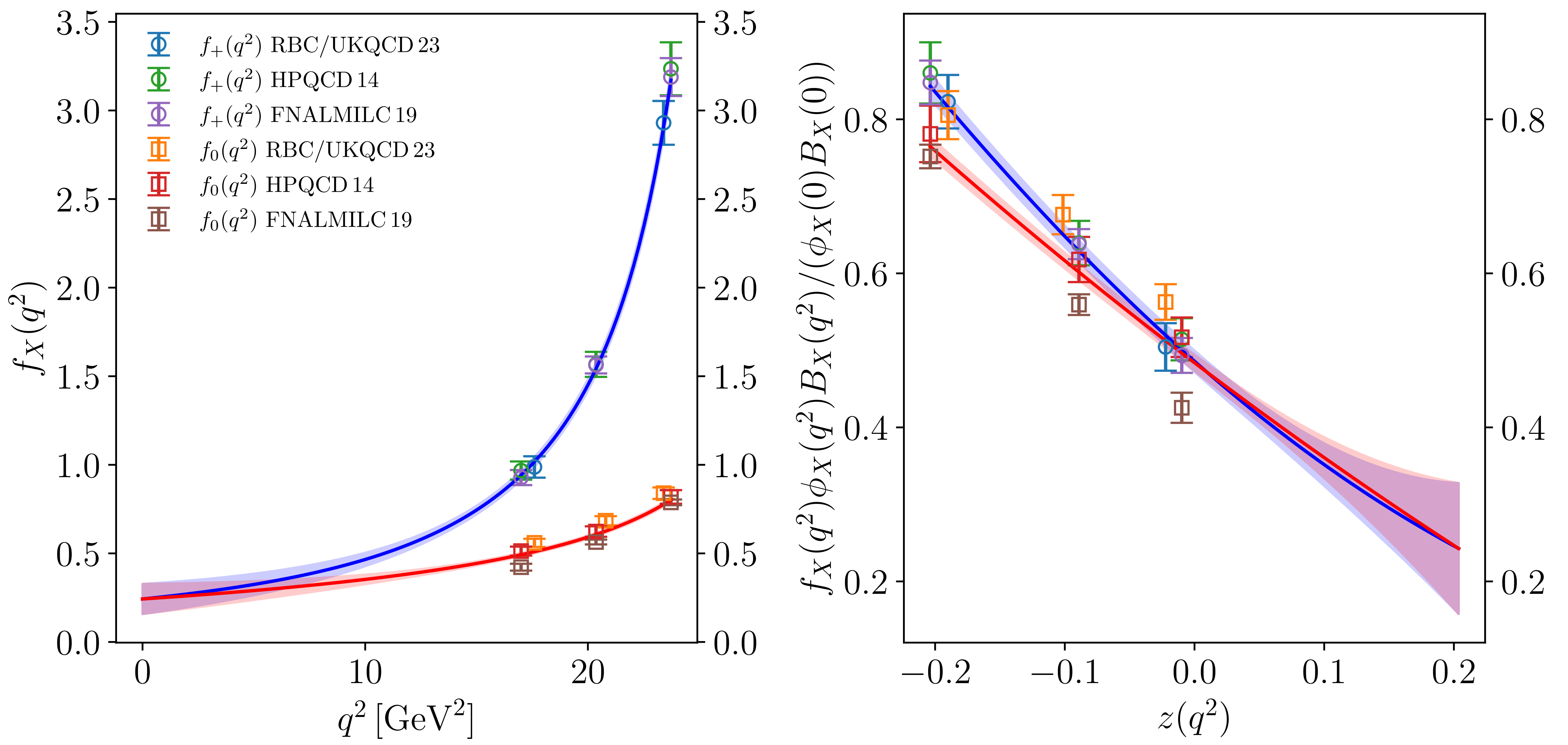
Combined fit over three lattice data sets — let's first look at **Frequentist**:

[HPQCD 14 PRD 90 (2014), RBC/UKQCD 23 PRD 107 (2023) FNAL/MILC 19 PRD 100 (2019)]

K_+	K_0	$a_{+,0}$	$a_{+,1}$	$a_{+,2}$	$a_{+,3}$	$a_{+,4}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
2	2	0.02641(58)	-0.0824(26)	-	-	-	0.00	5.15	14
2	3	0.02668(68)	-0.0811(31)	-	-	-	0.00	5.50	13
3	2	0.02477(68)	-0.0829(26)	0.054(12)	-	-	0.00	3.95	13
3	3	0.02534(73)	-0.0792(31)	0.062(12)	-	-	0.00	3.89	12
3	4	0.02534(73)	-0.0781(34)	0.067(14)	-	-	0.00	4.19	11
4	3	0.02535(73)	-0.0776(38)	0.074(20)	0.023(30)	-	0.00	4.19	11
4	4	0.02592(97)	-0.033(50)	0.69(69)	2.1(2.3)	-	0.00	4.53	10
5	5	0.0266(10)	0.052(65)	2.21(97)	11.1(5.6)	17.2(15.1)	0.00	5.04	8

K_+	K_0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
2	2	0.0854(17)	-0.2565(75)	-	-	-	0.00	5.15	14
2	3	0.0856(18)	-0.2527(91)	0.021(27)	-	-	0.00	5.50	13
3	2	0.0858(18)	-0.2501(77)	-	-	-	0.00	3.95	13
3	3	0.0864(18)	-0.2379(95)	0.061(28)	-	-	0.00	3.89	12
3	4	0.0869(19)	-0.231(13)	0.067(29)	-0.08(10)	-	0.00	4.19	11
4	3	0.0869(19)	-0.229(15)	0.091(48)	-	-	0.00	4.19	11
4	4	0.0887(27)	-0.08(17)	2.2(2.4)	7.0(7.9)	-	0.00	4.53	10
5	5	0.0887(28)	0.07(20)	6.1(3.3)	41.5(19.0)	93.3(44.0)	0.00	5.04	8

Bayesian fit works but doesn't describe data:



Example 1: $B_s \to K\ell\nu$ — global fit

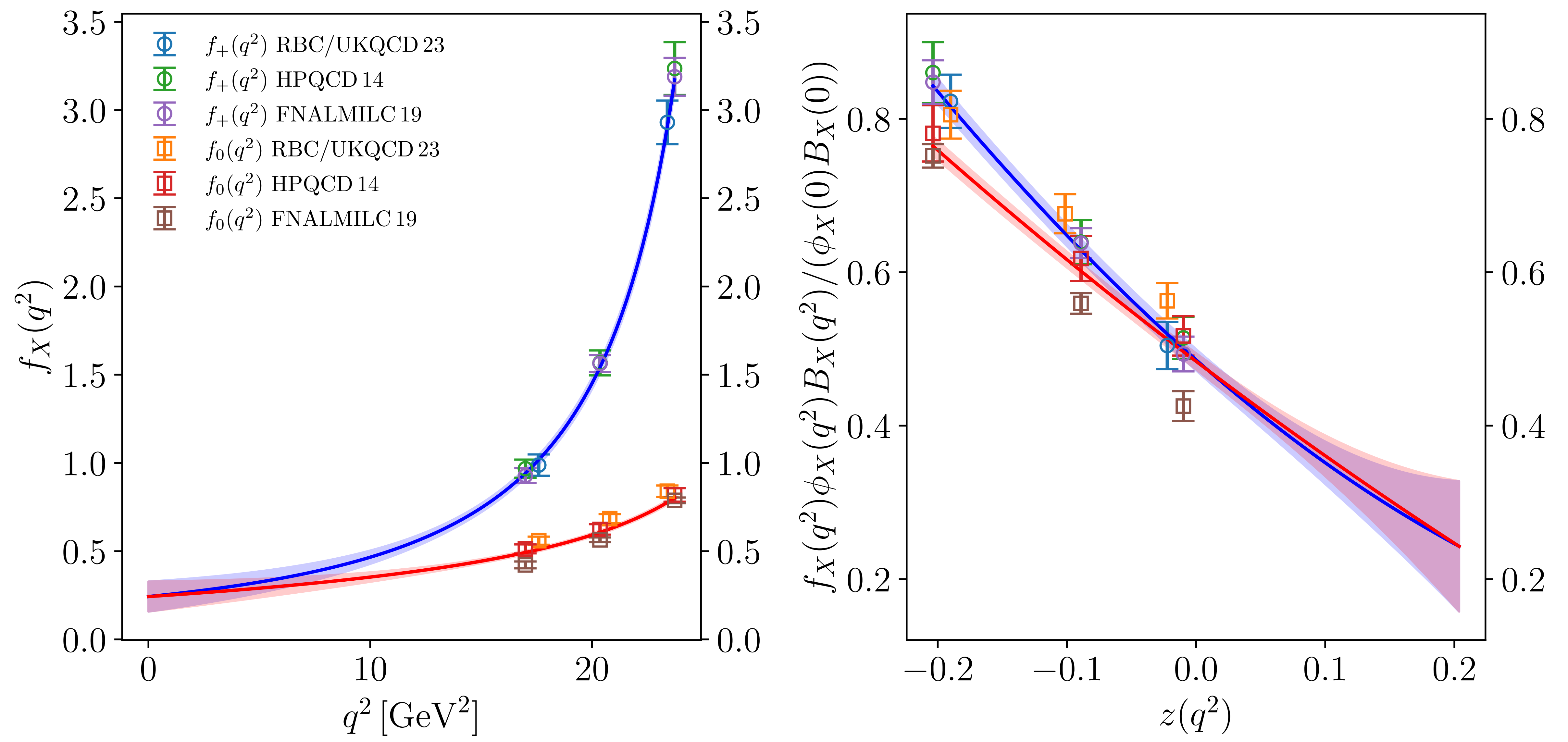
Combined fit over three lattice data sets — let's first look at **Frequentist**:

[HPQCD 14 PRD 90 (2014), RBC/UKQCD 23 PRD 107 (2023) FNAL/MILC 19 PRD 100 (2019)]

K_+	K_0	$a_{+,0}$	$a_{+,1}$	$a_{+,2}$	$a_{+,3}$	$a_{+,4}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
2	2	0.02641(58)	-0.0824(26)	-	-	-	0.00	5.15	14
2	3	0.02668(68)	-0.0811(31)	-	-	-	0.00	5.50	13
3	2	0.02477(68)	-0.0829(26)	0.054(12)	-	-	0.00	3.95	13
3	3	0.02534(73)	-0.0792(31)	0.062(12)	-	-	0.00	3.89	12
3	4	0.02534(73)	-0.0781(34)	0.067(14)	-	-	0.00	4.19	11
4	3	0.02535(73)	-0.0776(38)	0.074(20)	0.023(30)	-	0.00	4.19	11
4	4	0.02592(97)	-0.033(50)	0.69(69)	2.1(2.3)	-	0.00	4.53	10
5	5	0.0266(10)	0.052(65)	2.21(97)	11.1(5.6)	17.2(15.1)	0.00	5.04	8

K_+	K_0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
2	2	0.0854(17)	-0.2565(75)	-	-	-	0.00	5.15	14
2	3	0.0856(18)	-0.2527(91)	0.021(27)	-	-	0.00	5.50	13
3	2	0.0858(18)	-0.2501(77)	-	-	-	0.00	3.95	13
3	3	0.0864(18)	-0.2379(95)	0.061(28)	-	-	0.00	3.89	12
3	4	0.0869(19)	-0.231(13)	0.067(29)	-0.08(10)	-	0.00	4.19	11
4	3	0.0869(19)	-0.229(15)	0.091(48)	-	-	0.00	4.19	11
4	4	0.0887(27)	-0.08(17)	2.2(2.4)	7.0(7.9)	-	0.00	4.53	10
5	5	0.0887(28)	0.07(20)	6.1(3.3)	41.5(19.0)	93.3(44.0)	0.00	5.04	8

Bayesian fit works but doesn't describe data:



In the case at hand: World lattice data for $B_s \to K\ell\nu$ requires further scrutiny; Frequentist fit indicates problem!

Bayesian and frequentist provide complementary information — use both to gain comprehensive understanding of data and fit

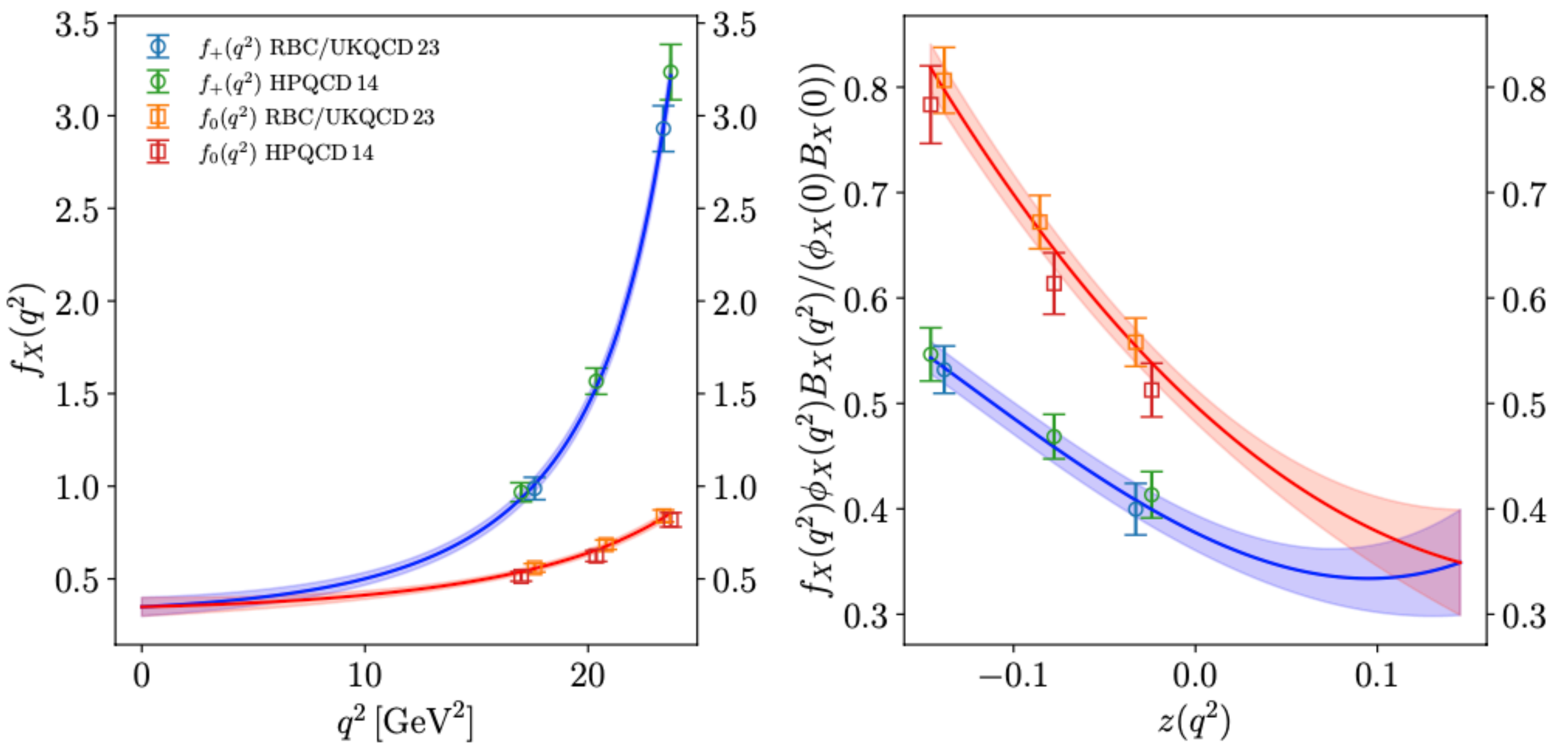
Example 1: $B_s \to K\ell\nu$ — global fit

Let's fit just HPQCD 14 and RBC/UKQCD 23 (FNAL/MILC 19 excluded):

Frequentist works:

K_+	K_0	$a_{+,0}$	$a_{+,1}$	$a_{+,2}$	$a_{+,3}$	$a_{+,4}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
3	3	0.01095(48)	-0.0202(43)	0.079(26)	-	-	0.14	1.61	6
5	5	0.0129(22)	0.11(13)	2.5(2.5)	16.1(20.3)	34.9(64.6)	0.25	1.38	2

K_+	K_0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
3	3	0.0752(24)	-0.250(24)	0.61(11)	-	-	0.14	1.61	6
5	5	0.0830(85)	0.29(55)	11.7(11.9)	92.9(115.0)	270.2(401.8)	0.25	1.38	2



Bayesian fit works:

K_+	K_0	$f(q^2=0)$	$R^{\rm impr}_{B_s\to K}$	$R_{B_s\to K}$	$\frac{\Gamma^\tau}{\lvert V_{ub}\rvert^2}\,[\frac{1}{\rm ps}]$	$\frac{\Gamma^\mu}{\lvert V_{ub}\rvert^2}\,[\frac{1}{\rm ps}]$	$V^{\rm low}_{\rm CKM}$	$V^{\rm high}_{\rm CKM}$	$V^{\rm full}_{\rm CKM}$
3	3	0.341(45)	1.650(34)	0.635(40)	5.11(47)	8.1(1.2)	0.00277(38)	0.00346(33)	0.00316(33)
5	5	0.349(50)	1.689(41)	0.642(47)	5.02(46)	7.9(1.2)	0.00279(41)	0.00354(35)	0.00322(35)
10	10	0.327(44)	1.687(40)	0.661(42)	4.87(43)	7.4(1.0)	0.00294(40)	0.00361(35)	0.00332(35)

K_+	K_0	$I[\mathcal{A}^\tau_{\rm FB}]\,[\frac{1}{\rm ps}]$	$I[\mathcal{A}^\mu_{\rm FB}]\,[\frac{1}{\rm ps}]$	$\bar{\mathcal{A}}^\tau_{\rm FB}$	$\bar{\mathcal{A}}^\mu_{\rm FB}$	$I[\mathcal{A}^\tau_{\rm pol}]\,[\frac{1}{\rm ps}]$	$I[\mathcal{A}^\mu_{\rm pol}]\,[\frac{1}{\rm ps}]$	$\bar{\mathcal{A}}^\tau_{\rm pol}$	$\bar{\mathcal{A}}^\mu_{\rm pol}$
3	3	1.47(15)	0.058(13)	0.2875(36)	0.00703(65)	0.34(12)	8.0(1.1)	0.066(22)	0.9805(17)
5	5	1.45(14)	0.059(14)	0.2891(37)	0.00744(79)	0.22(13)	7.7(1.2)	0.042(26)	0.9792(21)
10	10	1.40(13)	0.053(12)	0.2876(34)	0.00711(70)	0.22(13)	7.3(1.0)	0.045(25)	0.9801(19)

Example 2: $B \to D^{*} l \nu$

Combined fit of FNAL/MILC 21, HPQCD 23 and JLQCD 23 with 2 kinematic constraints

[FNAL/MILC Eur.Phys.J.C 82 (2022), HPQCD 23 arXiv:2304.03137, JLQCD 23 arXiv:2306.05657]

Results for F_2 — Frequentist

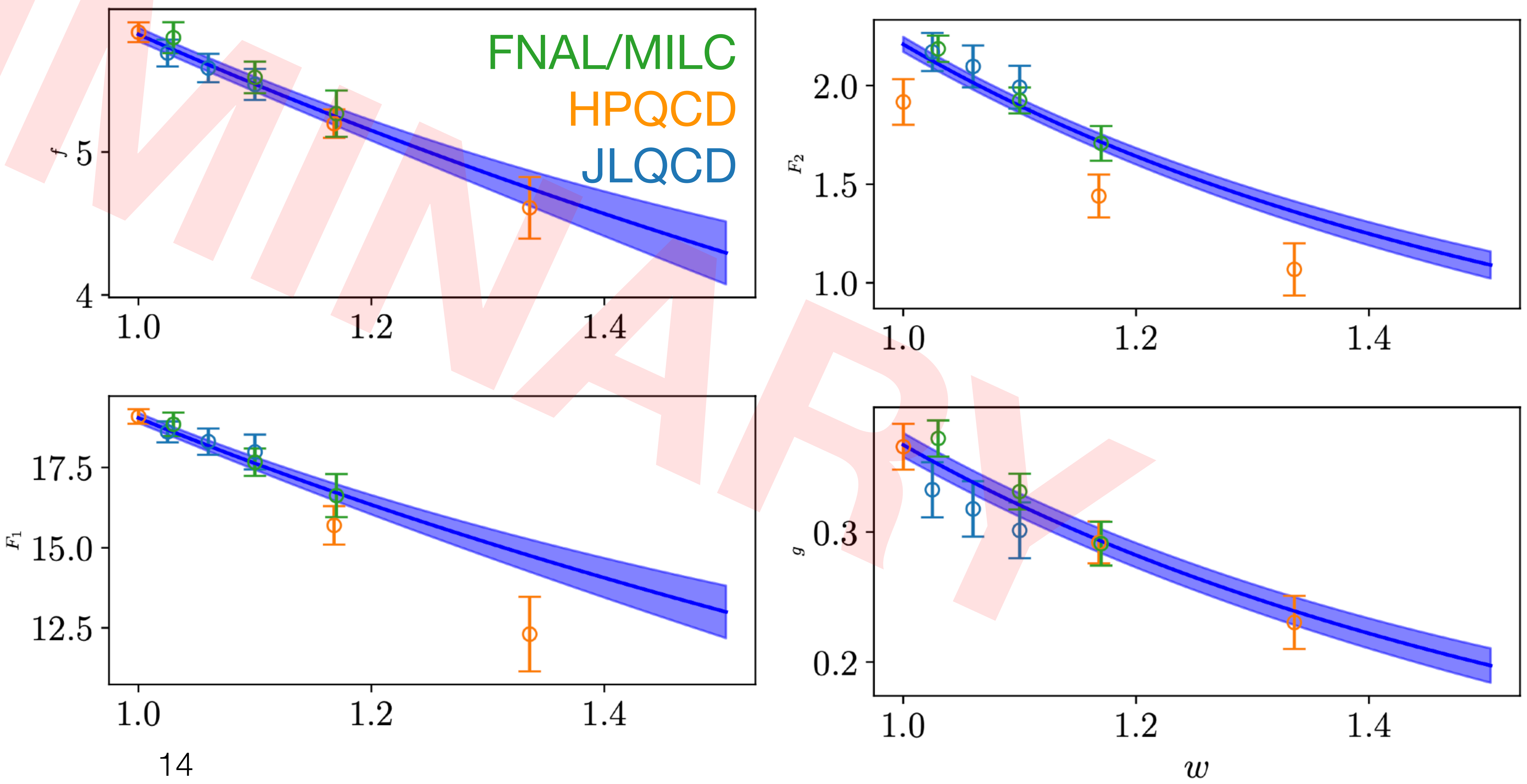
K_f	K_{F_1}	K_{F_2}	K_g	$a_{F_2,0}$	$a_{F_2,1}$	$a_{F_2,2}$	$a_{F_2,3}$	$a_{F_2,4}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
2	2	2	2	0.04908(83)	-0.173(26)	-	-	-	0.43	1.02	30
3	3	3	3	0.04901(84)	-0.195(42)	-0.5(1.2)		-	0.44	1.02	26
4	4	4	4	0.04880(85)	-0.212(64)	1.2(4.0)	-53.2(65.0)	-	0.32	1.12	22

Results for F_2 — Bayesian

K_f	K_{F_1}	K_{F_2}	K_g	$a_{F_2,0}$	$a_{F_2,1}$	$a_{F_2,2}$	$a_{F_2,3}$	$a_{F_2,4}$
5	5	5	5	0.04897(84)	-0.200(38)	-0.05(41)	-0.01(45)	0.01(41)
2	2	2	2	0.04909(81)	-0.174(25)	-	-	-
3	3	3	3	0.04905(87)	-0.201(38)	-0.06(54)	-	-
4	4	4	4	0.04895(80)	-0.200(36)	-0.05(49)	0.03(51)	-
5	5	5	5	0.04897(84)	-0.200(38)	-0.05(41)	-0.01(45)	0.01(41)

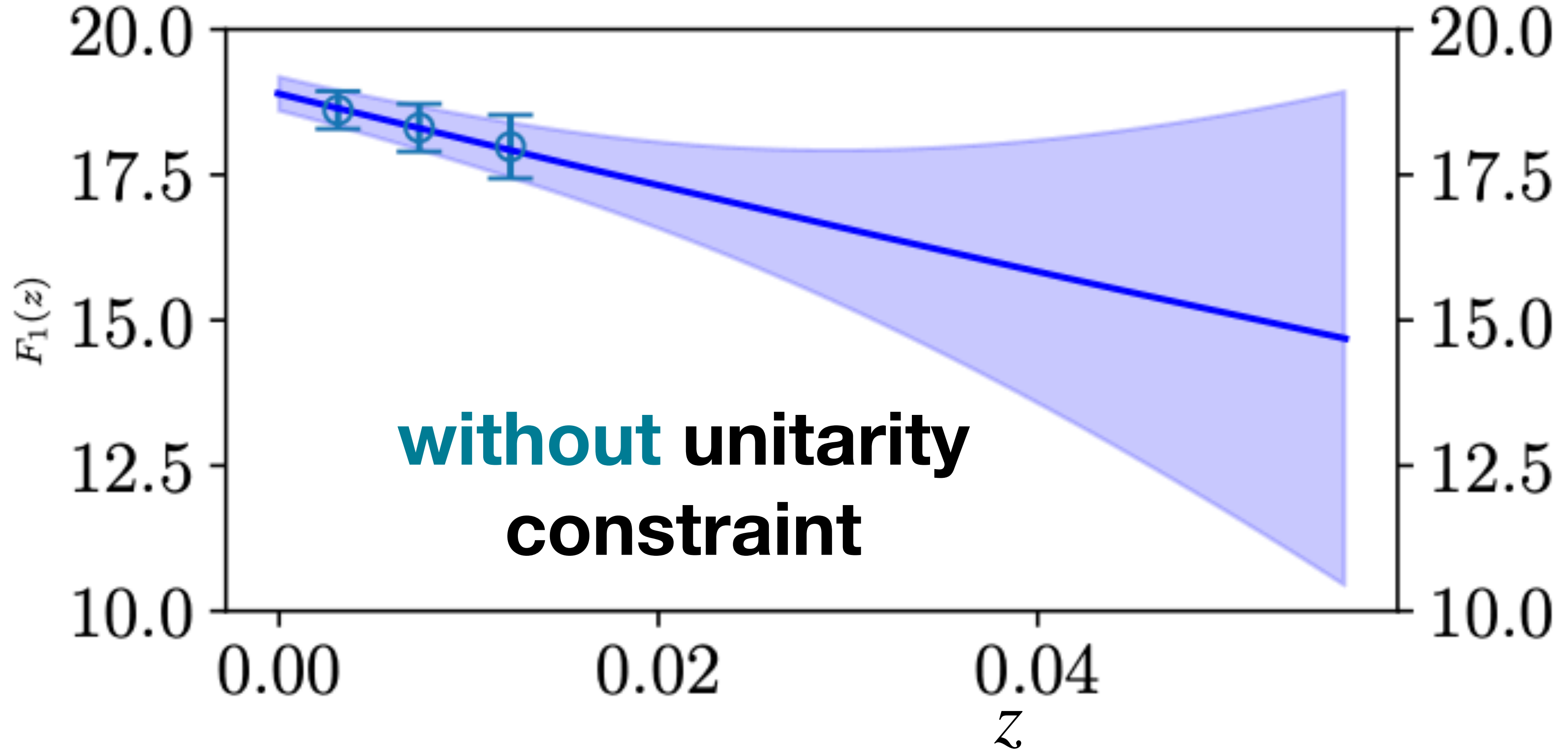
$(K_f, K_{F_1}, K_{F_2}, K_g) = (\mathbf{5,5,5,5})$

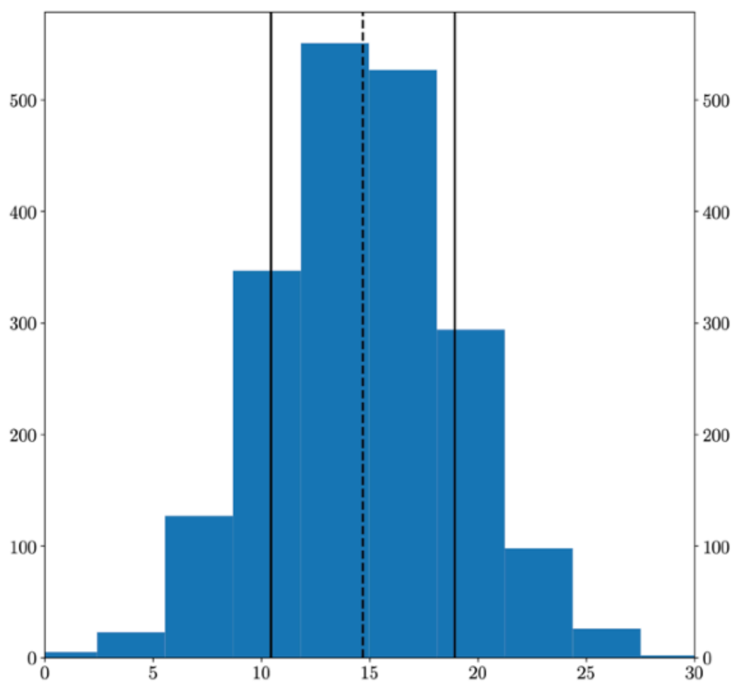
- Frequentist fit to all data acceptable (however: very correlated data)
- Bayesian fit shows convergence
- Unitarity constraint constrains error and central value of higher-order coefficients



Example 2: $B \to D^* l\nu$

Bayesian fit to JLQCD 23 data 2306.05657





$F_1(q^2 = 0)$

Combined fit over 3 form factors including two kinematic constraints

Example 2: $B \to D^{*} l \nu$

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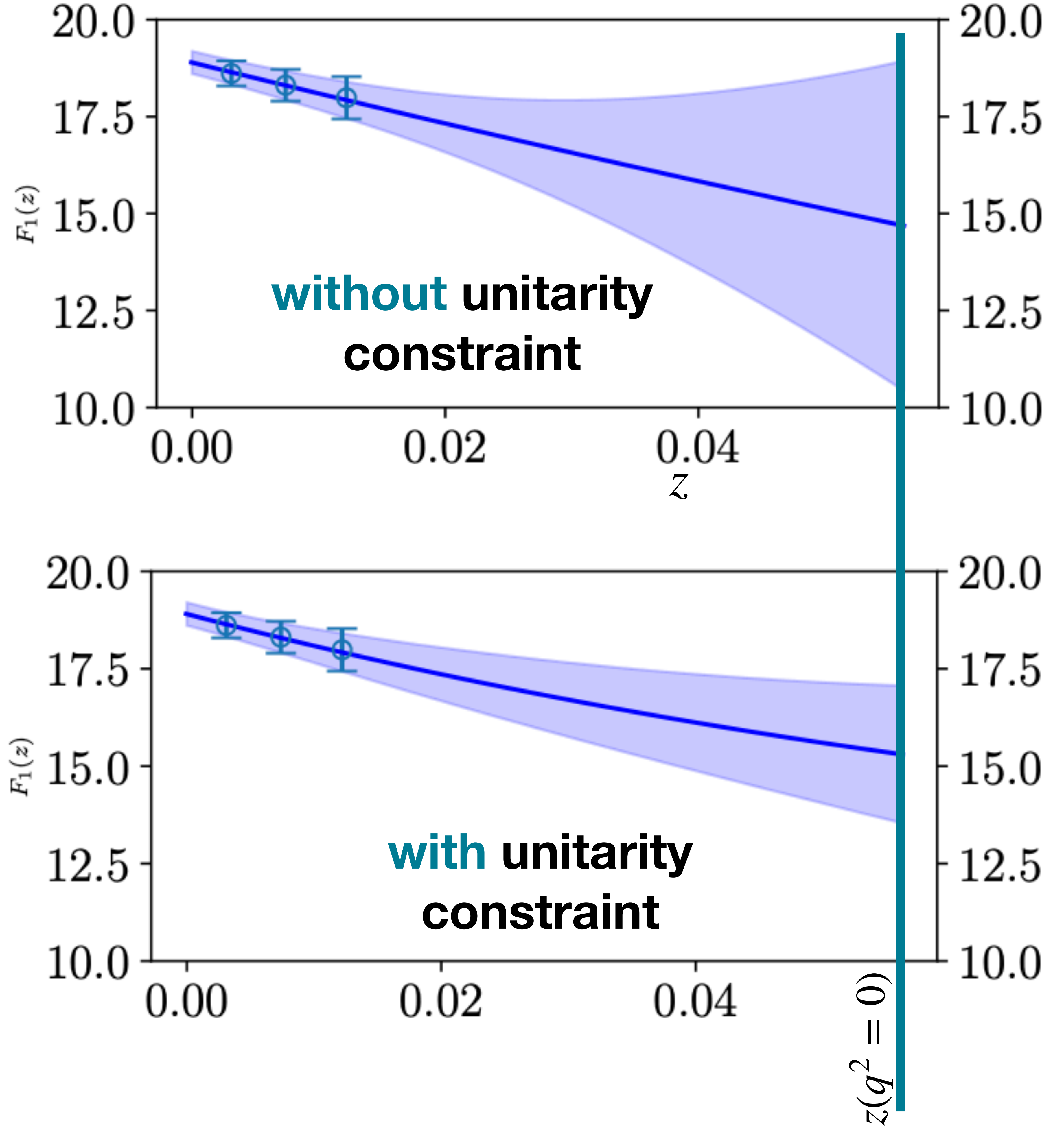
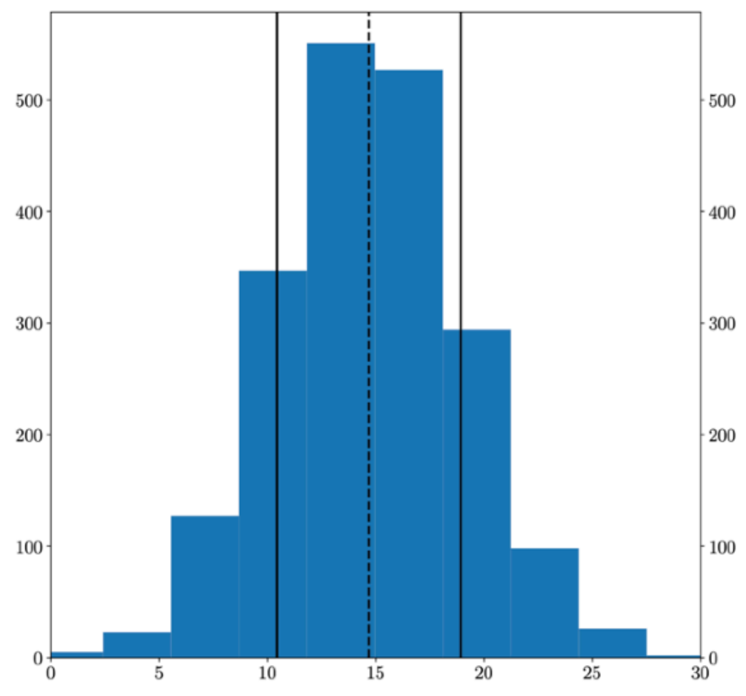
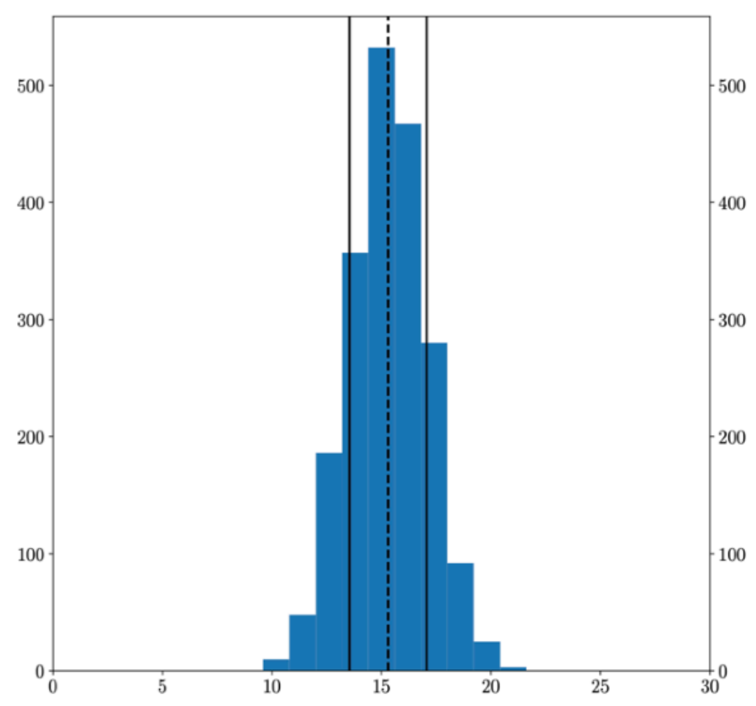




$F_1(q^2 = 0)$

Combined fit over 3 form factors including two kinematic constraints



$F_1(q^2 = 0)$

Unitarity constrains leads to substantial reduction in stat. error

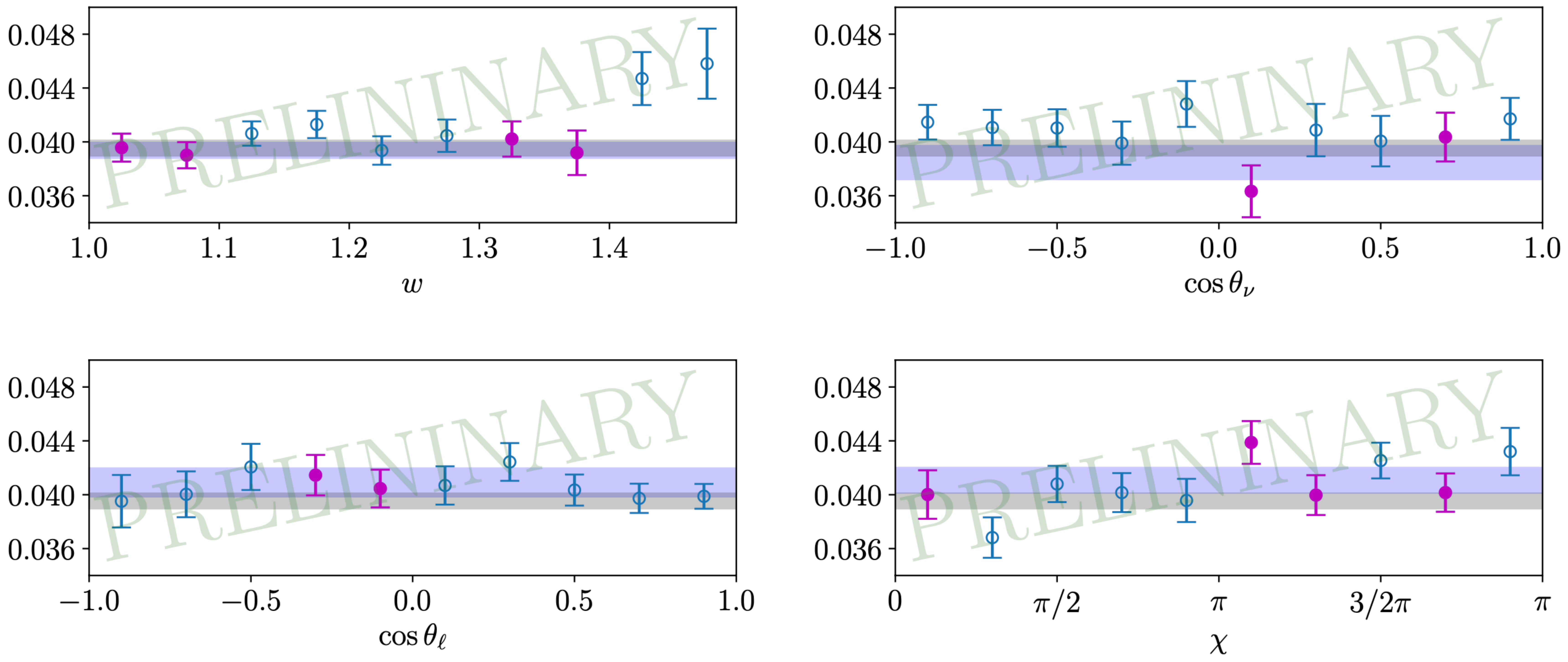
Example 2: $B \to D^* l\nu$

Combined fit of FNAL/MILC 21, HPQCD 23 and JLQCD 23 with 2 kinematic constraints

[FNAL/MILC Eur.Phys.J.C 82 (2022), HPQCD 23 arXiv:2304.03137, JLQCD 23 arXiv:2306.05657]

Experimental input — I use Belle 2023 **[Belle PRD 108 (2023)]**

diff. decay rate $d\Gamma/dx$ $(x \in \{w,\ \cos\theta_\nu,\ \cos\theta_\ell, \chi\})$, each with 10 bins



- $V_{cb} = \sqrt{\dfrac{d\Gamma/dx\ _{\text{exp}}}{d\bar{\Gamma}/dx\ _{\text{th}}}}$
- lattice data with Bayesian $K = 5$-BGL
- integrate ff for each bin
  → CKM result for each bin
- $N_{\text{lattice}} < N_{\text{bin}}$ — singular correlation matrix for $d\Gamma/dx\ _{\text{lat}}$; idea:
  → sample over fits over subset of bin results (e.g. magenta points)
  → systematic?
- work in progress

Example 3: $B \to \pi l \nu$ and $B_s \to K l \nu$

Idea: simultaneous Bayesian fit over both channels subject to combined unitarity constraint

dispersion relation

$$\chi_X \geq \int_{t_{B\pi}}^{\infty} dt \frac{W_{B\pi}(t) \left| f_X^{B\to\pi}(t) \right|^2}{(t-q^2)^{n_X}} + \int_{t_{B_sK}}^{\infty} dt \frac{W_{B_sK}(t) \left| f_X^{B_s\to K}(t) \right|^2}{(t-q^2)^{n_X}} \quad \to \quad 1 \geq \left| \vec{a}_X^{B\to\pi} \right|^2 + \left| \vec{a}_X^{B_s\to K} \right|^2 \qquad (X = +, 0)$$

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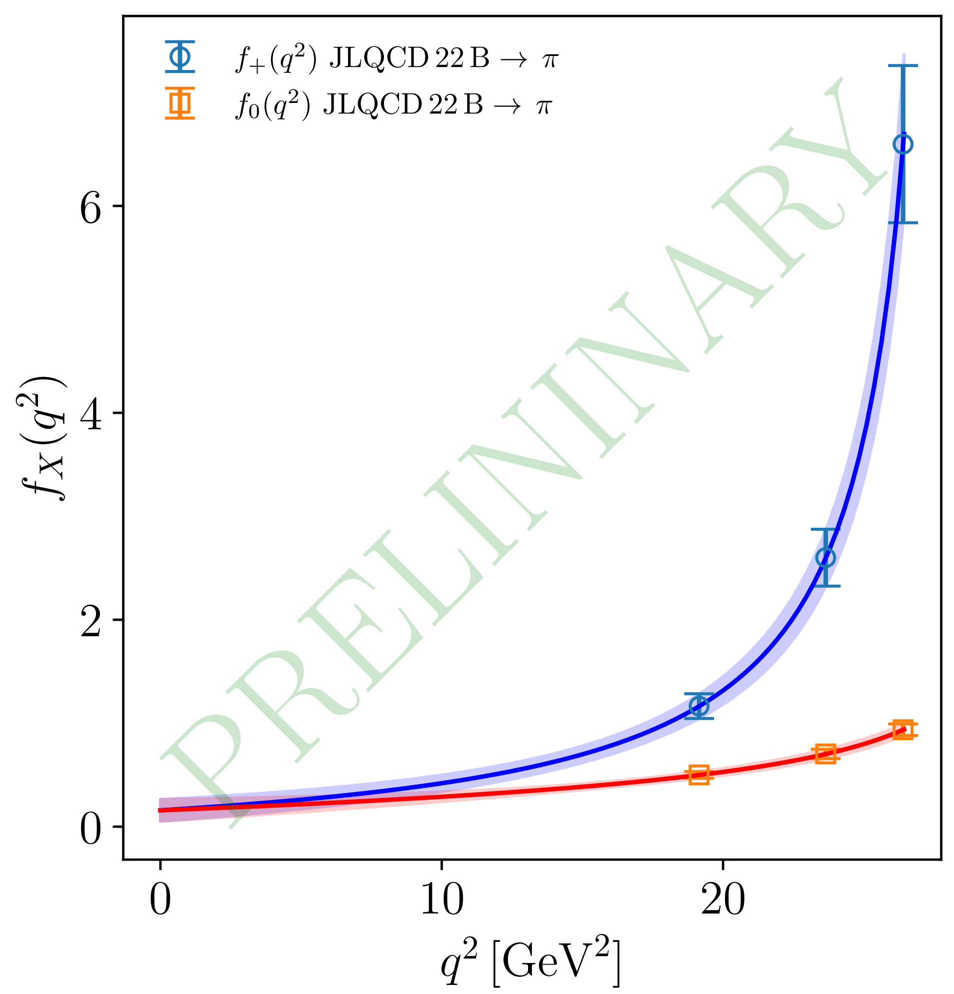
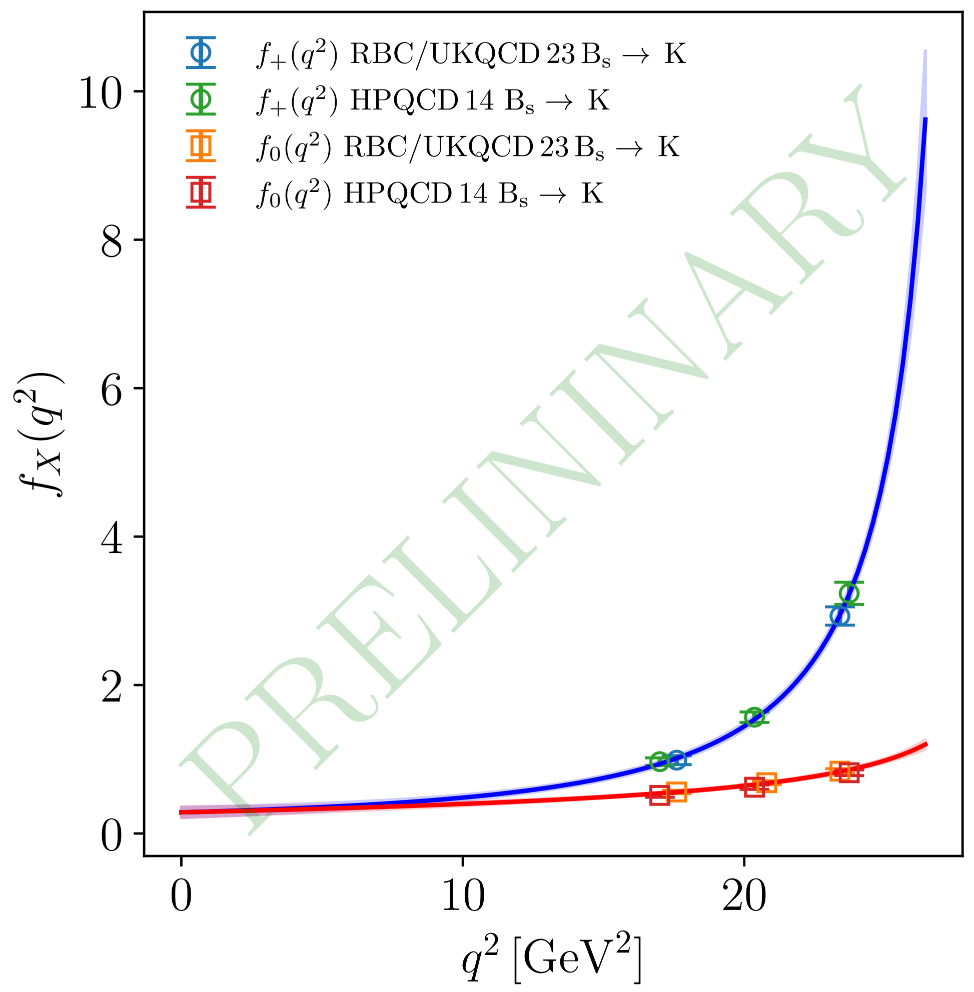
Lattice data: $B_s \to K$ [HPQCD 14 PRD 90 (2014), RBC/UKQCD 23 PRD 107 (2023)]

$B \to \pi$ [JLQCD 22 PRD 106 (2022)]





- we find simultaneous unitarity constraint stronger than individual
- effect will depend on channel
- in case at hand coefficients have noticeably smaller error
- work in progress

Code: BFF

Flynn, AJ, Tsang, arXiv:2303.11285

Python3 available via github/Zenodo

https://github.com/andreasjuettner/BFF   **https://zenodo.org/record/7799543#.ZEezTy8Ro80**

```
        }
#####################################################################################
# specify input for BGL fit
#####################################################################################
input_dict = {
         'decay':        'Btopi',
         'Mi':           pc.mBphys,      # initial-state mass
         'Mo':           pc.mpiphys,     # final-state mass
         'sigma':        .5,             # sigma for prior in algorithm
         'Kp':           4,              # target Kp (BGL truncation) - can be changed later
         'K0':           4,              # target K0 (BGL truncation) - can be changed later
         'tstar':        '29.349570696829012', # value of t*
         't0':           'self.tstar - np.sqrt(self.tstar*(self.tstar-self.tm))', # definition of t0
         'chip':         pc.chip_Btopi,  # susceptibility fp
         'chi0':         pc.chi0_Btopi,  # susceptibility f0
         'mpolep':       [pc.mBstar],    # fplus pole
         'mpole0':       [],             # fzero pole (no pole for BstoK)
         'N'     :       N,              # number of desired samples
         'outer_p':      [3./2,'48*np.pi',3,2], # specs for outer function fp
         'outer_0':      [3./2,'16*np.pi/(self.tp*self.tm)',1,1], # specs for outer function f0
         'seed':         123,            # RNG seed
```

```
input_data = {
          'RBCUKQCD 23 lat':
          {
           'data type':   'ff',
           'label'        :'RBC/UKQCD 23',
           'Np':          2,
           'N0':          3,
           'qsqp':        np.array([17.60,23.40]),
           'qsq0':        np.array([17.60,20.80,23.40]),
           'fp':          fparray,
           'f0':          f0array,
           'Cff':         cov_array
          }
}
```

Summary

Framework for truncation- and model-independent form-factor fitting
combining Frequentist and Bayesian statistics

Implemented and demonstrated for

- $P \to P$ and $P \to V$ transitions
- Framework imposes unitarity constraint with meaningful statistical interpretation
- Unitarity constraint acts as regulator for higher-order coefficients
- Results converge to stable and truncation independent values
- Combined fits over data sets straight-forwardly implemented
  with simultaneous unitarity constraint increasing constraining power
- Primary output are BGL-coefficients (central values and standard deviation or samples with distributions / covariance)

other

Bayesian form-factor fit — Algorithm

Flynn, AJ, Tsang, arXiv:2303.11285

In practice high-dimensional → low probability of drawing random number compatible with constraint

$$\pi_{\mathbf{a}}(\mathbf{a}\ \ \mathbf{f}_p, C_{\mathbf{f}_p})\pi_{\mathbf{a}}(\mathbf{a}\ \ \mathbf{a}_p, M) \propto \theta(\mathbf{a})\exp\left(-\frac{1}{2}(\mathbf{f}_p - Z\mathbf{a})^T C_{\mathbf{f}_p}^{-1}(\mathbf{f}_p - Z\mathbf{a}) - \frac{1}{2}\mathbf{a}^T M/\sigma^2\,\mathbf{a}\right)$$

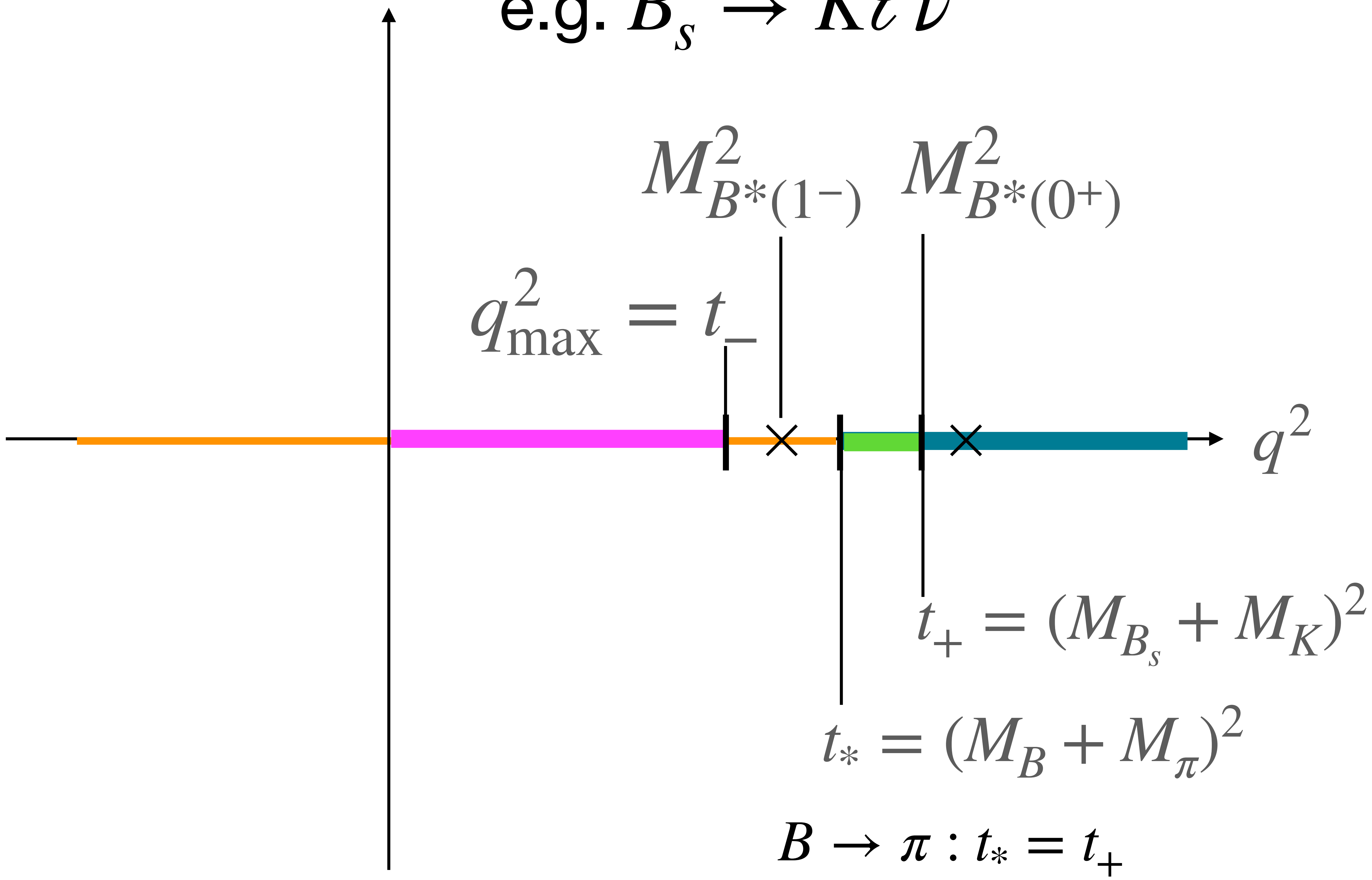
- Add ‘technical’ prior
- Choose M such that $\mathbf{a}^T M\mathbf{a} \leq 1$ in presence of kinematical constraint
- Correct towards ‘flat unitarity-only prior’ with accept-reject step with probability

$$p \leq \frac{\exp(-1/\sigma^2)}{\exp(-\mathbf{a}^T M\mathbf{a}/2\sigma^2)}$$

FINAL RESULT INDEPENDENT OF *TECHNICAL* PRIOR

An aside: correct BGL unitarity constraint for for $B_s \to K\ell\nu$

e.g. $B_s \to K\ell\nu$



$B \to \pi : t_* = t_+$

$B_s \to K : t_* < t_+$

An aside: correct BGL unitarity constraint for for $B_s \to K\ell\nu$

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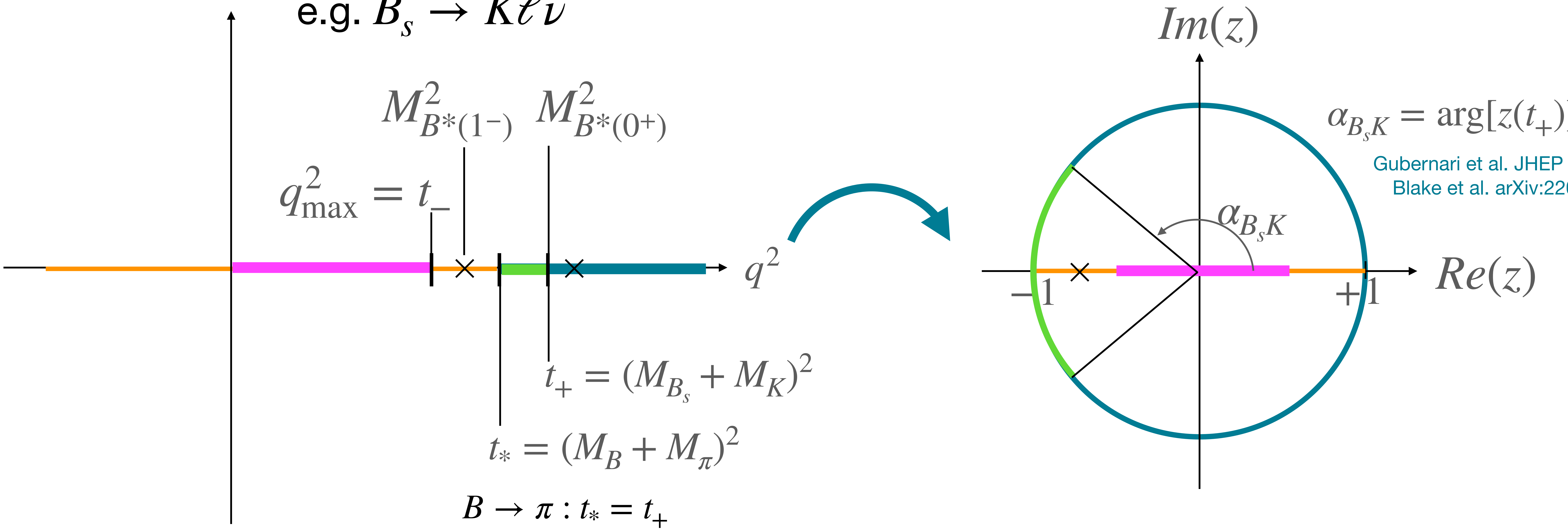

$B \to \pi : t_* = t_+$

$B_s \to K : t_* < t_+$

$\alpha_{B_sK} = \arg[z(t_+)]$

Gubernari et al. JHEP 2021,2022,
Blake et al. arXiv:2205.06041

Okubo, PRD 3, 2807 (1971), PRD 4, 725 (1971).
Okubo, Shih, PRD 4, 2020 (1971).
Boyd, Grinstein, Lebed, PLB 353, 306 (1995).
NPB461, 493 (1996). PRD 56, 6895 (1997).

An aside: correct BGL unitarity constraint for for $B_s \to K\ell\nu$

$$f_X(q_i^2) = \frac{1}{B_X(q_i^2)\phi_X(q_i^2, t_0)} \sum_{n=0}^{K_X-1} a_{X,n} z(q_i^2)^n \qquad X = +,0$$

Boyd, Grinstein, Lebed, PRL 74 (1995)

unitarity constraint

e.g. $$\frac{1}{\pi \chi_J^T(q^2)} \int_{t_+}^{\infty} dt \frac{W(t) \left| f_+(t) \right|^2}{(t-q^2)^3} \leq 1 \longrightarrow \frac{1}{2\pi i} \oint_C \frac{dz}{z} \theta_{B_sK} \left| B_X(q^2)\phi_X(q^2, t_0) f_X(q^2) \right|^2 \leq 1$$

An aside: correct BGL unitarity constraint for for $B_s \to K\ell\nu$

$$f_X(q_i^2) = \frac{1}{B_X(q_i^2)\phi_X(q_i^2, t_0)} \sum_{n=0}^{K_X-1} a_{X,n} z(q_i^2)^n \qquad X = +,0$$

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unitarity constraint

e.g. $$\frac{1}{\pi\chi_J^T(q^2)} \int_{t_+}^{\infty} dt \frac{W(t) \ f_+(t)^{\ 2}}{(t-q^2)^3} \le 1 \longrightarrow \frac{1}{2\pi i} \oint_C \frac{dz}{z} \theta_{B_sK} \ B_X(q^2)\phi_X(q^2, t_0) f_X(q^2)^{\ 2} \le 1$$

$$a_{X,i} \langle z^i \ \ z^j \rangle a_{X,j} \le 1$$

$$\langle z^i | z^j \rangle_\alpha = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} d\phi (z^i)^* z^j |_{z=e^{i\phi}} = \begin{cases} \dfrac{\sin(\alpha(i-j))}{\pi(i-j)} & i \neq j \\ \dfrac{\alpha}{\pi} & i = j \end{cases}$$

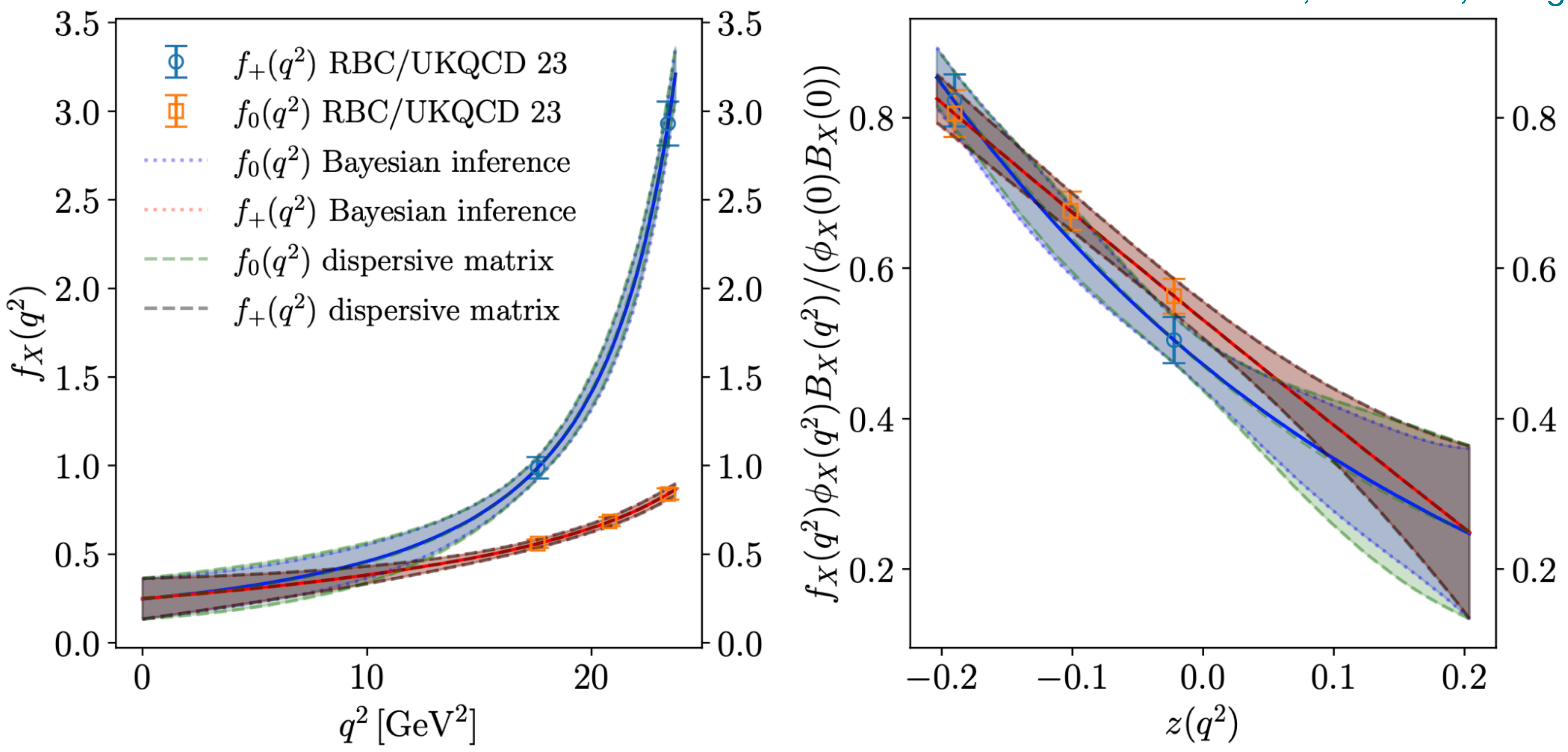
Flynn, AJ, Tsang, arXiv:2303.11285

modified unitarity constraint for channels where relevant threshold t_* not the lowest

Gubernari et al. JHEP 2021,2022,
Blake et al. arXiv:2205.06041

Relation to dispersive-matrix method?

Di Carlo, Martinelli, Naviglio et al. PRD 104 (2021) 054502



Bayesian inference and dispersive-matrix method produce very similar results.

Practical advantages of Bayesian inference:

- kinematical constraints exactly and cleanly implemented
- simultaneous fit over various (correlated) data sets possible
- clean statistical underpinning of unitarity constraint
- produces set of coefficients for further use
- simultaneous unitarity constraint