Flavor Symmetries and GUTs

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Basics of Model Building

- What is the Lagrangian of Nature?
- Rules:
    - Gauge symmetries of the Lagrangian
    - Representations of fermions and scalars under symmetries
    - Pattern of spontaneous symmetry breaking
- Assumptions:
    - Poincare invariance
    - QFT
- SM tested to high accuracy; still many deficiencies
- Goals: **Address deficiencies of SM with (hidden) Symmetries**

The Plan

- **Part I: Primer - Standard Model and its Deficiencies**
  - Symmetries, Particle content, and Lagrangian
  - Why Go beyond the SM?
- **Part II: Flavor Symmetries**
  - Flavor Puzzle
    - Problem of fermion masses and mixing
    - Neutrino mass generation
  - Froggatt-Nielsen Mechanism
  - Non-Abelian Discrete Symmetries
  - CP Violation ⇔ outer automorphism
- **Part III: GUT Symmetries** (Michael Ratz)
  - Motivations for GUTs
  - GUTs in 4D
  - Problems of GUTs in 4D
  - Orbifold GUTs
  - Modular Flavor Symmetries

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  - GUTs in 4D
  - Problems of GUTs in 4D
  - Orbifold GUTs
  - Modular Flavor Symmetries

Tools of symmetries can be applied to problems beyond Flavor and GUTs.

References

- **Flavor Symmetries:**
  - Ishimori, Kobayashi, Ohki, Shimizu, 1003.3552
  - P. Ramond, Group Theory: A Physicist's Survey (2010)
- **GUT Symmetries:**
  - Cheng and Li
  - R. Slansky, Group Theory for Unified Model Building, Phys. Rep. 79 (1981)
  - R. Mohapatra, TASI Lecture Notes on SUSY GUTs (1996)
  - S. Raby, Supersymmetric Grand Unified Theories, From Quarks to Strings via SUSY GUTs

Standard Model

- Gauge Symmetries

$$G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$$

- Three Fermion Generations

	$Q^I_{L_i}$	$U^I_{R_i}$	$D^I_{R_i}$	$\ell^I_{L_i}$	$E^I_{R_i}$
$SU(3)_C$	3	3	3	1	1
$SU(2)_L$	2	1	1	2	1
$U(1)_Y$	1/6	2/3	-1/3	-1/2	-1

(I: gauge interaction eigenstates; flavor index $i = 1,2,3$)

Group Work: Warm Up

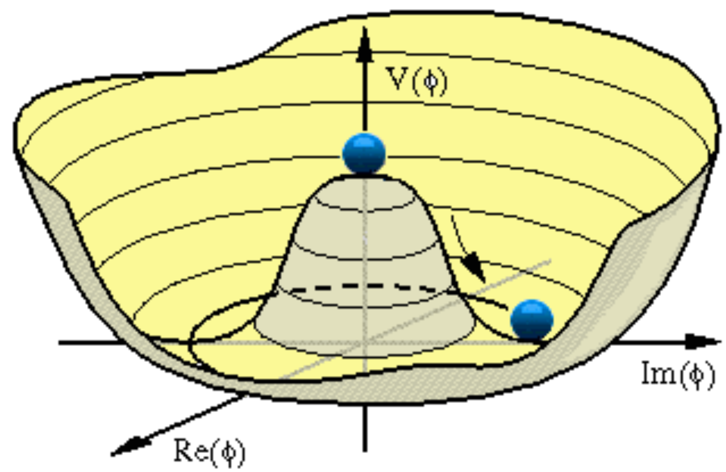
How many degrees of freedom are there for one generation of fermions in the Standard Model?

Standard Model

- Higgs Sector and Spontaneous Electroweak Symmetry Breaking

$$\phi(1,2)_{+1/2} \quad , \quad \langle \phi \rangle = \begin{pmatrix} 0 \\ \upsilon/\sqrt{2} \end{pmatrix}$$

$$G_{SM} \rightarrow SU(3)_C \times U(1)_{EM}$$



Standard Model - Kinetic Terms

- Standard Model Lagrangian

$$\mathscr{L}_{SM} = \mathscr{L}_{kinetic} + \mathscr{L}_{Higgs} + \mathscr{L}_{Yukawa}$$

Kinetic Terms: covariant derivative

$$D^\mu = \partial^\mu + ig_s G_a^\mu L_a + igW_b^\mu T_b + ig' B^\mu Y$$

G_a^μ : 8 gluons

W_b^μ : 3 weak gauge bosons

B^μ : 1 hypercharge boson

L_a : generators of $SU(3)$

T_b : generators of $SU(2)$

Standard Model - Kinetic Terms

- Examples:

$$\mathcal{L}_{kinetic}(Q_L) = i\overline{Q^I_{L_i}}\gamma_\mu\left(\partial^\mu + \frac{i}{2}g_s G^\mu_a \lambda_a + \frac{i}{2}g W^\mu_b \tau_b + \frac{i}{6}g' B^\mu\right)Q^I_{L_i}$$

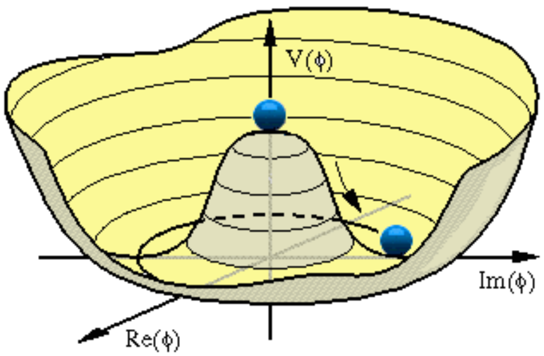
$$\mathcal{L}_{kinetic}(\ell_L) = i\overline{\ell^I_{L_i}}\gamma_\mu\left(\partial^\mu + \frac{i}{2}g W^\mu_b \tau_b - \frac{i}{2}g' B^\mu\right)\ell^I_{L_i}$$

Standard Model - Higgs Sector

Higgs Potential:

$$-\mathscr{L}_{Higgs} = V(\phi) = \mu^2\left(\phi^\dagger\phi\right) + \lambda\left(\phi^\dagger\phi\right)^2$$

- Two parameters: $\mu^2, \lambda \quad \leftrightarrow m_H, \upsilon$



- Vacuum stability $\Rightarrow \lambda > 0$
- Spontaneous symmetry breaking $\Rightarrow \mu^2 < 0$
- W and Z gauge bosons acquire masses; gluons, photons remain massless

Standard Model - Yukawa Sector

Yukawa interactions:

$$-\mathcal{L}^{lepton}_{Yukawa} = Y^e_{ij}\, \overline{\ell^I_{L_i}}\, \phi\, E^I_{R_j} + \text{h.c.}$$

$$-\mathcal{L}^{quark}_{Yukawa} = Y^d_{ij}\, \overline{Q^I_{L_i}}\, \phi\, D^I_{R_j} + Y^u_{ij}\, \overline{Q^I_{L_i}}\, \tilde{\phi}\, U^I_{R_j} + \text{h.c.}$$

Upon electroweak symmetry breaking:

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \Longrightarrow \text{charged lepton and quark masses}$$

Non diagonal Yukawa matrices Y^u and Y^d $\Rightarrow$ mismatch between quark mass eigenstates and weak gauge eigenstates

Standard Model - Counting Parameters

Yukawa Sector has many parameters, but not all physical

Symmetry argument for parameter counting:

Ex. Hydrogen Atom in uniform magnetic field (along $\hat{z}$)

$B = 0 : SO(3)$ symmetry $\Rightarrow$ 3 degenerate energy eigenvalues

$B \neq 0 : SO(3) \rightarrow SO(2)$ symmetry

1 unbroken generator → 2D rotation on xy plane

2 broken generators → allow to align $B \parallel \hat{z}$

$$O_{xz}O_{yz}\left(B_x, B_y, B_z\right) = \left(0, 0, B_z'\right)$$

Standard Model - Accidental Symmetries

Standard Model has the following accidental global symmetries

$$U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau$$

$U(1)_B$: baryon number

$U(1)_e$: L_e lepton number

$U(1)_\mu$: L_μ lepton number

$U(1)_\tau$: L_τ lepton number

Total lepton number

$$L = L_e + L_\mu + L_\tau$$

Standard Model - Counting Parameters

- Gauge theory w/ matter content:
  - Gauge kinetic terms: global symmetries G_f
  - Potential respecting gauge symmetries may break $G_f \to H_f$ (global symmetry of the entire model)
  - Breaking of G_f allow freedom to rotate away unphysical parameters

$$N_{phys} = N_{general} - N_{broken}$$

N_{broken} : number of broken generators

Group Work: Counting Parameters

- General complex $n \times n$ matrices
  - How many real parameters?
  - How many phases?
- For Hermitian Matrices
  - How many real parameters?
  - How many phases?
- For Unitary Matrices
  - How many real parameters?
  - How many phases?

Group Work: Counting Parameters

- General complex $n \times n$ matrices
  - n^2 Real parameters
  - n^2 Phases
- For Hermitian Matrices
  - $n(n+1)/2$ Real parameters
  - $n(n-1)/2$ Phases
- For Unitary Matrices
  - $n(n-1)/2$ Real parameters
  - $n(n+1)/2$ Phases

Standard Model - Counting Parameters

- Applying to Standard Model Quark Sector
- Gauge Kinetic terms:

$$G_f = U(3)_Q \times U(3)_U \times U(3)_D$$

  - $U(3)$: 9 generators (3 real, 6 imaginary)
  - Total number of generators of G_f = 27
- Yukawa Interactions, Y^u, Y^d : complex 3×3 matrices

Group Work: Counting Parameters

- How many general parameters do Y^u, Y^d have?
- What is the global symmetry of the entire model (quark sector only)?
- How many broken generators?
- How many physical parameters?

Group Work: Counting Parameters

- General parameters in Y^u, Y^d = 36
- Global symmetry: $U(3)_Q \times U(3)_U \times U(3)_D \to U(1)_B$

$U(1)_B$: 1 generator (1 phase)

$$N_{broken} = 27 - 1 = 26$$

$$N_{physical} = N_{general} - N_{broken} = 36 - 26 = 10$$

$$N^r_{physical} = 18 - 9 = 9$$

$$N^i_{physical} = 18 - 17 = 1$$

Group Work: Counting Parameters

- General parameters in Y^u, Y^d = 36
- Global symmetry: $U(3)_Q \times U(3)_U \times U(3)_D \rightarrow U(1)_B$

$U(1)_B$: 1 generator (1 phase)

$$N_{broken} = 27 - 1 = 26$$

$$N_{physical} = N_{general} - N_{broken} = 36 - 26 = 10$$

$$N^r_{physical} = 18 - 9 = 9$$ ⇒ 6 masses, 3 angles

$$N^i_{physical} = 18 - 17 = 1$$ ⇒ 1 CP phase

Standard Model - Discrete Symmetries

- Any local Lorentz invariant QFT respect CPT
- CPT invariance: T violation ⇔ CP violation
- In SM: C, P maximally violated
  - Violation independent of parameters
- SM can violate CP, depending on values of Yukawa couplings

$$Y_{ij}\,\overline{\psi_{L_i}}\,\phi\,\psi_{R_j} + Y^*_{ij}\,\overline{\psi_{R_j}}\,\phi^\dagger\,\psi_{L_i}$$

Under CP transformation: $\mathcal{O} \to \mathcal{O}^\dagger, \quad c \to c$

$\overline{\psi_{L_i}}\,\phi\psi_{R_j} \;\to\; \overline{\psi_{R_j}}\,\phi^\dagger\psi_{L_i}\,, \qquad Y_{ij}, Y^*_{ij}$ unchanged

- CP invariant?

Standard Model - Discrete Symmetries

- Any local Lorentz invariant QFT respect CPT
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- In SM: C, P maximally violated
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- CP invariant if $Y_{ij} = Y^*_{ij}$

Standard Model - Discrete Symmetries

- SM can violate CP, depending on values of Yukawa couplings

$$Y_{ij}\,\overline{\psi_{L_i}}\,\phi\,\psi_{R_j} \;+\; Y_{ij}^{*}\,\overline{\psi_{R_j}}\,\phi^{\dagger}\,\psi_{L_i}$$

  - CP invariant if $Y_{ij} = Y_{ij}^{*}$

- 1 CP phase in quark sector
- More precisely, CP is violated in the SM quark sector, iff

$$\Im\left(\det\left[Y^{d}Y^{d\dagger},\, Y^{u}Y^{u\dagger}\right]\right) \neq 0$$

Standard Model – CKM Matrix

- Most measurements done in mass basis: Higgs acquiring VEV

$$\Re(\phi^0) \to \left(v + H^0\right)/\sqrt{2}$$

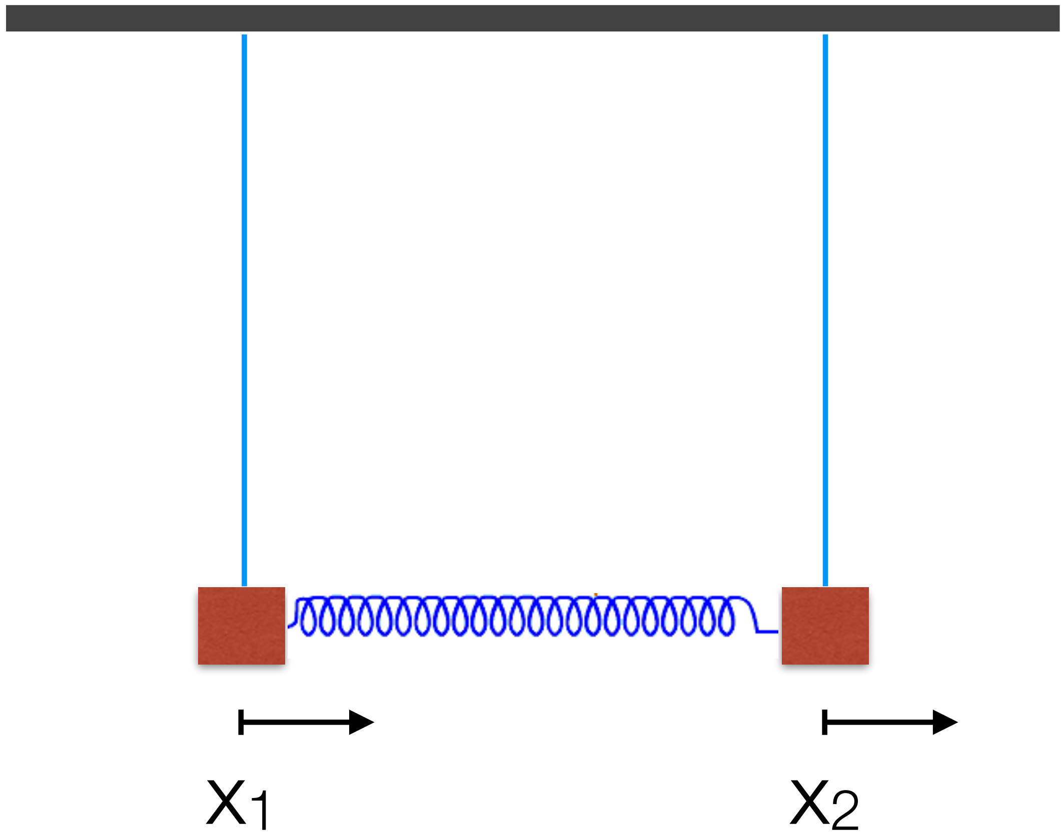
$$Q_{L_i}^I = \begin{pmatrix} U_{L_i}^I \\ D_{L_i}^I \end{pmatrix}$$

Yukawa interactions → mass terms

$$-\mathscr{L}_M^q = \left(M_d\right)_{ij} \overline{D_{L_i}^I}\, D_{R_j}^I + \left(M_u\right)_{ij} \overline{U_{L_i}^I}\, U_{R_j}^I + \text{ h.c.}, \quad M_q = \frac{v}{\sqrt{2}} Y^q$$

- Mismatch between weak eigenstates and mass eigenstates

Group Work: Normal Modes of Coupled Pendulums



Standard Model - CKM Matrix

- Interaction $\longrightarrow$ Mass basis: diagonal mass matrices

$$-\mathscr{L}_M^q = \overline{D_{L_i}^I}\left(M_d\right)_{ij} D_{R_j}^I + \overline{U_{L_i}^I}\left(M_u\right)_{ij} U_{R_j}^I + \text{h.c.}$$

- Diagonalization: bi-unitary transformation

$$V_{u_L} M_u V_{u_R}^\dagger = M_u^{\text{diag}}, \quad V_{d_L} M_d V_{d_R}^\dagger = M_d^{\text{diag}}$$

$$q_{L_i} = \left(V_{q_L}\right)_{ij} q_{L_j}^I, \quad q_{R_i} = \left(V_{q_R}\right)_{ij} q_{R_j}^I, \quad q = u, d$$

Standard Model - CKM Matrix

- Weak Charged Current Interactions:

$$q_{L_i} = \left(V_{q_L}\right)_{ij} q^I_{L_j}, \quad q_{R_i} = \left(V_{q_R}\right)_{ij} q^I_{R_j}, \quad q = u, d$$

$$-\mathscr{L}^q_{W^\pm} = \frac{g}{\sqrt{2}} \overline{u^I_{L_i}} \gamma^\mu \, d^I_{L_i} W^+_\mu \; + \text{h.c.}$$

$$= \frac{g}{\sqrt{2}} \overline{u_{L_i}} \gamma^\mu \left(V_{u_L} V^\dagger_{d_L}\right)_{ij} d_{L_j} W^+_\mu \; + \text{h.c.}$$

Cabibbo-Kobayashi-Maskawa (CKM) Matrix

$$V = V_{u_L} V^\dagger_{d_L} \qquad , \quad V V^\dagger = \mathbb{1}$$

Standard Model - CKM Matrix

Cabibbo-Kobayashi-Maskawa (CKM) Matrix $V = V_{u_L} V_{d_L}^{\dagger}$

- V not diagonal $\Rightarrow$ $W^{\pm}$ gauge bosons couple to mass eigenstates of quarks of different generations
- SM: Only flavor-changing quark interactions
- Elements of CKM matrix:

$$V = \begin{pmatrix} V_{ud} & V_{cd} & V_{td} \\ V_{us} & V_{cs} & V_{ts} \\ V_{ub} & V_{cb} & V_{tb} \end{pmatrix}$$

Standard Model - CKM Matrix

Parametrization not unique:

PDG convention

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

Standard Model - CKM Matrix

PDG convention

$$V_{CKM} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} =$$

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}$$

Standard Model - CKM Matrix

Parametrization not unique:

Wolfstein Rapametrization

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

$$\lambda \sim 0.22$$

Standard Model - CKM Matrix

Different parametrizations: freedom of phase rotation

Parametrization independent measure for CPV

Jarlskog invariant

$$\Im\left(V_{ij}V_{kl}V_{il}^{*}V_{kj}^{*}\right) = J_{CKM}\sum_{m,\,n=1}^{3}\epsilon_{ikm}\epsilon_{jln}, \quad (i, j, k, l = 1, 2, 3)$$

In terms of explicit parametrizations given above

$$J_{CKM} = c_{12}c_{23}c_{13}^{2}s_{12}s_{23}s_{13}\sin\delta \approx \lambda^{6}A^{2}\eta$$

Group Work: CP Violation

- In terms of flavor parameters, sufficient conditions for CPV in SM quark sector: $\Delta m_{ij}^2 = m_i^2 - m_j^2$

$$\Delta m_{tc}^2 \Delta m_{tu}^2 \Delta m_{cu}^2 \Delta m_{bs}^2 \Delta m_{bd}^2 \Delta m_{sd}^2 J_{CKM} \neq 0$$

Requirements on SM for CPV?

Group Work: CP Violation

- In terms of flavor parameters, sufficient conditions for CPV in SM quark sector: $\Delta m_{ij}^2 = m_i^2 - m_j^2$

$$\Delta m_{tc}^2 \Delta m_{tu}^2 \Delta m_{cu}^2 \Delta m_{bs}^2 \Delta m_{bd}^2 \Delta m_{sd}^2 J_{CKM} \neq 0$$

Requirements on SM for CPV

- Within each quark sector, no mass degeneracy
- None of the mixing angles should be 0 or $\pi/2$
- Phase should be neither 0 or π

Group Work: n Generations

For the CKM-like matrix describing the flavor couplings of n generations of up- and down-type quarks, how many free parameters are there?

How many angles and phases?

Group Work: n Generations

- CKM-like matrix for n generations:

$$2n^2 - n^2 - (2n-1) = (n-1)^2 \text{ free parameters}$$

- A general $n \times n$ orthogonal matrix:

$$\frac{1}{2}n(n-1) \text{ angles describing rotations among } n \text{ dimensions}$$

- Remaining free paraders are phases

$$(n-1)^2 - \frac{1}{2}n(n-1) = \frac{1}{2}(n-1)(n-2)$$

$\Rightarrow$ need at least 3 generations to have CPV in CKM matrix

Standard Model – Unitarity Triangle

- Unitarity Triangle of CKM Matrix: relations among matrix elements

$$\sum_i V_{id} V_{is}^* = 0$$

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0\,.$$

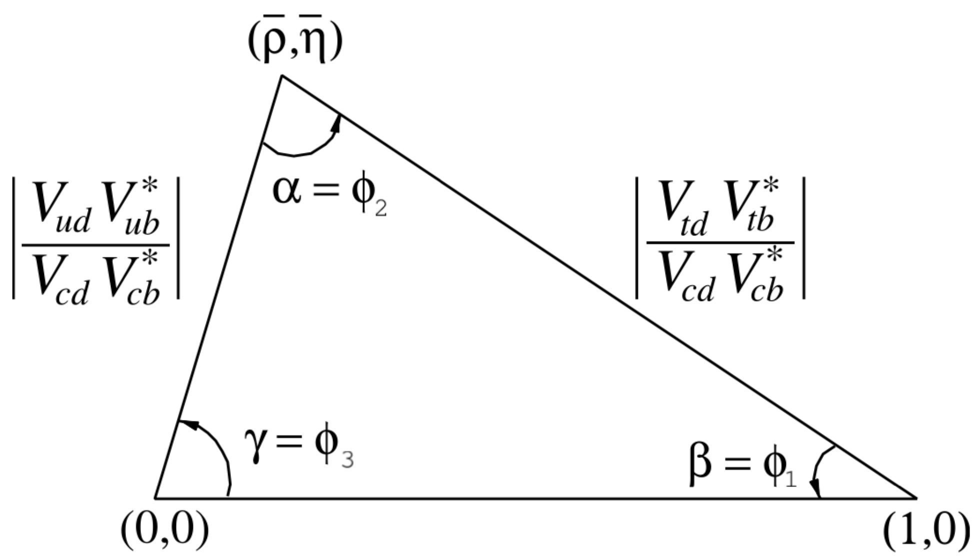
- CKM: all unitarity triangles have same areas = $J_{CKM}/2$

- Angles of unitarity triangles

$$\alpha \equiv \arg\left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right], \quad \beta \equiv \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right], \quad \gamma \equiv \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right]$$

Another common convention

$$\phi_1 = \beta, \quad \phi_2 = \alpha, \quad \phi_3 = \gamma$$



Flavor Changing Neutral Currents

- Flavor changing charged currents: only source of flavor violating interactions in SM
- No fundamental reason why there cannot be FCNCs
- Experimentally, FCNCs are highly suppressed
    - In SM: no tree level FCNCs; generated at loop level
    - **In NP: FCNCs place stringent constraints**
- Distinction:
    - Non-diagonal couplings
    - Diagonal couplings
    - Universal couplings: diagonal in any basis
    - Non-universal couplings: diagonal with different strengths; become non-diagonal in a different basis

Flavor Changing Neutral Currents

- Four neutral bosons that could mediate neutral currents:
  - Gluons, photons, Z-boson, (Higgs boson)
- Gluons, Photons (massless):
  - Exact gauge symmetries: Only couple to fermions through gauge kinetic terms
  - Canonical kinetic terms $\Rightarrow$ universal and flavor conserving couplings
  - Gauge symmetry protects FCNCs

Group Work: FCNCs

- Z-boson mediated neutral current:
  - Couplings to fermions $\propto \left(T_3 - q\sin^2\theta_w\right)$
  - In interaction basis

$$-\mathscr{L}_Z = \frac{g}{\cos\theta_w}\left[\overline{u^I_{L_i}}\gamma^\mu\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)u^I_{L_i} + \overline{u^I_{R_i}}\gamma^\mu\left(-\frac{2}{3}\sin^2\theta_w\right)u^I_{R_i}\right.$$
$$\left.+\overline{d^I_{L_i}}\gamma^\mu\left(\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)d^I_{L_i} + \overline{d^I_{R_i}}\gamma^\mu\left(\frac{1}{3}\sin^2\theta_w\right)d^I_{R_i}\right]Z_\mu + \text{h.c.}$$

- What happen when going to physical basis?

Group Work: FCNCs

- Z-boson mediated neutral current:
  - In interaction basis

$$-\mathscr{L}_Z = \frac{g}{\cos\theta_w}\left[\overline{u^I_{L_i}}\gamma^\mu\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)u^I_{L_i} + \overline{u^I_{R_i}}\gamma^\mu\left(-\frac{2}{3}\sin^2\theta_w\right)u^I_{R_i}\right.$$
$$\left.+\overline{d^I_{L_i}}\gamma^\mu\left(\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)d^I_{L_i} + \overline{d^I_{R_i}}\gamma^\mu\left(\frac{1}{3}\sin^2\theta_w\right)d^I_{R_i}\right]Z_\mu + \text{h.c.}$$

  - In physical basis

$$-\mathscr{L}_Z = \frac{g}{\cos\theta_w}\left[\overline{u_{L_i}}\left(V_{u_L}\right)_{ik}\gamma^\mu\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)\left(V^\dagger_{u_L}\right)_{kj}u_{L_j}\right]Z_\mu\,,$$
$$= \frac{g}{\cos\theta_w}\left[\overline{u_{L_i}}\gamma^\mu\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)u_{L_i}\right]Z_\mu$$

Flavor Changing Neutral Currents

- Z-boson mediated neutral current
    - $V_{u_L} V_{u_L}^\dagger = \mathbb{1}$,
    - compared to W-mediated charged current, $V_{u_L} V_{d_L}^\dagger = V_{CKM}$
- Generally, fields can mix if they belong to same representation under unbroken generators
- **Theorem:** To prevent FCNCs in gauge sector: particles with same unbroken gauge quantum numbers must also have same quantum numbers under the broken gauge group
    - Homework: SM satisfies this criterion

Probing the CKM Matrix

- Elements of CKM matrix

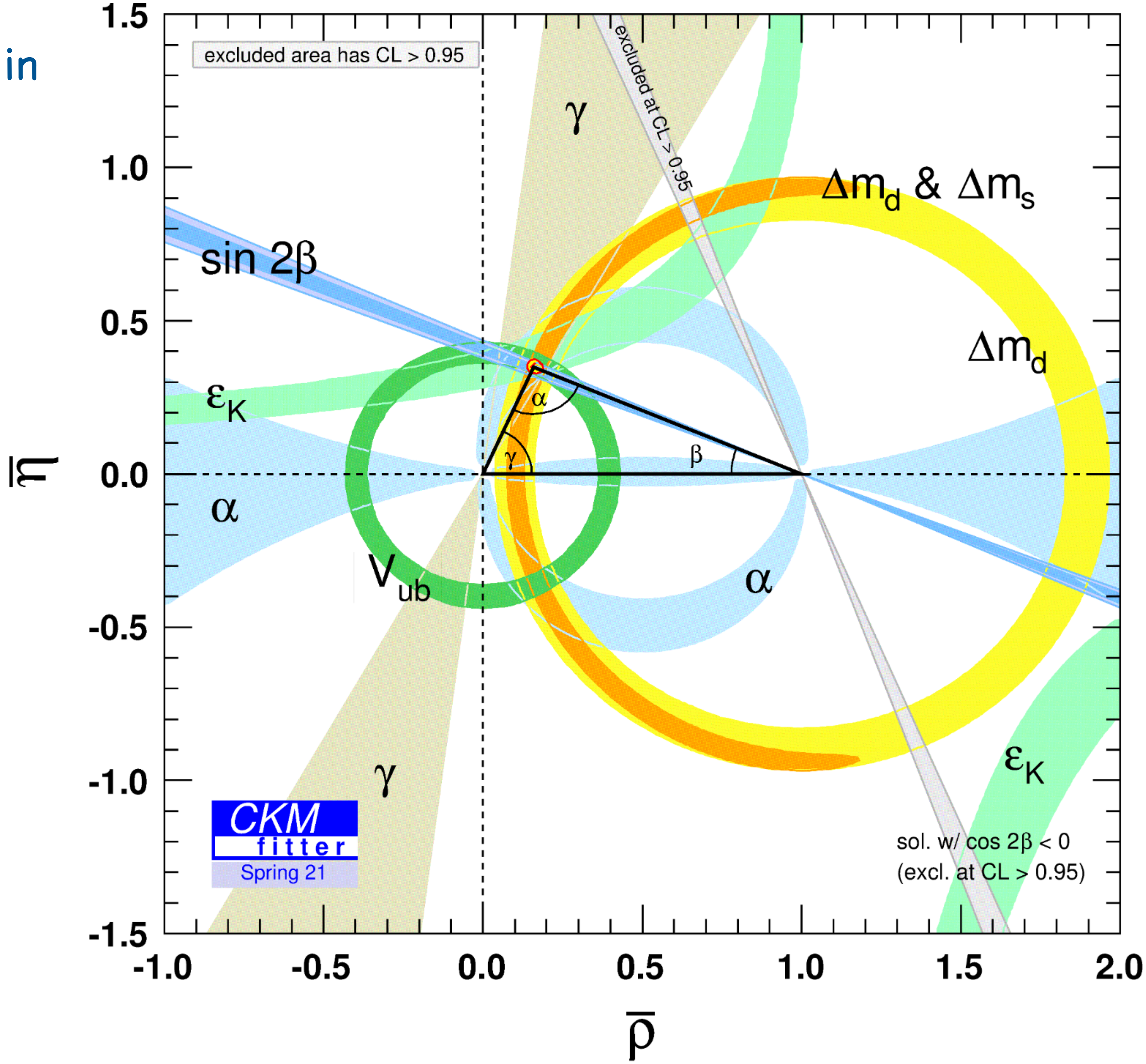
$$V_{CKM} = \begin{pmatrix} 0.97446 & 0.22452 & 0.00365 \\ 0.22438 & 0.97359 & 0.04214 \\ 0.00896 & 0.04133 & 0.99911 \end{pmatrix} \pm \begin{pmatrix} 0.00010 & 0.00044 & 0.00012 \\ 0.00044 & 0.00011 & 0.00076 \\ 0.00024 & 0.00974 & 0.00003 \end{pmatrix}$$

$$\left| V_{CKM} \right| \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}$$

$$J_q \sim 10^{-5}$$

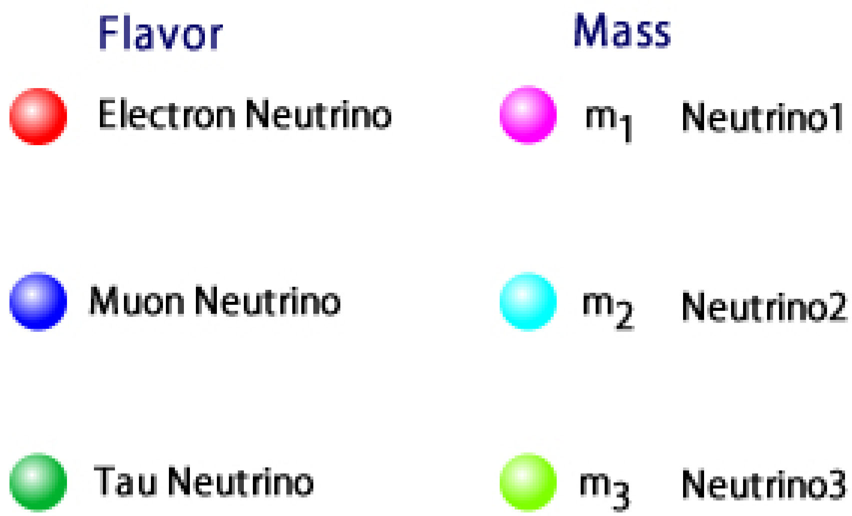
Consistency:

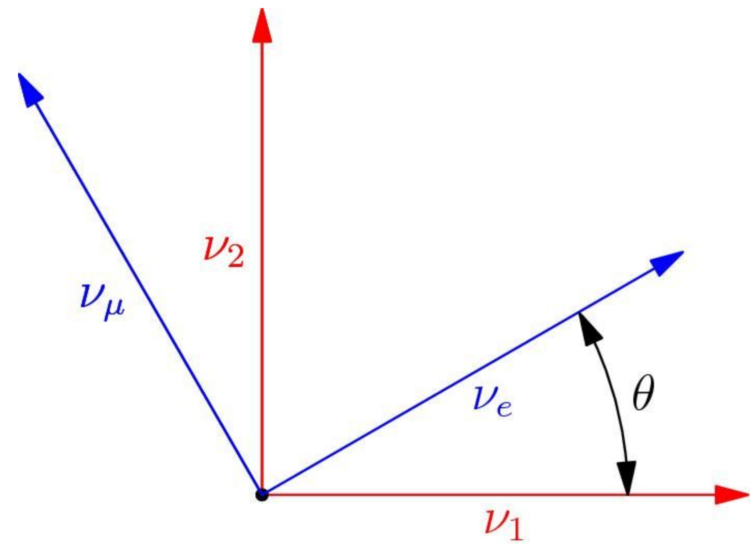
Impressive precision in the measurements in quark flavor sector.

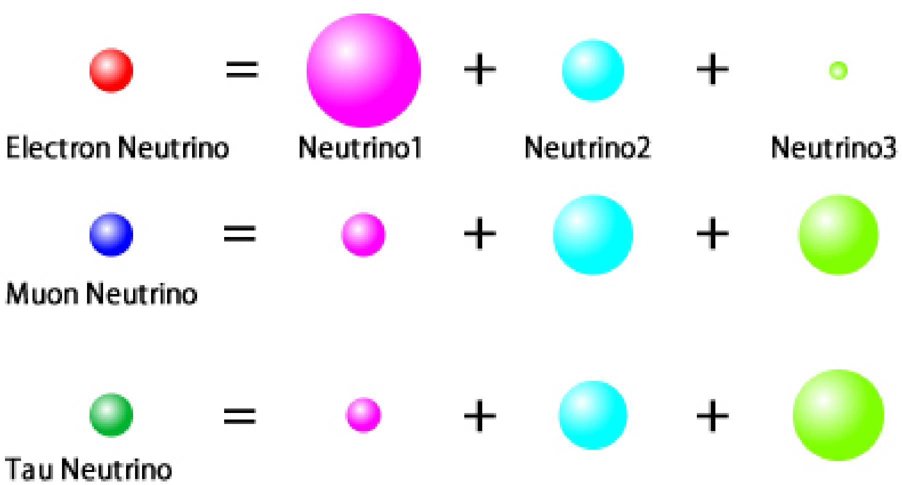


Why BSM?

- SM: tested to high accuracy, good low energy effective description of Nature
- Reasons for Beyond the Standard Model (BSM) New Physics
    - Neutrino Mass
    - Dark Matter
    - Matter-antimatter asymmetry of the Universe
    - Strong CP Problem
    - Gauge Hierarchy Problem
    - Flavor Puzzle
    - Gravity
    - Understanding of Charge quantization

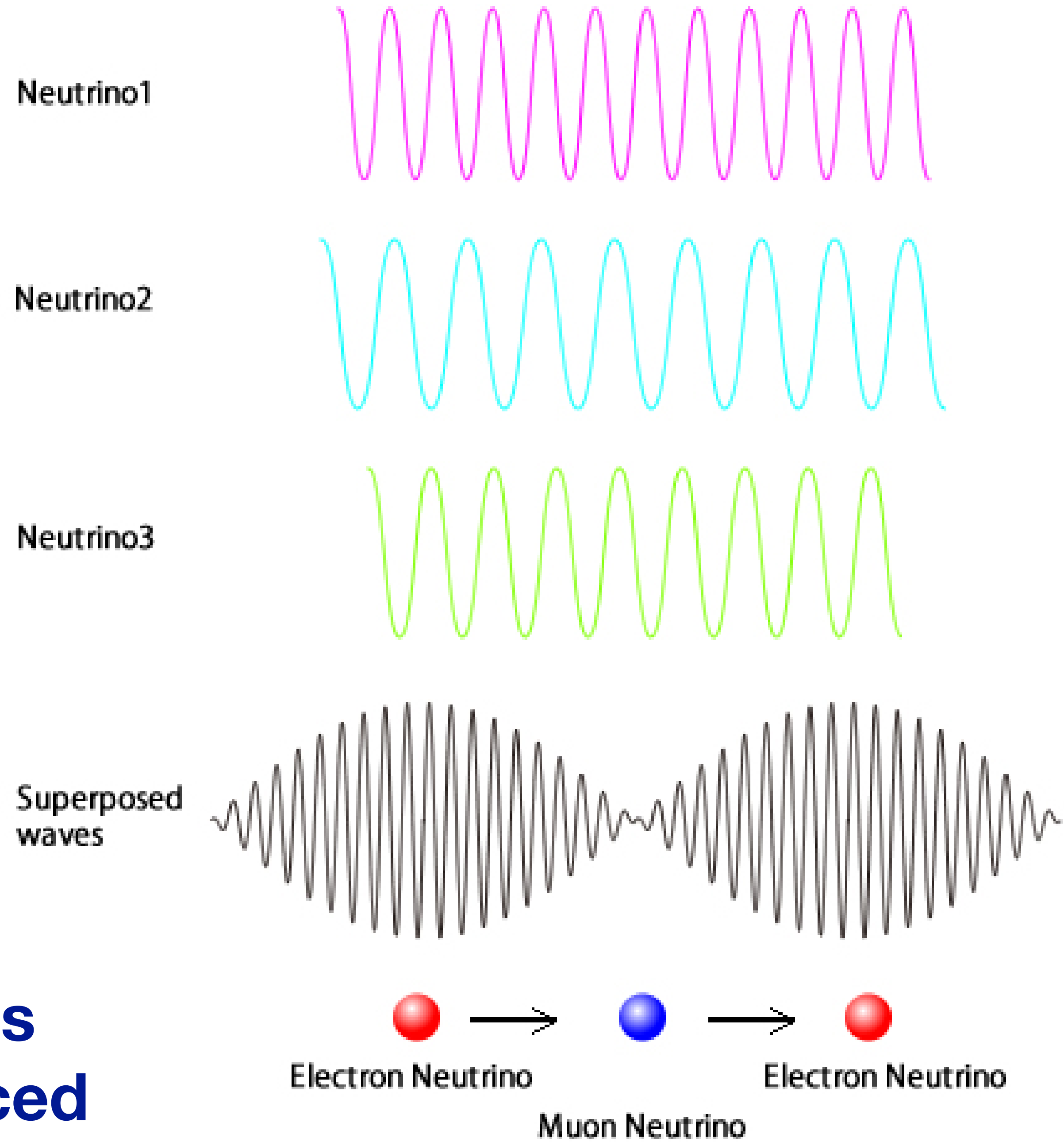






[Picture credit: Symmetry Magazine]

Macroscopic Quantum Mechanics at Work



states propagate

states produced

states detected

Standard Model of Particle Physics

Helicity of Neutrinos*

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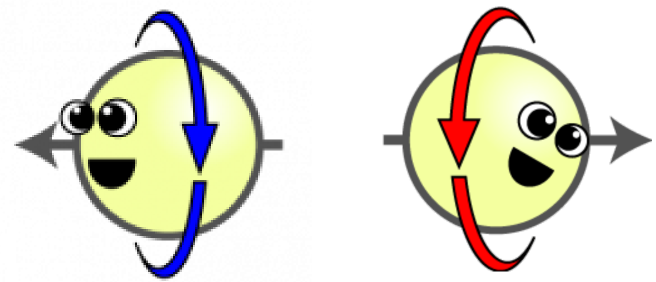
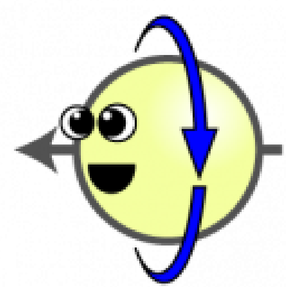
(Received December 11, 1957)

A COMBINED analysis of circular polarization and resonant scattering of γ rays following orbital electron capture measures the helicity of the neutrino. We have carried out such a measurement with Eu^{152m}, which decays by orbital electron capture. If we assume the most plausible spin-parity assignment for this isomer compatible with its decay scheme,[1] $0-$, we find that the neutrino is "left-handed," i.e., $\boldsymbol{\sigma}_\nu \cdot \hat{\mathbf{p}}_\nu = -1$ (negative helicity).



all particles have both left-handed and right-handed partners, except for neutrinos

only LH neutrinos have been observed



Fermion Mass Generation

- Two types of mass terms:

  - **Dirac masses**
    - couple left and right handed fields

$$m_D \overline{\psi_L} \psi_R + h.c.,$$

    - it always involve two different fields
    - the additive quantum numbers of the two fields are opposite
    - there are four d.o.f. with the same mass

Fermion Mass Generation

- **Majorana masses**
  - couple a left-handed or right-handed field to itself

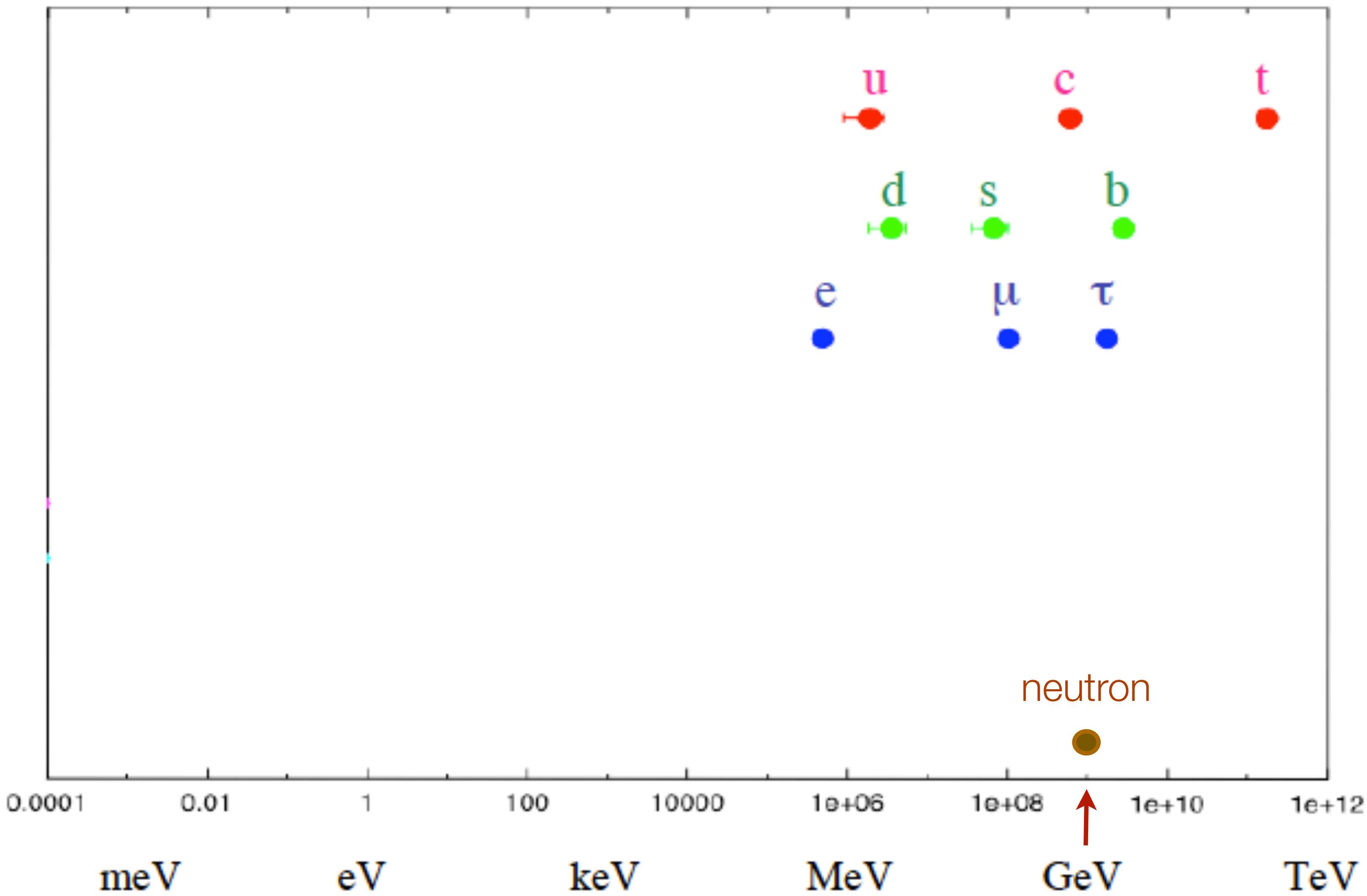
$$m_M \overline{\psi_R^c} \psi_R, \qquad \psi^c = C \, \overline{\psi}^T,$$

  - there can be only two d.o.f. with the same mass
  - additive quantum numbers of the two fields are the same ⇒ break all the U(1) symmetries
  - can only be written for neutral fermions

Neutrino Mass in the SM

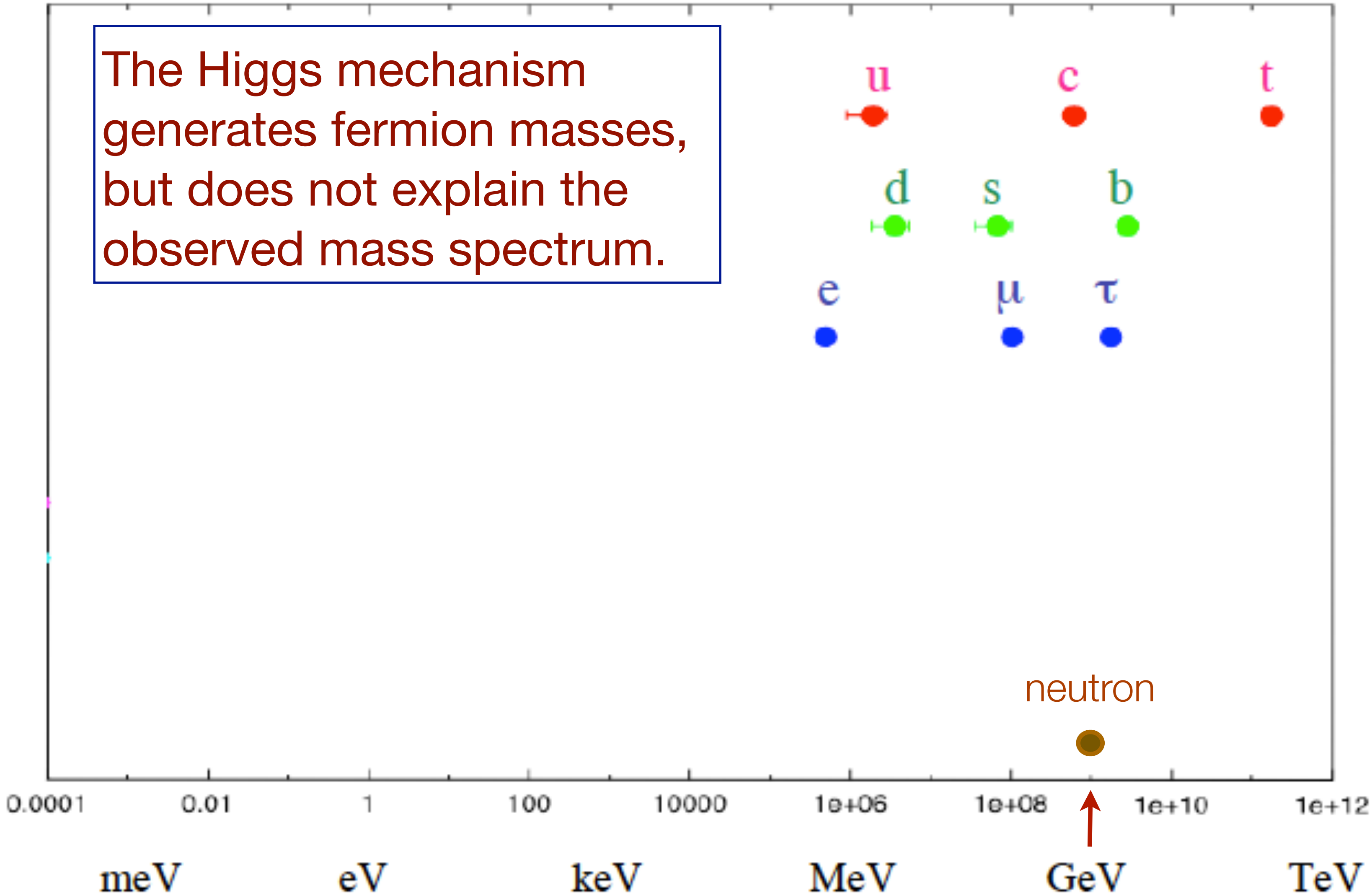
- SM implies exactly massless neutrinos
  - no ν_R ⇒ neutrinos are massless
  - no Higgs SU(2) triplet ⇒ no Majorana mass ΔLL
  - SM renormalizable ⇒ no Majorana mass term from dim-5 operator HHLL
- Unlike $m_\gamma = 0$ prediction, the $m_\nu = 0$ prediction is somewhat accidental

Mass Spectrum of Elementary Particles in SM

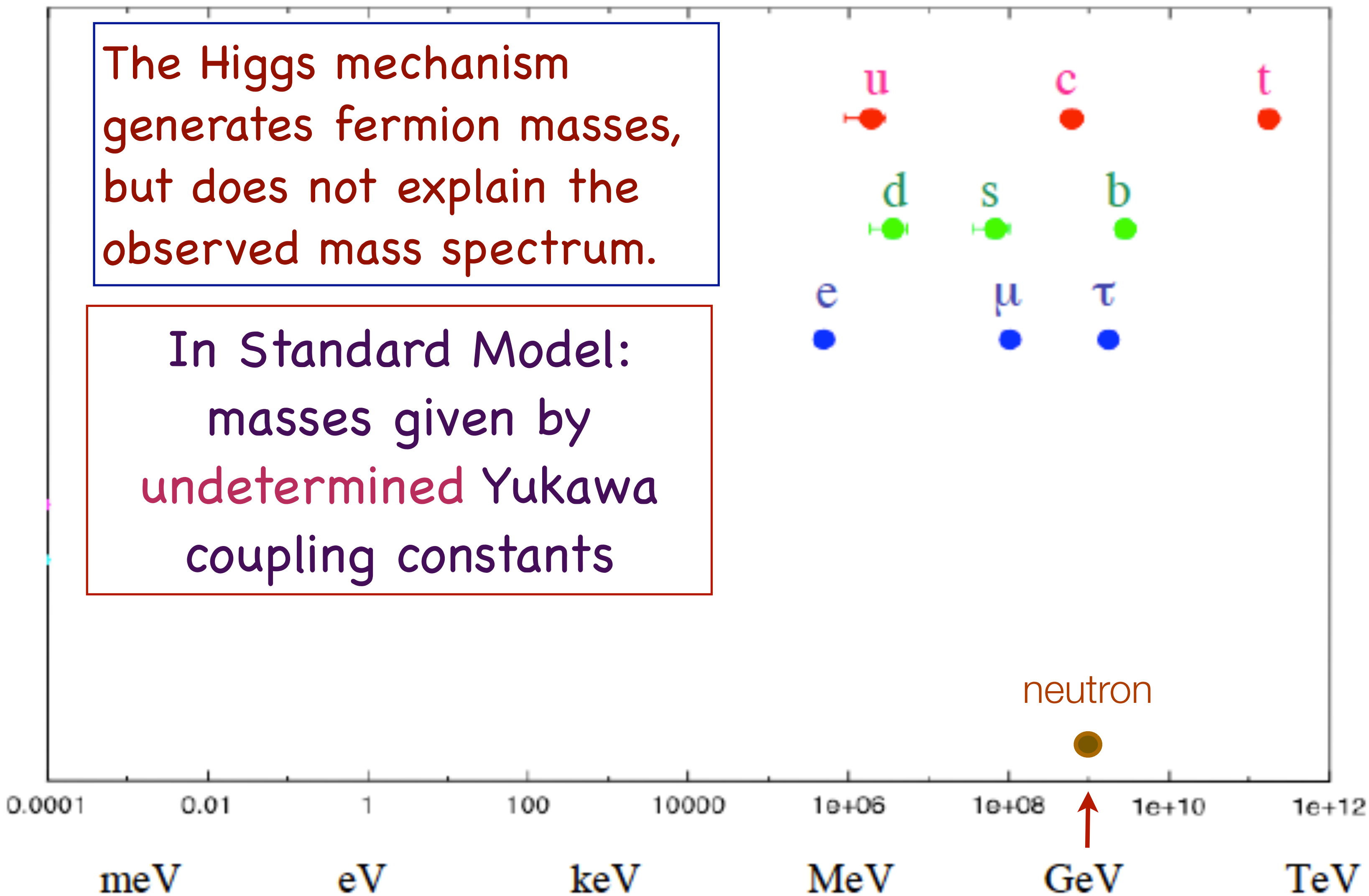


Mysteries of Masses in SM

The Higgs mechanism generates fermion masses, but does not explain the observed mass spectrum.



Mysteries of Masses in SM

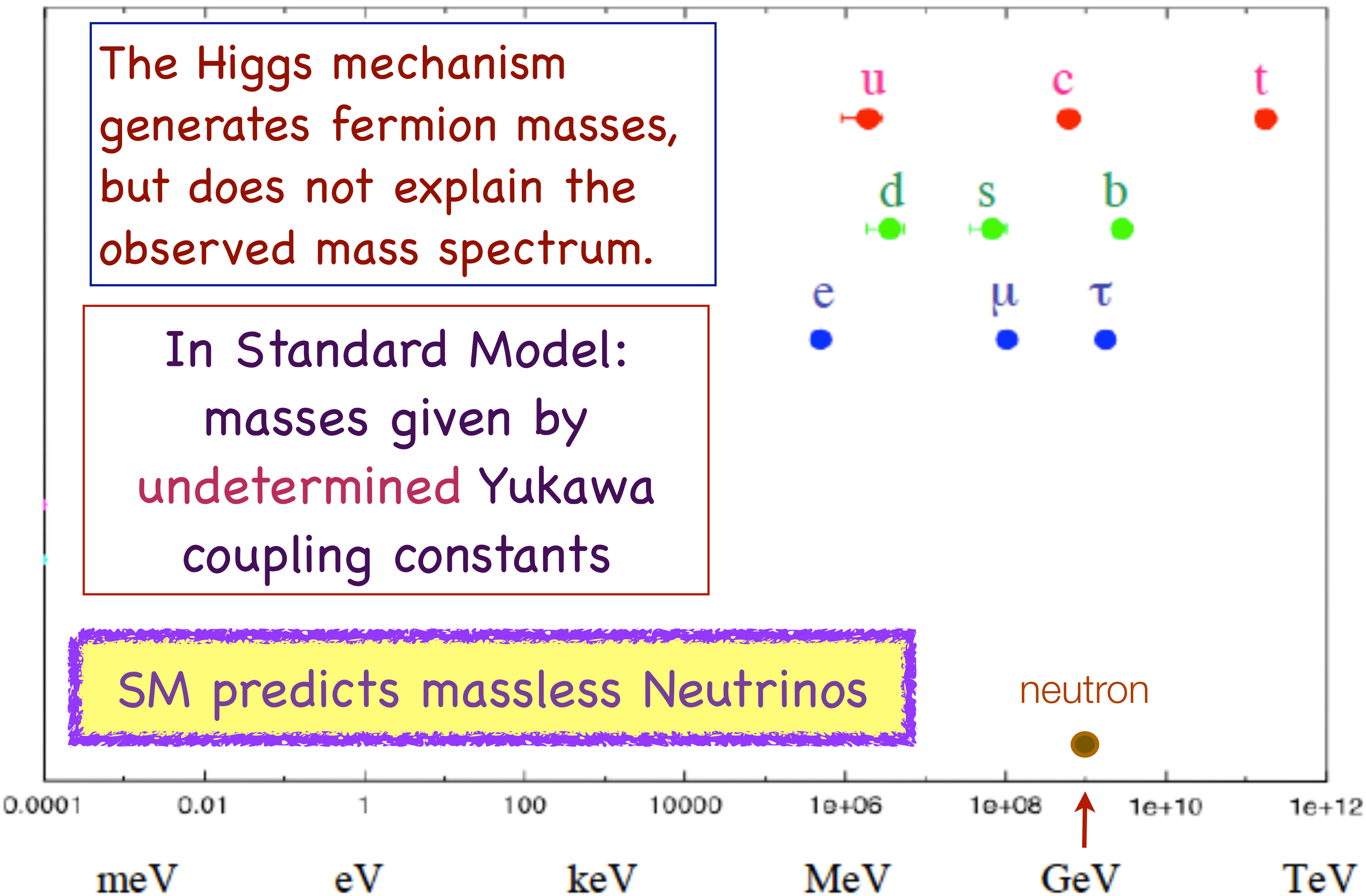


Mysteries of Masses in SM

The Higgs mechanism generates fermion masses, but does not explain the observed mass spectrum.

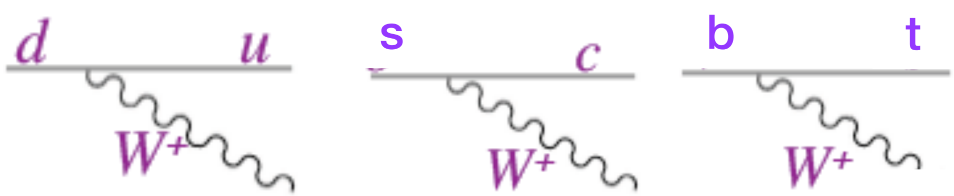
In Standard Model: masses given by undetermined Yukawa coupling constants

SM predicts massless Neutrinos



Mysteries of Masses and Flavor Mixing in SM

- Charged current weak interaction mediated by $W^{\pm}$ gauge boson:



$$\frac{g}{\sqrt{2}}(\overline{u_L})_i V_{ij} \gamma^\mu (d_L)_j W^+_\mu$$

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{cd} & V_{td} \\ V_{us} & V_{cs} & V_{ts} \\ V_{ub} & V_{cb} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

mass eigenstates — CKM matrix — weak eigenstates

CKM matrix → 3 mixing angles + 1 phase

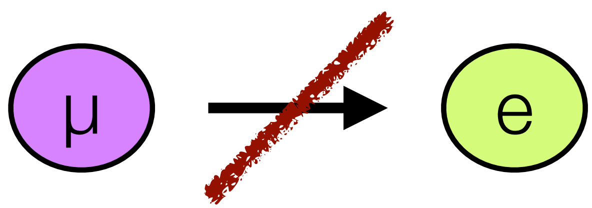
weak eigenstates = mixture of mass eigenstates

Cabibbo, 1963;
Kobayashi, Maskawa, 1973

Nobel prize to KM in 2008

Mysteries of Masses and Flavor Mixing in SM

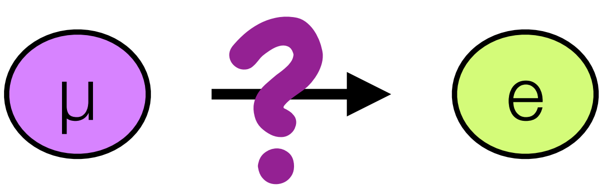
- Neutrino Masses are degenerate (all zero)
  - mass eigenstates = weak eigenstates
- Accidental symmetries in SM
  - lepton flavor numbers: L_e, L_μ, L_τ
  - no processes cross family lines in lepton sector
    - As a result
      - no neutrino oscillation
      - lepton flavor violation decays forbidden



  - total lepton number conserved: $L = L_e + L_\mu + L_\tau$

Neutrino Oscillation ⇒ Massive Neutrinos

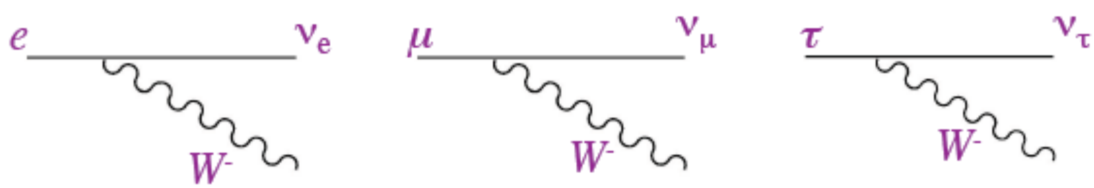
- Neutrino Masses are non-degenerate (at least two are non-zero)
  - mass eigenstates ≠ weak eigenstates
- Accidental symmetries in SM
  - Broken lepton flavor numbers: L_e, L_μ, L_τ
  - Processes cross family lines in lepton sector now possible
    - As a result
      - neutrino oscillation ✔
      - lepton flavor violation decays?



- total lepton number? $L \overset{?}{=} L_e + L_\mu + L_\tau$ ⇔

What if Neutrinos Have Mass?

- Similar to the quark sector, there can be a mismatch between mass eigenstates and weak eigenstates
- weak interactions eigenstates: ν_e, ν_μ, ν_τ



$$\frac{g}{\sqrt{2}}\,\overline{\ell_L}U_{\ell i}\gamma^\mu(\nu_L)_i W^-_\mu$$

- mass eigenstates: ν_1, ν_2, ν_3
- Pontecorvo-Maki-Nakagawa-Sakata (PMNS) Matrix

Maki, Nakagawa, Sakata, 1962 ;
Pontecorvo, 1967

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

→ 3 mixing angles
+ 1 (3) phase(s) for
Dirac (Majorana)
neutrinos

Recall: n Generations

- CKM-like matrix for n generations: For Dirac fermions

$$2n^2 - n^2 - (2n-1) = (n-1)^2 \text{ free parameters}$$

- A general $n \times n$ orthogonal matrix:

$$\frac{1}{2}n(n-1) \text{ angles describing rotations among } n \text{ dimensions}$$

- Remaining free parameters are phases

$$(n-1)^2 - \frac{1}{2}n(n-1) = \frac{1}{2}(n-1)(n-2)$$

$\Rightarrow$ need at least 3 generations to have CPV in CKM matrix

Recall: n Generations

What happens for Majorana neutrinos?

How many unphysical parameters can we rotate away by phase redefinition?

What is the smallest number of families in order to have CPV?

Recall: n Generations

- CKM-like matrix for n generations: For Majorana fermions?

$(2n-1) \to n$

$$2n^2 - n^2 - (2n-1) = (n-1)^2 \text{ free parameters}$$

$$n^2 - n$$

- A general $n \times n$ orthogonal matrix:

$\frac{1}{2}n(n-1)$ angles describing rotations among n dimensions

- Remaining free parameters are phases

$$n^2 - n$$

$$(n-1)^2 - \frac{1}{2}n(n-1) = \frac{1}{2}(n-1)(n-2) \quad \frac{1}{2}n(n-1)$$

$\Rightarrow$ need at least 2 generations to have CPV for Majorana neutrinos

Where Do We Stand?

- Latest 3 neutrino global analysis:

Gonzalez-Garcia, Maltoni, Schwetz (NuFIT), 2111.03086

with SK atmospheric data

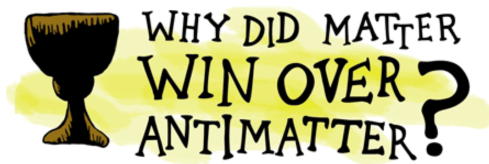
	Normal Ordering (Best Fit)		Inverted Ordering ($\Delta\chi^2 = 7.0$)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.304^{+0.012}_{-0.012}$	$0.269 \to 0.343$	$0.304^{+0.013}_{-0.012}$	$0.269 \to 0.343$
$\theta_{12}/^\circ$	$33.45^{+0.77}_{-0.75}$	$31.27 \to 35.87$	$33.45^{+0.78}_{-0.75}$	$31.27 \to 35.87$
$\sin^2\theta_{23}$	$0.450^{+0.019}_{-0.016}$	$0.408 \to 0.603$	$0.570^{+0.016}_{-0.022}$	$0.410 \to 0.613$
$\theta_{23}/^\circ$	$42.1^{+1.1}_{-0.9}$	$39.7 \to 50.9$	$49.0^{+0.9}_{-1.3}$	$39.8 \to 51.6$
$\sin^2\theta_{13}$	$0.02246^{+0.00062}_{-0.00062}$	$0.02060 \to 0.02435$	$0.02241^{+0.00074}_{-0.00062}$	$0.02055 \to 0.02457$
$\theta_{13}/^\circ$	$8.62^{+0.12}_{-0.12}$	$8.25 \to 8.98$	$8.61^{+0.14}_{-0.12}$	$8.24 \to 9.02$
$\delta_{CP}/^\circ$	230^{+36}_{-25}	$144 \to 350$	278^{+22}_{-30}	$194 \to 345$
$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.510^{+0.027}_{-0.027}$	$+2.430 \to +2.593$	$-2.490^{+0.026}_{-0.028}$	$-2.574 \to -2.410$

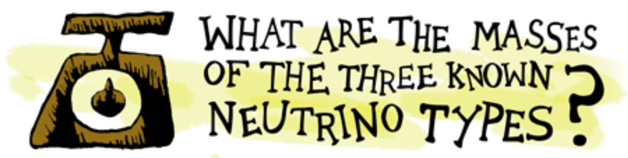
➡ hints of $\theta_{23} \neq \pi/4$

➡ expectation of Dirac CP phase δ

➡ slight preference for normal mass ordering

Open Questions - Neutrino Properties

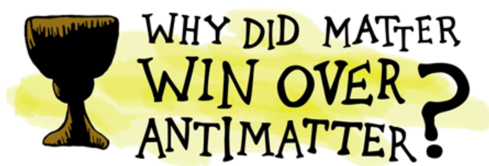


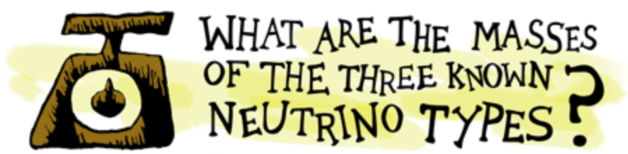


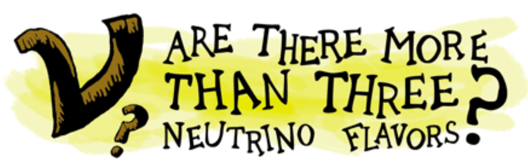
- ☞ Majorana vs Dirac?
- ☞ CP violation in lepton sector?
- ☞ Absolute mass scale of neutrinos?
- ☞ Mass ordering: sign of (Δm_{13}^2)?
- ☞ Sterile neutrino(s)?
- ☞ Precision: $\theta_{23} > \pi/4$, $\theta_{23} < \pi/4$, $\theta_{23} = \pi/4$?

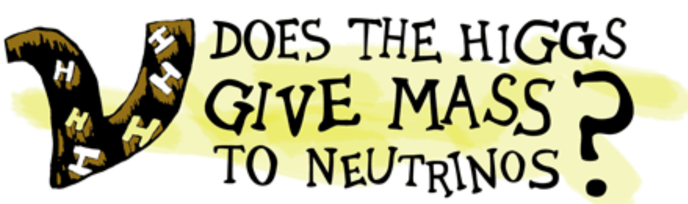
a suite of current and upcoming experiments to address these puzzles

Open Questions – Neutrino Properties









- ☞ **Majorana vs Dirac?**
- ☞ **CP violation in lepton sector?**
- ☞ **Absolute mass scale of neutrinos?**
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- ☞ **Sterile neutrino(s)?**
- ☞ **Precision: $\theta_{23} > \pi/4$, $\theta_{23} < \pi/4$, $\theta_{23} = \pi/4$?**

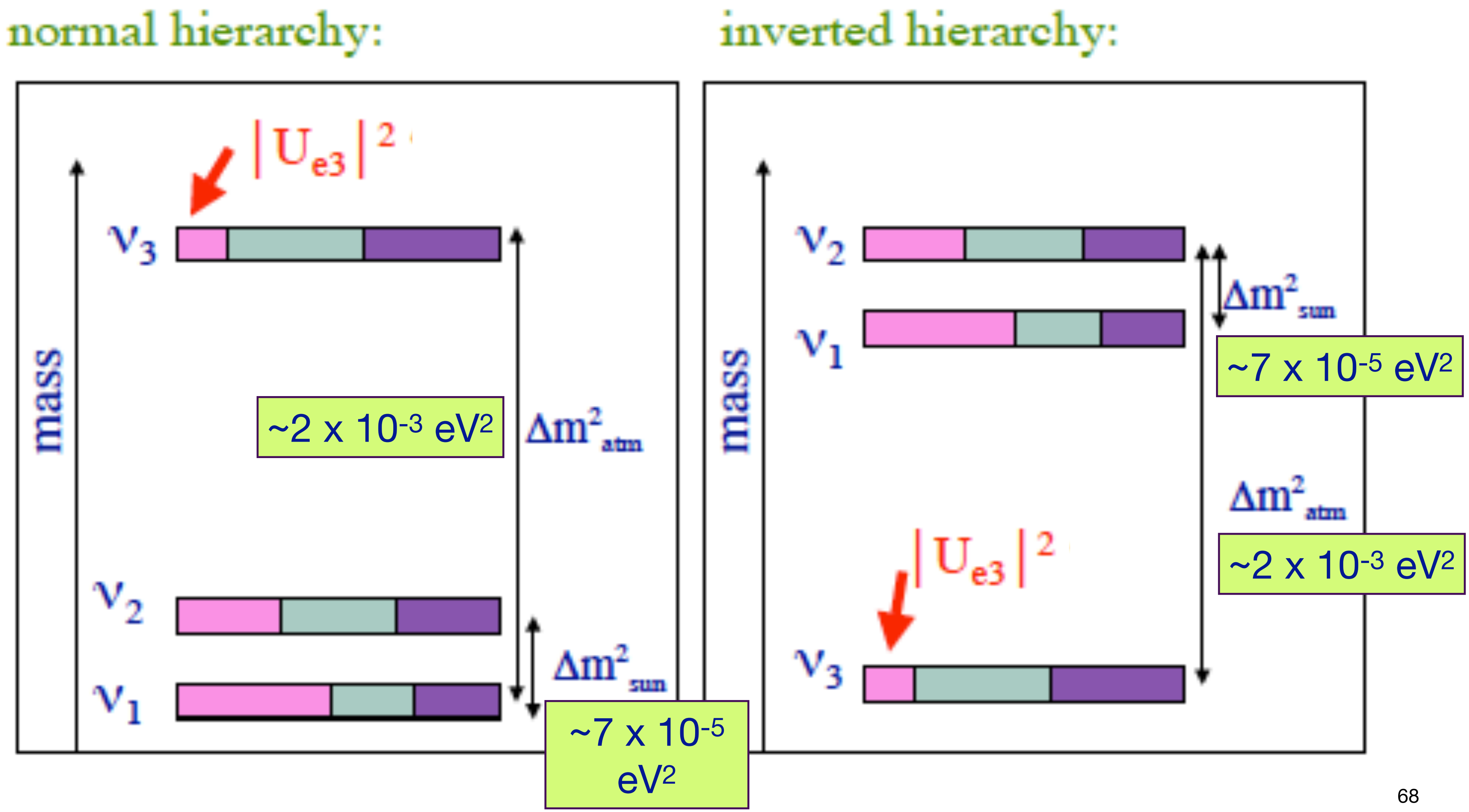
a suite of current and upcoming experiments to address these puzzles

To understand these properties
⇒ BSM Physics

Part II: Flavor Symmetries

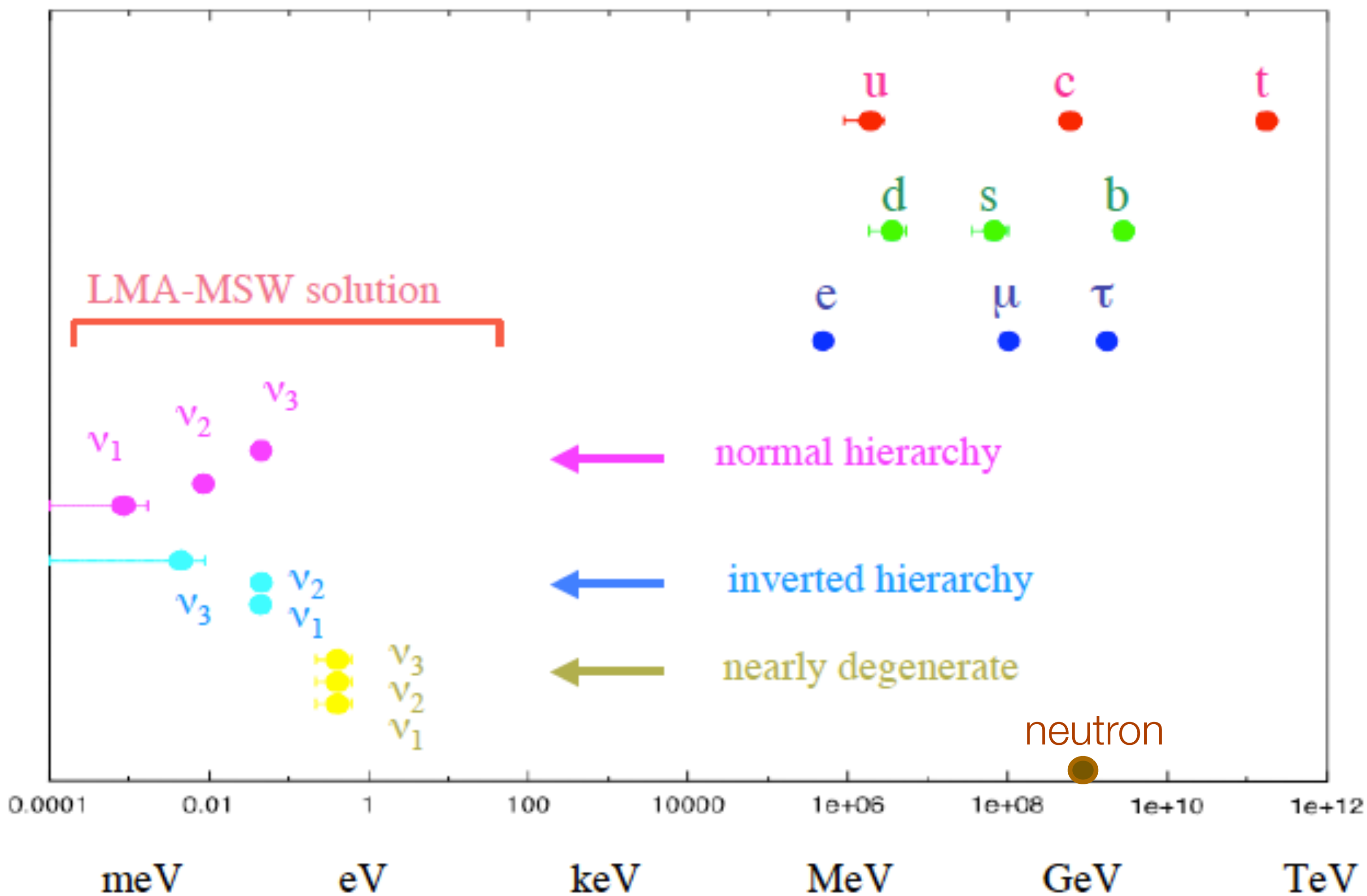
Where Do We Stand?

The known knowns:



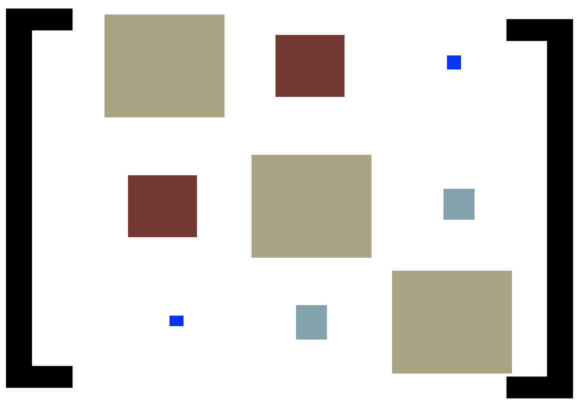
Open Questions - Theoretical

☞ Smallness of neutrino mass: $m_{\nu} \ll m_{e,\,u,\,d}$

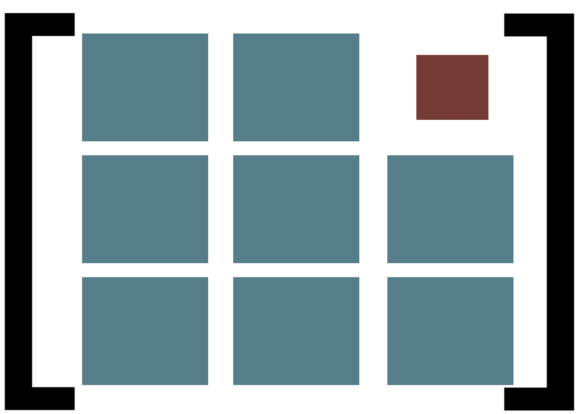


Open Questions - Theoretical

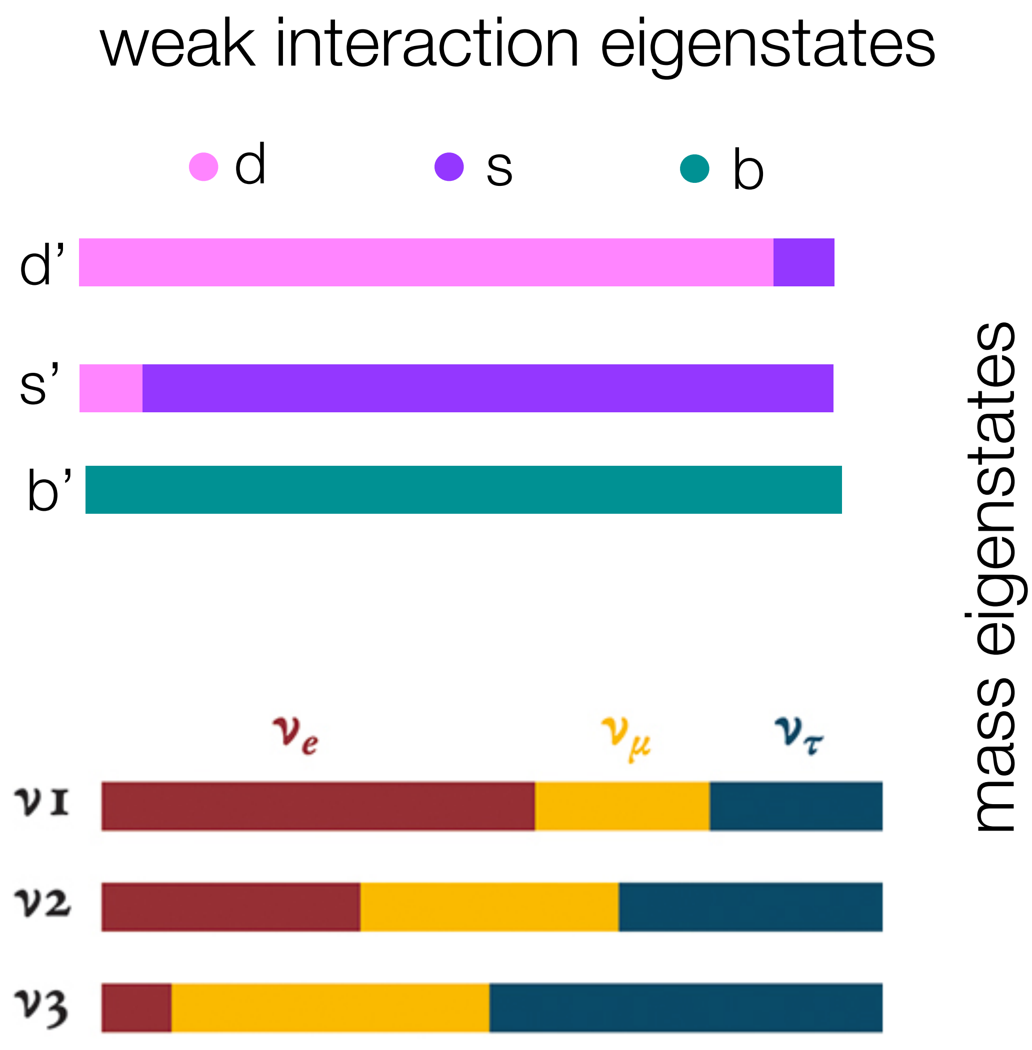
☞ Flavor structure:



quark mixing



leptonic mixing



Group Work: Parameter Counting in SM

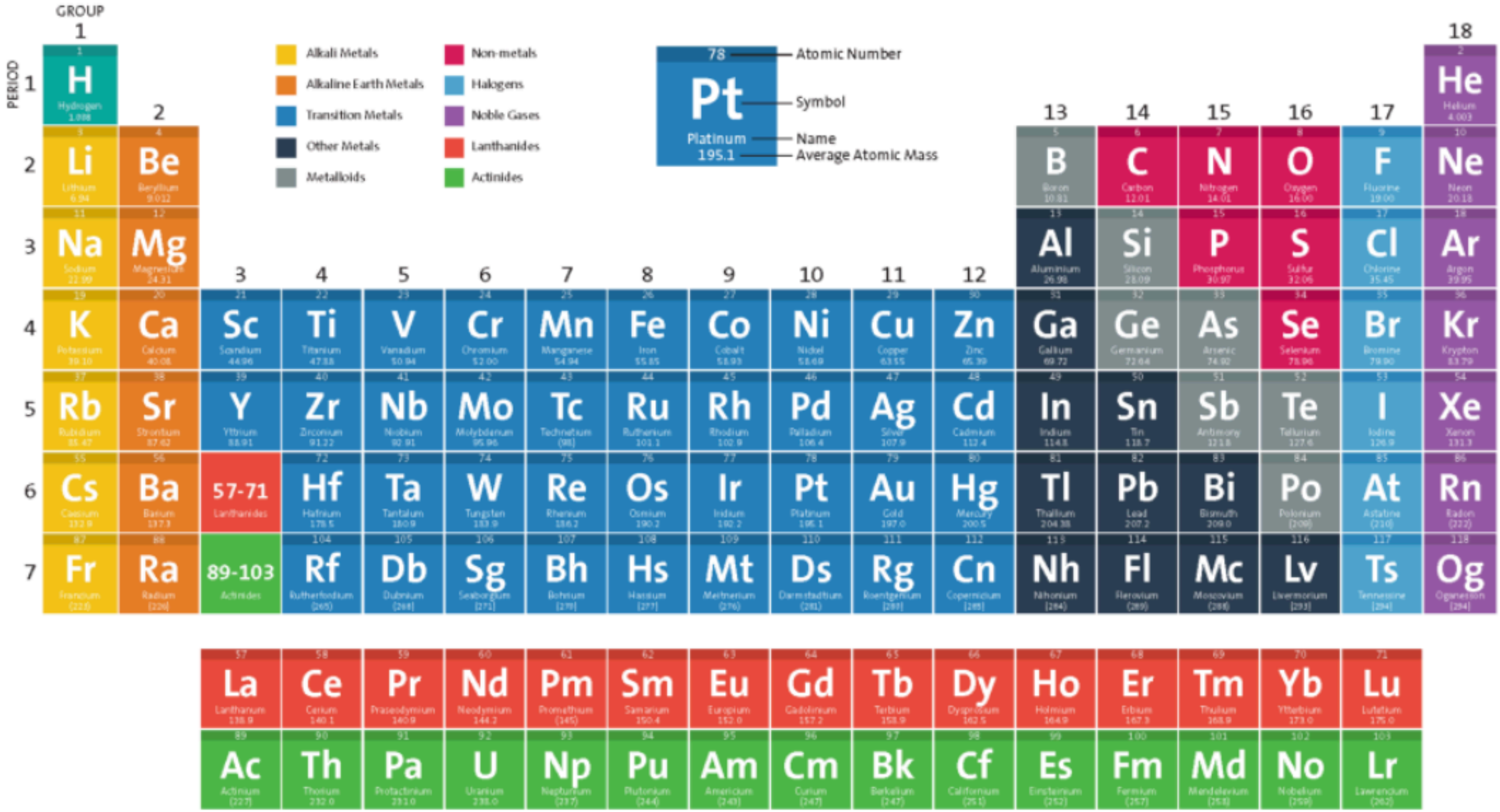
How many free physical parameters are there in the Yukawa sector of SM w/ 3 RH neutrinos (assuming Majorana neutrinos)?

Fermion mass and hierarchy problem ➡ Many (22) free parameters (out of 28) in the Yukawa sector of SM

Where do fermion mass hierarchy, flavor mixing, and CP violation come from?

Where do fermion mass hierarchy, flavor mixing, and CP violation come from?

Is there a simpler organization principle?


GROUP
1
2
PERIOD
Alkali Metals
Alkaline Earth Metals
Transition Metals
Other Metals
Metalloids
Non-metals
Halogens
Noble Gases
Lanthanides
Actinides
78
Pt
Platinum
195.1
Atomic Number
Symbol
Name
Average Atomic Mass
3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
H He
Li Be B C N O F Ne
Na Mg Al Si P S Cl Ar
K Ca Sc Ti V Cr Mn Fe Co Ni Cu Zn Ga Ge As Se Br Kr
Rb Sr Y Zr Nb Mo Tc Ru Rh Pd Ag Cd In Sn Sb Te I Xe
Cs Ba 57-71 Hf Ta W Re Os Ir Pt Au Hg Tl Pb Bi Po At Rn
Fr Ra 89-103 Rf Db Sg Bh Hs Mt Ds Rg Cn Nh Fl Mc Lv Ts Og
La Ce Pr Nd Pm Sm Eu Gd Tb Dy Ho Er Tm Yb Lu
Ac Th Pa U Np Pu Am Cm Bk Cf Es Fm Md No Lr

Where do fermion mass hierarchy, flavor mixing, and CP violation come from?

Is there a simpler organization principle?

Where do neutrinos get their masses?

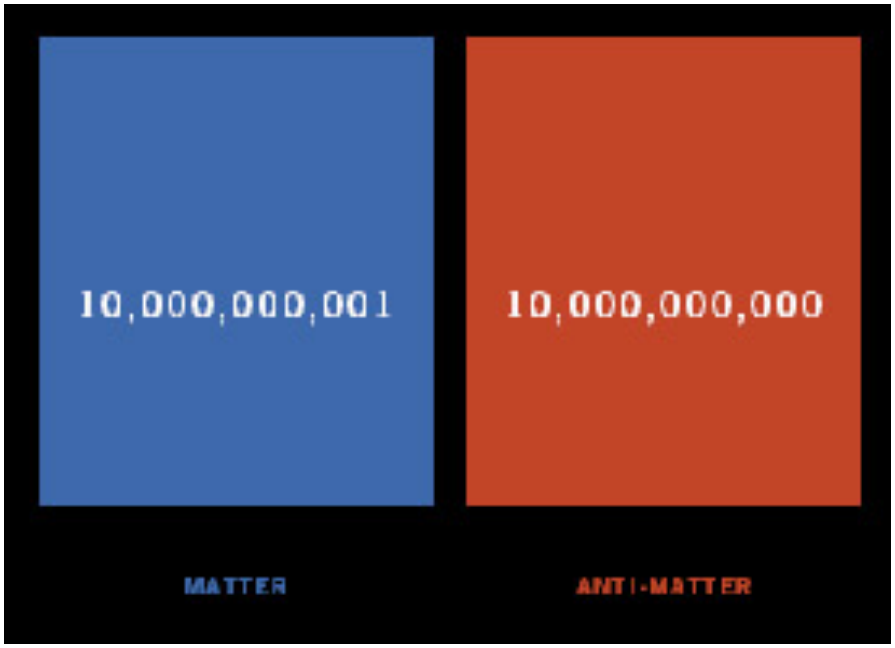
Where do fermion mass hierarchy, flavor mixing, and CP violation come from?

Is there a simpler organization principle?

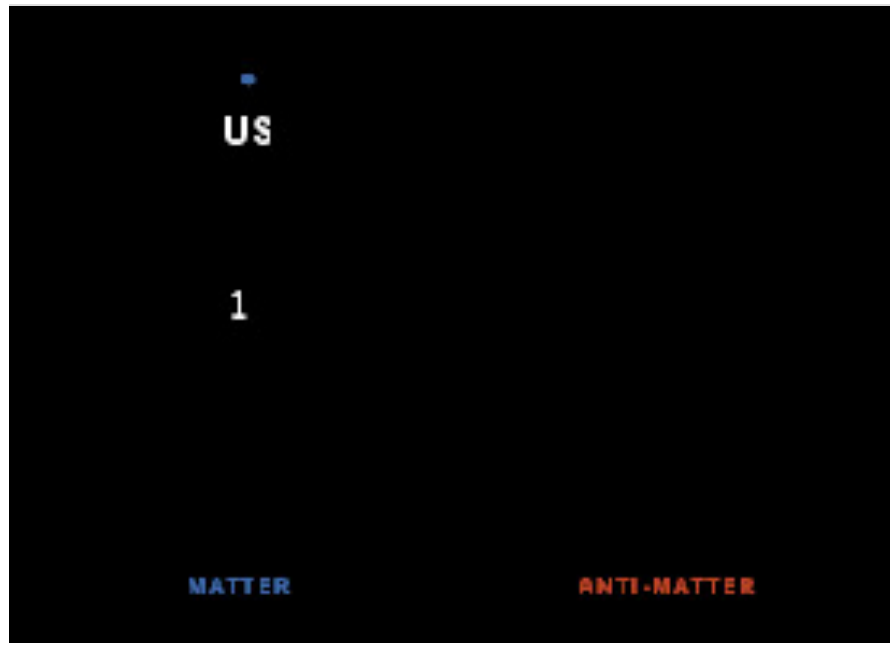
Where do neutrinos get their masses?

Is it the Higgs or something else that gives neutrino masses?

Matter-Antimatter Asymmetry



Early Universe



Universe Now

[Picture credit: H. Murayama]

What is the origin of matter antimatter asymmetry? Why do we exist?

Neutrino Mass beyond the SM

- Two options:
  - add RH neutrinos $N(1,1)_0 \Rightarrow$ Dirac mass $m_D \bar{\nu}_L \nu_R$
    - why Yukawa couplings are small
    - why there are no large Majorana mass terms for RH neutrinos
  - add Higgs SU(2) triplet $\Delta \Rightarrow$ Majorana mass ΔLL
    - why $\langle \Delta \rangle \ll \langle H \rangle$

- Generally, in these models
  - new fields are introduced only to generate neutrino mass
  - there is no understanding of why neutrinos are light

Neutrino Mass beyond the SM

- The SM is an effective low energy theory, with NR terms. NP effects are suppressed by powers of small parameter

$$\frac{M_W}{\Lambda}$$

- Neutrino masses are generated by so-called Weinberg operator

$$HH\ell_i\ell_j$$

Group Work: Mass Dimension

What is the mass dimension of the Weinberg Operator, $HH\ell_i\ell_j$?

How does it appear in the Lagrangian?

Neutrino Mass beyond the SM

- The SM is an effective low energy theory, with NR terms. NP effects are suppressed by powers of small parameter

$$\frac{M_W}{\Lambda}$$

- Neutrino masses are generated by so-called Weinberg operator, dim-5

$$\frac{\lambda_{ij}}{\Lambda} H H \ell_i \ell_j \Rightarrow m_\nu = \lambda_{ij} \frac{v^2}{\Lambda}$$

λ_{ij} are dimensionless couplings

Λ is some high scale

Group Work: Weinberg Operator

What symmetry/symmetries does the Weinberg operator break?

$$\frac{HH\ell_i\ell_j}{\Lambda}$$

Neutrino Mass beyond the SM

$$\frac{\lambda_{ij}}{\Lambda} H H \ell_i \ell_j \Rightarrow m_\nu = \lambda_{ij} \frac{\upsilon^2}{\Lambda}$$

- promoting SM to an effective field theory implies

$$m_\nu \neq 0$$

- m_ν is small because it arises from NR terms (Λ is high)
- Neutrino mass therefore probe the high energy physics
- Both total lepton number and family lepton numbers are broken

➡ lepton mixing and CP violation expected

Seesaw Mechanism

- Consider one generation SM with an additional singlet $N(1,1)_0$

$$\mathcal{L}_{m_\nu} = \frac{1}{2} M_N N N + Y_\nu H L N$$

$M_N \gg v$ is a Majorana mass of the RH neutrino

the 2nd term: Dirac mass term

- In the (ν_L, N_R) basis, the neutrino mass matrix is

$$m_\nu = \begin{pmatrix} 0 & m_D \\ m_D & M_N \end{pmatrix} \qquad m_D = Y_\nu v$$

Group Work: Seesaw Mechanism

$$m_\nu = \begin{pmatrix} 0 & m_D \\ m_D & M_N \end{pmatrix}$$

Assuming $M_N \gg v$, what are the two eigenvalues at leading order?

Seesaw Mechanism

$$m_\nu = \begin{pmatrix} 0 & m_D \\ m_D & M_N \end{pmatrix}$$

- Assuming $M_N \gg \upsilon$ to first order,

$$m_{N_R} = M_N \;, \; m_{\nu_L} = \frac{m_D^2}{M_N} \;, \qquad \text{(EFT:} \quad m_\nu = \lambda \frac{v^2}{M})$$

  - the new physics scale M is identified with M_N
  - the seesaw scale can be generalized to three generations
  - seesaw is realized in, e.g. Left-Right, Pati-Salam, and GUT models

Group Work: Seesaw Mechanism

$$m_\nu = \begin{pmatrix} 0 & m_D \\ m_D & M_N \end{pmatrix}$$

What is needed to populate the (1 1) entry with a non-vanishing contribution?

Neutrino Mass beyond the SM

- SM: effective low energy theory

$$\mathcal{L} = \mathcal{L}_{\rm SM} + \boxed{\frac{\mathcal{O}_{5D}}{M} + \frac{\mathcal{O}_{6D}}{M^2} + \ldots} \longrightarrow \text{new physics effects}$$

- only one dim-5 operator: most sensitive to high scale physics

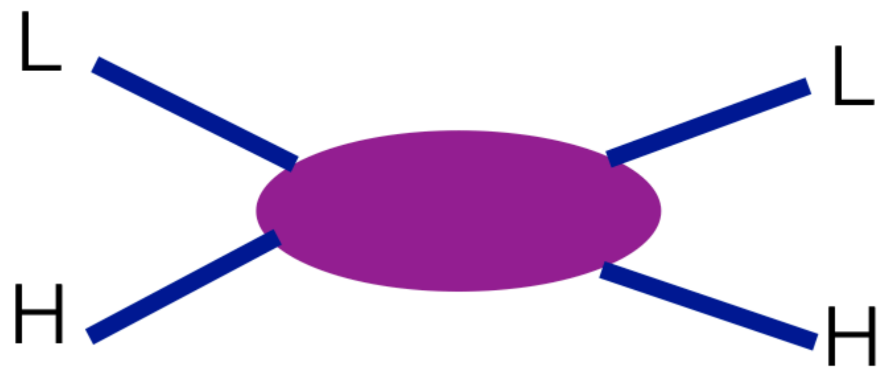
Weinberg, 1979

$$\frac{\lambda_{ij}}{M} H H L_i L_j \quad \Rightarrow \quad m_\nu = \lambda_{ij} \frac{v^2}{M}$$

- m_ν ~ $(\Delta m^2_{atm})^{1/2}$ ~ 0.1 eV with v ~ 100 GeV, λ ~ O(1) ⇒ M ~ 10^{14} GeV
- Lepton number violation $\Delta L = 2$ ⇨ Majorana fermions

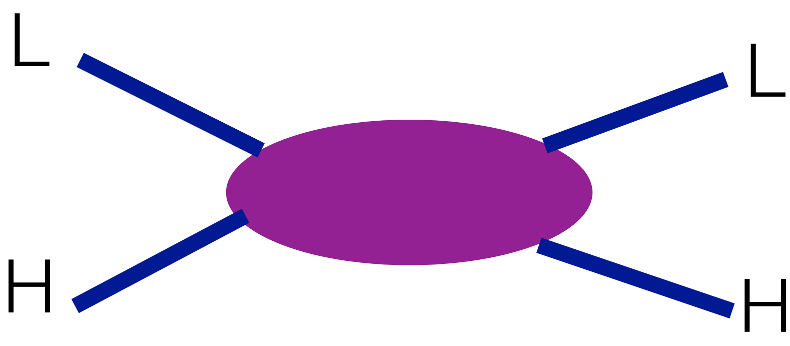
↕ GUT scale

Group Work: UV Completion for Weinberg Operator



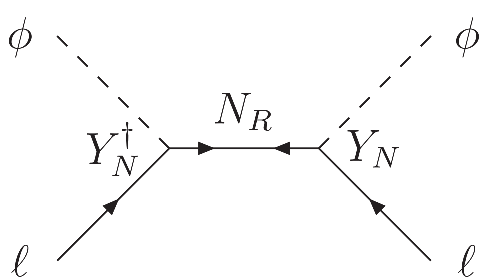
Schematically, how do you UV complete the Weinberg Operator? What can be the "portal" particles?

Neutrino Mass beyond the SM



3 possible portals

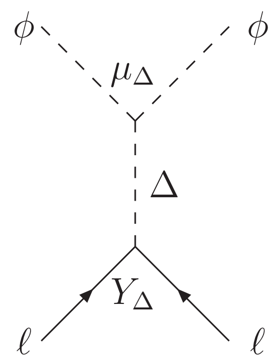
Type-I seesaw



N_R: $SU(3)_c$ x $SU(2)_w$ x $U(1)_Y$ ~(1,1,0)

Minkowski, 1977; Yanagida, 1979; Glashow, 1979;
Gell-mann, Ramond, Slansky,1979;
Mohapatra, Senjanovic, 1979;

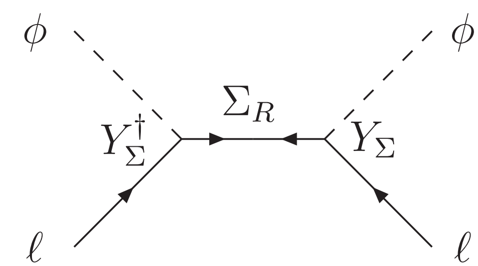
Type-II seesaw



Δ: $SU(3)_c$ x $SU(2)_w$ x $U(1)_Y$ ~(1,3,1)

Lazarides, 1980; Mohapatra, Senjanovic, 1980

Type-III seesaw



$$\Sigma = (\Sigma^+, \Sigma^0, \Sigma^-)$$

Σ_R: $SU(3)_c$ x $SU(2)_w$ x $U(1)_Y$ ~(1,3,0)

Foot, Lew, He, Joshi, 1989; Ma, 1998

Why are neutrinos light? Seesaw Mechanism

- Adding the right-handed neutrinos:

Minkowski, 1977; Yanagida, 1979; Gell-Mann, Ramond, Slansky, 1979; Mohapatra, Senjanovic, 1981

$$\begin{pmatrix} \nu_L & \nu_R \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D & M_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R \end{pmatrix}$$

$$m_\nu \sim m_{light} \sim \frac{m_D^2}{M_R} << m_D$$

$$m_{heavy} \sim M_R$$

For $m_{\nu_3} \sim \sqrt{\Delta m^2_{atm}}$

If $m_D \sim m_t \sim 180\ GeV$

$\Rightarrow$ $M_R \sim 10^{15}\ GeV$ (GUT !!)

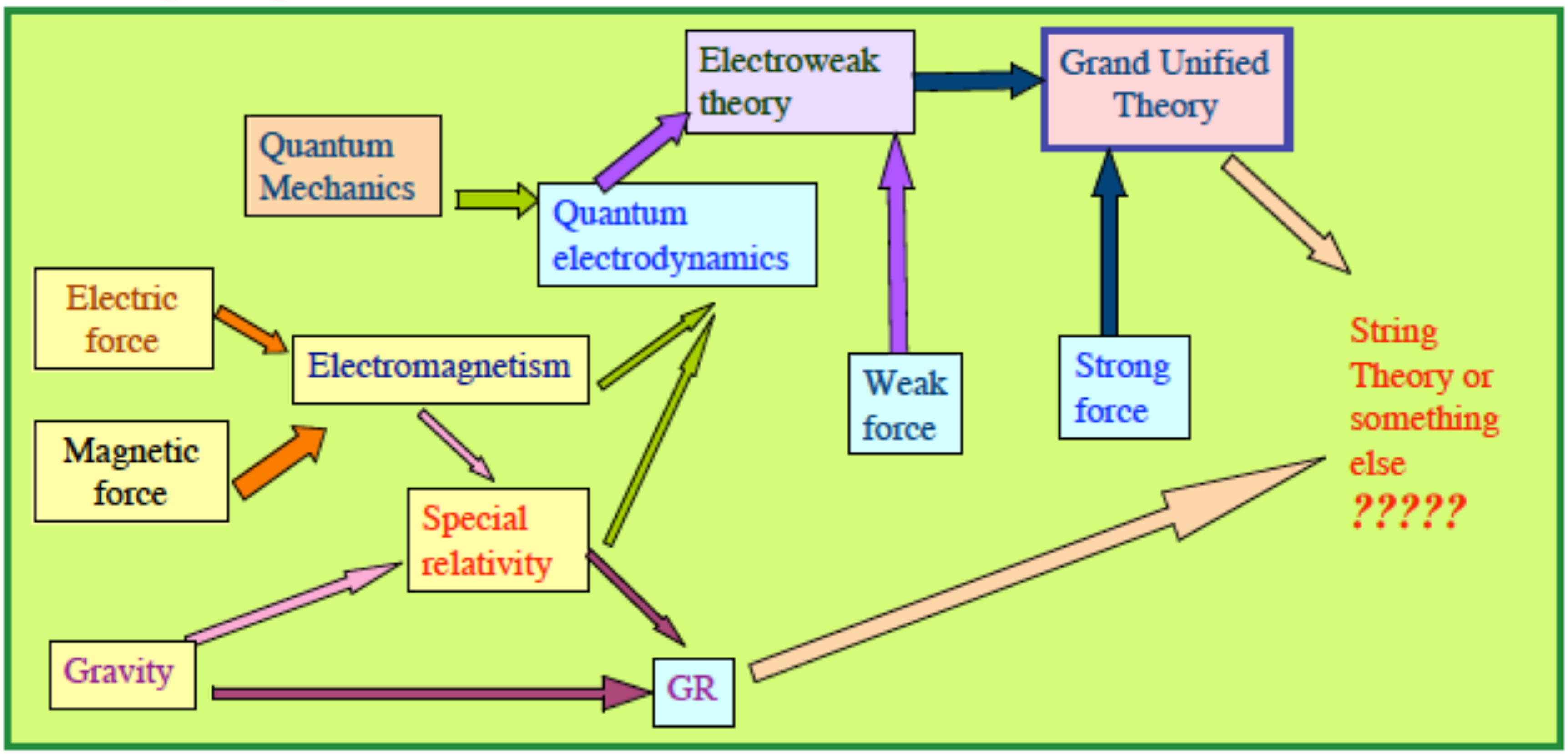
Ultimate Goal of Grand Unification

- Maxwell: electric and magnetic forces are different aspects of electromagnetism
- Einstein: early attempt to unify electric force and gravity

Weinberg, Salam, Glashow: electroweak Theory

Nobel prize 1979

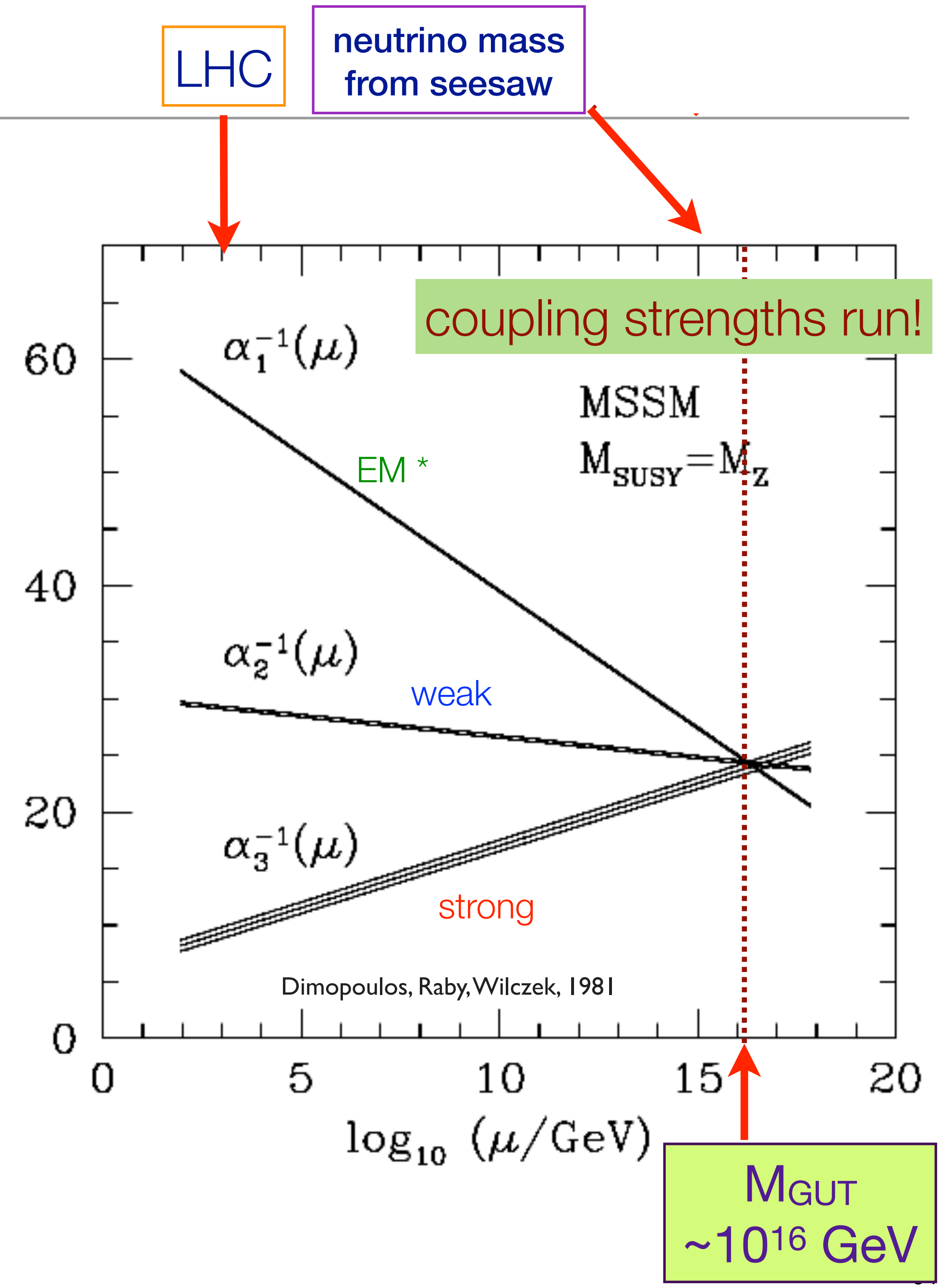
We are getting there.....



Grand Unification

- Motivations:
  - Electromagnetic, weak, and strong forces have very different strengths
  - But their strengths become the same at 10^{16} GeV if there is supersymmetry

    10^{16} GeV ~ 10^{-30} meters
  - To obtain $m_\nu \sim (\Delta m^2_{atm})^{1/2}$, $m_D \sim m_{top}$, $M_R \sim 10^{15}$ GeV
  - **Neutrino oscillations probe physics at unification scale!**



Grand Unification

Georgi, Glashow, 1974

SO(10):

Fritzsch, Minkowski, 1975

quarks and leptons
are close relatives

matter fields come
in 3 copies

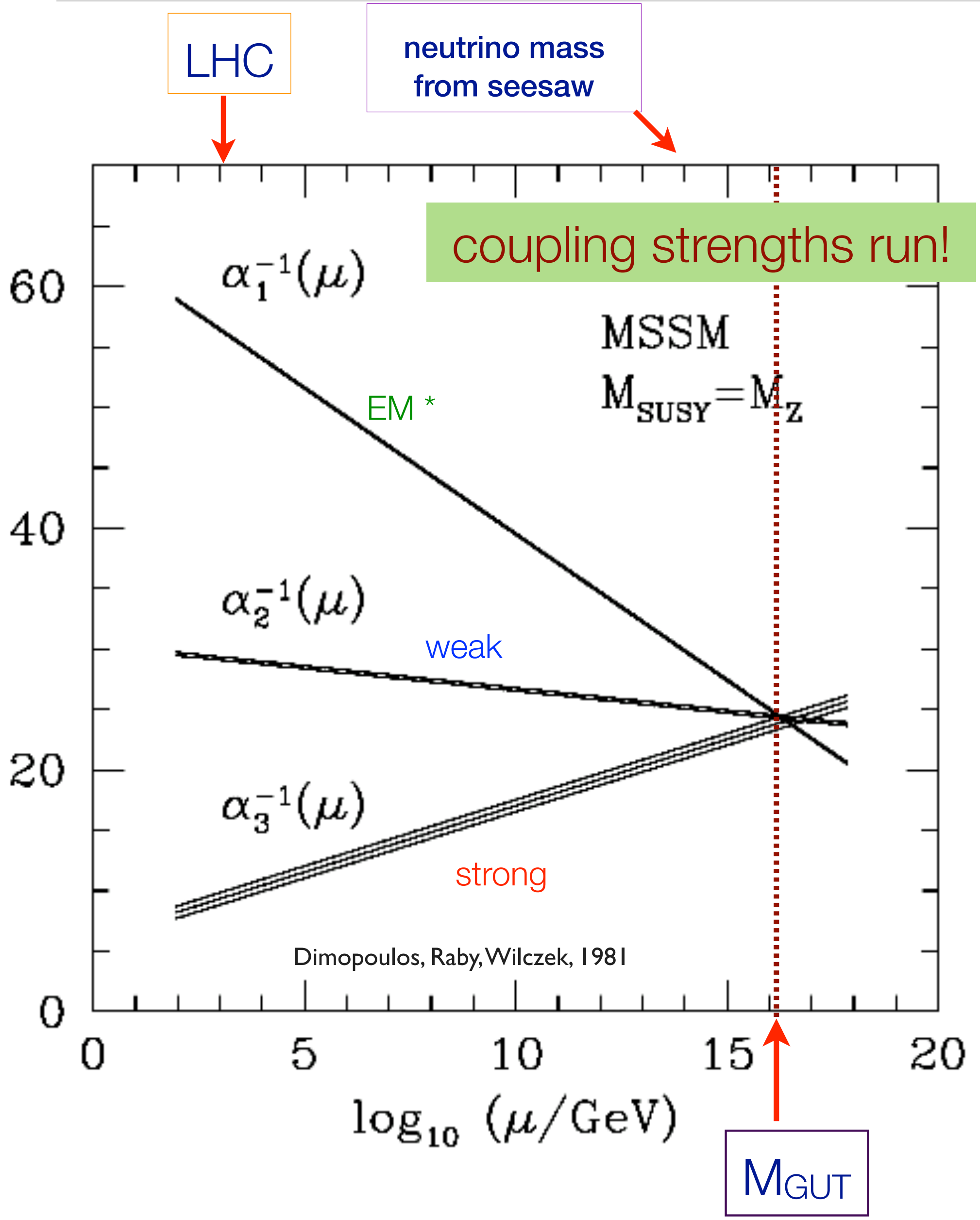
charge quantization
can be understood

$16 = 10 + 5^* + 1$

RH neutrino
predicted

u :	↑	↓	↑	↑	↓
u :	↑	↓	↑	↓	↑
u :	↑	↓	↓	↑	↑
d :	↓	↑	↑	↑	↓
d :	↓	↑	↑	↓	↑
d :	↓	↑	↓	↑	↑
u^c :	↓	↓	↑	↓	↓
u^c :	↓	↓	↓	↑	↓
u^c :	↓	↓	↓	↓	↑
d^c :	↑	↑	↑	↓	↓
d^c :	↑	↑	↓	↑	↓
d^c :	↑	↑	↓	↓	↑
e :	↓	↑	↓	↓	↓
ν_e :	↑	↓	↓	↓	↓
e^c :	↓	↓	↑	↑	↑
ν_e^c :	↑	↑	↑	↑	↑

Grand Unification Naturally Accommodates Seesaw



- ☞ origin of the heavy scale ⇒ $U(1)_{B-L}$
- ☞ exotic mediators ⇒ predicted in many GUT theories, e.g. SO(10)
- ☞ exotic mediators for Type II, III harder to get from string theory

Dienes, March-Russell, 1996

$$16 = (3, 2, 1/6) \sim \begin{bmatrix} u & u & u \\ d & d & d \end{bmatrix}$$

$$+ (3^*, 1, -2/3) \sim (u^c\;\; u^c\;\; u^c)$$

$$+ (3^*, 1, 1/3) \sim (d^c\;\; d^c\;\; d^c)$$

$$+ (1, 2, -1/2) \sim \begin{bmatrix} \nu \\ e \end{bmatrix}$$

$$+ (1, 1, 1) \sim e^c$$

$$+ (1, 1, 0) \sim \nu^c$$

Fritzsch, Minkowski, 1975

Flavor Structure

Flavor Structure

- there are parametrically small numbers
  - $m_2/m_3 < 1$, $\theta_{13} < 1$
- In general, large mixing $\Leftrightarrow$ no hierarchy

$$m = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

  - $a >> b, c \Rightarrow \sin^2\theta << 1, \; m_1/m_2 << 1$
  - $a,b,c \sim 1 \Rightarrow \det(m) \sim 1 \Rightarrow \sin^2\theta \sim 1, \; m_1/m_2 \sim 1$
  - $a,b,c \sim 1 \Rightarrow \det(m) << 1 \Rightarrow \sin^2\theta \sim 1, \; m_1/m_2 << 1$

Texture	Hierarchy	$\lvert U_{e3}\rvert$	$\lvert\cos 2\theta_{23}\rvert$ (n.s.)	$\lvert\cos 2\theta_{23}\rvert$	Solar Angle
$\frac{\sqrt{\Delta m^2_{13}}}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	Normal	$\sqrt{\frac{\Delta m^2_{12}}{\Delta m^2_{13}}}$	O(1)	$\sqrt{\frac{\Delta m^2_{12}}{\Delta m^2_{13}}}$	O(1)
$\sqrt{\Delta m^2_{13}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$	Inverted	$\frac{\Delta m^2_{12}}{\lvert\Delta m^2_{13}\rvert}$	–	$\frac{\Delta m^2_{12}}{\lvert\Delta m^2_{13}\rvert}$	O(1)
$\frac{\sqrt{\Delta m^2_{13}}}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	Inverted	$\frac{\Delta m^2_{12}}{\lvert\Delta m^2_{13}\rvert}$	O(1)	$\frac{\Delta m^2_{12}}{\lvert\Delta m^2_{13}\rvert}$	$\lvert\cos 2\theta_{12}\rvert \sim \frac{\Delta m^2_{12}}{\lvert\Delta m^2_{13}\rvert}$
$\sqrt{\Delta m^2_{13}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	Normal[a]	> 0.1	O(1)	–	O(1)

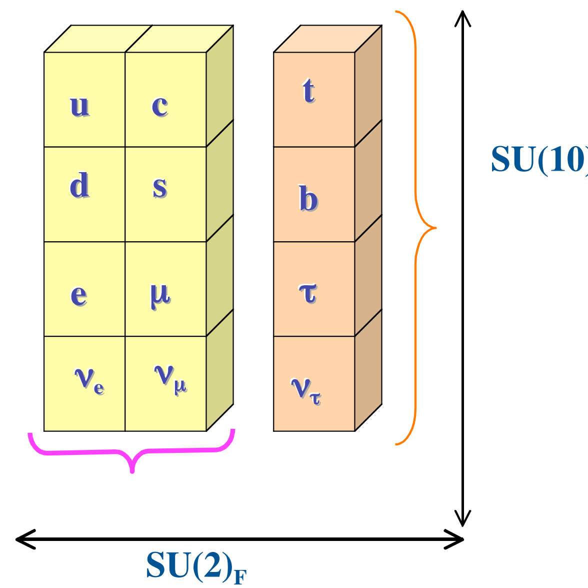
Flavor structure

anarchy vs symmetry

Symmetry Relations

Grand Unified Theories: GUT symmetry (vertical)

Quarks ↔ Leptons



Family Symmetry: (horizontal)

e-family ↔ muon-family ↔ tau-family

Symmetry Relations

Symmetry ⇒ relations among parameters

⇒ reduction in number of fundamental parameters

Symmetry ⇒ experimentally testable correlations among physical observables

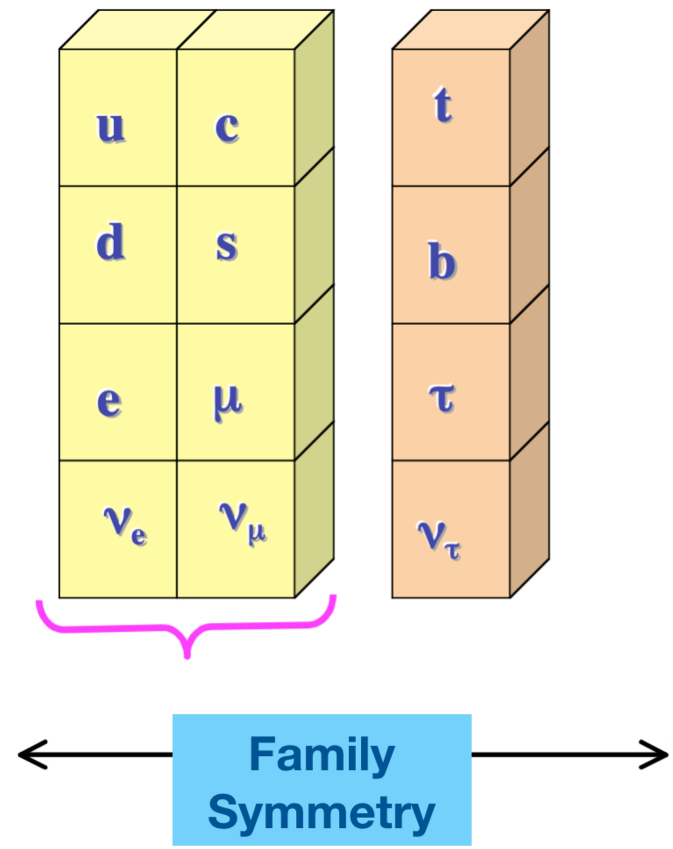
Froggatt-Nielsen Mechanism

☞ Popular scenario for addressing flavor hierarchies: Froggatt–Nielsen scenario

Froggatt and Nielsen [1979]

☞ E.g. effective Lagrangean for charged lepton masses

$$\mathscr{L} \supset y_0^{f\,g} \left(\frac{\widetilde{S}}{\Lambda}\right)^{n_{f\,g}} \overline{e}_{\mathrm{R}}^{g} \cdot \phi^* \cdot \ell_{\mathrm{L}}^{f} + \text{h.c.}$$



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$$n_{fg} = q_{\mathrm{R}}^{(f)} - q_{\mathrm{L}}^{(g)}$$

$$\mathcal{O}(1)$$

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flavon

- Assume $\widetilde{S}$ acquires VEV $\upsilon_{\tilde{S}} \sim \lambda\Lambda$

$$\begin{pmatrix} \lambda^{n_{11}} & \lambda^{n_{12}} & \lambda^{n_{13}} \\ \lambda^{n_{21}} & \lambda^{n_{22}} & \lambda^{n_{23}} \\ \lambda^{n_{31}} & \lambda^{n_{32}} & \lambda^{n_{33}} \end{pmatrix}.$$

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☞ Assume $\widetilde{S}$ acquires VEV $v_{\tilde{S}} \sim \lambda\Lambda$

➥ Hierarchical Yukawa couplings and nontrivial mixing angles

$$\begin{pmatrix} \lambda^{n_{11}} & \lambda^{n_{12}} & \lambda^{n_{13}} \\ \lambda^{n_{21}} & \lambda^{n_{22}} & \lambda^{n_{23}} \\ \lambda^{n_{31}} & \lambda^{n_{32}} & \lambda^{n_{33}} \end{pmatrix} .$$

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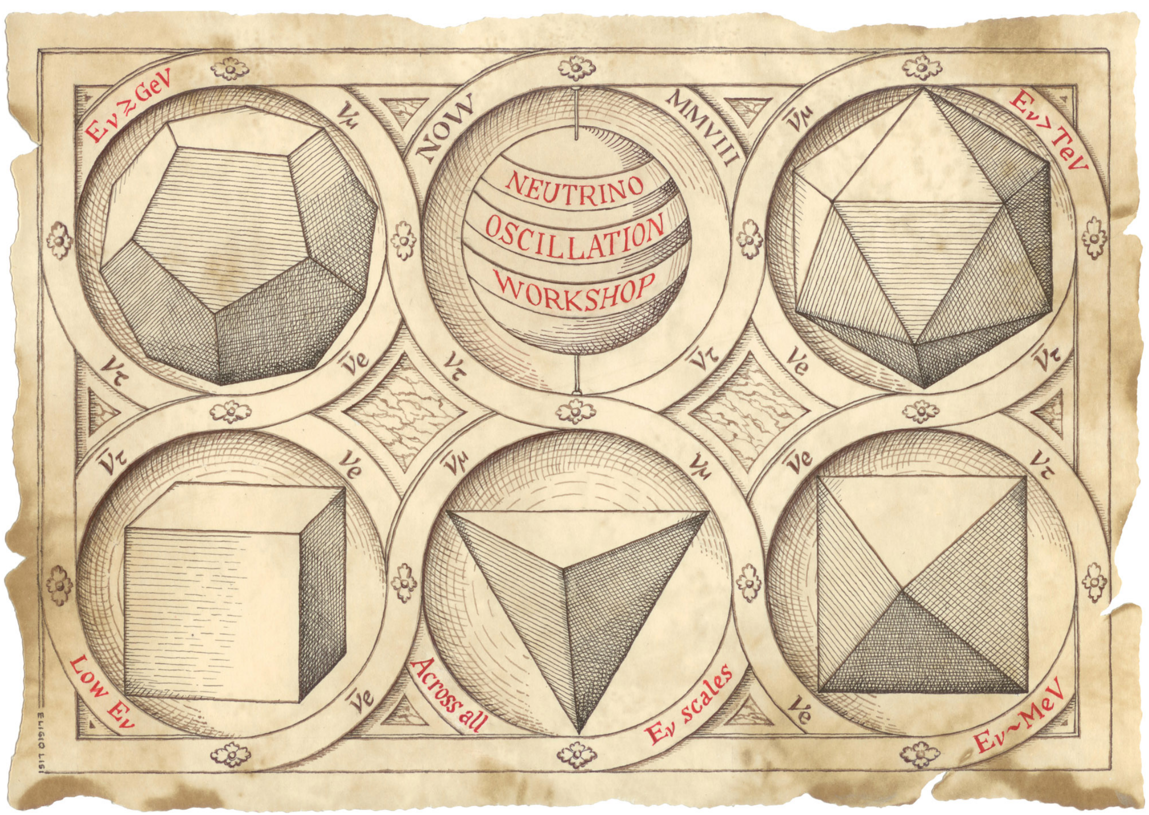
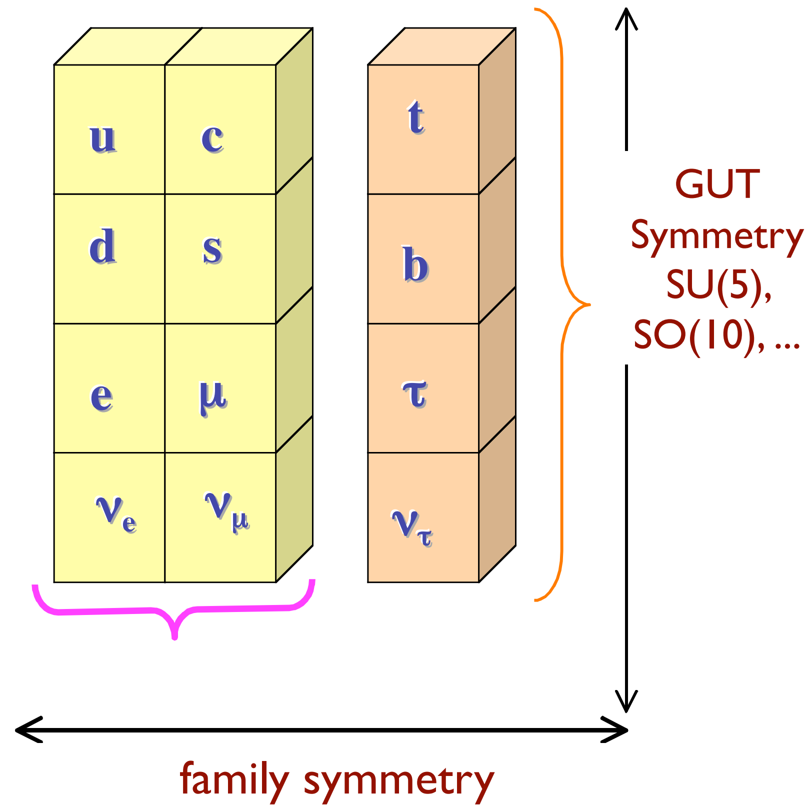
Gauging the U(1) symmetry, what constraints do we have on the model?

Origin of Flavor Mixing and Mass Hierarchy

Large neutrino mixing
$\Rightarrow$ discrete family symmetry

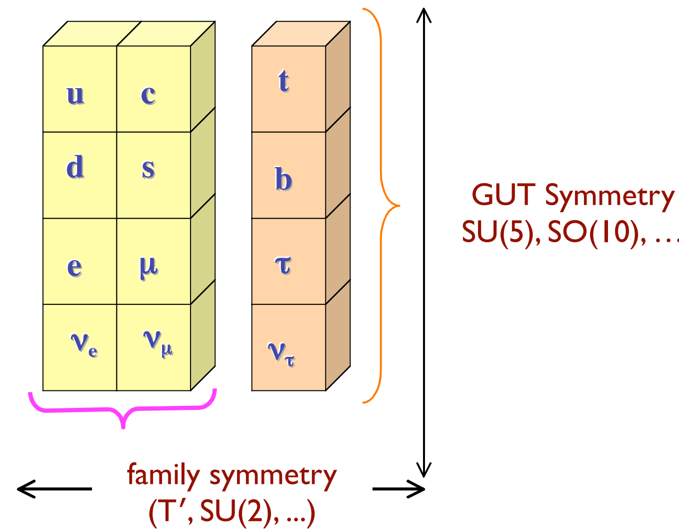


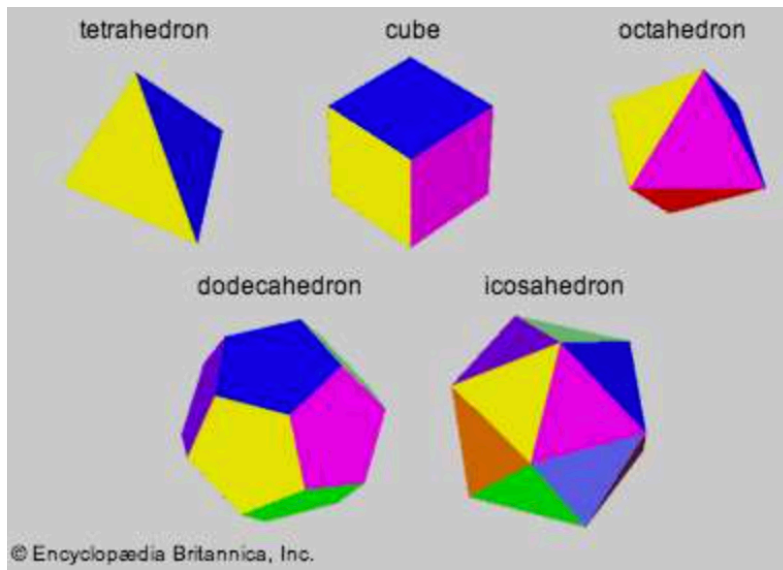
[Eligio Lisi for NOW2008]



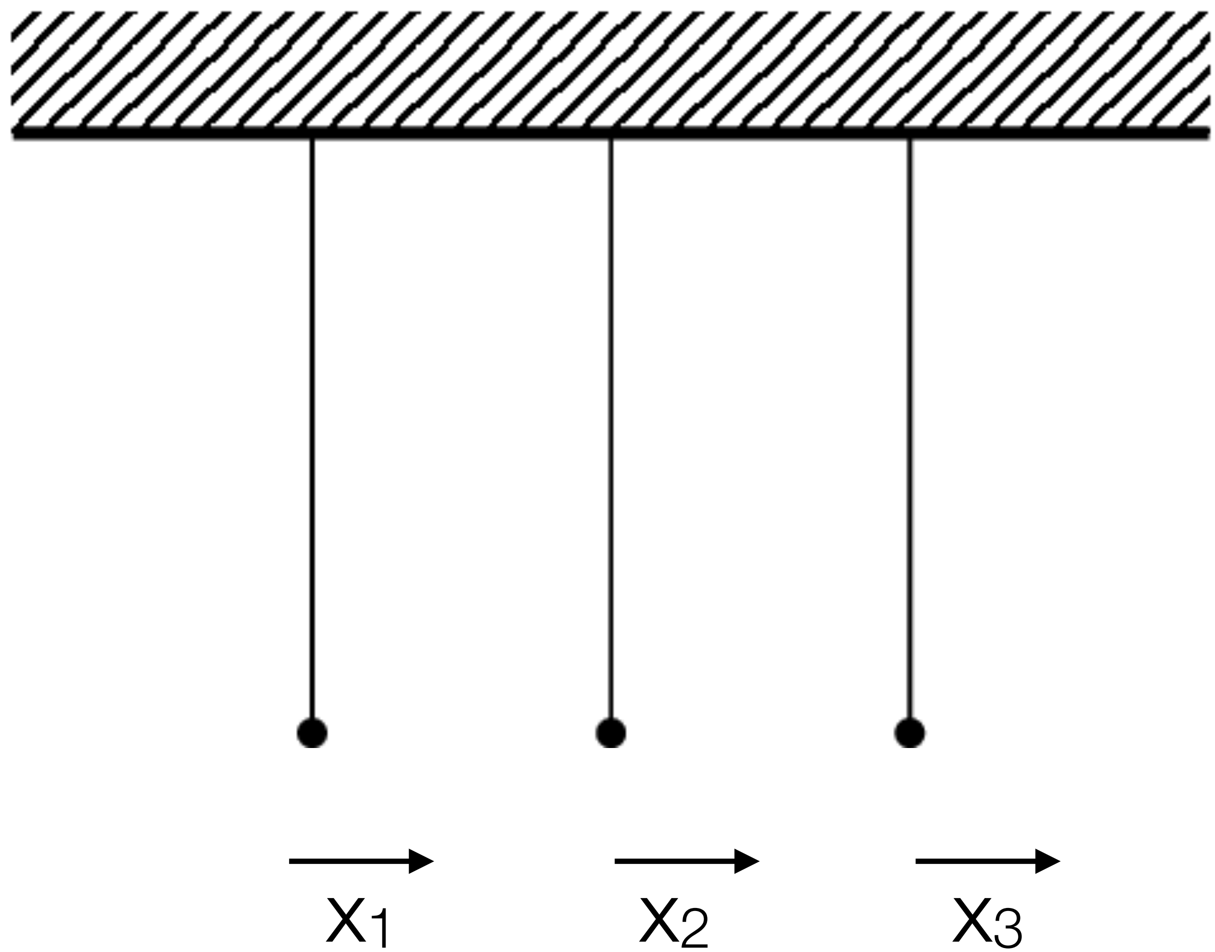
Origin of Flavor Mixing and Mass Hierarchies

- several models have been constructed based on
    - GUT Symmetry [SU(5), SO(10)] ⊕ Family Symmetry G_F
- models based on discrete family symmetry groups have been constructed
    - A_4 (tetrahedron)
    - T´ (double tetrahedron)
    - S_3 (equilateral triangle)
    - S_4 (octahedron, cube)
    - A_5 (icosahedron, dodecahedron)
    - Δ_{27}
    - Q_6

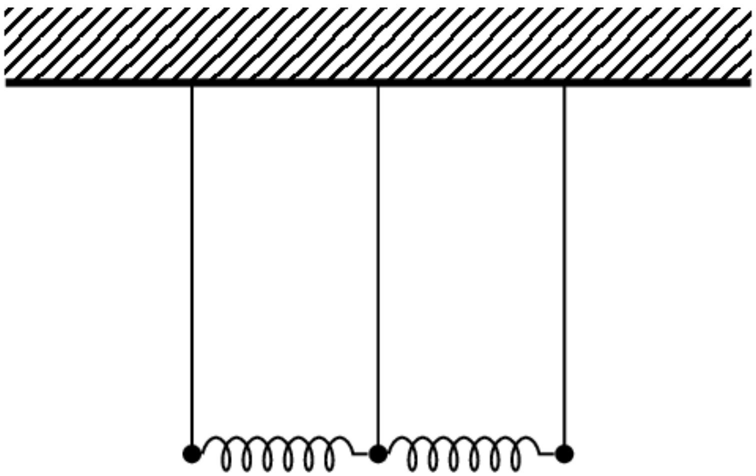




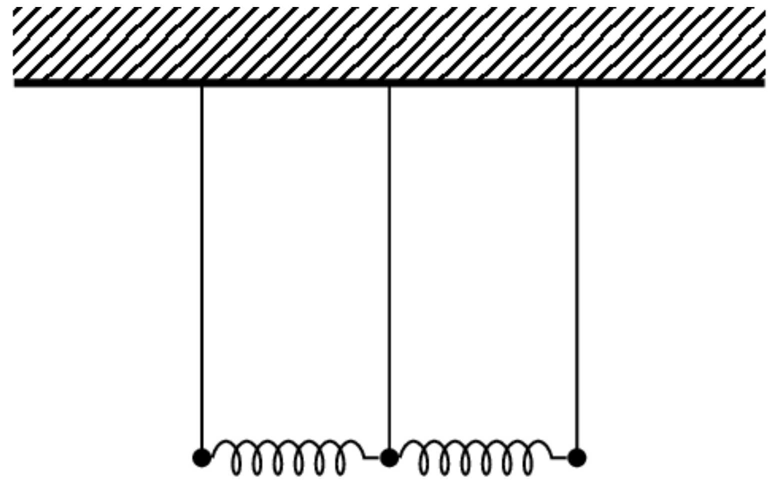
TBM and Coupled Pendulums



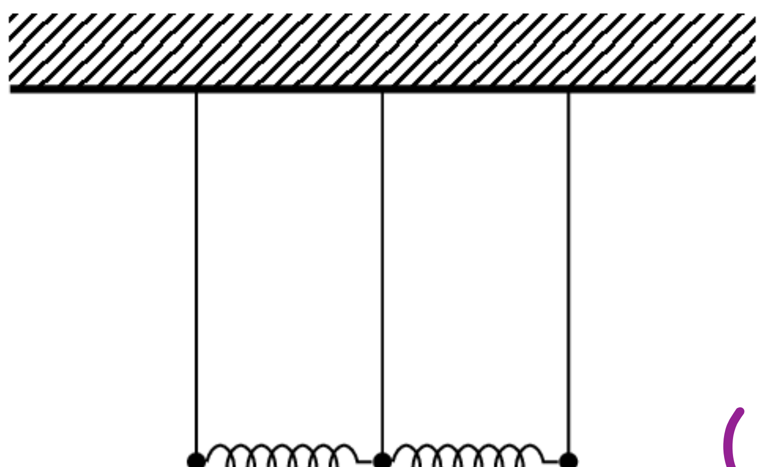
TBM and Coupled Pendulums



(1, 0, -1)



(1, 1, 1)



(1, -2, 1)

Tri-bimaximal Neutrino Mixing

- **Latest Global Fit (3σ)**

Esteban, Gonzalez-Garcia, Maltoni, Schwetz, Zhou (2020)

$$\sin^2\theta_{23} = 0.437\ (0.374 - 0.626)$$ $[\theta^{lep}_{23} \sim 49.2°]$

$$\sin^2\theta_{12} = 0.308\ (0.259 - 0.359)$$ $[\theta^{lep}_{12} \sim 33.4°]$

$$\sin^2\theta_{13} = 0.0234\ (0.0176 - 0.0295)$$ $[\theta^{lep}_{13} \sim 8.57°]$

- **Tri-bimaximal Mixing Pattern**

Harrison, Perkins, Scott (1999)

$$U_{TBM} = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/6} & \sqrt{1/3} & -\sqrt{1/2} \\ -\sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \end{pmatrix}$$

$$\sin^2\theta_{\mathrm{atm,\,TBM}} = 1/2 \qquad \sin^2\theta_{\odot,\mathrm{TBM}} = 1/3$$

$$\sin\theta_{13,\mathrm{TBM}} = 0.$$

Non-Abelian Finite Family Symmetry A_4

- TBM mixing matrix: can be realized with finite group family symmetry based on A_4 Ma, Rajasekaran (2001); Babu, Ma, Valle (2003); ...

- A_4: even permutations of 4 objects

- How many such permutations (i.e. group elements)?

Non-Abelian Finite Family Symmetry A_4

- TBM mixing matrix: can be realized with finite group family symmetry based on A_4 Ma, Rajasekaran (2001); Babu, Ma, Valle (2003); ...

- A_4: even permutations of 4 objects

  S: (1234) → (4321)

  T: (1234) → (2314)

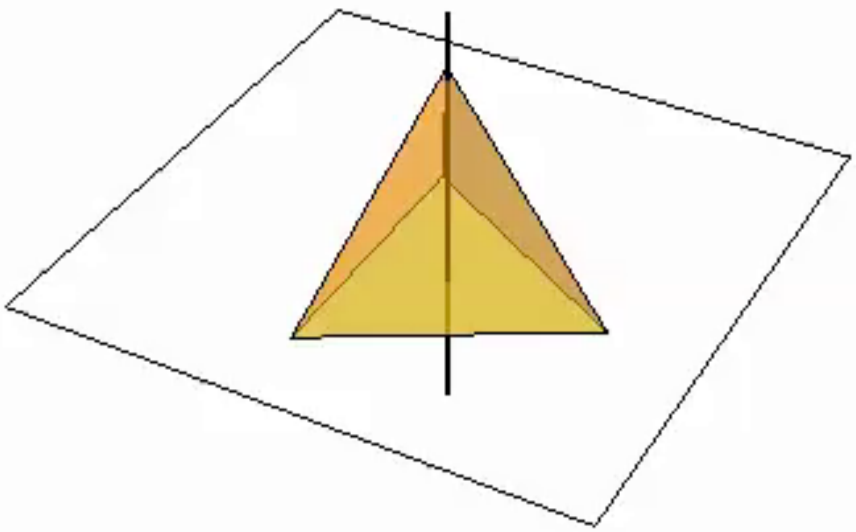
- Group of order 12
- Invariant group of tetrahedron

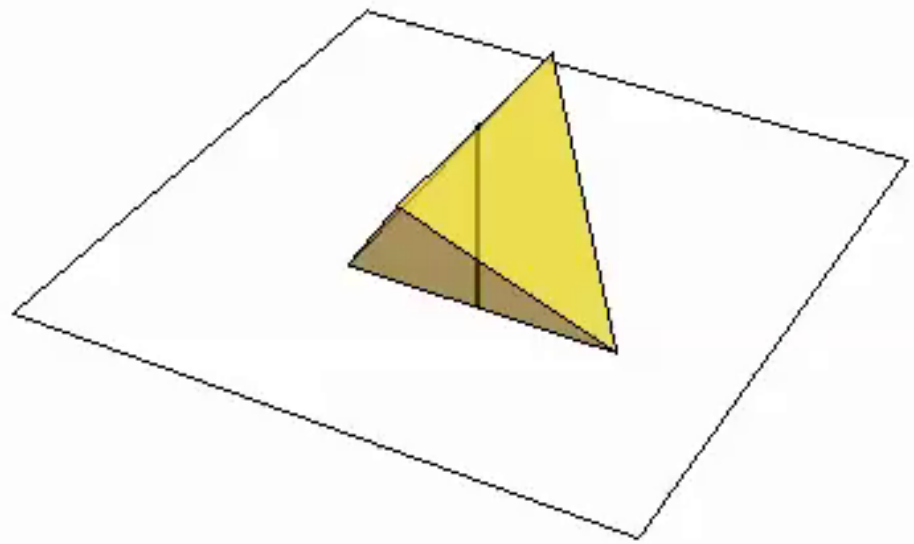
TBM from A4 Group

T: (1234) → (2314)

S: (1234) →(4321)

$$S^2 = 1, \quad (ST)^3 = 1, \quad T^3 = 1$$





A4 Group Theory

Irreps: $1, 1', 1'', 3$

$S^2 = 1, \quad (ST)^3 = 1, \quad T^3 = 1$

$$1: \quad S = 1, T = 1$$

$$1': \quad S = 1, T = e^{4\pi i/3} \equiv \omega^2$$

$$1'': \quad S = 1, T = e^{2\pi i/3} \equiv \omega$$

$$3: \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

12 group elements: $1, S, T, ST, TS, T^2, ST^2, STS, TST, T^2S, TST^2, T^2ST$

Tri-bimaximal Neutrino Mixing

Altarelli, Feruglio (2005)

- fermion charge assignments:

$$\begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix}_L \sim 3, \quad e_R \sim 1, \quad \mu_R \sim 1'', \quad \tau_R \sim 1'$$

- SM Higgs ~ singlet under A4
- operators for neutrino masses: $\frac{HHLL}{M}\left(\frac{\langle\xi\rangle}{\Lambda} + \frac{\langle\eta\rangle}{\Lambda}\right)$
- two scalar (flavon) fields for neutrino sector: $\xi \sim 3, \quad \eta \sim 1$

$$A_4 \to G_{TST^2}: \quad \langle\xi\rangle = \xi_0\Lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad A_4 - \text{invariant:} \quad \langle\eta\rangle = u\Lambda$$

- product rules: $3 \otimes 3 = 3 \oplus 3 \oplus 1 \oplus 1' \oplus 1''$

Tri-bimaximal Neutrino Mixing

- Neutrino Masses: triplet flavon contribution

Altarelli, Feruglio (2005)

$$3_S = \frac{1}{3}\begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1 \end{pmatrix} \qquad 1 = \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2$$

- Neutrino Masses: singlet flavon contribution $\quad 1 = \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2$
- resulting mass matrix:

$$M_\nu = \frac{\lambda v^2}{M_x}\begin{pmatrix} 2\xi_0 + u & -\xi_0 & -\xi_0 \\ -\xi_0 & 2\xi_0 & u - \xi_0 \\ -\xi_0 & u - \xi_0 & 2\xi_0 \end{pmatrix} \qquad U_{\mathrm{TBM}} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -\sqrt{1/6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -\sqrt{1/6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

$$V_\nu^{\mathrm{T}} M_\nu V_\nu = \mathrm{diag}(u + 3\xi_0,\, u,\, -u + 3\xi_0)\frac{v_u^2}{M_x}$$

Form diagonalizable:
-- no adjustable parameters
-- neutrino mixing from CG coefficients!

Tri-bimaximal Neutrino Mixing from A$_4$

Altarelli, Feruglio (2005)

- charged lepton sector -- without quarks
  - operators for charged lepton masses

$$(\ell\phi)_1 e_R(1) + (\ell\phi)_{1'}\mu_R(1'') + (\ell\phi)_{1''}\tau_R(1')$$

  - scalar sector: flavon triplet for charged lepton masses

$$\begin{aligned} 1 &= \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 \\ 1' &= \alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1 \\ 1'' &= \alpha_2\beta_2 + \alpha_1\beta_3 + \alpha_3\beta_1 \end{aligned}$$

$$A_4 \to G_T : \quad \langle\phi\rangle = \phi_0\Lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- resulting charged lepton mass matrix = diagonal

$$V_{MNS} = V_{e,L}^{\dagger} V_{\nu} = \mathcal{I}\cdot U_{TBM} = U_{TBM}$$

Group Work: A4 Neutrino Mass Model

$$M_\nu = \frac{\lambda v^2}{M_x}\begin{pmatrix} 2\xi_0 + u & -\xi_0 & -\xi_0 \\ -\xi_0 & 2\xi_0 & u-\xi_0 \\ -\xi_0 & u-\xi_0 & 2\xi_0 \end{pmatrix}$$

$$U_{\mathrm{TBM}} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -\sqrt{1/6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -\sqrt{1/6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

Show that M_ν is always diagonalizable by U_{TBM}

What are the mass eigenvalues?

Group Work: A4 Neutrino Mass Model

$$M_\nu = \frac{\lambda v^2}{M_x} \begin{pmatrix} 2\xi_0 + u & -\xi_0 & -\xi_0 \\ -\xi_0 & 2\xi_0 & u - \xi_0 \\ -\xi_0 & u - \xi_0 & 2\xi_0 \end{pmatrix}$$

$$U_{\mathrm{TBM}} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -\sqrt{1/6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -\sqrt{1/6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

$$M^{\nu}_{eff}{}^{\mathrm{diag}} = U^T_{\mathrm{TBM}} M^{\nu}_{eff} U_{\mathrm{TBM}} = \mathrm{diag}(3\alpha_s + \alpha_0,\ \alpha_0,\ 3\alpha_s - \alpha_0) \cdot \frac{v^2}{\Lambda_L} \equiv (m_1,\ m_2,\ m_3)$$

Neutrino Mass Matrix from A4

Ma, Rajasekaran (2001); Babu, Ma, Valle (2003); Altarelli, Feruglio (2005)

$$M_\nu = \frac{\lambda v^2}{M_x} \begin{pmatrix} 2\xi_0 + u & -\xi_0 & -\xi_0 \\ -\xi_0 & 2\xi_0 & u - \xi_0 \\ -\xi_0 & u - \xi_0 & 2\xi_0 \end{pmatrix}$$

2 free parameters

relative strengths ⇒ CG's

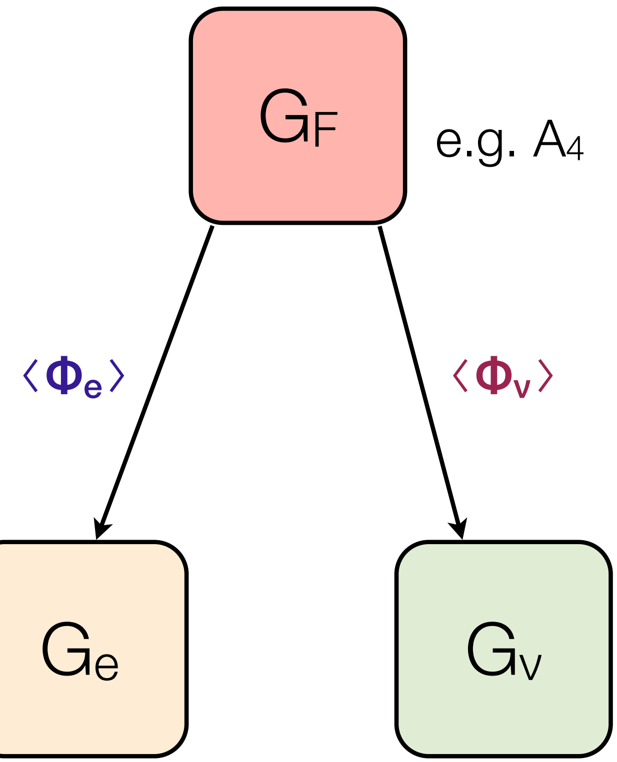
- always diagonalized by TBM matrix, independent of the two free parameters

$$U_{\rm TBM} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -\sqrt{1/6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -\sqrt{1/6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

Neutrino Mixing Angles from Group Theory

- 2 independent parameters for 3 masses ⇒ 1 relation

General Structure



charged lepton sector e.g. Z_3 subgroup of A_4

neutrino sector e.g. Z_2 subgroup of A_4

$\langle \Phi_e \rangle \propto (1,0,0)$

$\langle \Phi_\nu \rangle \propto (1,1,1)$

Example: SU(5) Compatibility ⇒ T′ Family Symmetry

M.-C.C, K.T. Mahanthappa (2007, 2009)

- Double Tetrahedral Group T´: double covering of A4
- Symmetries ⇒ 10 parameters in Yukawa sector ⇒ 22 physical observables
- Symmetries ⇒ **correlations among quark and lepton mixing parameters**

$$\theta_{13} \simeq \theta_c / 3\sqrt{2}$$

← CG's of SU(5) & T´

no free parameters!

$$\tan^2\theta_\odot \simeq \tan^2\theta_{\odot,TBM} + \frac{1}{2}\theta_c \cos\delta$$

neutrino solar mixing

1/2

quark Cabibbo mixing

leptonic CP phase

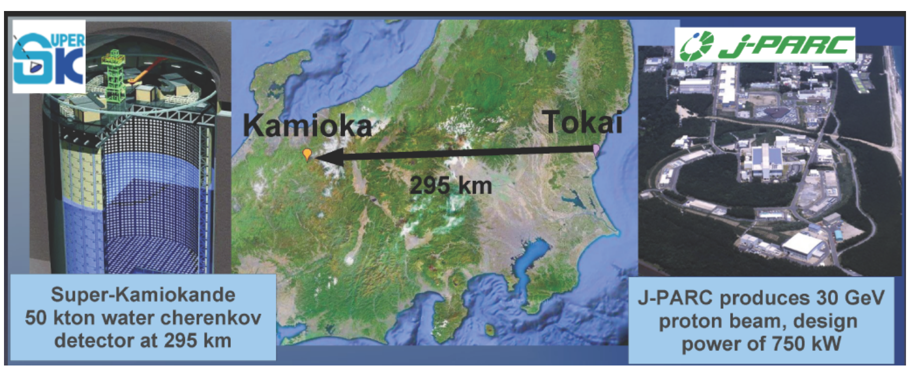
CP Violation in Neutrino Oscillation

- With leptonic Dirac CP phase $\delta \neq 0$ ➜ leptonic CP violation
- Predict different transition probabilities for neutrinos and antineutrinos

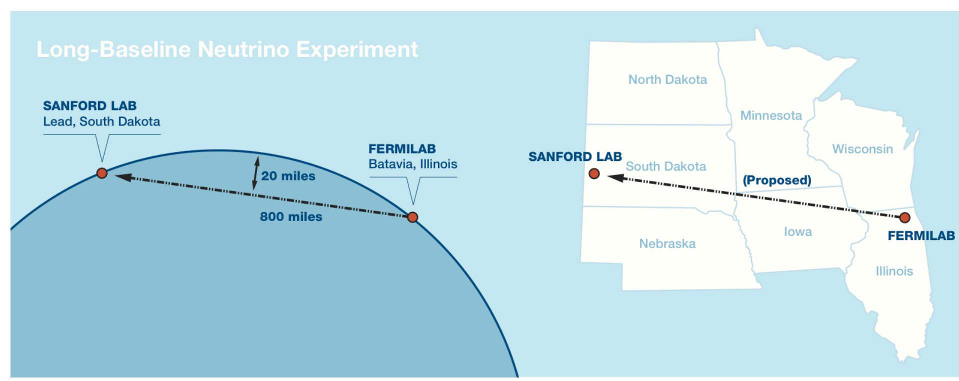
$$P\,(\nu_\alpha \rightarrow \nu_\beta) \neq P\,(\overline{\nu_\alpha} \rightarrow \overline{\nu_\beta})$$

- One of the major scientific goals at current and planned neutrino experiments

T2K



DUNE/LBNF

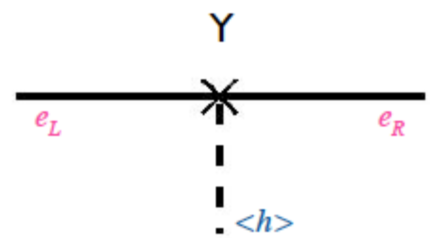


Origin of CP Violation

- CP violation ⇔ complex mass matrices

$$\overline{U}_{R,i}(M_u)_{ij}Q_{L,j} + \overline{Q}_{L,j}(M_u^\dagger)_{ji}U_{R,i} \xrightarrow{\mathcal{CP}} \overline{Q}_{L,j}(M_u)_{ij}U_{R,i} + \overline{U}_{R,i}(M_u)^*_{ij}Q_{L,j}$$

- Conventionally, CPV arises in two ways:
    - Explicit CP violation: complex Yukawa coupling constants Y
    - Spontaneous CP violation: complex scalar VEVs <h>



- **Complex CG coefficients in certain discrete groups ⇒ explicit CP violation**
    - CPV in quark and lepton sectors purely from complex CG coefficients

M.-C.C., K.T. Mahanthappa, Phys. Lett. B681, 444 (2009)

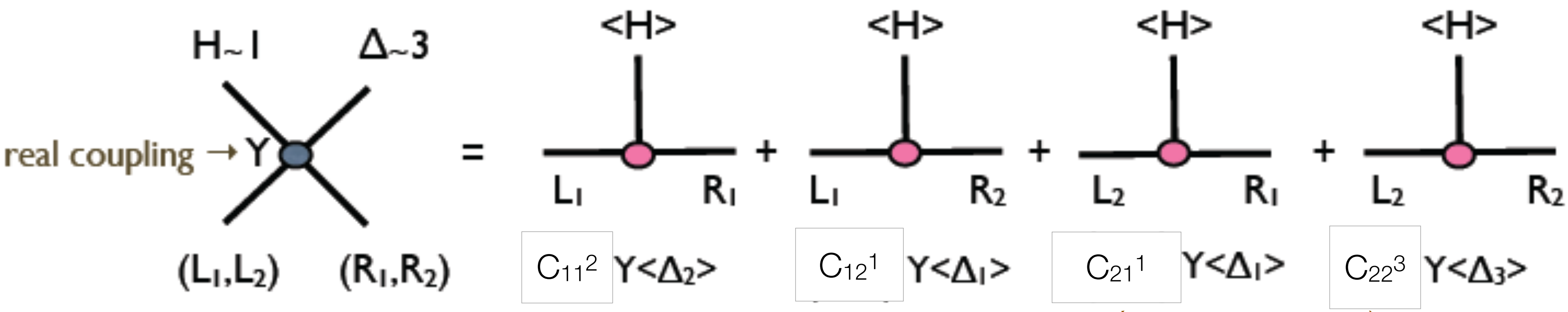
CG coefficients in non-Abelian discrete symmetries
➪ relative strengths and phases in entries of Yukawa matrices
➪ mixing angles and phases (and mass hierarchy)

Group Theoretical Origin of CP Violation

M.-C.C., K.T. Mahanthappa
Phys. Lett. B681, 444 (2009)

Basic idea

Discrete symmetry ***G***



$C_{ij}{}^{k}$: complex CG coefficients of ***G***

- if Z_3 symmetric ⇒ $\langle \Delta_1 \rangle = \langle \Delta_2 \rangle = \langle \Delta_3 \rangle \equiv \langle \Delta \rangle$ real
- Complex effective mass matrix: **phases determined by group theory**

$$(L_1 \quad L_2)\; M = \begin{pmatrix} C_{11}{}^{2} & C_{21}{}^{1} \\ C_{12}{}^{1} & C_{22}{}^{3} \end{pmatrix} Y \langle \Delta \rangle \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

CP Transformation

- Canonical CP transformation

$$\phi(x) \overset{\mathcal{CP}}{\longmapsto} \eta_{\mathcal{CP}}\ \phi^*(\mathcal{P}x)$$

freedom of re–phasing fields

- Generalized CP transformation

Ecker, Grimus, Konetschny (1981); Ecker, Grimus, Neufeld (1987); Grimus, Rebelo (1995)

$$\Phi(x) \overset{\widetilde{\mathcal{CP}}}{\longmapsto} U_{\mathrm{CP}}\, \Phi^*(\ \mathcal{P}\ \ x)$$

unitary matrix

Generalized CP Transformation

- setting w/ discrete symmetry G

G and CP transformations do not commute

- generalized CP transformation

Feruglio, Hagedorn, Ziegler (2013); Holthausen, Lindner, Schmidt (2013)

- invariant contraction/coupling in A_4 or T'

$$\left[\phi_{\mathbf{1}_2} \otimes (x_{\mathbf{3}} \otimes y_{\mathbf{3}})_{\mathbf{1}_1}\right]_{\mathbf{1}_0} \propto \phi \left(x_1\, y_1 + \omega^2\, x_2\, y_2 + \omega\, x_3\, y_3\right)$$

$\omega = \mathrm{e}^{2\pi\,\mathrm{i}/3}$

- canonical CP transformation maps A_4/T' invariant contraction to something non–invariant

Group Work: Generalized CP Transformation

$$\left[\phi_{\mathbf{1}_2} \otimes (x_{\mathbf{3}} \otimes y_{\mathbf{3}})_{\mathbf{1}_1}\right]_{\mathbf{1}_0} \propto \phi \left(x_1 y_1 + \omega^2 x_2 y_2 + \omega x_3 y_3\right)$$

$\omega = \mathrm{e}^{2\pi\,\mathrm{i}/3}$

Is there a unitary transformation that can "repair" the A4 invariance once a naïve CP transformation is performed?

Generalized CP Transformation

- ☞ setting w/ discrete symmetry G

G and CP transformations do not commute

- ☞ generalized CP transformation

Feruglio, Hagedorn, Ziegler (2013); Holthausen, Lindner, Schmidt (2013)

- ☞ invariant contraction/coupling in A_4 or T'

$$\left[\phi_{\mathbf{1}_2} \otimes (x_{\mathbf{3}} \otimes y_{\mathbf{3}})_{\mathbf{1}_1}\right]_{\mathbf{1}_0} \propto \phi\,\left(x_1 y_1 + \omega^2 x_2 y_2 + \omega\, x_3 y_3\right)$$

$\omega = \mathrm{e}^{2\pi\,\mathrm{i}/3}$

- ☞ canonical CP transformation maps A_4/T' invariant contraction to something non–invariant

➥ need generalized CP transformation $\widetilde{\mathcal{CP}}$: $\phi \overset{\widetilde{\mathcal{CP}}}{\longmapsto} \phi^*$ as usual but

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \overset{\widetilde{\mathcal{CP}}}{\longmapsto} \begin{pmatrix} x_1^* \\ x_3^* \\ x_2^* \end{pmatrix} \quad \& \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \overset{\widetilde{\mathcal{CP}}}{\longmapsto} \begin{pmatrix} y_1^* \\ y_3^* \\ y_2^* \end{pmatrix}$$

How (Not) to Generalize CP

proper CP transformations

- ☞ map field operators to *their own* Hermitean conjugates
- ☞ violation of physical CP is prerequisite for a non–trivial

$$\varepsilon_{i\to f} = \frac{\left|\Gamma\left(i\to f\right)\right|^2 - \left|\Gamma\left(\bar{\imath}\to \bar{f}\right)\right|^2}{\left|\Gamma\left(i\to f\right)\right|^2 + \left|\Gamma\left(\bar{\imath}\to \bar{f}\right)\right|^2}$$

➥ connection to observed ~~CP~~, baryogenesis & . . .

CP–like transformations

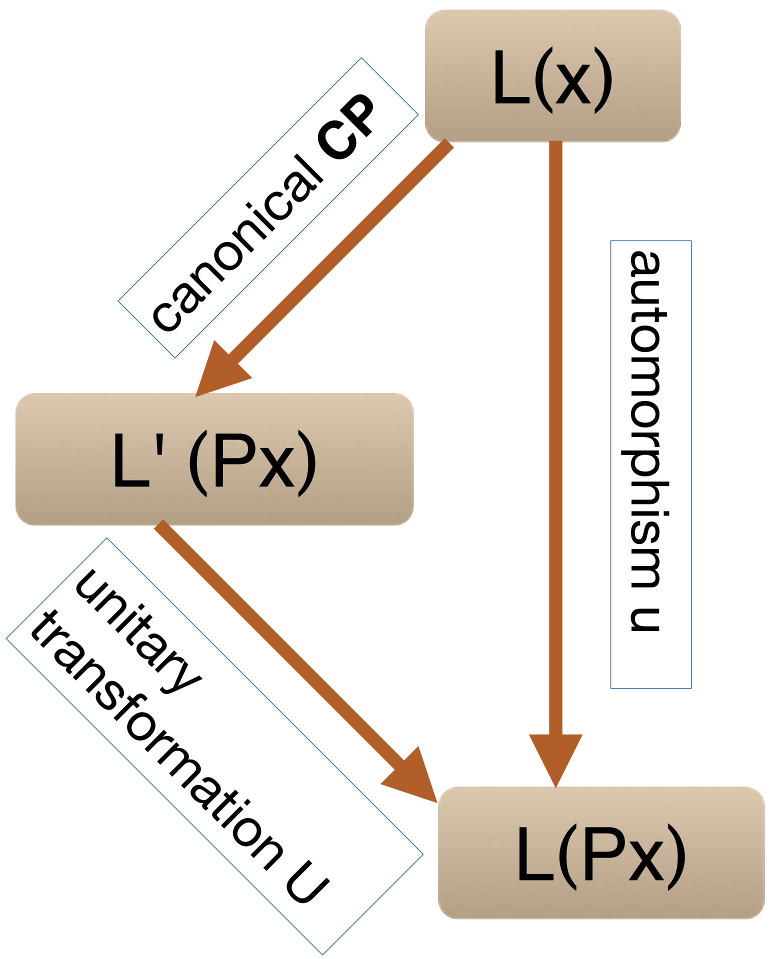
- ☞ map some field operators to some other operators
- ☞ such transformations have sometimes been called “generalized CP transformations” in the literature
- ☞ however, imposing CP–like transformations does **not** imply physical CP conservation

➥ **NO** connection to observed ~~CP~~, baryogenesis & . . .

Group Theoretical Origin of CP Violation

M.-C.C, M. Fallbacher, K.T. Mahanthappa, M. Ratz, A. Trautner, NPB (2014)

complex CGs ➪ G and physical CP transformations do not commute



$$\Phi(x) \xmapsto{\widetilde{CP}} U_{\mathrm{CP}}\, \Phi^*(\mathcal{P}\ x)$$

$$\rho_{\boldsymbol{r}_i}(u(g)) = U_{\boldsymbol{r}_i}\, \rho_{\boldsymbol{r}_i}(g)^*\, U^\dagger_{\boldsymbol{r}_i} \quad \forall\, g \in G \text{ and } \forall\, i$$

u has to be a **class-inverting,**
involutory automorphism of G

➪ non-existence of such automorphism in certain groups

➪ calculable physical CP violation in generic setting

examples: T_7, $\Delta(27)$, …..

The Bickerstaff-Damhus automorphism (BDA)

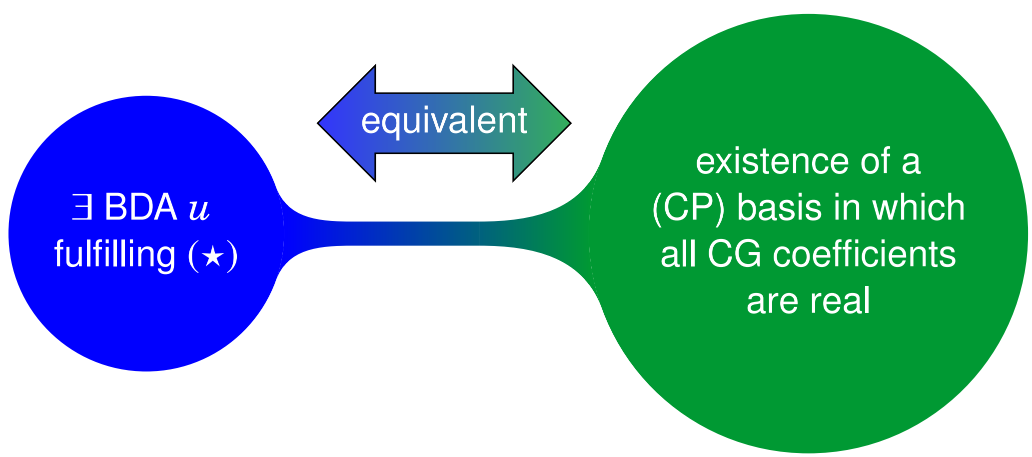
- Bickerstaff-Damhus automorphism (BDA) $\boldsymbol{u}$

Bickerstaff, Damhus (1985)

$$\rho_{\boldsymbol{r}_i}(u(g)) \;=\; U_{\boldsymbol{r}_i}\;\rho_{\boldsymbol{r}_i}(g)^*\,U_{\boldsymbol{r}_i}^\dagger \quad \forall\, g \in G \text{ and } \forall\, i \qquad (\,\star\,)$$

unitary & symmetric

- BDA vs. Clebsch-Gordan (CG) coefficients



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Twisted Frobenius-Schur Indicator

- How can one tell whether or not a given automorphism is a BDA?
- Frobenius-Schur indicator:

$$\mathrm{FS}(\boldsymbol{r}_i) := \frac{1}{|G|} \sum_{g \in G} \chi_{\boldsymbol{r}_i}(g^2) = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}\left[\rho_{\boldsymbol{r}_i}(g)^2\right]$$

$$\mathrm{FS}(\boldsymbol{r}_i) = \begin{cases} +1, & \text{if } \boldsymbol{r}_i \text{ is a real representation,} \\ 0, & \text{if } \boldsymbol{r}_i \text{ is a complex representation,} \\ -1, & \text{if } \boldsymbol{r}_i \text{ is a pseudo–real representation.} \end{cases}$$

- Twisted Frobenius-Schur indicator

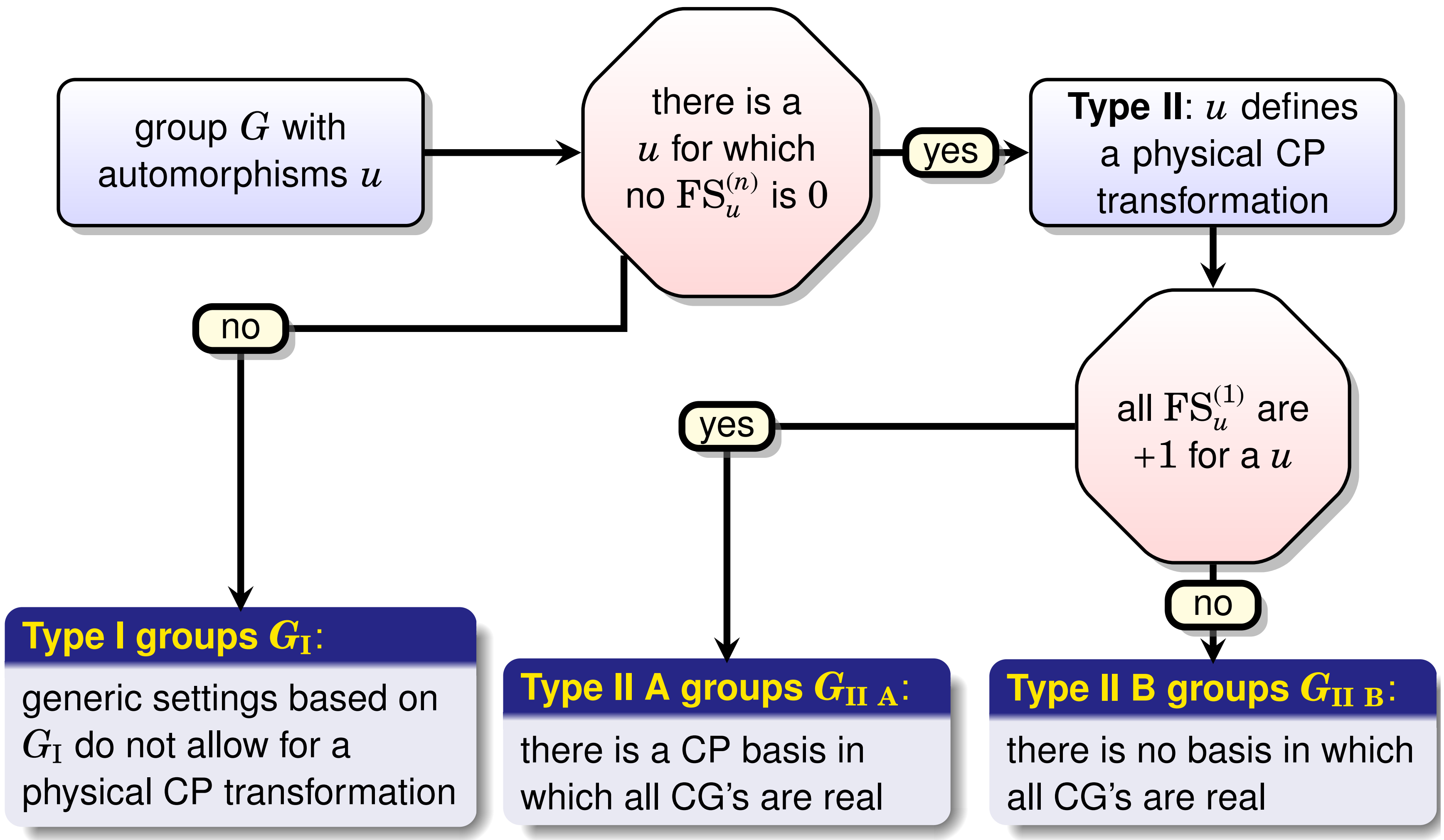
Bickerstaff, Damhus (1985); Kawanaka, Matsuyama (1990)

$$\mathrm{FS}_u(\boldsymbol{r}_i) = \frac{1}{|G|} \sum_{g \in G} \left[\rho_{\boldsymbol{r}_i}(g)\right]_{\alpha\beta} \left[\rho_{\boldsymbol{r}_i}(u(g))\right]_{\beta\alpha}$$

- $$\mathrm{FS}_u(\boldsymbol{r}_i) = \begin{cases} +1 \quad \forall\, i, & \text{if } u \text{ is a BDA,} \\ +1 \text{ or } -1 \quad \forall\, i, & \text{if } u \text{ is class–inverting and involutory,} \\ \text{different from } \pm 1, & \text{otherwise.} \end{cases}$$

Three Types of Finite Groups

M.-C.C., M. Fallbacher, K.T. Mahanthappa, M. Ratz, A. Trautner (2014)

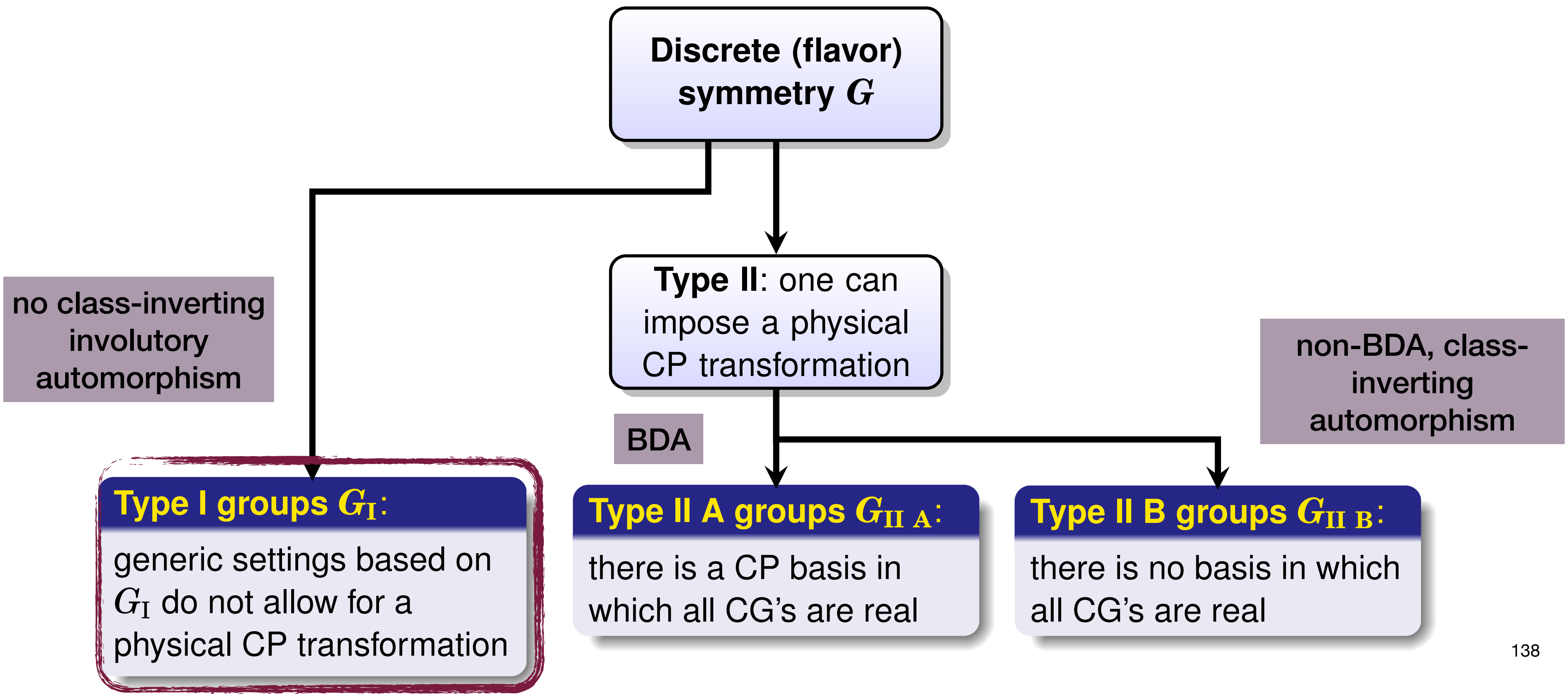


A Novel Origin of CP Violation

M.-C.C, M. Fallbacher, K.T. Mahanthappa, M. Ratz,
A. Trautner, NPB (2014)

- **For discrete groups that do not have class-inverting, involutory automorphism, CP is generically broken by complex CG coefficients (Type I Group)**
- **Non-existence of such automorphism ⇔ Physical CP violation**

CP Violation from Group Theory!



Examples

M.-C.C., M. Fallbacher, K.T. Mahanthappa, M. Ratz, A. Trautner (2014)

- Type I: all odd order non-Abelian groups

group	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	T_7	$\Delta(27)$	$\mathbb{Z}_9 \rtimes \mathbb{Z}_3$
SG	(20,3)	(21,1)	(27,3)	(27,4)

- Type IIA: dihedral and all Abelian groups

group	S_3	Q_8	A_4	$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	T′	S_4	A_5
SG	(6,1)	(8,4)	(12,3)	(24,1)	(24,3)	(24,12)	(60,5)

- Type IIB

group	$\Sigma(72)$	$((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$
SG	(72,41)	(144,120)

Novel Origin of CP (Time Reversal) Violation

complex CGs ⇨ CP symmetry cannot be defined for certain groups

CP Violation from Group Theory!