INVISIBLES 2023 : ANOMALOUS SYMMETRIES

① Key Ideas

, Informal defⁿ: anomaly = a quantum obstruction to a classical symmetry

- Classical symmetry: $\mathcal{L} \to \mathcal{L} + \partial_\mu k^\mu$ (1)

  $\Rightarrow$ EOMs are invariant. [local property]

- Quantum symmetry: $Z \to Z$ (2)

  for simple theories, $Z = \int [D\phi] \exp\left(-2\pi i\, S(\phi)\right)$, $S(\phi) = \int_M \mathcal{L}(\phi)$

- Anomalies are possible b/c (1) $\not\Rightarrow$ (2): can be global (topological) obstructions
- " = 'explicit' symmetry breaking (vs. 'spontaneous' breaking)
- Example: 4d Yang-Mills. Classical th. is scale invariant.
  Quantize, $\beta(g) \neq 0 \Rightarrow$ scale Λ_{QCD} emerges in IR.
  (strong coupling)
  (β-fⁿ ~ 'conformal anomaly')

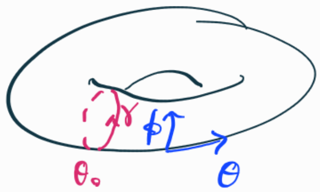
What can go wrong?

①. $\delta\mathcal{L} = \partial_\mu k^\mu \;\not\Rightarrow\; \delta\left(e^{2\pi i S(\phi)}\right) = 0$

topological obstruction.

Example: QM on T^2

$$\mathcal{L} = \frac{1}{2}(\dot\theta^2 + \dot\phi^2) - \underbrace{\frac{N}{4\pi^2}}_{\text{magnetic field}} \frac{1}{2}(\theta\dot\phi - \phi\dot\theta)$$



Classical $U(1)\times U(1)$ translation symmetry

EOMs: $\ddot\theta = \frac{N}{4\pi^2}\dot\phi$, $\ddot\phi = -\frac{N}{4\pi^2}\dot\theta$

$(\theta, \phi) \to (\theta + a, \phi + b)$, $\mathcal{L} \to \left(\mathcal{L} - \underbrace{\frac{N}{8\pi^2} a\dot\phi}_{\text{total deriv}},\ \mathcal{L} + \frac{N}{8\pi^2} b\dot\theta\right)$

But S not invariant if worldline γ wraps non-trivial cycles!

$$S[\gamma] = S_{kin} - \frac{N}{4\pi^2}\theta_0 \int_0^{2\pi} d\phi = S_{kin} - \frac{N\theta_0}{2\pi}$$

not $U(1)_\theta$-inv.!

Anomaly is an explicit breaking: $\underbrace{U(1)\times U(1)}_{\text{class. symm.}} \to \underbrace{\mathbb{Z}_N \times \mathbb{Z}_N}_{\text{qu. symm.}}$

This "basic" anomaly occurs also in higher-dim QFTs.

e.g. CHM on $\frac{SO(5)\times U(1)_X}{SO(4)} \simeq S^4 \times S^1$, $U(1)_X$ can have this anomaly.

2. Measure $\int (D\phi)$ not invariant

Measure over ∞-dim. field space needs regulating:

not always possible to regulate while preserving classical symmetry.

This happens for chiral symmetries of massless fermions.

(main subject of this lecture).

Example: 4d QED, $U(1)_A$: $\psi \to e^{i\alpha\gamma^5}\psi$, $Z \to e^{\frac{i\alpha}{16\pi^2}\int F\tilde{F}} Z$

Explicitly broken; cannot gauge chiral $U(1)_A$

Disaster? SM has chiral EW gauge interaction!!

Anomalies can cancel exactly b/c they are not renormalised ($\because$ quantised: 1-loop exact)

$\to$ This must occur for G in a chiral gauge theory (e.g. SM)

[= a mathematical consistency constraint – more later...]

Plan:

- Maths (2/3): $\mathbb{C}$ ψ's ; $\mathbb{R}$ ψ's ; general classification
- Physics (1/3): (B)SM anomaly cancellation ; EFTs ; QCD ; axion

② Chiral Fermion Anomalies

2.1 Preliminaries: Differential forms.

k-forms:

- totally-antisymmetric $(0,k)$ tensors. Given basis 1-forms $\{dx^\mu\}$ (dual vectors)

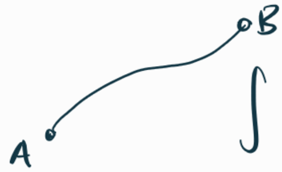
$$\alpha = \frac{1}{k!}\alpha_{[\mu_1 \ldots \mu_k]}\, dx^{\mu_1} \ldots dx^{\mu_k}$$

Examples:

- 0-forms f = smooth f^n
- 1-form $\alpha = \alpha_\mu(x)\, dx^\mu$

  d-dim basis $\{dx^\mu\}$

  e.g. gauge field $A = A_\mu(x)\, dx^\mu$



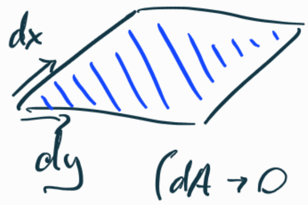
Line integrals: $\int \alpha$

- 2-form $\omega = \frac{1}{2}\omega_{[\mu\nu]}\, dx^\mu dx^\nu$

  $\frac{d(d-1)}{2}$-dim basis $\{dx^\mu \wedge dx^\nu\}$

  e.g. field strength $F = \frac{1}{2}F_{\mu\nu}\, dx^\mu \wedge dx^\nu$

$dx \wedge dy$ = oriented area element



($dA \to 0$ as $dy \to dx$)

Wedge product:

- $(p\text{-form}) \wedge (q\text{-form}) = (p+q)\text{-form}$
- Properties follow from those of signed volume:

$$\alpha_p \wedge \beta_q = (-1)^{pq}\, \beta_q \wedge \alpha_p$$

$$(\alpha + \beta)\wedge \gamma = \alpha\wedge\gamma + \beta\wedge\gamma \qquad \text{etc.}$$

- In components:

$$(\alpha_p \wedge \beta_q)_{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_{p+q}} = \frac{(p+q)!}{p!\,q!}\,\alpha_{[\mu_1 \ldots \mu_p}\beta_{\mu_{p+1} \ldots \mu_{p+q}]}$$

Exterior derivatives:

$$d_k: \ k\text{-form} \to (k+1)\text{-form}, \qquad d_{k+1} \circ d_k = 0$$
$$(\text{"}d^2 = 0\text{"})$$

$$d_k \alpha_k = \frac{1}{k!}\left(\frac{\partial}{\partial x^\nu} \alpha_{\mu_1 \dots \mu_k}\right) dx^\nu \wedge dx^{\mu_1} \dots \wedge dx^{\mu_k}$$

- Example: 3d vector calculus

0-form: $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$ grad

1-form: $d\alpha = \left(\frac{\partial \alpha}{\partial x} - \frac{\partial \alpha}{\partial y}\right) dx \wedge dy + \dots$ curl

2-form: $d\omega = \left(\frac{\partial \omega_{yz}}{\partial x} + \frac{\partial \omega_{zx}}{\partial y} + \frac{\partial \omega_{xy}}{\partial z}\right) dx \wedge dy \wedge dz$ div

- Example: Gauge field

$$F = dA \Rightarrow dF = (d_2 \circ d_1) A = 0,$$

Bianchi identity

Stokes' theorem:

$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

In 3d, $M_2 = 2d$ submfd. $\Rightarrow \int_{M_2} \underline{\nabla} \times \underline{F} = \oint_{\partial M_2} \underline{F} \cdot d\underline{x}$

$M_3 = 3d$ $\Rightarrow \int_{M_3} \underline{\nabla} \cdot \underline{F} = \oint_{\partial M_3} \underline{F} \cdot d\underline{l}$

c.f. Exercise 1

2.2 4d Chiral fermions

▸ Ingredients for chiral gauge theory

- Spacetime Σ (smooth, compact, Riemannian mfd. Σ).
- Spinors $\psi \to -\psi$ under 2π rotation: must "lift" $\underbrace{SO(d) \to Spin(d)}$

requires a "spin structure" σ on Σ_d.
hence $w_1 = w_2 = 0$
(w_i = Stiefel-Whitney classes)

Index theorems easier in Euclidean signature!

- Euclideanized, "spun" Lorentz group in 4d:

$$\boxed{\begin{array}{l} Spin(4) = SU(2)_L \times SU(2)_R \\ \text{4d Weyl spinors} \begin{cases} \psi_L \sim (\underline{2}, \underline{1}) \\ \psi_R \sim (\underline{1}, \underline{2}) \end{cases} \end{array}}$$

- 4d Dirac spinor $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, composed of two Weyls.

- A unitary symmetry G is chiral if it rotates ψ_L & ψ_R differently.

$$\boxed{\begin{array}{l} \text{Dirac mass term} \quad m\bar{\psi}\psi = m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \\ \text{breaks all chiral symmetries.} \end{array}}$$

- Fermion kinetic term, allowing a (possibly non-dynamical) gauge field coupling is built from a Dirac operator:

$$\mathcal{L} = \bar{\psi} i \not{D} \psi, \qquad D = d + A$$

- Fermion partition function

$$Z_\psi[A] = \int D\psi D\bar\psi \, e^{\int_\Sigma \bar\psi i \not{D} \psi} = \mathrm{Det}(i\not{D})$$

product of Gaussian integrals (Grassmann)

▷ The trouble w/. regulating Z_ψ

Two typical regulators for UV divergences:

(i). Pauli-Villars (PV):

- bosonic ghost $\tilde\psi$ for each Weyl ψ, v. high mass Λ
- BUT, mass term $\Lambda \bar{\tilde\psi}\tilde\psi$ breaks any chiral symmetry!

(ii). Dim reg?

- γ^5 in $4-\epsilon$ dimensions?

Converse statement: (any dimension)

$\boxed{\text{if } \psi \text{ admits a mass term, } \exists \text{ PV regulator that manifestly preserves all } G. \iff \text{any massive (gappable) fermion is anomaly-free}}$ (*)

has very direct consequences, esp. in 4d.

Some rep. theory: A rep $\underline{R}$ of symmetry group K is:

1. real iff $\exists$ a K-invariant, symmetric bilinear form h

   e.g. $\underline{N}$ of $SO(N)$; $h(u,v) = \delta_{ij} u^i v^j = u \cdot v$

2. pseudo-real iff $\exists$ a K-inv. AS bilinear ω

   e.g. $\underline{2N}$ of $Sp(2N)$; $\Omega(u,v) = u^T \begin{pmatrix} 0 & -\mathbb{I}_N \\ \mathbb{I}_N & 0 \end{pmatrix} v$

3. complex if there is no invariant bilinear form e.g. $\underline{N}$ of $SU(N \geq 3)$

Invariant mass term requires antisymm. bilinear b/c fermions anticommute (Grassmann)

- 4d, Symm. $K = \mathrm{Spin}(4) \times \underset{\uparrow\ \text{internal}}{G}$

$$\psi_L \sim (\underline{2}, \underline{1}), \quad \psi_R \sim (\underline{1}, \underline{2}) \quad \text{both pseudo-real reps of } \mathrm{Spin}(4)$$

$$\omega(\psi, \psi) = \epsilon_{\alpha\beta}\, \psi^\alpha \psi^\beta \quad , \quad \alpha, \beta = SU(2)_L \text{ indices.}$$

Corollaries of (*):

1. $\underline{R}_G$ real $\Rightarrow$ $\underline{R}_K$ pseudo-real $\Rightarrow$ $\exists$ inv. mass term $\Rightarrow$ anomaly-free

$\mathcal{L} \supset \epsilon_{\alpha\beta}\, h(\psi^\alpha, \psi^\beta)$ e.g. $\epsilon_{\alpha\beta}\, \delta^{ij}\, \psi^\alpha_i \psi^\beta_j$ for $\underline{N}$ of $SO(N)$

e.g. $\underline{3}$ of $G = A_4$

2. $\underline{R}_G$ p-real $\Rightarrow$ $\underline{R}_K$ real $\Rightarrow$ mass term vanishes by Fermi stats, but can form non-vanishing mass if $\exists \times 2$ copies ψ, χ:

$$\mathcal{L} \supset \epsilon_{\alpha\beta}\, w^{ij}\, \psi_i^\alpha \chi_j^\beta$$

$\therefore$ can give mass in pairs $\Rightarrow$ at most a mod-2 anomaly!

Example: $SU(2)$ anomaly for $\psi_L \sim \underline{2}$ (Witten '1982)

3. $\underline{R}_G$ $\mathbb{C}$ $\Rightarrow$ all bets are off!

Perturbative anomalies (seen at weak coupling, $Z[A^{g\simeq 1}] \neq Z[A]$) are possible. e.g. $\psi_L \sim \underline{3}$ of $SU(3)$ [LH quarks]

$$\boxed{\text{4. The anomaly is always a PHASE}}$$

Any rep; any dimension!

Proof:

$\rightarrow$ consider $\psi \oplus \bar{\psi}$, $S[\bar{\psi}] = S[\psi]^*$, always gappable ("vector-like")

$\rightarrow$ $Z_{\bar{\psi}} = Z_\psi^* \Rightarrow Z_{\psi \oplus \bar{\psi}} = Z_\psi Z_\psi^* = |Z_\psi|^2$

$\rightarrow$ $|Z_\psi|$ anomaly-free!

Witten-Yonekura thm for $\arg(Z_\psi)$ $\rightsquigarrow$ rigorous classification of all anomalies !!

2.3 4d $\mathbb{C}$ chiral ψ : Perturbative anomaly

- Massless 4d Dirac $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, + $U(1)$ bkg. g. field on Σ.

$$\mathcal{L} = \underbrace{\bar\psi}_{\psi^\dagger\gamma_0} i \underbrace{\not{D}}_{d+A} \psi \quad , \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \ \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Global symms : $\begin{cases} U(1)_V : & \psi \to e^{i\alpha}\psi & j^\mu_V = i\bar\psi\gamma^\mu\psi \\ U(1)_A : & \psi \to e^{i\alpha\gamma^5}\psi & j^\mu_A = i\bar\psi\gamma^\mu\gamma^5\psi \end{cases}$

- $$Z[A_\mu] = \int \underbrace{D\psi D\bar\psi}_{\text{inv. under } U(1)_A \,?} e^{\int_\Sigma \bar\psi i \not{D}\psi} = \operatorname{Det}(i\not{D})$$

▷ Method: heat kernel regularisation (Fujikawa '79, 80, 86)

- $i\not{D}$ is a hermitian operator. Take Σ compact (discrete spectrum)
- complete basis of eigenspinors: $i\not{D}\phi_n = \lambda_n\phi_n$, $\lambda_n \in \mathbb{R}$. $\Rightarrow$ $\psi = \sum a_n \phi_n$
  [orthog. : $\int_M \bar\phi_n \phi_m = \delta_{nm}$]
  $-i D_\mu \bar\phi_n \gamma^\mu = \lambda_n \bar\phi_n$ $\qquad \bar\psi = \sum \bar b_n \bar\phi_n$
  $a_n, \bar b_n$ = Grassmann
- Action $\int_\Sigma i\bar\psi\not{D}\psi = \sum_n \lambda_n \bar b_n a_n$
- Measure $\int D\psi = \prod_n \int da_n$
- Axial $U(1)_A$: $\psi \to e^{i\alpha\gamma^5}\psi \simeq 1 + i\alpha\gamma^5\psi$, $\bar\psi \to \bar\psi + i\alpha\bar\psi\gamma^5$ ($\gamma^0\gamma^5 = -\gamma^5\gamma^0$)

$$\delta\psi = \sum \delta a_n \phi_n = i\alpha\gamma^5 \sum a_m \phi_m$$

$$\Rightarrow \ \delta a_n = X_{nm} a_m \ , \quad X_{nm} = i\int_\Sigma \alpha \bar\phi_n \gamma^5 \phi_m$$

Jacobian $J = \det^{-1}(1 + X)$ from $\prod_n a_n$'s; same from $\prod_n \bar b_n$'s.
($\det^{-1}$: b/c Grassmann)

$$\Rightarrow \quad \int D\psi D\bar\psi \;\to\; \underbrace{\left[\det{}^{-1}(1+X)\right]^2}\int D\psi D\bar\psi$$

$$\approx \det(1-2X) \approx \det e^{-2X} \approx e^{-2\,\mathrm{Tr}\,X} = \exp\Big(-2i\int_\Sigma \alpha(x)\sum_n \bar\phi_n \gamma^5\phi_n\Big)$$

▷ $\mathrm{Tr}(\gamma^5)=0$? Not so fast!!

Must regulate trace:

$$\int \alpha \sum_n \bar\phi_n \gamma^5 \phi_n = \lim_{\Lambda\to\infty}\int \alpha \sum_n \bar\phi_n\left(\gamma^5 e^{-(i\slashed{D})^2/\Lambda^2}\right)\phi_n$$

$$\xrightarrow[\text{basis}]{\text{change to momentum}} \lim_{\Lambda\to\infty}\int_\Sigma \alpha \int\frac{d^4k}{(2\pi)^4}\,\underbrace{\mathrm{Tr}}_{\text{spinor trace}}\left(\gamma^5 e^{-ik\cdot x} e^{\slashed{D}^2/\Lambda^2} e^{ik\cdot x}\right)$$

$$\sim \Lambda^4\left[\cancelto{0}{\mathrm{Tr}(\gamma^5)} + \cancelto{0}{\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu)}\,\mathcal{O}\!\left(\frac{\slashed{D}^2}{\Lambda^2}\right) + \underbrace{\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)}_{=4\epsilon^{\mu\nu\rho\sigma}}\,\mathcal{O}\!\left(\frac{\slashed{D}^4}{\Lambda^4}\right)\right]$$

∴ finite piece !!

- $\slashed{D}^2 = D_\mu D^\mu - \frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$
- $\mathrm{Tr}\left(\gamma^5 e^{-ik\cdot x} e^{\slashed{D}^2/\Lambda^2} e^{ik\cdot x}\right) = e^{-k^2/\Lambda^2}\cdot 4\epsilon^{\mu\nu\rho\sigma}\cdot \frac{1}{2}\left(\frac{-ie}{2\Lambda^2}\right)^2 F_{\mu\nu}F_{\rho\sigma}$
- $\int \frac{d^4k}{(2\pi)^4} e^{-k^2/\Lambda^2} = \frac{\Lambda^4}{16\pi^2}$

$$\Rightarrow \quad \int D\psi D\bar\psi \;\to\; \exp\left(\frac{ie^2}{16\pi^2}\int \alpha\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right)\int D\psi D\bar\psi$$

$$\boxed{\underset{\text{4d Dirac}}{Z_\psi[A]} \xrightarrow{e^{i\alpha}\in U(1)_A} \exp\left(\frac{ie^2}{4\pi^2}\int \alpha\, F\wedge F\right) Z_\psi[A]}$$

Adler, Bell, Jackiw (ABJ' 1969)

c.f. Exercise 2 for 2d version

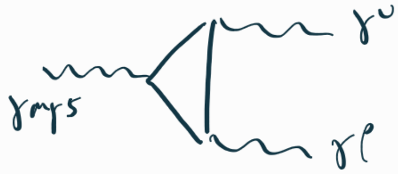
Comments:

1. ABJ anomaly is gauge-invariant!
   We lost the global symmetry, but not the gauge symmetry.

2. Anomalous Ward id.
   Take $\alpha(x)$ local. Noether procedure $\Rightarrow$

$$\langle \partial^\mu j_\mu^5 \rangle = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\langle F_{\mu\nu}F_{\rho\sigma}\rangle \qquad \left(d * j^5 = -\frac{e^2}{4\pi^2} F\wedge F \right)$$

3. $\frac{1}{16\pi^2} \rightsquigarrow$ 1-loop interpretation? Yes! $\therefore$ "perturbative" anomaly (seen @ weak coupling)
   Ward id $\Rightarrow$ can produce 2-photons from vacuum.

$$\langle p, \epsilon_\nu | \langle k, \epsilon_\rho | \partial^\mu j_\mu^5 | 0 \rangle \sim$$  + crossed

- divergent loop integral
- use e.g. 't Hooft-Veltmann prescription for γ^5

$$\sim -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\langle p, \epsilon_\nu | \langle k, \epsilon_\rho | F_{\mu\nu}F_{\rho\sigma} | 0 \rangle$$

4. Non-renormalisation & Atiyah-Singer index th$^{\underline{m}}$

Higher loop corr$^{\underline{n}}$s? None. Anomaly is secretly an integer!

- Non-zero e'values λ_n of $i\not{D}$ come in pairs $(\lambda_n, -\lambda_n)$:
  - $i\not{D}\phi_n = \lambda_n\phi_n \Rightarrow i\not{D}(\gamma^5\phi_n) = -i\gamma^5\not{D}\phi_n = -\lambda_n(\gamma^5\phi_n)$
  - $i\not{D}$ real $\Rightarrow$ ϕ_n & $\gamma^5\phi_n$ orthogonal $\Rightarrow \int \bar{\phi}_n\gamma^5\phi_n = 0$ $\quad \lambda_n \neq 0$
  - So actually only zero modes $(i\not{D}\phi_0 = 0)$ contributed above to the Jacobian!

- ZMs are e'states of γ^5, w/. e'value ± 1 $[(\gamma^5)^2 = 1]$
(can simultaneously diagonalise $i\not{D}$ & γ^5 because both ϕ_{ZM} & $\gamma^5\phi_{ZM}$ have same $i\not{D}$ eigenvalue i.e. zero)

- $\therefore$ Our trace simplifies to

$$\int_x \sum_{i \in ZMs} \bar{\phi}_i \gamma^5 \phi_i = \underset{\substack{\uparrow \\ \#\,ZMs \\ w\ \gamma^5 = +1}}{n_+} - n_- = \text{Index}(i\not{D}) \in \mathbb{Z}$$

We learn: (i). Anomaly is quantized
(ii). $\therefore$ cannot get higher loop corrections (would vary continuously w/. RG)
(iii). We (Fujikawa) "proved":

$$\boxed{\underset{\substack{\uparrow \\ \text{top. quantity}}}{\text{Index}(i\not{D})} = \underset{\substack{\uparrow \\ \text{integral !}}}{\frac{1}{4\pi^2}\int_M F \wedge F}}$$

Atiyah - Singer index theorem
(bridge global $\leftrightarrow$ local)

▹ 2.4 4d iR chiral fermion : non-perturbative anomaly

General claim: 4d iR fermion has at most mod-2 anomaly.
(i.e. $\psi \oplus \psi$ anomaly-free)

Finite order (here $\mathbb{Z}_2$) implies:
- → cannot be seen for infinitesimal gauge transformation
- → " at weak coupling / in perturbation theory
- → not determined by $\mathrm{Lie}(G)$ alone (as for local anomalies).

= a global / non-perturbative anomaly.

▹ 4d SU(2) anomaly (Witten '1982)

- $G = SU(2)$, 4d Weyl $\psi_L \sim \underline{2}$ of $SU(2)$.
- No inv. mass term / condensate b/c $\epsilon_{ij}\, \epsilon^{\alpha\beta}\, \psi^i_\alpha \psi^j_\beta = 0$.

→ fermion stays massless in IR despite confinement ?!

Answer: Sick gauge theory!

▹ Derivation #1 (sketch)

- $SU(2)$ g. transf. $\begin{cases} \psi \to U(x)\psi \\ A \to A^U = U^{-1} A U - i U^{-1} dU \end{cases}$

- $U(x)$ is a map $U : \Sigma \to SU(2)$
- Take $\Sigma = S^4$ ($\mathbb{R}^4$, fields → 0 at ∞ for finite action)
- $\pi_4(SU(2)) = \mathbb{Z}_2 \Rightarrow \exists$ two classes of $U(x)$ that cannot be deformed into one another. Take $[U(x)] = 1 \mod 2$.

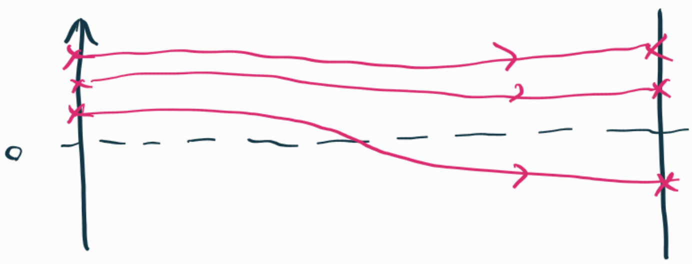
- $Z_4[A] \to Z_4[A^U] = ?$
- $Z_4[A] = \underset{\text{Weyl}}{\mathrm{Det}(i\not{D}^+)} = \underset{\substack{\text{full Dirac} \\ \text{(gappable)}}}{\left(\mathrm{Det}(i\not{D})\right)}^{1/2}$ ← choose + or − root?

- Choose + roots for all eigenvalues λ of $i\not{D}$
- But this choice is not "invariant" under $A \to A^U$!!

Compute this "spectral flow" by forming a 5d mapping torus: $A^t = (1-t)A + tA^U \quad t \in [0,1)$

can glue $t=0$ to $t=1$ b/c $A^U \sim A$ gauge equivalent

$S^4 \times S^1$



$\#$ (+ e'values → − e'values)
$= \#$ ZMs of $i\not{D}_5$
$= \mathrm{Index}(i\not{D}_5)$
$\downarrow$ mod-2 index theorem (magic!)
$= 1 \bmod 2.$

BUT: - how to actually compute the index?
- was $\pi_4(SU(2))$ actually important?

[A: no - $\pi_d(G) \neq 0$ is neither nec. nor suff. for global anomalies]

Derivation # 2 (Wang, Wen, Witten '18)

- $\psi_L \sim$ isospin j rep of $SU(2)$
- Consider $U(x) = -\mathbb{1}_{2j+1}$ constant g. transf.
- this is equivalent to $(-1)^F$. So, $Z[A] \xrightarrow{(-1)^F} ?$
- We previously saw that non-ZMs come in pairs ϕ_n & $\gamma^5 \phi_n$ w. e'values $\pm \lambda_n$

$\Rightarrow$ only ZMs can contribute to $(-1)^F$!

- $\therefore Z \to (-1)^{n_+ + n_-} Z = (-1)^{n_+ - n_-} Z = (-1)^{\mathrm{Ind}(i\cancel{D})} Z$
- Compute $\mathrm{Ind}(i\cancel{D})$ by A-S index thm :

$$\boxed{Z_4[A] \xrightarrow{U(x)=-1} \exp\left[i\pi \int_\Sigma \frac{\mathrm{Tr}_R\, F\wedge F}{8\pi^2} \right] Z_4[A]}$$

~ perturbative anomaly formula but at fixed (discrete) angle

$$= \exp\left[i\pi\, T(j)\, p_1(F) \right] Z_4[A]$$

Dynkin index : s.t. $\mathrm{Tr}_R (t^a_j t^b_j) = \frac{1}{2} T(j) \delta^{ab}$

$T(j) = \frac{2}{3} j(j+1)(2j+1)$

instanton number : must be odd to see anomaly.

$\boxed{\text{So, for e.g. 1-instanton bkg. field (on e.g. } S^4\text{), we see the mod 2 anomaly for reps } j = 2r + \tfrac{1}{2} = \tfrac{1}{2}, \tfrac{5}{2}, \tfrac{9}{2}, \dots \text{ (a subset of the pseudoreal reps !)}}$

4d $SU(2)$ anomaly cancellation : $N_2 + N_6 + N_{10} + \dots \in 2\mathbb{Z}$

example (thankfully): SM [3 + 1 doublets per generation]

Q: how to compute global anomalies systematically?
Cannot compute F. diagrams!

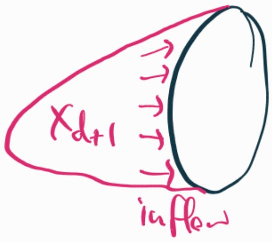
2.5 General classification of chiral fermion anomalies

▷ Anomaly Inflow

- Recall corollary #4 above: ANOMALY IS A PHASE.
- Phase of Z_ψ is now precisely understood using anomaly inflow & Witten-Yonekura th^{m}.

INFLOW:

$$Z_\psi = \underset{\substack{\uparrow \\ \text{anomaly-free}}}{|Z_\psi|}\, e^{-2\pi i S_{anom}[X_{d+1}]}$$



$(\Sigma, \sigma, A, \dots)$

$\Sigma = \partial X_{d+1}$ ← fictitious bulk

n.b. inflow is a physical phenomenon in condensed matter: see eg. quantum Hall effect.

Example: 4d $U(1)_A$ in QED.

global bkg.-g. field; QED $f = da$

$$S_{anom} = \frac{1}{4\pi^2}\int A \wedge f \wedge f$$

$$e^{i\alpha(x)} \in U(1)_A : A \to A + d\alpha : \quad S \to S + \frac{1}{8\pi^3}\int_{X_{d+1}} d(\alpha \wedge f \wedge f)$$

$$= S + \frac{1}{8\pi^3}\int_{\partial X = \Sigma} \alpha\, f \wedge f$$

So the $(d+1)$ CS in bulk reproduce exactly the chiral anomaly!

Conjecture: any anomaly can be realised in this way.

▸ General perturbative anomaly

$$Z_\psi = |Z_\psi| e^{-2\pi i \int_{X_{d+1}} CS_{d+1}} , \quad \text{where}$$

$$\boxed{\Phi_{d+2} \equiv d\, CS_{d+1} = \hat{A}(R)\, \mathrm{Tr}_{\underline{R}} \exp\left(\frac{F}{2\pi}\right)\Big|_{d+2}}$$

is 'anomaly polynomial', the gauge-invariant object that properly measures local anomaly.

- $\hat{A}(R) = 1 - \frac{1}{24} \underbrace{p_1(R)}_{\substack{= \frac{1}{8\pi^2} \mathrm{Tr}\, R \wedge R, \\ \text{4-form}}} + \frac{1}{5760}(-4p_2 + 7p_1^2) + \ldots$

  this bit gives all (mixed) gravitational anomalies

- $\mathrm{Tr}_{\underline{R}} \exp\left(\frac{F}{2\pi}\right) = 1 + \mathrm{Tr}\, \frac{F}{2\pi} + \frac{1}{2} \mathrm{Tr}\, \frac{F \wedge F}{4\pi^2} + \frac{1}{3!} \mathrm{Tr}\, \frac{F^3}{8\pi^3} + \ldots$

c.f. Exercise 3 to unpack lots of physics!

▸ Non-perturbative generalisation (rough sketch)

$$Z_\psi = |Z_\psi| \exp\left(-2\pi i \eta_{X_{d+1}}\right)$$

proof: Witten-Yonekura '2019

where $\eta_X = \lim_{\epsilon \to 0^+} \sum_k e^{-\epsilon |\lambda_k|} \frac{\mathrm{sign}(\lambda_k)}{2}$ = APS η-invariant

- $\exp(-2\pi i \eta)$ is the 'anomaly theory' for chiral fermions.
- If $\exp(-2\pi i \eta) = 1$ on all closed $(d+1)$-mflds, theory is completely anomaly-free (of local & global anomalies); on any possible spacetime

To see this:

- Can compare phases of $Z[A]$ vs. $Z[A^g]$ for any $g(x)$: (on any Σ)

$$\frac{Z[A]}{Z[A^g]} = \frac{\exp(-2\pi i \eta_{X'})}{\exp(-2\pi i \eta_X)} \underset{\text{(cut)}}{\overset{\text{Dai-Freed}}{=\!=}} \exp(-2\pi i \eta(\Sigma \times [0,1]))$$

$$\underset{\text{(glue)}}{\overset{\text{Dai-Freed}}{=\!=}} \exp(-2\pi i \eta(\underbrace{\Sigma \times S^1}_{\text{mapping torus } \bar{X}}))$$

- So anomaly $= \exp(-2\pi i \eta)$ on mapping torus $\bar{X}$

APS index th^{m}: If $\bar{X}_{d+1} = \partial Y_{d+2}$ itself a bdy, then

$$\eta(\bar{X}) = -\mathrm{Ind}(i\not{D}_Y) + \int_Y \Phi_{d+2}$$

Perturbatively: can take $X = \partial Y$,

$$e^{-2\pi i S_{anom}[X]} = e^{-2\pi i \int_Y \Phi_{d+2}} = e^{-2\pi i \int_{\bar{X}} CS_{d+1}} \text{ as before.}$$

Non-perturbatively:

first cancel local anomalies $\Rightarrow \Phi_{d+2} = 0$.

APS $\Rightarrow$ $e^{-2\pi i \eta_X} = 1$ on all boundaries.

This defines a $(d+1)$-dim bordism invariant

→ "Bordism groups" $\Omega_{d+1}^{\mathrm{Spin}\times G}$ classify all global anomalies

→ can compute given any $\mathrm{Spin}(d)\times G$ using techniques from algebraic topology.

→ a systematic procedure (rigorous) for computing all possible global anomalies.

BSM Example:

$$SL(2,\mathbb{F}_3) \simeq \mathbb{Z}_2 \rtimes A_4 \text{ flavour model.}$$

Feruglio et al. '07

Leptons ~ $\underline{1}$, $\underline{3}$ (anomaly-free)

Quarks ~ $\underline{2}^{\oplus 12}$ ⟶ Mod 3 anomaly!

(1st + 2nd gens) ↑ a $\mathbb{C}$ doublet rep

∴ the $SL(2,\mathbb{F}_3)$ gauge symm. is actually anomalous (broken).

③ PARTICLE PHYSICS APPLICATIONS

- From physics perspective, it helps to distinguish bkg vs. dyn. gauge fields.
- E.g. for local anomalies, same maths, different physics:

FFF	FFf	Fff	fff
't Hooft	't Hooft (recently recast as "2-group")	ABJ, "broken symm" (recently recast as "non-inv. symmetry" if f is abelian)	Gauge anomaly: inconsistent theory! 1. non-unitary (-ve norm states) 2. non-renormalisable (... but see later...)

$\underbrace{\text{FFF, FFf}}_{}$ → constraints on UV/IR by anomaly matching! (topological ∴ not renormalised)

Important physics effects e.g.

$\boxed{fff}$

▷ 3.1 Anomaly cancellation for 'UV theories'

- Gauge anomalies must cancel: local & global.
- 4d local anomalies

$$\Phi_6 \sim \underbrace{\#\, p_1(R)\, \mathrm{tr}(F)}_{} + \underbrace{\#\, \mathrm{tr}(F^3)}_{}$$

- mixed U(1)-gravitational

- $U(1)^3$ anomaly
- non-abelian simple G e.g. $\underline{N}$ of $SU(N\geq 3)$
- mixed U(1) 

Ex: SM. $G = SU(3)\times SU(2)_L \times U(1)_Y$

		Q_L	L_L	U_R	d_R	e_R	
Local anomalies:	$U(1)_Y$-grav :	$+6(1)$	$+2(-3)$	$-3(+4)$	$-3(-2)$	$-1(-6)$	$=0$
	$U(1)_Y^3$:	$6(1^3)$	$+2(-3)^3$	$-3(4)^3$	$-3(-2)^3$	$-1(-6)^3$	$=0$
	$U(1)_Y - SU(3)$:	$2(+1)$		$-1(4)$	$-1(-2)$		$=0$
	$U(1)_Y - SU(2)_L$:	$3(+1)$	$+1(-3)$				$=0$

Global anomaly: $\#\, SU(2)_L$ doublets $= n_g \times (3+1) = 4n_g \in 2\mathbb{Z}$.

Example 2: BSM

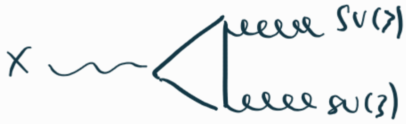
see Exercise 5

$G = G_{SM} \times U(1)_X \Rightarrow$ 6 new $\triangle$ anomalies to cancel:

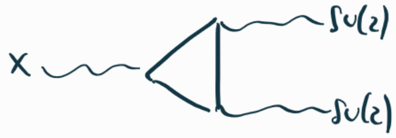
$\sim \sum x_i^3$

$\sim \sum x_i^2$

$\sim \sum x_i$



$\sim \sum x_i$



$\sim \sum x_i$

(gravity)

$\sim \sum x_i$

6 polynomial eq$^{\text{ns}}$ over integer variable (the charges), including $\times 1$ cubic & $\times 1$ quadratic. Difficult to solve in general!

<u>B-L</u>: $X_q = X_U = X_D = 1, \quad X_L = X_E = -3$

- has XXX and X-grav anomalies
- <u>cancelled</u> by including RH ν w/. $X_\nu = -3$
- "Invisible particle" (RHν) can be "useful" in model-building for absorbing gauge anomalies.

<u>$L_e - L_\mu$ etc</u>:

- all anomalies that are linear/cubic in x_i vanish automatically (e & μ contribute w/ opposite sign)
- only need to check YXX anomaly $\propto \sum_{i=1}^{3} \left(-X_{L_i}^2 + X_{E_i}^2\right) = 0$

(anomaly-free w/. or w/o. RH ν's).

▸ 3.2 Anomalies in EFT

Trivial example: SMEFT. $\mathcal{L} = \mathcal{L}^{d=4} + \sum_{n,i} \frac{c_n^i}{\Lambda^{n-4}} \mathcal{O}_n^i$

→ Anomaly is topological, quantized

→ Determined by $i\not{D}$, ψ kinetic term.

→ Index $(i\not{D})$ not affected by weak int$^{\text{ns}}$ $\sim \frac{1}{\Lambda^{n-4}}$;

if it was, it could not remain quantized as I continuously vary Λ! Same argument as for the non-renormalisation of the anomaly.

("EFT = renormalisation" after all...)

Less trivial example

- take anomaly free G gauge theory, set $\{\psi_i\}$ of fermions in reps $\{\underline{R}_i\}$. Well-defined in UV.
- Higgs a chiral fermion w/ Yukawa coupling:

$$\mathcal{L} \supset y\, \bar{\psi}_L^i \phi\, \psi_R^j \qquad \underline{R}_L^i \neq \underline{R}_R^j$$

- Integrate out ψ_i, ψ_j DOFs. $\rightsquigarrow$ G anomalous!!
- BUT, ϕ nec. non-singlet $\Rightarrow$ Higgses G

$\Rightarrow$ massive gauge boson.

- $M_\psi \sim y\langle\phi\rangle$, $M_A \sim g\langle\phi\rangle$, $M_H \sim \sqrt{\lambda}\langle\phi\rangle$
- If $g \ll y$, $M_\psi \gg M_A$ & makes sense to integrate out ψ ($\&\ \phi$) while retaining A.

UV

$- M_\psi$

$- M_\phi$

- - - - - - - anomalous EFT scale

$- M_A$

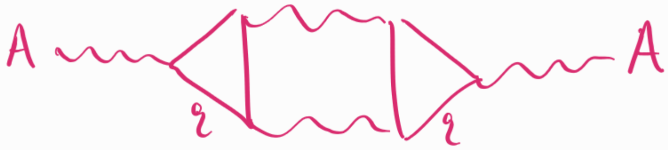
IR

$$\boxed{\text{From bottom-up, an anomalous EFT makes sense, but the gauge boson is necessarily massive}}$$

▷ Unitarity

Can explicitly see there is a radiative contribution to gauge boson mass M_A in the anomalous EFT (Preskill '91)

e.g. for $G = U(1)$



$$\Rightarrow m_A \sim \frac{e^3 q^3}{64\pi^3} \Lambda \sim \Phi_6 \Lambda$$

theory has an EFT cut-off !
(is non-renormalisable)

$M_A \neq 0$ cures unitarity failure

▷ Gauge-inv. in the EFT?

- Integrating out chiral ψ means $Z_\chi \xrightarrow{G} e^{\frac{i\#}{16\pi^2}\int \alpha f \wedge f} Z_\chi$

  χ: remaining light chiral fermions in EFT

- b/c UV th. is anomaly-free, there must be a term in the EFT that cancels out this variation.

- To write this term, choose gauge in which

  long. mode of (massive) A $\rightsquigarrow$ Goldstone $a(x)$, shift symmetry under G

  [Under $G = U(1)$, $a \to a + f_a \alpha$]

  Then
  $$\mathcal{L}_{EFT} \supset \frac{\#}{16\pi^2} \frac{a}{f_a} F \wedge F \equiv \mathcal{L}_{GS}$$

  "Green-Schwarz"

(see Preskill '1991)

Under G, $\mathcal{L}_{GS} \to \mathcal{L}_{GS} + \frac{\#}{16\pi^2} \alpha f \wedge f$

which exactly compensates the anomaly due to χ

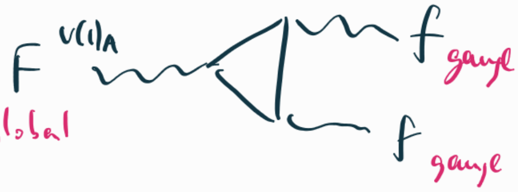
- Physical effects:

$Z' \to ZZ$ decay, diagnostic of "anomalous" gauge theory EFT.

Fff

▷ 3.3 Axions

- Take anomalous EFT of § 3.2, & change 'physics context' by replacing one gauge current f by a global symm. brkg. current F:

$\Phi_6 \supset$ 

(ABJ anomaly)

Example: f = gluon of $SU(3)$;

Consider $U(1)_A \equiv U(1)_{PQ}$ to be a global chiral symm. that enjoys an ABJ anomaly w/. QCD (various choices e.g. DFSZ, KSVZ)

i.e. $\partial^\mu j_\mu^{PQ} = \frac{g_s^2}{16\pi^2} \# \, G \wedge G + \frac{e^2}{16\pi^2} \# \, f_{em} \wedge f_{em}$

↑ need this to solve strong CP

↑ allow this anomaly also (gives $a \, \underline{E} \cdot \underline{B}$ coupling) in general produced by mixing of axion with π_0

> In the EFT, we have same $\mathcal{L}_{GS} = \# \frac{g_s^2}{16\pi^2} \frac{a(x)}{f_a} G \wedge G + \dots$ as above, except now the mode $a(x)$ is physical (light pseudoscalar) = AXION

▷ 3.4 QCD : Two consequences of ABJ anomalies

UV DOFs = $q_L^{i=u,d,s}, q_R^i$

$\downarrow$ χSB

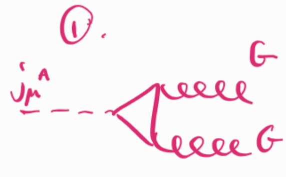
IR DOFs = χf pions on coset ~ $SU(3)$

$G_{global} = SU(3)_L \times SU(3)_R \times U(1)_V \times \cancel{U(1)_A}$

("baryon number" — $U(1)_V$)

ABJ w. $G \wedge G$

⇒ instanton contributions to the would-be pNGB (η') mass

① 

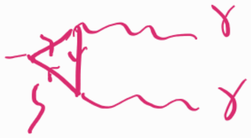
$H_{global} = SU(3)_V \times U(1)_B$

flavour symmetry (u, d, s)

[$SU(3)_A$ axial flav. symm is spontaneously broken]

2. A 2^{nd} ABJ anomaly is activated when we gauge $U(1)_{em} \subset SU(3)_V$

generator $Q = \begin{array}{c} \begin{pmatrix} +2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} \end{array} \begin{array}{c} u \\ d \\ s \end{array}$

$U(1)_A^{t_3} \subset SU(3)_A$  $\Rightarrow$ $\pi_0 \to \gamma\gamma$ decay w. $\underline{BR \propto N_c}$

$j_A^{t_3}$ generate π_0 from vacuum

trace over colours

→ So can determine # QCD colours in the IR pion EFT by measuring $\pi_0 \to \gamma\gamma$ decay!

→ Similar phenomena can occur in BSM e.g.

1. Dark sectors (e.g. 'SIMP DM')
2. Composite Higgs (e.g. $SO(6)/SO(5)$ model).

In such theories w/ anomalous symmetries (like in QCD), anomaly matching gives an EFT ⟷ UV bridge that is uniquely robust b/c anomaly is not renormalised w/ energy scale, ultimately b/c it is topological.

The END //.