Reconstruction and interpretation of propagators with complex singularities

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Nonperturbative QFT in the complex momentum space

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Ref. Y.H. and K.-I. Kondo,
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Introduction

- Analytic structure of a propagator: **states and spectrum**
  Physical case: **Källén-Lehmann spectral representation**

$$D(k^2) = \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - k^2},$$

$$\theta(k_0)\rho(k^2) := (2\pi)^d \sum_n |\langle 0|\phi(0)|P_n\rangle|^2 \delta^D(P_n - k),$$

singularities on complex k^2-plane
$\longleftrightarrow$ **states** non-orthogonal to $\phi(0)\,|0\rangle$

- **Analytic structures of the QCD propagators** would be helpful to understand fundamental aspects (e.g., **confinement**) and real-time dynamics.

Based on the progress on the Landau-gauge gluon, ghost, and quark propagators, there has been an increasing interest in their analytic structures.

An interesting possibility – complex singularity

An interesting possibility has been consistently suggested by independent (old and recent) analyses: the gluon propagator may have **complex singularities**, invalidating the Källén-Lehmann spectral representation.

1. Modeling gluon propagator to fit lattice results
   - (refined-)Gribov-Zwanziger model [Dudal et. al. 2008]...
   - Massive-like gluon model [Siringo 2016] [YH and Kondo, 2018]...
   - Numerical reconstruction techniques [Binosi and Tripolt 2019] [Falcão, Oliveira, and Silva 2020] [Lechien and Dudal 2022] [Boito et. al. 2022]...
2. Dyson-Schwinger equation on the complex momentum plane

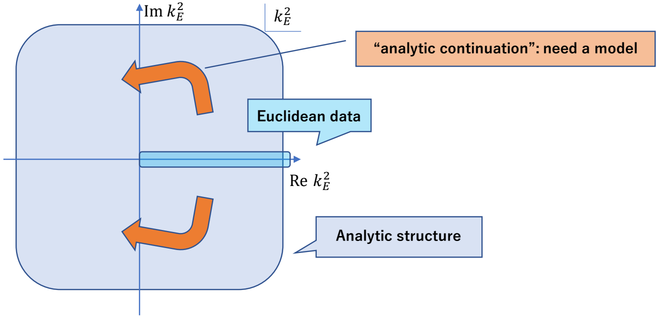
   [Strauss, Fischer, and Kellermann 2012] [Binosi and Tripolt 2019] [Huber and Fischer 2020]

However, the interpretation of complex singularities has not been established and deserves attention.
(short-lived gluon? non-locality? [Stingl 1986] etc.)

Plan

How to investigate analytic structures

- Aim: investigating analytic structures of the propagators from Euclidean data through "analytic continuation".
- The "analytic continuation" from finite data is in principle an ill-posed problem: **use models consistent with the Euclidean data with some theoretical backgrounds**.

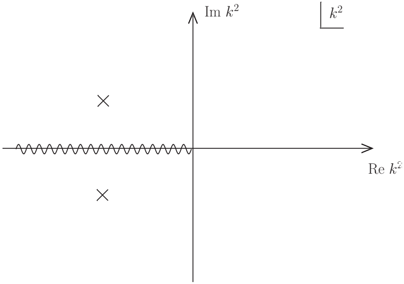


What is "complex singularity" exactly?

Complex singularity: singularity off the real axis in the complex Euclidean momentum plane k_E^2 of an analytically continued **Euclidean propagator** $D(k_E^2)$.

e.g.) complex poles: poles not on the real axis of analytically continued Euclidean propagator

$$D(k_E^2) = \frac{Z}{w + k_E^2} + \frac{Z^*}{w^* + k_E^2} + \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 + k_E^2},$$



Reconstruction from Euclidean field theory to QFT

[Osterwalder and Schrader 1973, 1975]

Euclidean
Schwinger functions $\{\mathcal{S}_n\}$

standard:
OS reconstruction

(α) reconstruction $\mathcal{S}_2 \to \mathcal{W}_2$

Minkowski
Wightman functions $\{\mathcal{W}_n\}$

standard: Wightman reconstruction

(β) a possibility is discussed

Relativistic QFT
states and operators

Main Questions

In the presence of complex singularities, natural questions on this procedure are,

(α) Is it possible to reconstruct a Wightman function $W(\xi^0, \vec{\xi})$ on the Minkowski spacetime from the Schwinger function? Which conditions of the Wightman/OS axioms are preserved/violated?

(β) Does there exist a quantum theory reproducing the reconstructed Wightman function $W(\xi^0, \vec{\xi})$ as a vacuum expectation value: $W(\xi) = \langle 0|\phi(\xi)\phi(0)|0\rangle$? If it exists, what states cause complex singularities?

We will answer these questions affirmatively.

Warm-up: Wick Rotation? (1)

Usually, we do not care about the reconstruction and instead use the inverse Wick rotation $k_E^2 \to -k^2$ (Euclidean propagator $\to$ time-ordered propagator).

However, *this is not applicable* in the presence of complex singularities.

As a warm-up example, let us see the Gribov-type propagator in a $(0+1)$ dimensional theory:

$$D(k_E^2) = \frac{k_E^2}{k_E^4 + \gamma^4}.$$

In terms of the imaginary time τ, the Euclidean propagator (Schwinger function) $S(\tau)$ is,

$$S(\tau) = \frac{1}{2\gamma} e^{-\frac{\gamma|\tau|}{\sqrt{2}}} \sin\left(-\frac{\gamma|\tau|}{\sqrt{2}} + \frac{\pi}{4}\right),$$

Warm-up: Wick Rotation? (2)

We reconstruct the Wightman function by identifying the Schwinger function as imaginary-time data of the holomorphic Wightman function [$S(\tau) = W(-i\tau)$ for $\tau > 0$].

Then, the reconstructed Wightman function is,

$$W(t) = \frac{i}{2\gamma} e^{i\frac{\gamma t}{\sqrt{2}}} \sinh\left(\frac{\gamma t}{\sqrt{2}} - \frac{i\pi}{4}\right),$$

which diverges exponentially in both limits $t \to \pm\infty$. The time-ordered propagator $\theta(t)W(t) + \theta(-t)W(-t)$ cannot be Fourier-transformed. Therefore, the simple prescription $k_E^2 \to -k^2$ does not give a correct answer.

From this example,

- we cannot rely on the inverse Wick rotation $k_E^2 \to -k^2$, and
- we need to consider the reconstruction procedure carefully.

General properties of complex singularities

Wightman function $W(t, \vec{x})$ is reconstructed from Schwinger function $S(\tau, \vec{x})$ by identifying $S(\tau, \vec{x}) = W(-i\tau, \vec{x})$ $(\tau > 0)$.
In the presence of complex singularities (bounded in k_E^2-plane), we rigorously prove:

List of properties

- Holomorphy of $W(t, \vec{x})$ in the tube $\mathbb{R}^4 - iV_+$ [V_+: forward lightcone]
- Existence of the boundary value $W(t, \vec{x}) = \lim_{\tau \to +0} W(t - i\tau, \vec{x})$ as a distribution.
- $W(t, \vec{x})$ satisfies Lorentz symmetry and locality (i.e. spacelike commutativity).
- ◦ Non-temperedness of the boundary value $W(t, \vec{x})$
- ◦ Violation of the positivity of $W(t, \vec{x})$.

Sketches of proofs: (0)holomorphy

Reconstruction: $S(\tau, \vec{x}) \to W(\xi = (t, \vec{x}))$

$$S(\tau, \vec{x}) \overset{\tau \geq 0}{=} W(-i\tau, \vec{x}) \;\to\; W(\xi - i\eta) \text{ for } \xi \in \mathbb{R}^4,\ \eta \in V_+$$
$$\to\; W(\xi) = \lim_{\eta \to 0,\ \eta \in V_+} W(\xi - i\eta)$$

e.g.) Complex poles ($E_{\vec{p}} := \sqrt{\vec{p}^2 + M^2}$ with $\operatorname{Re} E_{\vec{p}} > 0$)

$$D(k_E^2) = \frac{Z}{k_E^2 + M^2} + \frac{Z^*}{k_E^2 + (M^*)^2}$$

$$\to S(\tau, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \left[\frac{Z e^{-E_{\vec{p}}\tau}}{2E_{\vec{p}}} + \frac{Z e^{-E_{\vec{p}}^*\tau}}{2E_{\vec{p}}^*} \right]$$

$$\to W(\xi - i\eta) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{\xi} - i\vec{\eta})} \left[\frac{Z e^{-iE_{\vec{p}}(\xi^0 - i\eta^0)}}{2E_{\vec{p}}} + \frac{Z^* e^{-iE_{\vec{p}}^*(\xi^0 - i\eta^0)}}{2E_{\vec{p}}^*} \right]$$

converges (and is holomorphic in $\xi - i\eta$) for $\eta^0 > |\vec{\eta}|$, i.e., $\eta \in V_+$.

Sketches of proofs: (1) boundary value and non-temperedness

Reconstruction: $S(\tau, \vec{x}) \to W(\xi = (t, \vec{x}))$

$$S(\tau,\vec{x}) \overset{\tau\geq 0}{=} W(-i\tau,\vec{x}) \;\to\; W(\xi - i\eta) \text{ in } \mathbb{R}^4 - iV_+$$

$$\to\; W(\xi) = \lim_{\eta\to 0,\ \eta\in V_+} W(\xi - i\eta)$$

e.g.) a pair of complex conjugate poles (cont'd)

$$W(\xi - i\eta) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{\xi}-i\vec{\eta})} \left[\frac{Z e^{-iE_{\vec{p}}(\xi^0 - i\eta^0)}}{2E_{\vec{p}}} + \frac{Z^* e^{-iE^*_{\vec{p}}(\xi^0 - i\eta^0)}}{2E^*_{\vec{p}}} \right]$$

- ${}^\exists W(\xi) = \lim_{\eta\to 0,\ \eta\in V_+} W(\xi - i\eta)$ as a distribution [$\because$ the limit exists if smeared by a smooth compactly-supported function of ξ].
- Since $E_{\vec{p}}$ is complex, $W(\xi)$ grows exponentially for $\xi^0 \to \pm\infty$. $W(\xi)$ is **not tempered**.

Sketches of proofs: (2) violation of positivity

The positivity is violated due to the non-temperedness.

For this, we show

$$\text{positivity} \Longrightarrow \text{temperedness}$$

Rough idea:

- Positivity of 2pt.-function $\to$ the sector $\{\phi(x)\,|0\rangle\}_{x\in\mathbb{R}^4}$ has a positive metric.
- translational invariance $\to$ translation operator $U(a)$: $U(a)\phi(x)\,|0\rangle = \phi(x+a)\,|0\rangle$ is unitary.

Therefore, the Wightman function $W(a) = \langle 0|\phi(0)U(-a)\phi(0)|0\rangle$ will be bounded above $\Rightarrow$ tempered.

Sketches of proofs: (3) Lorentz symmetry

The Schwinger function has the Euclidean rotation $SO(4)$ invariance.

- $\rightarrow$ $W(\xi - i\eta)$ is invariant under infinitesimal Euclidean rotations.
- $\rightarrow$ $W(\xi - i\eta)$ is invariant under infinitesimal complex Lorentz transformations.
- $\rightarrow$ [1] $W(\xi - i\eta)$ is invariant under the proper complex Lorentz symmetry $L_+(\mathbb{C})$ (within its domain of definition), where $L_+(\mathbb{C}) := \{\Lambda \in \mathbb{C}^{4\times 4} \ ; \ \Lambda^T G \Lambda = G, \det \Lambda = 1\}$ with the metric $G = \mathrm{diag}(1, -1, -1, -1)$.
- $\rightarrow$ The Wightman function $W(\xi)$ is invariant under the restricted Lorentz transformation[2].

[1] An argument similar to Bargmann-Hall-Wightman theorem is used here.

[2] In the case of a scalar field, the invariance under Lorentz boosts can be explicitly checked by a contour deformation.

Sketches of proofs: (4) locality

The spacelike commutativity $[W_{ij}(\xi) = (-1)^\sigma W_{ji}(-\xi)]$[3] follows from

- permutation symmetry of Schwinger function $S_{ij}(x-y) = (-1)^\sigma S_{ji}(y-x)$,
- single-valued continuation $W_{ij}(\xi - i\eta)$ in the 'extended tube' $L_+(\mathbb{C})[\mathbb{R}^4 - iV_+]$ including spacelike points ('Jost points').
- $W_{ij}(z) = (-1)^J W_{ij}(-z)$ from $-1 \in SO(4) \subset L_+(\mathbb{C})$.[4]

$$\begin{tikzcd}
W_{ij}(\xi - i\eta) = (-1)^{\sigma+J} W_{ji}(\xi - i\eta) \arrow[d, "\begin{array}{l}\eta \to 0 \\ (\eta \in V_+)\end{array}"] & = (-1)^\sigma W_{ji}(-\xi + i\eta) \arrow[d, "\begin{array}{l}\eta \to 0 \\ \text{holomorphy at spacelike } \xi\end{array}"] \\
W_{ij}(\xi) \arrow[r, equal] & (-1)^\sigma W_{ji}(-\xi)
\end{tikzcd}$$

[3] $(-1)^\sigma$: statistical factor

[4] $(-1)^J$: the number of dotted indices in the correlator

Summary: answer to the question (α)

Main question (α)

Is it possible to reconstruct a Wightman function $W(\xi^0, \vec{\xi})$ on the Minkowski spacetime from the Schwinger function?
Which conditions of the Wightman/OS axioms are preserved/violated?

In this section, we have seen that it is possible to reconstruct the Wightman function as a distribution.

The violated/preserved conditions of the Wightman/OS axioms are summarized in the next slide.

Summary of Wightman/OS axioms

Minkowski: Wightman axioms for Wightman functions

[W0] Temperedness	violated ✗
[W1] Poincaré Symmetry	preserved ✓
[W2] Spectral Condition	violated ✗
[W3] Spacelike Commutativity	preserved ✓
[W4] Positivity	violated ✗
[W5] Cluster property	irrelevant

Euclidean: Osterwalder-Schrader axioms for Schwinger functions

[OS0] Temperedness	assumed ✓
[OS1] Euclidean Symmetry	assumed ✓
[OS2] Reflection Positivity	violated ✗
[OS3] Permutation Symmetry	assumed ✓
[OS4] Cluster property	irrelevant
[OS0'] Laplace transform condition	violated (but irrelevant)

Interpretation in an indefinite metric state space (0)

Main question (β)

Does there exist a quantum theory reproducing the reconstructed Wightman function $W(\xi)$ as a vacuum expectation value: $W(\xi) = \langle 0|\phi(\xi)\phi(0)|0\rangle$?
If exists, what states cause complex singularities?

In this section, we argue that complex singularities can be realized in indefinite-metric QFTs and correspond to pairs of zero-norm eigenstates of complex energies.

Interpretation in an indefinite metric state space (1)

An important observation is that complex-energy spectra can appear in an indefinite metric state space.

- Complex conjugate eigenvalues of a hermitian Hamiltonian can be realized by zero-norm pairs:

$$(|E\rangle, |E^*\rangle) \begin{cases} H|E\rangle = E|E\rangle, \quad H|E^*\rangle = E^*|E^*\rangle \\ \langle E|E\rangle = \langle E^*|E^*\rangle = 0, \quad \langle E|E^*\rangle \neq 0 \end{cases}$$

- This pair contributes to the Wightman function,

$$\langle 0|\phi(t)\phi(0)|0\rangle \supset (\langle E^*|E\rangle)^{-1} e^{-iEt} \langle 0|\phi(0)|E\rangle \langle E^*|\phi(0)|0\rangle + (\langle E|E^*\rangle)^{-1} e^{-iE^* t} \langle 0|\phi(0)|E^*\rangle \langle E|\phi(0)|0\rangle .$$

- By preparing such states for all momentum $\vec{p}$, we can reproduce the Wightman function reconstructed from complex poles.

Example: Lee-Wick model (1)

To show this more concretely, we consider an example, the covariant operator formulation of Lee-Wick model [Nakanishi1972]. The essential ingredients are as follows.

- Annihilation operators $\alpha(\vec{p}), \beta(\vec{p})$:
  $[\alpha(\vec{p}), \beta^\dagger(\vec{q})] = [\beta(\vec{p}), \alpha^\dagger(\vec{q})] = (2\pi)^3 \delta(\vec{p} - \vec{q})$, and other commutators vanish.
- The field operator is

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[\alpha(\vec{p}) e^{i\vec{p}\cdot\vec{x} - iE_{\vec{p}}t} + \beta^\dagger(\vec{p}) e^{-i\vec{p}\cdot\vec{x} + iE_{\vec{p}}t} \right].$$

  where $E_{\vec{p}} := \sqrt{M^2 + \vec{p}^2}$ and $M^2 \in \mathbb{C}$.
- Vacuum state $|0\rangle$: $\alpha(\vec{p})\,|0\rangle = \beta(\vec{p})\,|0\rangle = 0$.
- Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3} \left[E_{\vec{p}} \beta^\dagger(\vec{p}) \alpha(\vec{p}) + E^*_{\vec{p}} \alpha^\dagger(\vec{p}) \beta(\vec{p}) \right].$$

Example: Lee-Wick model (2)

The states $|\vec{p},\alpha\rangle := \alpha^\dagger(\vec{p})\,|0\rangle$ and $|\vec{p},\beta\rangle := \beta^\dagger(\vec{p})\,|0\rangle$ form the pair of complex-energy zero-norm states for every $\vec{p} \in \mathbb{R}^3$:

$$
\begin{aligned}
& H\,|\vec{p},\alpha\rangle = E^*_{\vec{p}}\,|\vec{p},\alpha\rangle\,, \quad H\,|\vec{p},\beta\rangle = E_{\vec{p}}\,|\vec{p},\beta\rangle\,, \\
& \langle\vec{p},\alpha|\vec{q},\alpha\rangle = \langle\vec{p},\beta|\vec{q},\beta\rangle = 0, \quad \langle\vec{p},\alpha|\vec{q},\beta\rangle = (2\pi)^3\delta(\vec{p}-\vec{q}).
\end{aligned}
$$

From these pairs, the Wightman functions are

$$
\begin{aligned}
\langle 0|\phi(x)\phi(0)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{i\vec{p}\cdot\vec{x}-iE_{\vec{p}}t}, \\
\langle 0|\phi(x)\phi^\dagger(0)|0\rangle &= \langle 0|\phi^\dagger(x)\phi(0)|0\rangle = 0, \\
\langle 0|\phi^\dagger(x)\phi^\dagger(0)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E^*_{\vec{p}}}e^{i\vec{p}\cdot\vec{x}-iE^*_{\vec{p}}t},
\end{aligned}
$$

which are indeed Wightman functions reconstructed from complex poles $\frac{1}{k^2+M^2}$ and $\frac{1}{k^2+(M^*)^2}$.

Remarks on the interpretation

Thus, complex singularities can be realized in an indefinite-metric QFT and understood as pairs of zero-norm states.

- Complex-energy states violate the spectral condition and make asymptotic states ill-defined.
- Complex singularity of the gluon propagator suggests

$$A_\mu^A(0)\,|0\rangle = |E\rangle + |E^*\rangle + \cdots$$

- Such zero-norm states should be **confined**. In the Kugo-Ojima scenario, $|E\rangle$ and $|E^*\rangle$ should be in BRST quartets.
- Both states $|E\rangle$ and $|E^*\rangle$ should contain BRST-parent states $\to$ complex singularities in ghost-gluon bound states?
- Different choice of in- and out- vacuum $\to$ short lifetime gluon? [Siringo and Comitini '22] [Siringo's talk]

Summary

Many studies suggest that the Landau-gauge gluon propagator has complex singularities. Therefore, we have considered the reconstruction procedure in the presence of complex singularities

- The 2pt. Wightman function can be reconstructed as a distribution.
- Complex singularities lead to **non-temperedness** of the Wightman function $\Rightarrow$ **violation of the positivity and spectral condition**
- Complex singularities are **consistent with Lorentz symmetry and locality**.
- Complex singularities in a propagator can be realized in an indefinite-metric QFT and understood as **pairs of zero-norm confined states**.