Real-time functional renormalization group and critical dynamics

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Based on
JR, L. von Smekal, arXiv:2303.11817
JR, D. Schweitzer, L. J. Sieke, L. von Smekal, Phys. Rev. D **105**, 116017 (2022)

Motivation: Why real time?

Study **QCD phase diagram** through heavy-ion collisions:

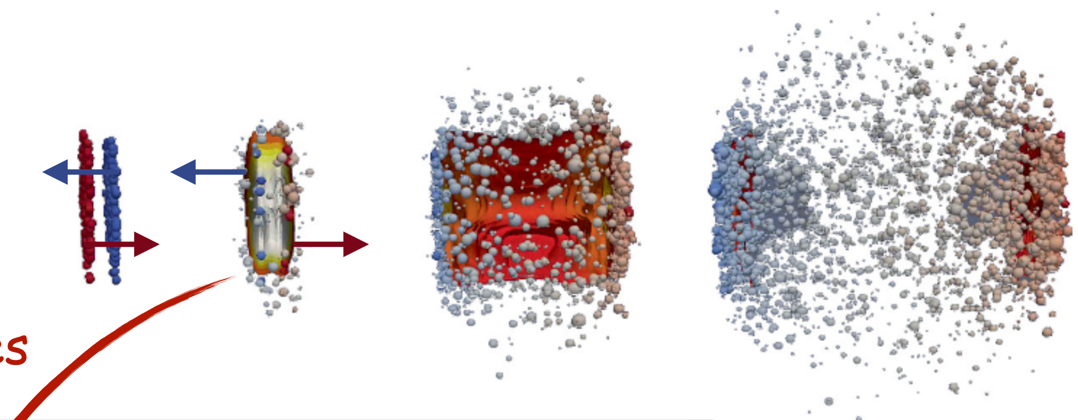

Figure from MADAI collaboration

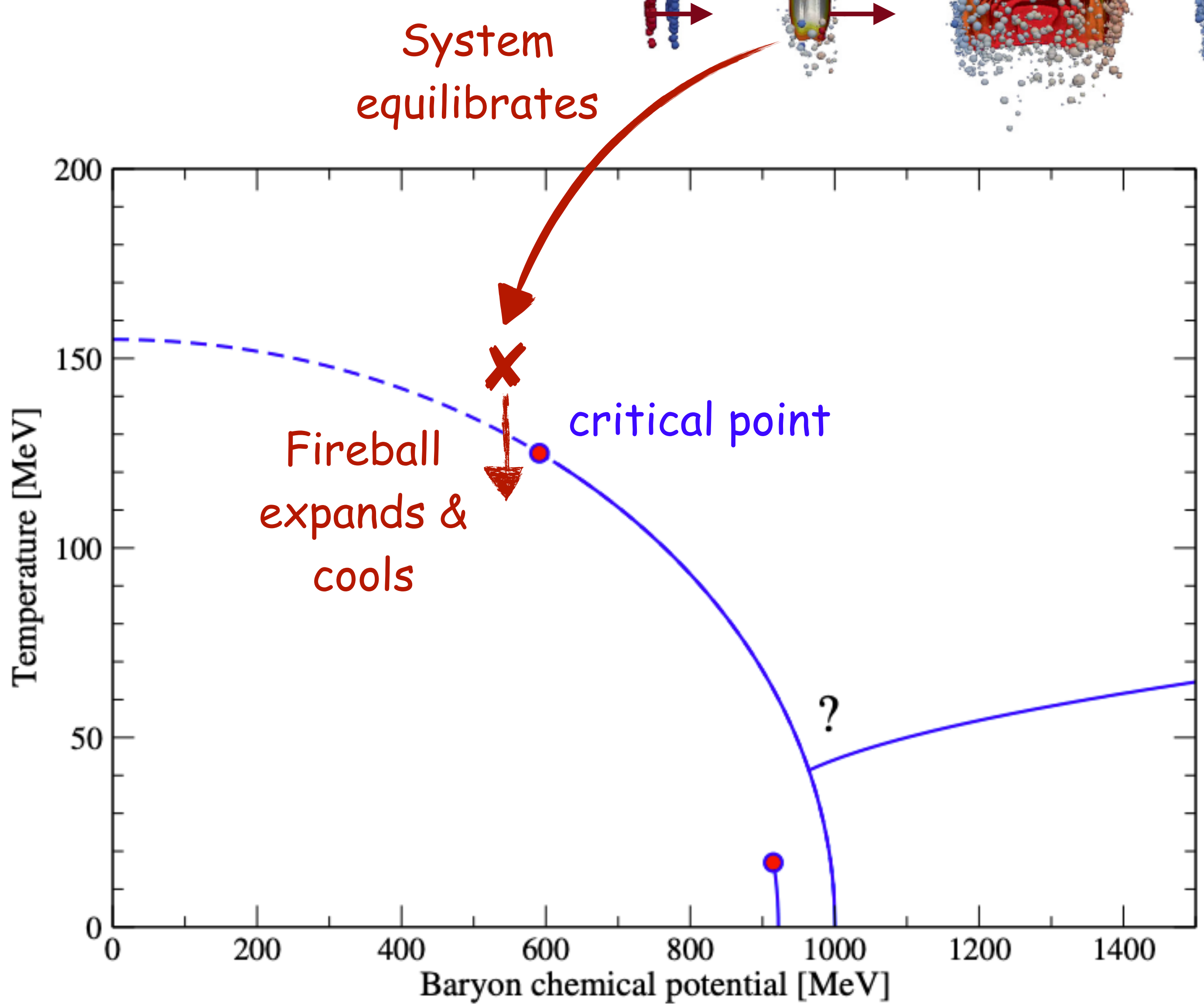

Figure adapted from C. S. Fischer, Prog. Part. Nucl. Phys. **105**, 1 (2019)

Fireball passes critical point:

Critical slowing down ⤳
System falls out of equilibrium

- non-perturbative
- off-equilibrium

⤳ proper description needs genuine **real-time methods**

Outline

1. The Schwinger-Keldysh contour
2. Renormalization in Minkowski spacetime
3. Field theory applications

The Schwinger-Keldysh contour

The Schwinger-Keldysh contour

von Neumann equation: $$i\frac{d}{dt}\rho(t) = [H(t), \rho(t)]$$ (Schrödinger picture)

- formal solution: $\rho(t) = U(t, -\infty)\rho(-\infty)U(-\infty, t)$

$$U(t, t_0) = T\exp\left\{-i\int_{t_0}^{t} dt'\, H(t')\right\}$$

- expectation value of observable:

$$\langle O(t)\rangle = \frac{\mathrm{tr}\left(U(-\infty, t)OU(t, -\infty)\rho(-\infty)\right)}{\mathrm{tr}\rho(-\infty)}$$

(Heisenberg picture)

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extend evolution to t = +∞

$$= \frac{\mathrm{tr}\left(U(-\infty,+\infty)U(+\infty,t)OU(t,-\infty)\rho(-\infty)\right)}{\mathrm{tr}\,\rho(-\infty)}$$

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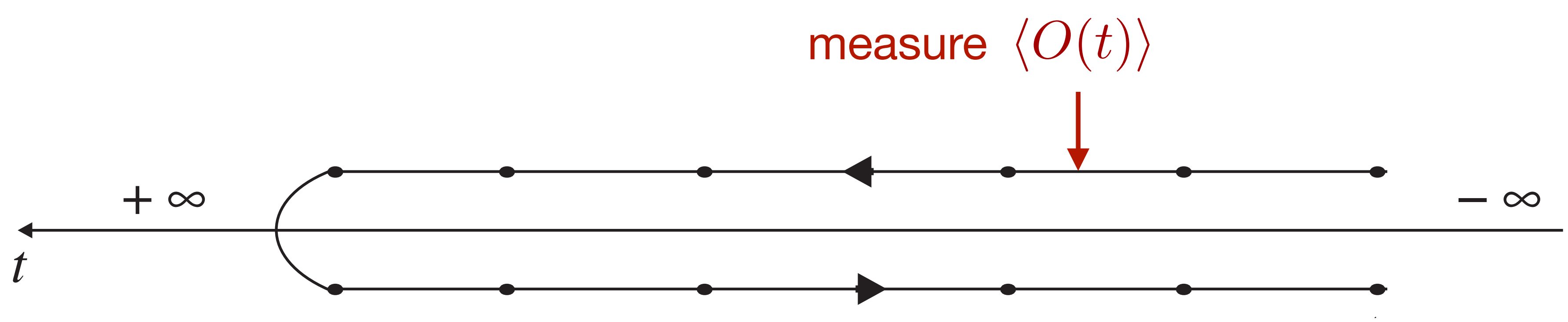

Figure taken from A. Kamenev, *Field Theory of Non-Equilibrium Systems* (Cambridge University Press, 2011)

The Schwinger-Keldysh contour

Partition function:

$$Z \equiv \frac{\mathrm{tr}\left[U(-\infty,+\infty)U(+\infty,-\infty)\rho(-\infty)\right]}{\mathrm{tr}\,\rho(-\infty)} = 1$$

Suzuki-Trotter decomposition along contour (here for scalar field theory)

$$\rightsquigarrow \quad Z = \int_{\rho_0} \mathcal{D}\phi^+\,\mathcal{D}\phi^-\;e^{iS[\phi^+,\phi^-]}$$

Keldysh action:

$$S = \int_x \big(\mathcal{L}(\phi^+) - \mathcal{L}(\phi^-)\big)$$

path-integral description of non-equilibrium systems

(requires doubling number of fields in comparison to Matsubara formalism)

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initial state

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Suzuki-Trotter decomposition along contour (here for scalar field theory)

$$\rightsquigarrow \quad \langle O \rangle = \int_{\rho_0} \mathcal{D}\phi^+ \, \mathcal{D}\phi^- \, e^{iS[\phi^+,\phi^-]} \, O[\phi^+,\phi^-]$$

initial state non-equilibrium dynamics insert observable here

Keldysh action:

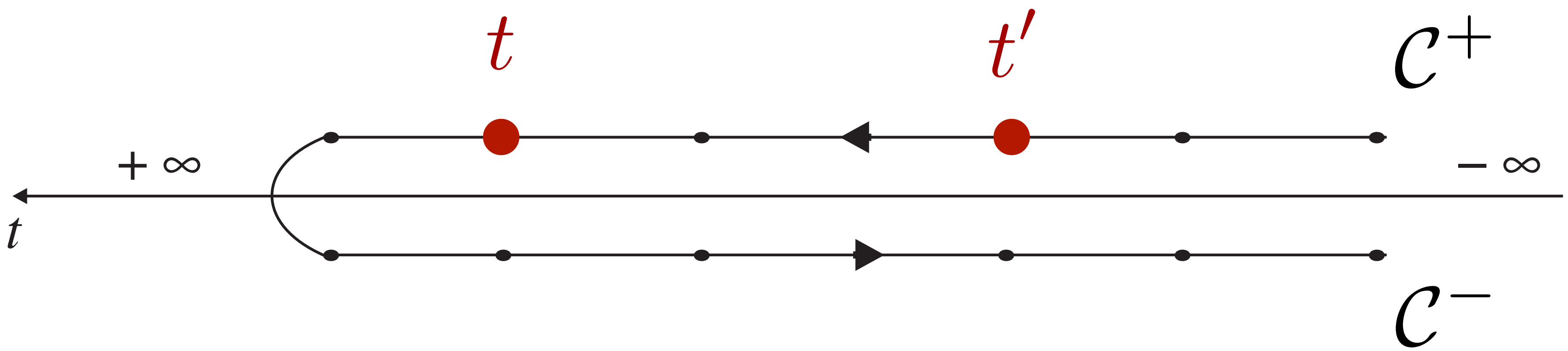
$$S = \int_x \left(\mathcal{L}(\phi^+) - \mathcal{L}(\phi^-) \right)$$

path-integral description of non-equilibrium systems

(requires doubling number of fields in comparison to Matsubara formalism)

- Which correlation functions can we access?

both times on forward (+) branch:



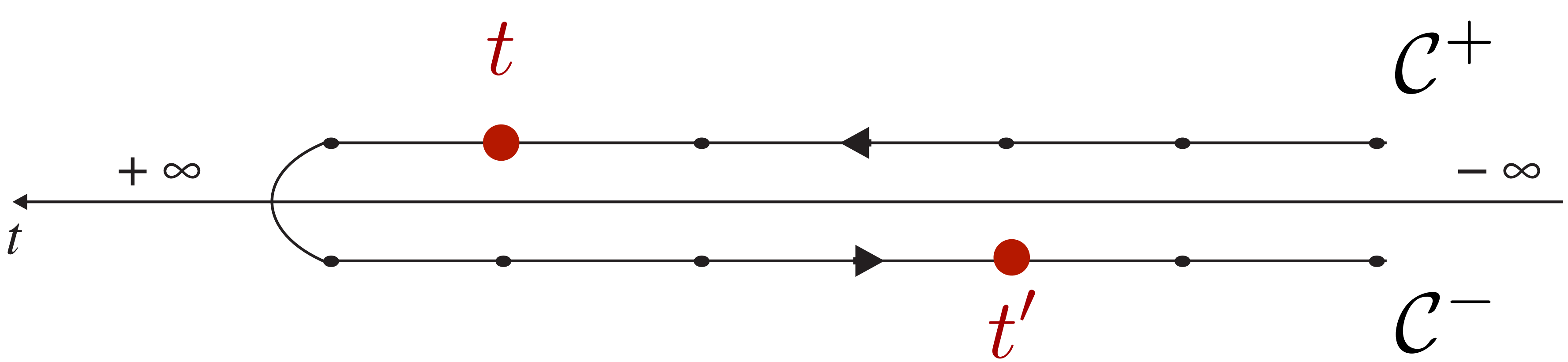
$$G^{++}(t,t') = i\langle T\phi(t)\phi(t')\rangle = G^T(t,t')$$

time ordered

The Schwinger-Keldysh contour

- Which correlation functions can we access?

times on different branches:



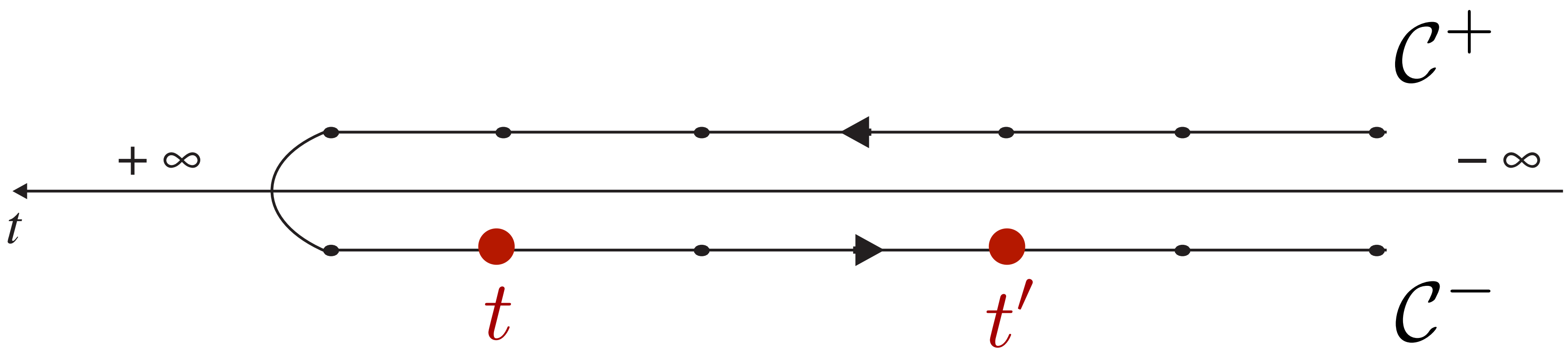
$$G^{+-}(t,t') = i\langle\phi(t')\phi(t)\rangle = G^{<}(t,t')$$
$$G^{-+}(t,t') = i\langle\phi(t)\phi(t')\rangle = G^{>}(t,t')$$

lesser/greater

The Schwinger-Keldysh contour

- Which correlation functions can we access?

both times on backward (–) branch:



$$G^{--}(t,t') = i\langle \widetilde{T}\phi(t)\phi(t')\rangle = G^{\widetilde{T}}(t,t')$$

anti-time ordered

The Schwinger-Keldysh contour

- not independent:

$$G^T(t,t') + G^{\widetilde{T}}(t,t') - G^>(t,t') - G^<(t,t') = 0$$

- exploit through Keldysh rotation

two popular conventions

$$\phi^c(t) \equiv \frac{1}{\sqrt{2}}\left(\phi^+(t) + \phi^-(t)\right)$$
$$\phi^q(t) \equiv \frac{1}{\sqrt{2}}\left(\phi^+(t) - \phi^-(t)\right)$$

$$\phi(t) \equiv \frac{1}{2}\left(\phi^+(t) + \phi^-(t)\right)$$
$$\tilde{\phi}(t) \equiv \phi^+(t) - \phi^-(t)$$

- 'rotate' propagators:

time ordered, lesser, greater, anti-time ordered → statistical function, retarded, advanced

$$\begin{pmatrix} G^T(t,t') & G^<(t,t') \\ G^>(t,t') & G^{\widetilde{T}}(t,t') \end{pmatrix} \rightarrow \begin{pmatrix} G^K(t,t') & G^R(t,t') \\ G^A(t,t') & 0 \end{pmatrix}$$

The Schwinger-Keldysh contour

- after Keldysh rotation: **causal structure** manifest

statistical/distribution function

$$G^K(t,t') = i\langle\{\phi(t),\phi(t')\}\rangle$$

retarded propagator

$$G^R(t,t') = i\theta(t-t')\langle[\phi(t),\phi(t')]\rangle$$

$$\begin{pmatrix} G^K(t,t') & G^R(t,t') \\ G^A(t,t') & 0 \end{pmatrix}$$

advanced propagator

$$G^A(t,t') = i\theta(t'-t)\langle[\phi(t'),\phi(t)]\rangle$$

Causality: System can only respond **after** source is applied!

(Functional) renormalization in Minkowski spacetime

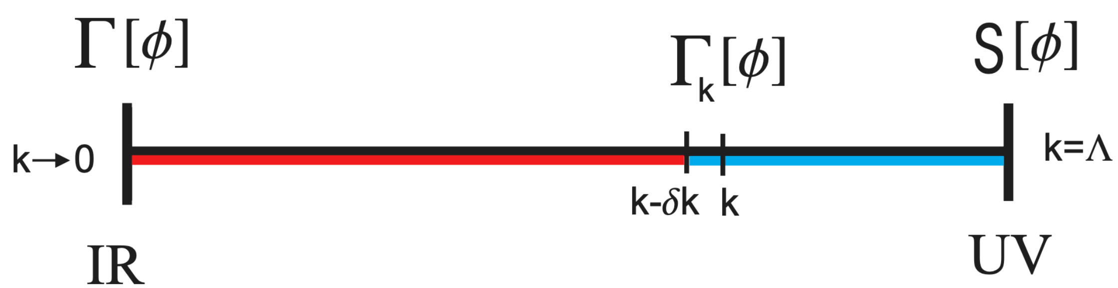
Functional RG (flow) equations

Wilson: introduce **infrared cutoff** to suppress fluctuations with $p \lesssim k$

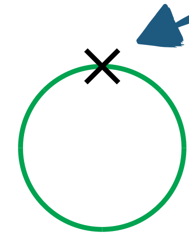
$$\Delta S_k[\phi] = \frac{1}{2}\int_{xx'} \phi^T(x) R_k(x,x')\phi(x') \qquad \phi = (\phi^c, \phi^q)^T$$

(scalar field theory)

Integrate fluctuations 'momentum shell by momentum shell'

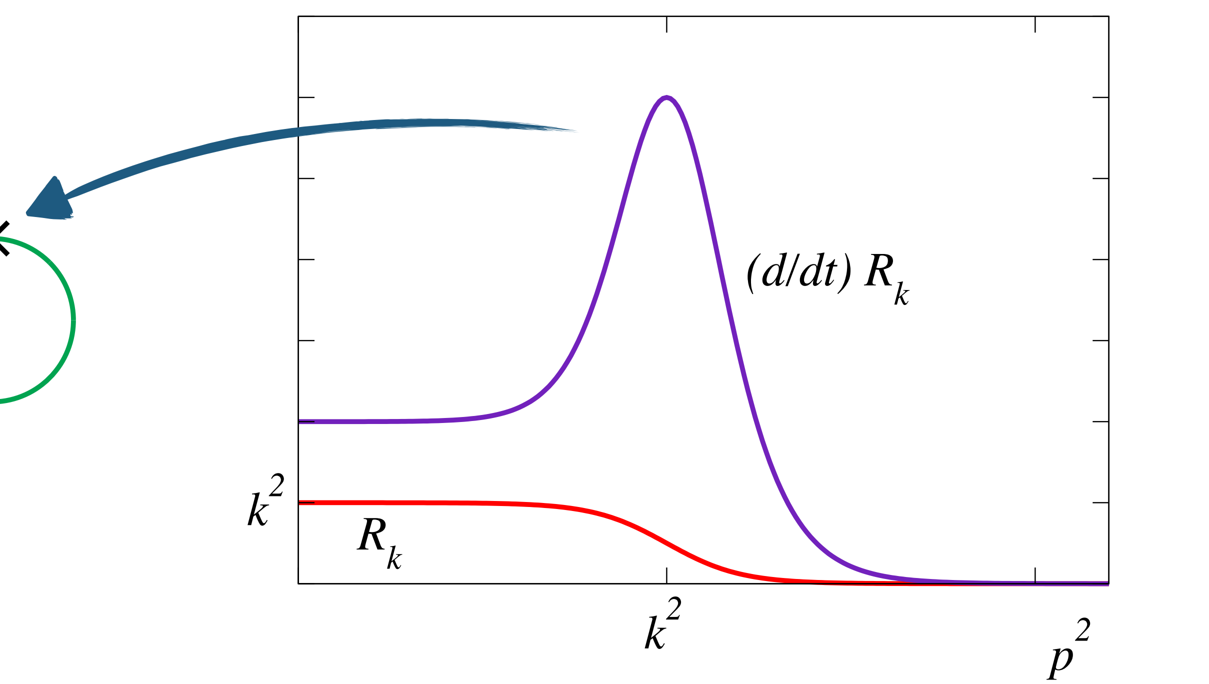


$$\partial_k \Gamma_k = \frac{i}{2}\mathrm{tr}\left\{\partial_k R_k \circ \left(R_k + \Gamma_k^{(2)}\right)^{-1}\right\} = -\frac{i}{2}$$



(exact 'flow' equation)

Typical form:

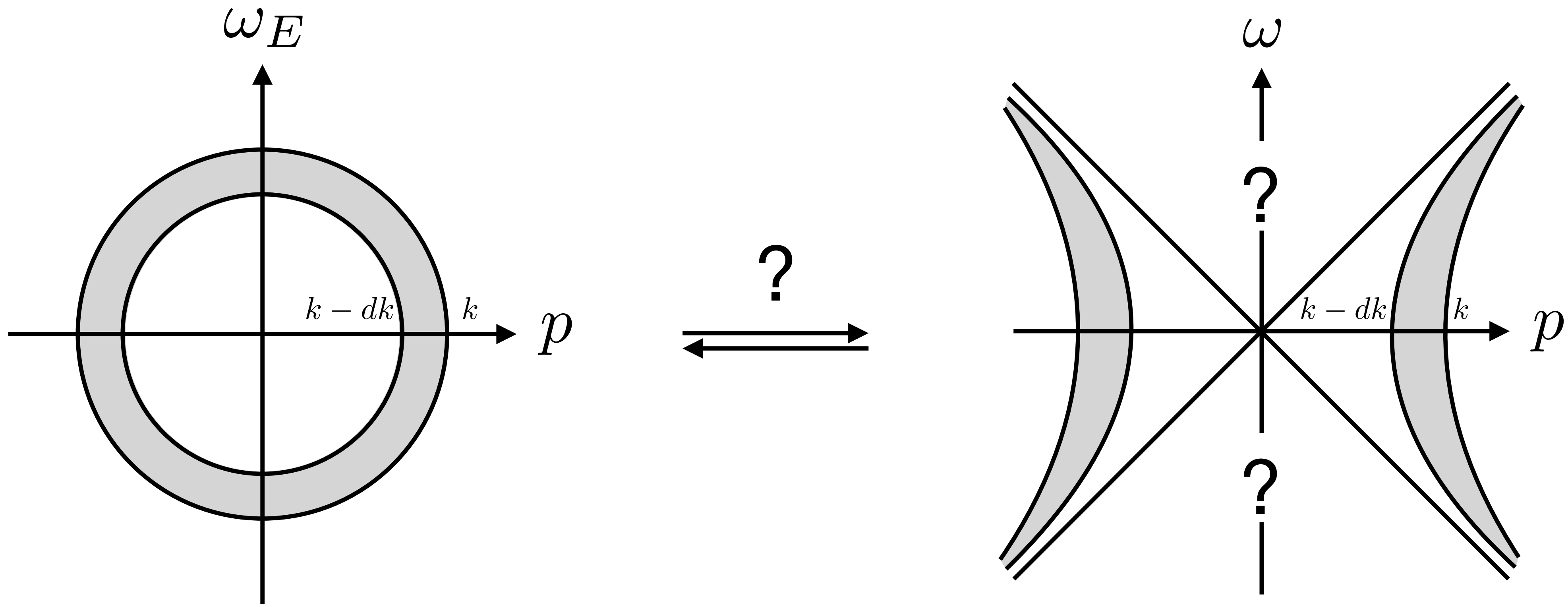


First derived by
C. Wetterich, Phys. Lett. B **301** (1993) 90-94

Real time:
J. Berges, D. Mesterházy, Nucl. Phys. B Proc. Suppl. **228** (2012) 37-60

Figure taken from H. Gies, Lect. Notes Phys. **852** (2012) 287-348

Regulators in Minkowski spacetime?



Wilsonian renormalization in Euclidean spacetime

vs.

Wilsonian renormalization in Minkowski spacetime

Conceptually easy:
integrate out (hyper-)spheres
no need to worry about causality

Conceptually intricate:
integrate hyperboloids?
timelike momenta?
causal structure of propagators?
...

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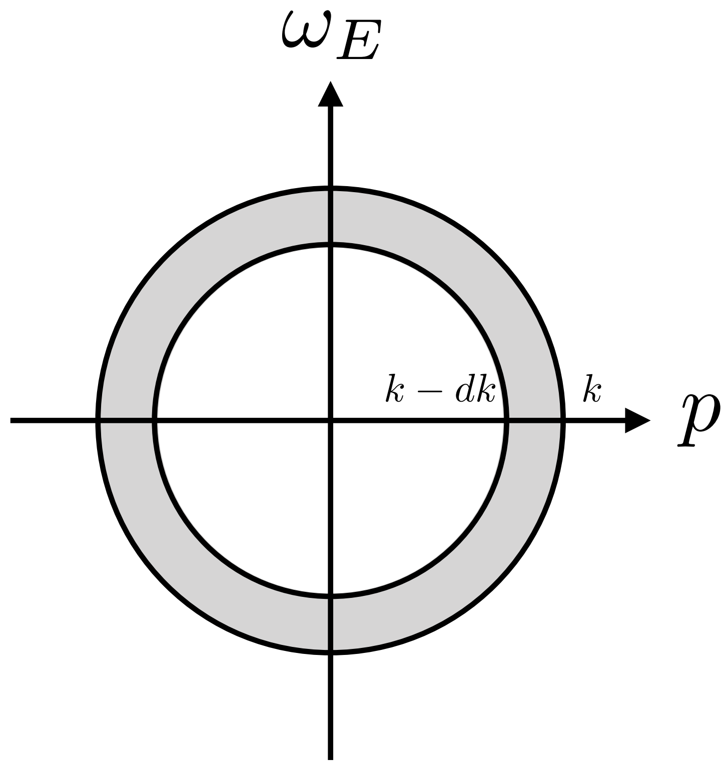
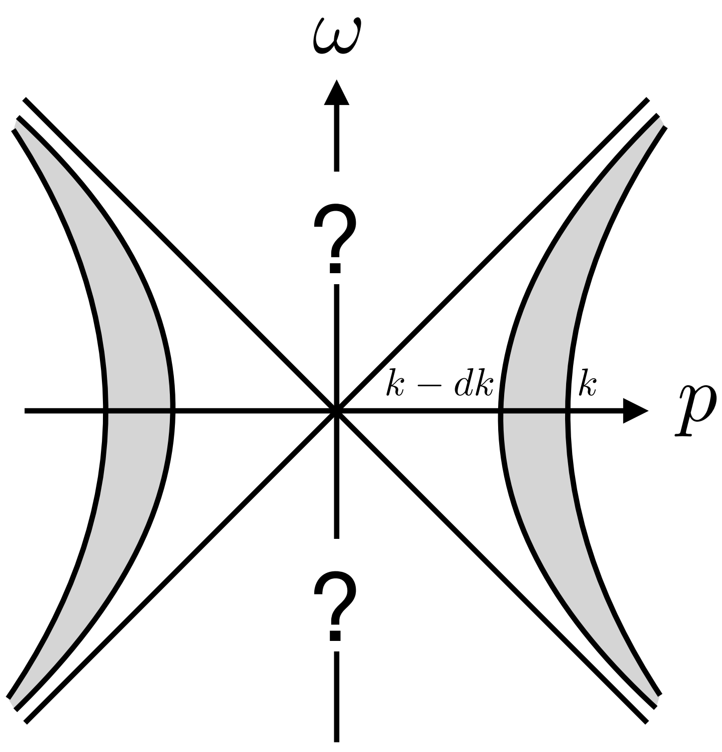


?



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Find: Frequency-dependent regulators usually violate **causal structure**

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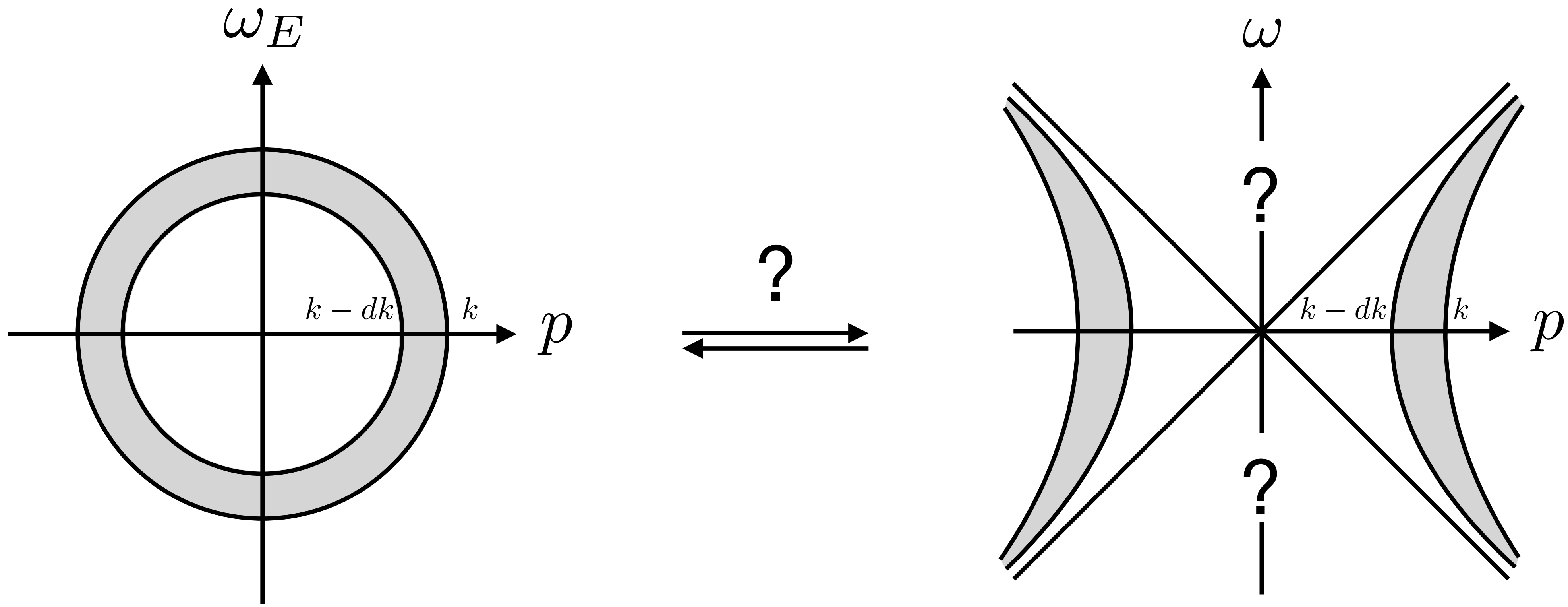

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Find: Frequency-dependent regulators usually violate **causal structure** $\rightsquigarrow$ General construction scheme which **guarantees** causality?

Causal regulators

Solution: Observe that regulator is a self-energy

- Self-energies generally inherit **causal structure**

  ⇝ **Spectral representation** from (subtracted) Kramers-Kronig relations

mass-like part (trivially causal)

'spectral density'

$$J_k(\omega, \boldsymbol{p}) = 2\,\mathrm{Im}\,R_k^R(\omega, \boldsymbol{p})$$

$$R_k^{R/A}(\omega, \boldsymbol{p}) = R_k^{R/A}(0, \boldsymbol{p}) - \int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega^2 J_k(\omega', \boldsymbol{p})}{\omega'((\omega \pm i\varepsilon)^2 - \omega'^2)}$$

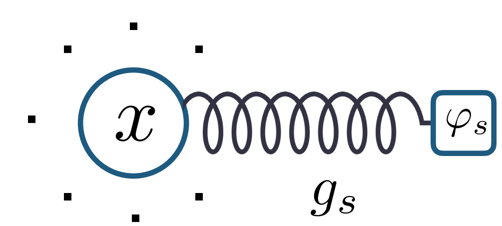
- Interpret as coupling to fictitious heat bath (Hubbard-Stratonovich transformation):

  ⇝ $$J_k(\omega) = \pi \sum_s \frac{g_s^2(k)}{\omega_s(k)} \left(\delta(\omega - \omega_s(k)) - \delta(\omega + \omega_s(k))\right)$$

  encodes spectrum of bath oscillators

- **Physical** only for **positive-semidefinite** spectral densities $J_k(\omega, \boldsymbol{p}) \geq 0 \quad (\omega > 0)$

QM example (Caldeira-Leggett model)



$$\Delta H = \sum_s \left[\frac{\pi_s^2}{2} + \frac{\omega_s^2}{2}\left(\varphi_s - \frac{g_s}{\omega_s^2} x\right)^2\right]$$

QM example for causal regulator

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$$R_k^{R/A}(\omega) = R_k^{R/A}(0) - \int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega^2 J_k(\omega')}{\omega'((\omega \pm i\varepsilon)^2 - \omega'^2)} \quad \text{in} \quad {\Gamma_k^{(2)}}^R(\omega) = (\omega + i\varepsilon)^2 - m^2 + R_k^R(\omega)$$

- spectral density: $\rightsquigarrow$ **Regulator** (retarded part):

$$J_k(\omega) = 2k\omega\, e^{-\omega^2/k^2} = 2\,\mathrm{Im}\, R_k^R(\omega)$$

- assume UV finiteness:

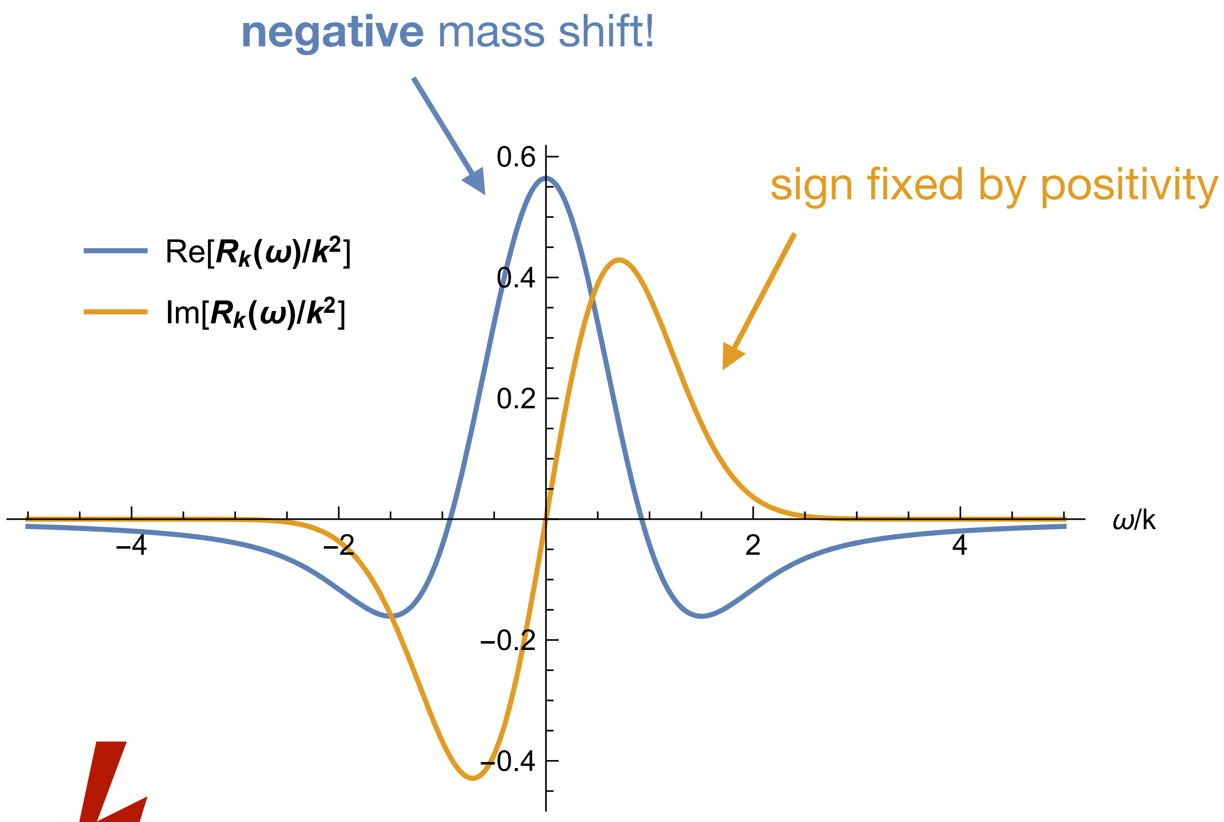
$$\Delta M_{UV}^2(k) = -R_k^{R/A}(0) + \underbrace{\int_0^\infty \frac{d\omega'}{\pi} \frac{J_k(\omega')}{\omega'}}_{\geq 0} \stackrel{!}{=} 0$$

(positivity)

$\Rightarrow$ IR mass shift:

$$\Delta M_{IR}^2(k) = -R_k^{R/A}(0) < 0$$

is negative!

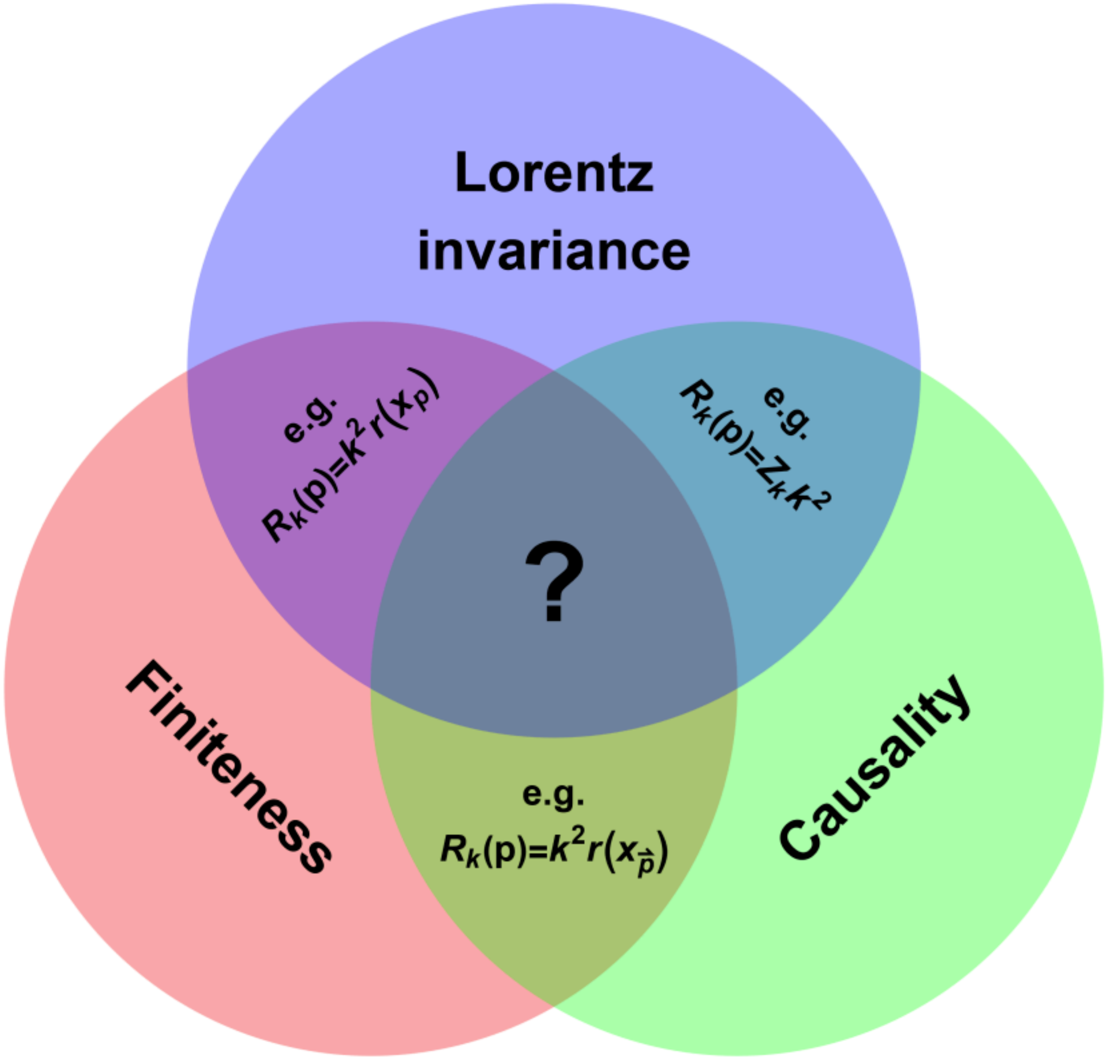


Solution: choose IR mass shift $\Delta M_{IR}^2(k) > 0$ positive (at cost of **UV finiteness**)

Regulator trinity in real-time FRG flows

requires invariant spectral distribution & momentum-independent mass shift

$$J_k(\omega, \boldsymbol{p}) = 2\pi\,\mathrm{sgn}(\omega)\,\theta(p^2)\,\widetilde{J}_k(p^2)\,, \qquad \Delta M_k^2(\boldsymbol{p}) = \Delta M_k^2$$



requires vanishing spectral density and mass shift in the UV

$$J_k(\omega, \boldsymbol{p}) \to 0 \text{ for } \omega \to \infty$$
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Figure adapted from arXiv:2206.10232 (fQCD collaboration)

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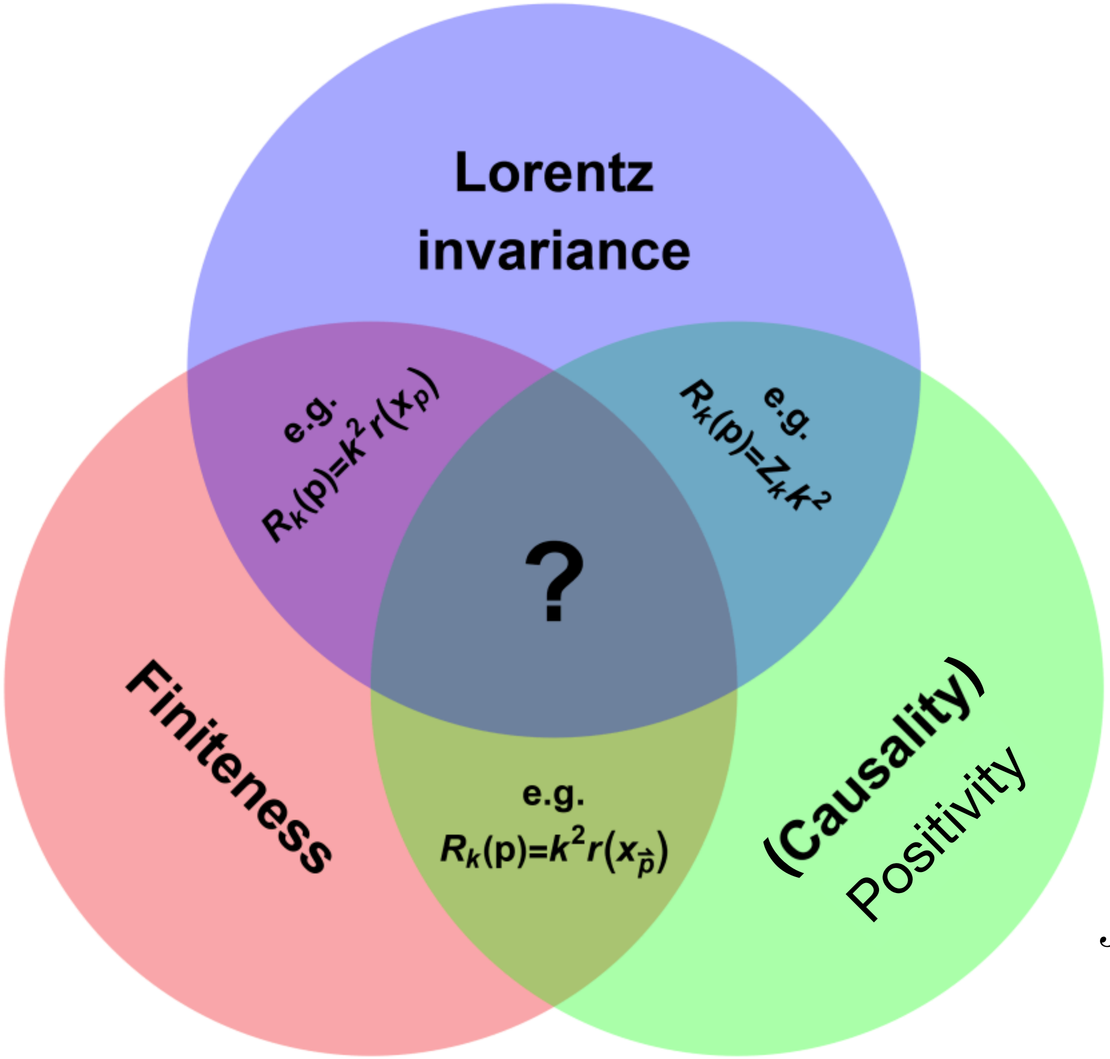

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Lorentz invariance & **Finiteness**
in conflict with **Positivity**

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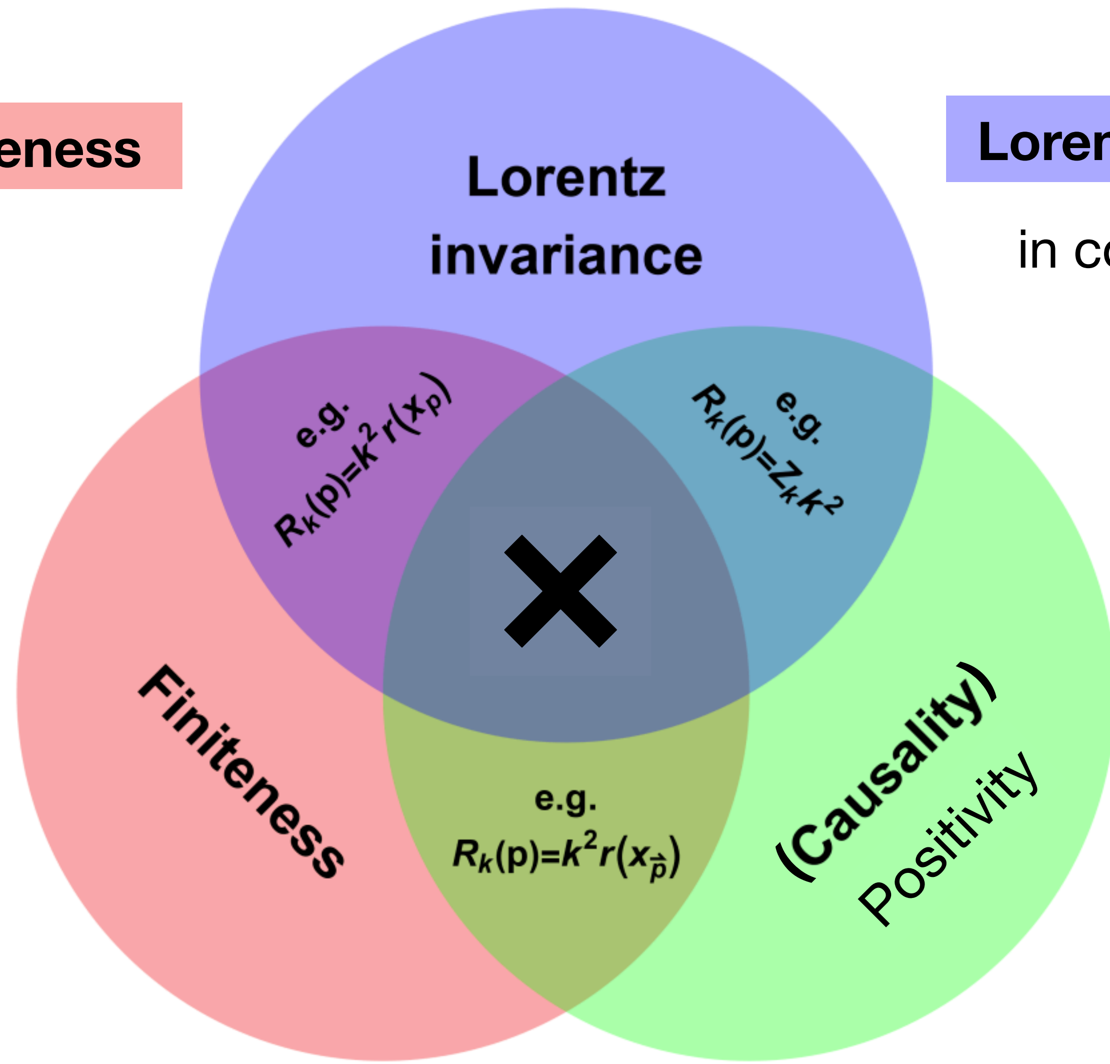

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JR, von Smekal, arXiv:2303.11817

Finiteness & **Positivity**
in conflict with **Lorentz invariance**

Figure adapted from arXiv:2206.10232 (fQCD collaboration)

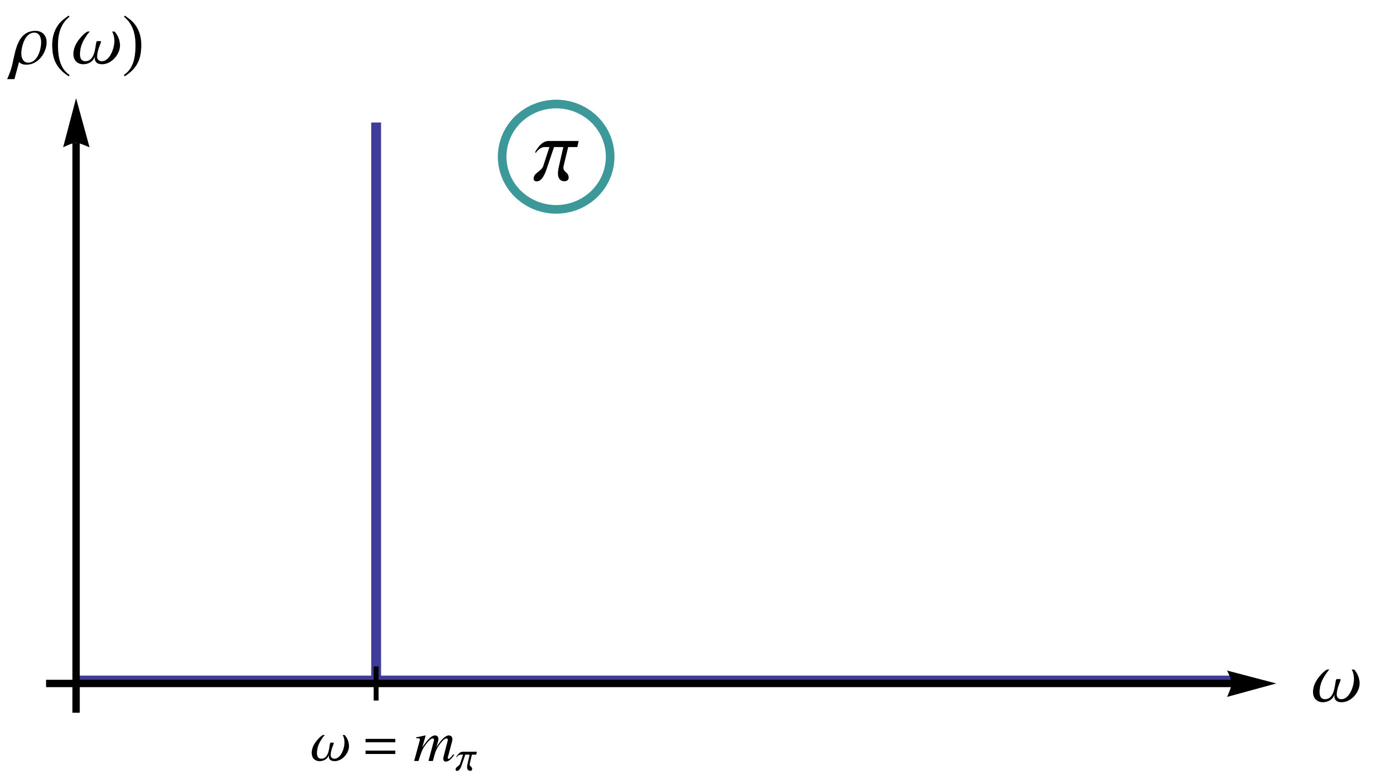
Field theory applications: Critical dynamics

Observable: Spectral function

Commutator of interacting fields (here order parameter):

$$\rho(\omega) = \frac{1}{2\pi i}\int dt\; e^{i\omega t}\int d^d x\; i\langle[\phi(t,\boldsymbol{x}),\phi(0,\mathbf{0})]\rangle$$

free fields (stable particle):



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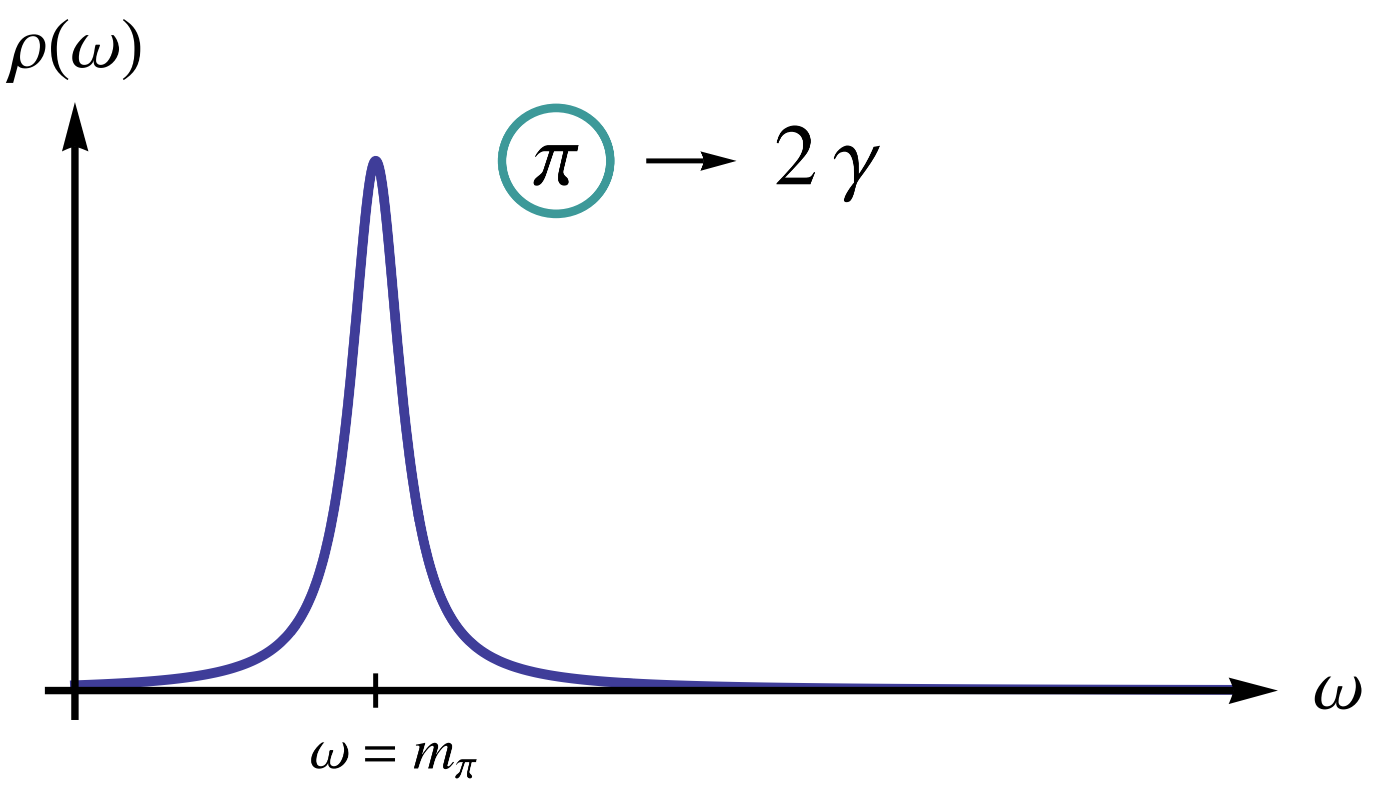
finite lifetime:



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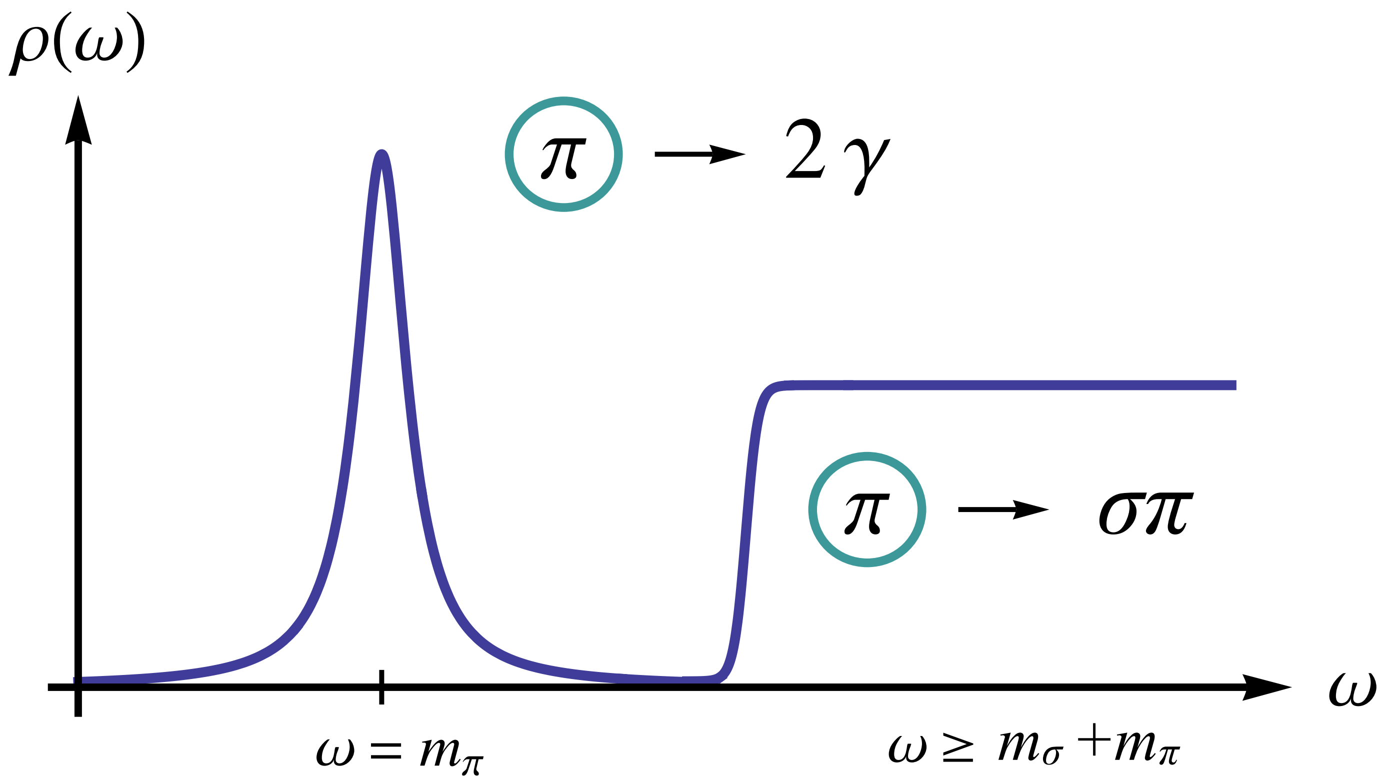
two-particle thresholds:



Critical spectral functions

Commutator of interacting fields (here order parameter):

$$\rho(\omega) = \frac{1}{2\pi i} \int dt\; e^{i\omega t} \int d^d x\; i\langle[\phi(t,\boldsymbol{x}), \phi(0,\boldsymbol{0})]\rangle$$

- typically at criticality: $\rho(\omega) \sim \omega^{-\sigma}$
- scaling exponent: $\sigma = (2-\eta)/z$
- related to dynamic critical exponent z: $\xi_t \sim \xi^z$ **critical slowing down**

  correlation time (ξ_t) correlation length (ξ)

- z determined by **dynamic** universality class

Dynamic universality classes

Static universality classes split up into **dynamic** universality classes:

classified into 'Models':
Hohenberg and Halperin, Rev. Mod. Phys. **49**, 435 (1977)

Static Z_2 Ising universality class

Model A
No conserved quantities
No Poisson brackets

Model B
Order parameter conserved
No Poisson brackets

Model C
Conserved energy density
No Poisson brackets

Model H
Order parameter conserved
Poisson bracket $\{\phi, \vec{j}\}$ with shear modes

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Dynamic universality class of QCD's critical endpoint!

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Dynamic universality classes

Model A
$z = 2 + c\eta$

Consider classical ϕ^4-theory with Landau-Ginzburg-Wilson functional

$$F = \int d^dx \, \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) \qquad \right\}$$

equilibrium distribution:

$$P[\varphi] \sim e^{-\beta F}$$

- and Langevin equations of motion

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi$$

Gaussian white noise

describes particle submerged in heat bath

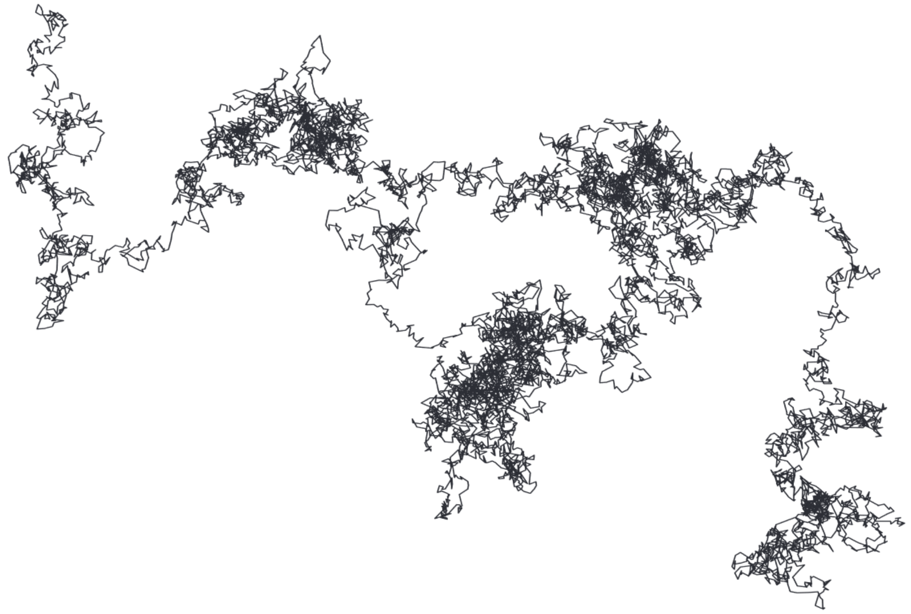


Image taken from P. Mörters, Y. Peres, *Brownian Motion* (Cambridge University Press, 2010)

- No conservation laws here! ⇝ **Model A**
- **Slow modes** determine critical dynamics

(e.g. densities of conserved quantities) (generally true)

Dynamic universality classes

Model B
$z = 4 - \eta$

Consider classical ϕ^4-theory with Landau-Ginzburg-Wilson functional

$$F = \int d^d x \, \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) + B\varphi n + \frac{n^2}{2\chi_0} \right\}$$

equilibrium distribution:

$$P[\varphi, n] \sim e^{-\beta F}$$

- and Langevin equations of motion

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi$$

Gaussian white noises

$$\partial_t n = \bar{\lambda} \vec{\nabla}^2 \frac{\delta F}{\delta n} + \vec{\nabla} \cdot \vec{\zeta}$$

diffusive!

- Critical dynamics dominated by diffusion ⤳ **Model B**
- Include hydrodynamic shear modes of energy-momentum tensor ⤳ **Model H**

Dynamic universality classes

Model C
$z = 2 + \alpha/\nu$

Consider classical ϕ^4-theory with Landau-Ginzburg-Wilson functional

$$F = \int d^d x \, \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) \qquad + \frac{n^2}{2\chi_0} + \frac{g}{2}\varphi^2 n \right\}$$

equilibrium distribution:

$$P[\varphi, n] \sim e^{-\beta F}$$

- and Langevin equations of motion

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi$$

Gaussian white noises

$$\partial_t n = \bar{\lambda} \vec{\nabla}^2 \frac{\delta F}{\delta n} + \vec{\nabla} \cdot \vec{\zeta}$$

diffusive!

- Order parameter not conserved but interacts non-linearly with conserved (energy) density ⇝ **Model C**

Critical dynamics – truncation

1PI vertex expansion around scale-dependent minimum $\phi_{0,k}$:

- effective average action:

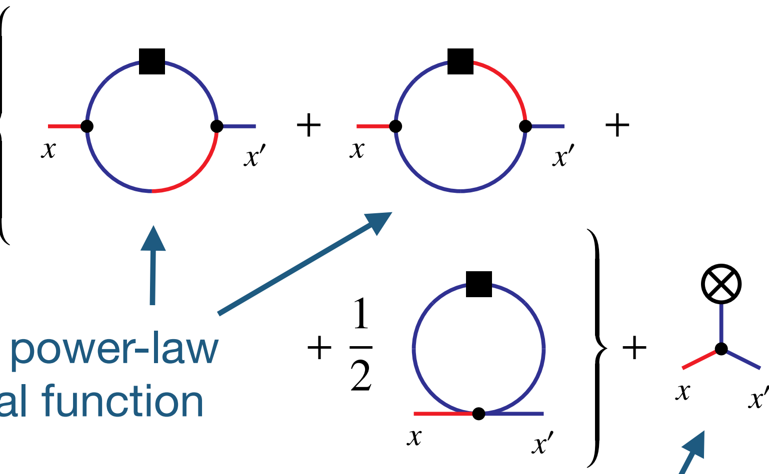
$$\Gamma_k = \frac{1}{2}\int_{xx'} \left(\phi^c - \phi^c_{0,k}, \phi^q\right)_x \begin{pmatrix} 0 & \Gamma^{cq}_k(x,x') \\ \Gamma^{qc}_k(x,x') & \Gamma^{qq}_k(x,x') \end{pmatrix} \begin{pmatrix} \phi^c - \phi^c_{0,k} \\ \phi^q \end{pmatrix}_{x'} - \frac{\kappa_k}{\sqrt{8}}\int_x \left(\phi^c - \phi^c_{0,k}\right)^2 \phi^q - \frac{\lambda_k}{12}\int_x \left(\phi^c - \phi^c_{0,k}\right)^3 \phi^q$$

expand 2-point function in spatial gradients, but keep full frequency dependence:

$$\Gamma^{qc}_k(\omega, \boldsymbol{p}) = \Gamma^{qc}_{0,k}(\omega) - Z^{\perp}_k \boldsymbol{p}^2 + \cdots$$
$$\Gamma^{cq}_k(\omega, \boldsymbol{p}) = \Gamma^{cq}_{0,k}(\omega) - Z^{\perp}_k \boldsymbol{p}^2 + \cdots$$
$$\Gamma^{qq}_k(\omega, \boldsymbol{p}) = \frac{2T}{\omega}\left(\Gamma^{qc}_{0,k}(\omega) - \Gamma^{cq}_{0,k}(\omega)\right)$$

- flow of 2-point function:

$$\partial_k \Gamma^{qc}_k(x,x') = -i$$

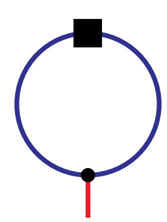


generate non-local power-law behavior in spectral function

'interaction' with scale-dependent minimum

- flow of effective potential:

$$\partial_k V'_k(\varphi) = -\frac{i}{\sqrt{8}}$$



use for squared mass and quartic coupling

- flow of couplings to density: (Model B)

vanish!
(coupling is linear ⤳ mixing)

for color coding and diagrammatic conventions, see
S. Huelsmann, S. Schlichting, P. Scior, Phys. Rev. D **102**, 096004 (2020)

Critical dynamics – truncation

1PI vertex expansion around $\phi = 0$:

- effective average action:

$$\Gamma_k = \frac{1}{2}\int_{xx'} (\phi^c, \phi^q)_x \begin{pmatrix} 0 & \Gamma_k^{cq}(x,x') \\ \Gamma_k^{qc}(x,x') & \Gamma_k^{qq}(x,x') \end{pmatrix} \begin{pmatrix} \phi^c \\ \phi^q \end{pmatrix}_{x'} + \frac{3\cdot 2^2}{4!}\int_{xx'} \phi^q(x)\phi^c(x) V_k^{an}(x,x')\phi^q(x')\phi^c(x') + \frac{3\cdot 2}{4!}\int_{xx'} \phi^q(x)\phi^c(x) V_k^{cl,R}(x,x')\phi^c(x')\phi^c(x') + \frac{3\cdot 2}{4!}\int_{xx'} \phi^c(x)\phi^c(x) V_k^{cl,A}(x,x')\phi^q(x')\phi^c(x')$$

expand 2- and 4-point functions in spatial gradients,
but keep full frequency dependence:

$$\Gamma_k^{qc}(\omega,\boldsymbol{p}) = \Gamma_{0,k}^{qc}(\omega) - Z_k^{\perp}\boldsymbol{p}^2 + \cdots$$

$$\Gamma_k^{cq}(\omega,\boldsymbol{p}) = \Gamma_{0,k}^{cq}(\omega) - Z_k^{\perp}\boldsymbol{p}^2 + \cdots$$

$$\Gamma_k^{qq}(\omega,\boldsymbol{p}) = \frac{2T}{\omega}\left(\Gamma_{0,k}^{qc}(\omega) - \Gamma_{0,k}^{cq}(\omega)\right)$$

$$V_k^{cl,A}(\omega,\boldsymbol{p}) = V_{0,k}^{cl,A}(\omega) + V_{1,k}^{cl,A}(0)\boldsymbol{p}^2 + \cdots$$

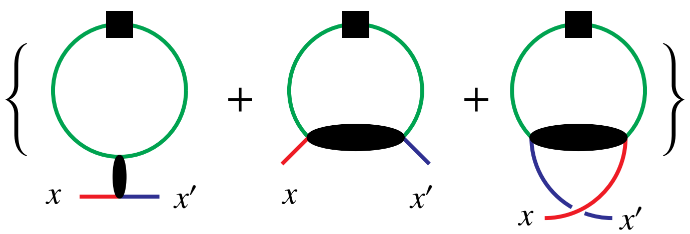
$$V_k^{cl,R}(\omega,\boldsymbol{p}) = V_{0,k}^{cl,R}(\omega) + V_{1,k}^{cl,R}(0)\boldsymbol{p}^2 + \cdots$$

$$V_k^{an}(\omega,\boldsymbol{p}) = \frac{2T}{\omega}\left(V_k^{cl,R}(\omega,\boldsymbol{p}) - V_k^{cl,A}(\omega,\boldsymbol{p})\right)$$

for the QM case, see
S. Huelsmann, S. Schlichting, P. Scior, Phys. Rev. D **102**, 096004 (2020)
JR, D. Schweitzer, L. J. Sieke, L. von Smekal, Phys. Rev. D **105**, 116017 (2022)

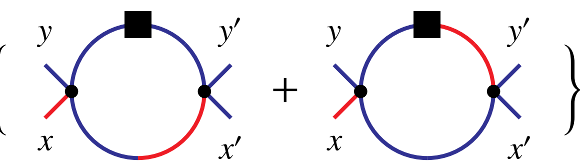
- flow of 2-point and 4-point functions:

$$\partial_k \Gamma_k^{qc}(x,x') = -\frac{i}{2}\Big\{$$



$$\Big\}$$

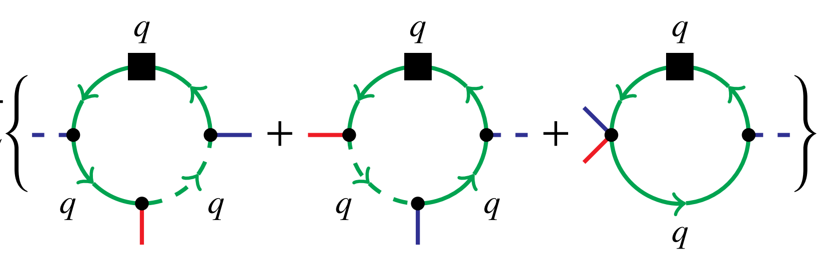
$$\partial_k V_k^{cl,R}(x,x') = -i \int_{\substack{x-y,\\ x'-y'}} \Big\{$$



$$\Big\}$$

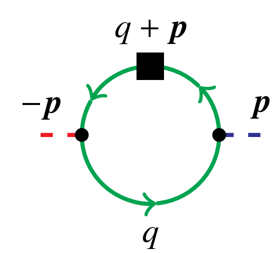
- flow of couplings to density: (Model C)

$$\partial_k g_k = i\sqrt{2}\Big\{$$



$$\Big\}$$

$$\partial_k \chi_{0,k}^{-1} = \frac{i}{\lambda} \lim_{p\to 0} \frac{1}{\boldsymbol{p}^2}$$

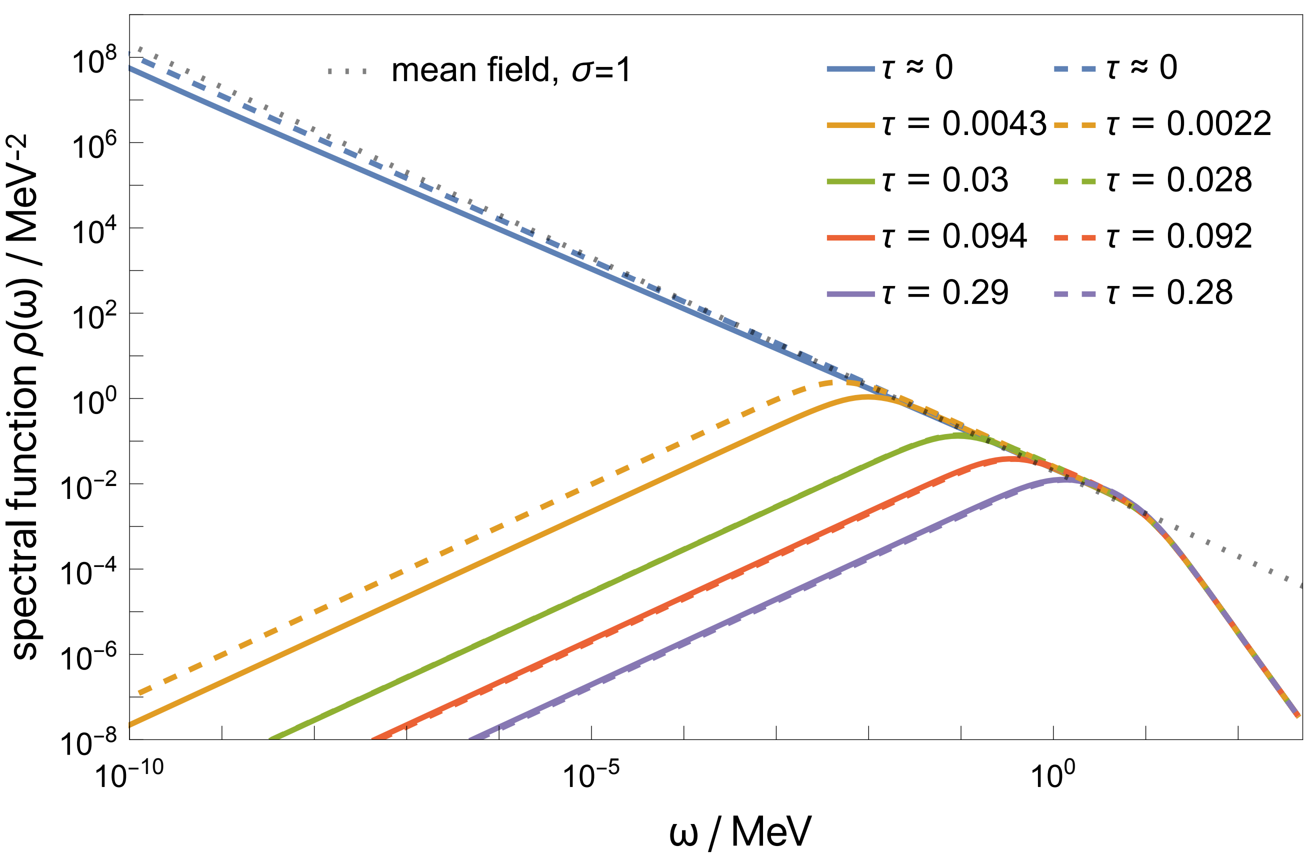


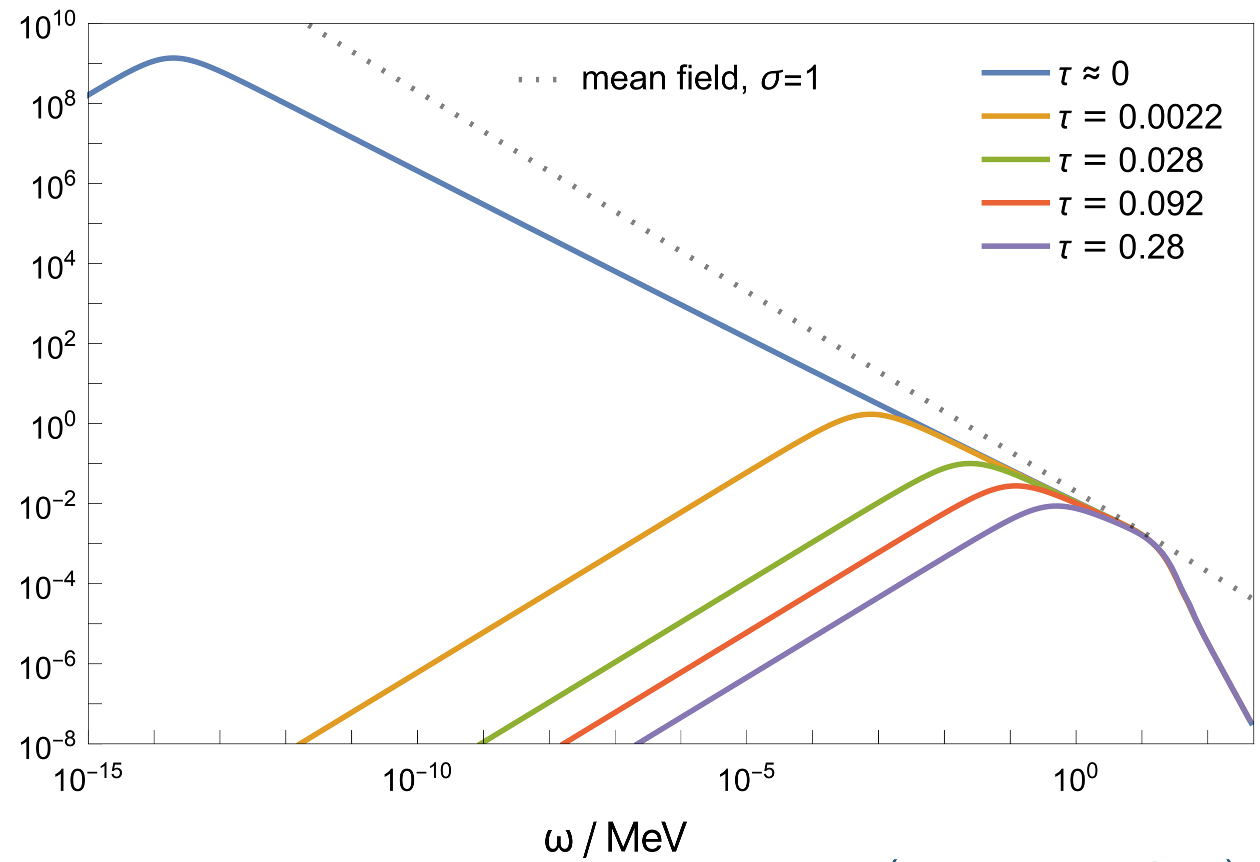
Results for critical spectral functions

Model A
$z = 2 + c\eta$

$$\rho(\omega) \sim \omega^{-\sigma} \quad \text{with} \quad \sigma = \frac{2-\eta}{z}$$

Model C
$z = 2 + \alpha/\nu$





(zero momentum)

$d = 3$

$z \approx 2.042$ (dashed)

$z \approx 2.035$ (solid)

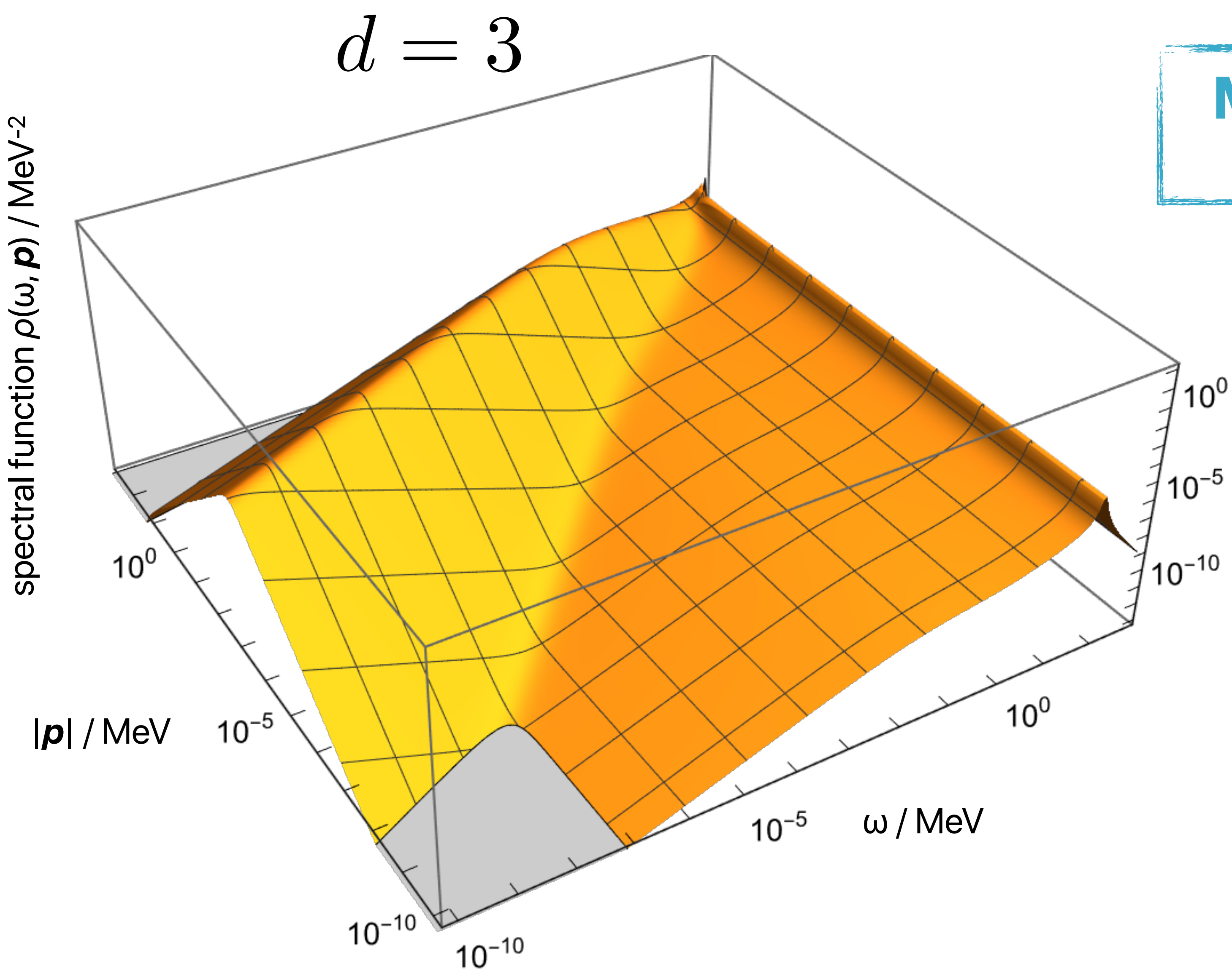
$z \approx 2.31$

[reduced temperature $\tau = (T - T_c)/T_c$]

JR, L. von Smekal, arXiv:2303.11817

Results for critical spectral functions

CRC–TR 211



Model B

$z = 4 - \eta$

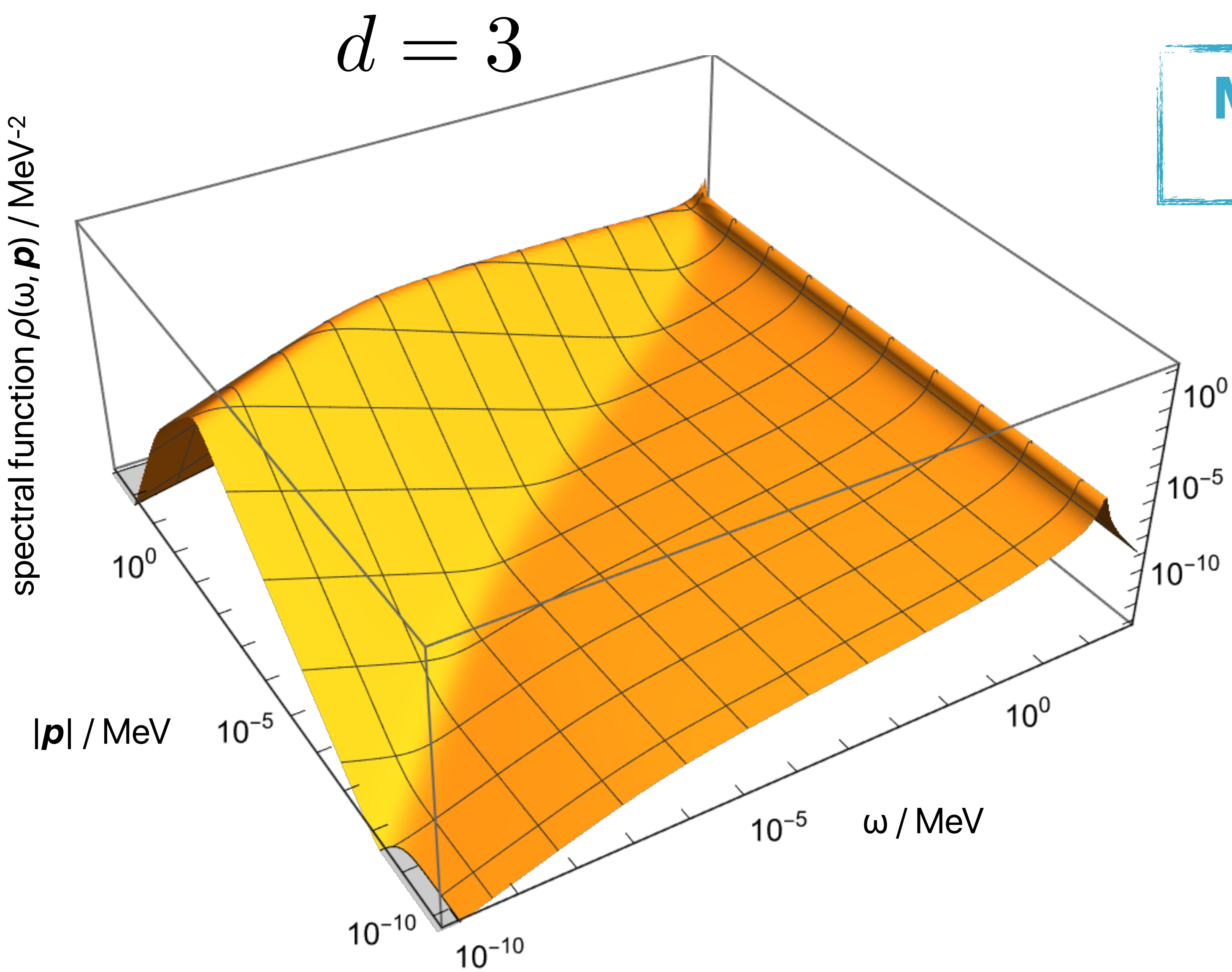
$\tau = -0.356$

(non-critical)

[reduced temperature $\tau = (T - T_c)/T_c$]

JR, L. von Smekal, arXiv:2303.11817

Results for critical spectral functions



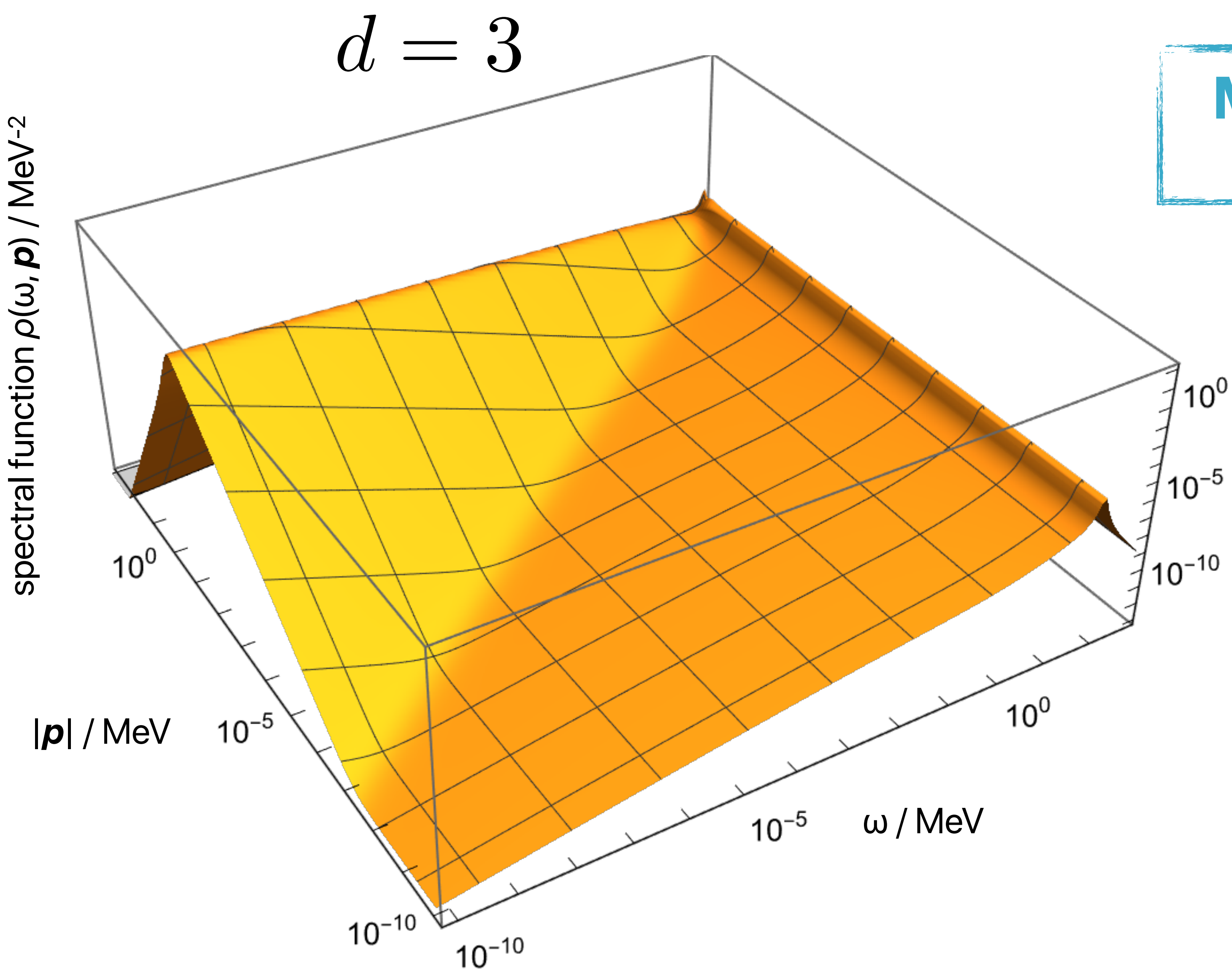
Model B
$z = 4 - \eta$

$\tau = -0.0021$
(near criticality)

[reduced temperature $\tau = (T - T_c)/T_c$]

JR, L. von Smekal, arXiv:2303.11817

Results for critical spectral functions



Model B
$z = 4 - \eta$

$\tau \approx 0$
(critical)

[reduced temperature $\tau = (T - T_c)/T_c$]

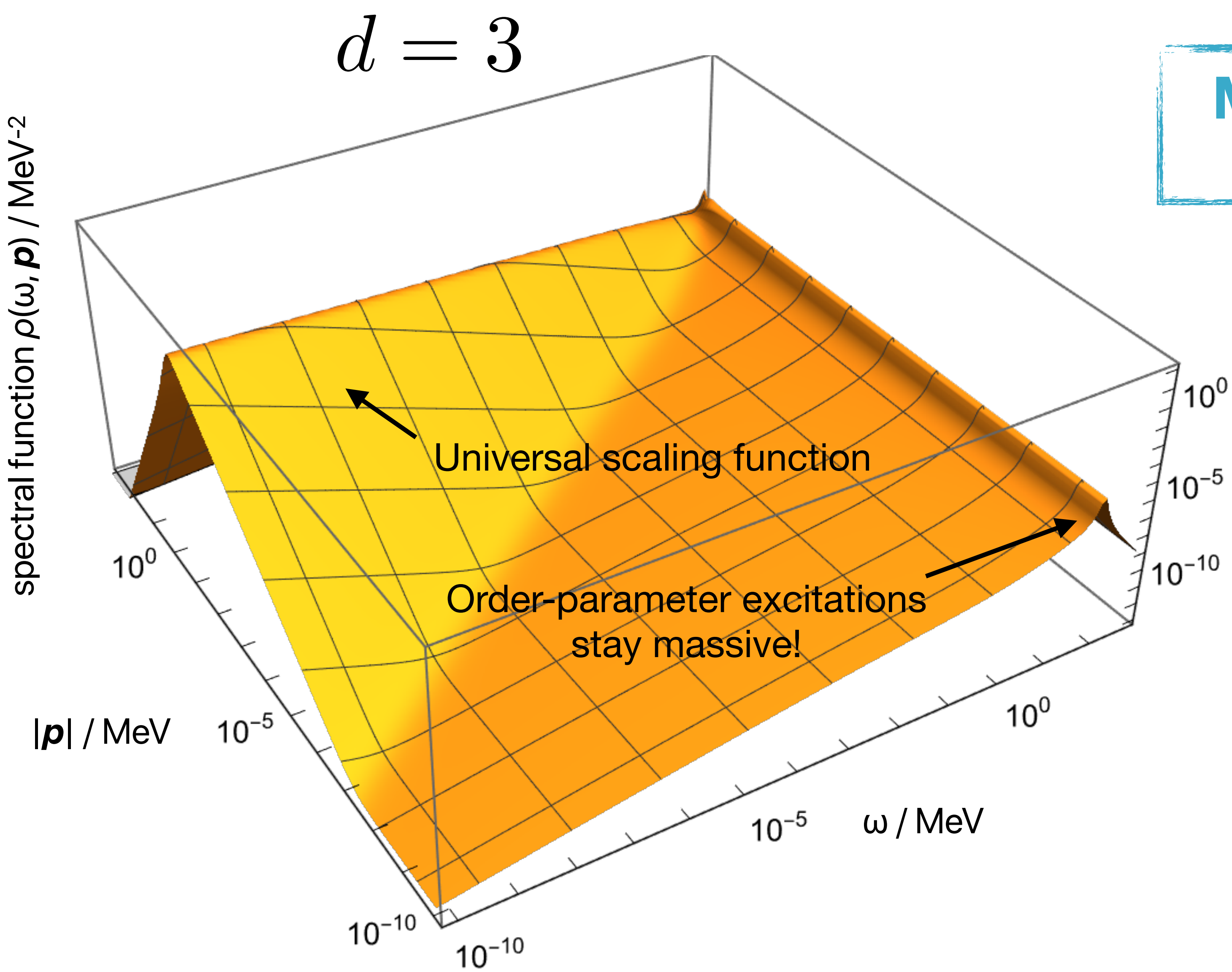
JR, L. von Smekal, arXiv:2303.11817

Results for critical spectral functions

$$\omega \sim |\boldsymbol{p}|^z$$

$$z = 4$$

(here)



Model B

$z = 4 - \eta$

$\tau \approx 0$

(critical)

Universal scaling functions:
Model A, C: Schweitzer, Schlichting, von Smekal, Nucl. Phys. B **960**, 115165 (2020)
Model B, BC: Schweitzer, Schlichting, von Smekal, Nucl. Phys. B **984**, 115944 (2022)

[reduced temperature $\tau = (T - T_c)/T_c$]

JR, L. von Smekal, arXiv:2303.11817

Summary & Outlook

CRC-TR 211

Summary:

- causal regulators for real-time FRG — JR, Schweitzer, Sieke, von Smekal, Phys. Rev. D **105**, 116017 (2022)
- critical spectral functions of **Models A, B and C** — JR, von Smekal, arXiv:2303.11817

Outlook:

- dynamic critical exponent & scaling functions of **Model G**
- real-time dynamics of **Model H** — JR, Schlichting, von Smekal, Ye, in preparation
- new dynamic scaling functions
- non-equilibrium phase transitions (Kibble-Zurek scaling)

Thank you!

Backup

Dynamic universality classes

Static universality classes split up into **dynamic** universality classes:

classified into 'Models':
Hohenberg and Halperin, Rev. Mod. Phys. **49**, 435 (1977)

Static *O*(4) universality class

...

Model G
Order parameter ϕ not conserved
Conserved isovector and isoaxial charges n
Non-vanishing Poisson brackets {ϕ, n}, {n, n}

...

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…

Dynamic universality class of QCD's chiral phase transition in chiral limit

Rajagopal and Wilczek,
Nucl. Phys. B **399** (1993) 395-425

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Conserved isovector and isoaxial charges n
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Dynamic universality class of QCD's chiral phase transition in chiral limit

Rajagopal and Wilczek,
Nucl. Phys. B **399** (1993) 395-425

Dynamic universality classes

Model G
z = *d*/2

Consider classical $O(N)$-theory with Landau-Ginzburg-Wilson functional

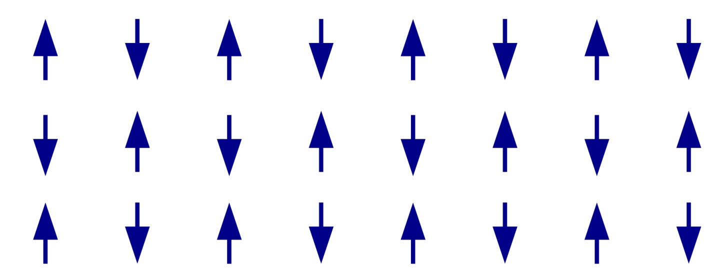
$$F = \int_{\vec{x}} \left\{ \frac{1}{2}(\partial^i \phi_a)(\partial^i \phi_a) + \frac{m^2}{2}\phi_a\phi_a + \frac{\lambda}{4!N}(\phi_a\phi_a)^2 + \frac{1}{2\chi}n_{ab}n_{ab} \right\}$$

equilibrium distribution:

$$P[\varphi, n] \sim e^{-\beta F}$$

- *N*-component order parameter $\phi_a(x)$
  (**not** conserved, staggered magnetization)

N = 3: **Heisenberg antiferromagnet**

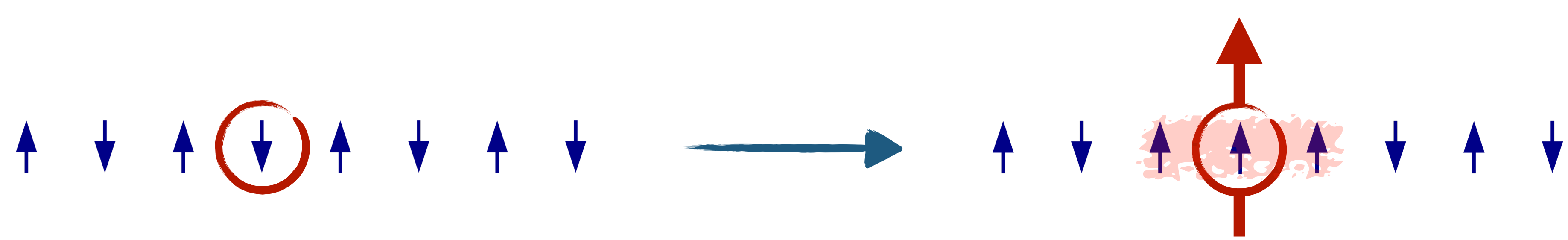


Low temperature:
Antiferromagnetic order, ϕ ≠ 0

- *N*(*N*-1)/2 densities of charges $n_{ab}(x)$
  (**conserved**, magnetization)

  can be non-zero due to fluctuations:

no ‘macroscopic’ **magnetization**



‘macroscopic’ **magnetization!**

Dynamic universality classes

Model G
$z = d/2$

Consider classical $O(N)$-theory with Landau-Ginzburg-Wilson functional

$$F = \int_{\vec{x}} \left\{ \frac{1}{2}(\partial^i \phi_a)(\partial^i \phi_a) + \frac{m^2}{2}\phi_a\phi_a + \frac{\lambda}{4!N}(\phi_a\phi_a)^2 + \frac{1}{2\chi} n_{ab}n_{ab} \right\}$$

equilibrium distribution:

$$P[\varphi, n] \sim e^{-\beta F}$$

- equations of motion:

N-component order parameter:

$$\frac{\partial \phi_a}{\partial t} = -\Gamma_0 \frac{\delta F}{\delta \phi_a} + g[\phi_a, n_{bc}]\frac{\delta F}{\delta n_{bc}} + \theta_a$$

$N(N$-1)/2 charge densities: (Generalized angular momenta)

$$\frac{\partial n_{ab}}{\partial t} = \gamma \vec{\nabla}^2 \frac{\delta F}{\delta n_{ab}} + g[n_{ab}, \phi_c]\frac{\delta F}{\delta \phi_c} + g[n_{ab}, n_{cd}]\frac{\delta F}{\delta n_{cd}} + \vec{\nabla}\cdot\vec{\zeta}_{ab}$$

Charge densities (on operator level):

$$n_{ab} = \phi_a \frac{\partial}{\partial t}\phi_b - \phi_b \frac{\partial}{\partial t}\phi_a$$

$\rightsquigarrow$ Calculate Poisson brackets:

$$[\phi_a, n_{bc}] = \phi_b \delta_{ac} - \phi_c \delta_{ab}$$

$$[n_{ab}, n_{cd}] = -\delta_{ad}n_{bc} - \delta_{bc}n_{ad} + \delta_{ac}n_{bd} + \delta_{bd}n_{ac}$$

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Model G
z = d/2

Consider classical $O(N)$-theory with Landau-Ginzburg-Wilson functional

$$F = \int_{\vec{x}} \left\{ \frac{1}{2}(\partial^i \phi_a)(\partial^i \phi_a) + \frac{m^2}{2}\phi_a\phi_a + \frac{\lambda}{4!N}(\phi_a\phi_a)^2 + \frac{1}{2\chi} n_{ab}n_{ab} \right\}$$

equilibrium distribution:

$$P[\varphi, n] \sim e^{-\beta F}$$

- equations of motion:

N-component order parameter:

$$\frac{\partial \phi_a}{\partial t} = -\Gamma_0 \frac{\delta F}{\delta \phi_a} + g[\phi_a, n_{bc}]\frac{\delta F}{\delta n_{bc}} + \theta_a$$

N(*N*-1)/2 charge densities: (Generalized angular momenta)

$$\frac{\partial n_{ab}}{\partial t} = \gamma \vec{\nabla}^2 \frac{\delta F}{\delta n_{ab}} + g[n_{ab}, \phi_c]\frac{\delta F}{\delta \phi_c} + g[n_{ab}, n_{cd}]\frac{\delta F}{\delta n_{cd}} + \vec{\nabla}\cdot\vec{\zeta}_{ab}$$

dissipative damping towards equilibrium

Charge densities (on operator level):

$$n_{ab} = \phi_a \frac{\partial}{\partial t}\phi_b - \phi_b \frac{\partial}{\partial t}\phi_a$$

$\leadsto$ Calculate Poisson brackets:

$$[\phi_a, n_{bc}] = \phi_b \delta_{ac} - \phi_c \delta_{ab}$$

$$[n_{ab}, n_{cd}] = -\delta_{ad} n_{bc} - \delta_{bc} n_{ad} + \delta_{ac} n_{bd} + \delta_{bd} n_{ac}$$

Dynamic universality classes

Model G
$z = d/2$

Consider classical $O(N)$-theory with Landau-Ginzburg-Wilson functional

$$F = \int_{\vec{x}} \left\{ \frac{1}{2}(\partial^i \phi_a)(\partial^i \phi_a) + \frac{m^2}{2}\phi_a\phi_a + \frac{\lambda}{4!N}(\phi_a\phi_a)^2 + \frac{1}{2\chi} n_{ab}n_{ab} \right\}$$

equilibrium distribution:

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- equations of motion:

N-component order parameter:

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dissipative damping towards equilibrium

reversible mode couplings (Poisson brackets!)

Charge densities (on operator level):

$$n_{ab} = \phi_a \frac{\partial}{\partial t}\phi_b - \phi_b \frac{\partial}{\partial t}\phi_a$$

$\rightsquigarrow$ Calculate Poisson brackets:

$$[\phi_a, n_{bc}] = \phi_b\delta_{ac} - \phi_c\delta_{ab}$$

$$[n_{ab}, n_{cd}] = -\delta_{ad}n_{bc} - \delta_{bc}n_{ad} + \delta_{ac}n_{bd} + \delta_{bd}n_{ac}$$

Dynamic universality classes

CRC-TR 211

Model G
$z = d/2$

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equilibrium distribution:

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- equations of motion:

N-component order parameter:

$$\frac{\partial \phi_a}{\partial t} = -\Gamma_0 \frac{\delta F}{\delta \phi_a} + g[\phi_a, n_{bc}] \frac{\delta F}{\delta n_{bc}} + \theta_a$$

N(*N*-1)/2 charge densities: (Generalized angular momenta)

$$\frac{\partial n_{ab}}{\partial t} = \gamma \vec{\nabla}^2 \frac{\delta F}{\delta n_{ab}} + g[n_{ab}, \phi_c] \frac{\delta F}{\delta \phi_c} + g[n_{ab}, n_{cd}] \frac{\delta F}{\delta n_{cd}} + \vec{\nabla} \cdot \vec{\zeta}_{ab}$$

dissipative damping towards equilibrium — reversible mode couplings (Poisson brackets!) — thermal noise

Charge densities (on operator level):

$$n_{ab} = \phi_a \frac{\partial}{\partial t}\phi_b - \phi_b \frac{\partial}{\partial t}\phi_a$$

$\rightsquigarrow$ Calculate Poisson brackets:

$$[\phi_a, n_{bc}] = \phi_b \delta_{ac} - \phi_c \delta_{ab}$$

$$[n_{ab}, n_{cd}] = -\delta_{ad} n_{bc} - \delta_{bc} n_{ad} + \delta_{ac} n_{bd} + \delta_{bd} n_{ac}$$

Structure of reversible mode couplings

Simpler example: O(2) model (e.g. planar antiferromagnets)

Order parameter
$$\frac{\partial \phi_a}{\partial t} = -\Gamma_0 \frac{\delta F}{\delta \phi_a} + g[\phi_a, m] \frac{\delta F}{\delta m} + \theta_a$$

Magnetization (conserved)
$$\frac{\partial m}{\partial t} = \gamma \vec{\nabla}^2 \frac{\delta F}{\delta m} + g[m, \phi_a] \frac{\delta F}{\delta \phi_a} + \vec{\nabla} \cdot \vec{\zeta}$$

Verify: eom's symmetric under displacement of m &
corresponding precession of ϕ

$$m(t, \vec{x}) \to m(t, \vec{x}) + \delta m$$

$$\phi(t, \vec{x}) \to e^{i\omega t} \phi(t, \vec{x})$$

Larmor frequency: $\omega = \frac{g}{\chi} \delta m$

here $\phi = \phi_1 + i\phi_2$

exact symmetry! ⇝ Ward identities

Structure of reversible mode couplings

Simpler example: O(2) model (e.g. planar antiferromagnets)

Order parameter
$$\frac{\partial \phi_a}{\partial t} = -\Gamma_0 \frac{\delta F}{\delta \phi_a} + g[\phi_a, m]\frac{\delta F}{\delta m} + \theta_a$$

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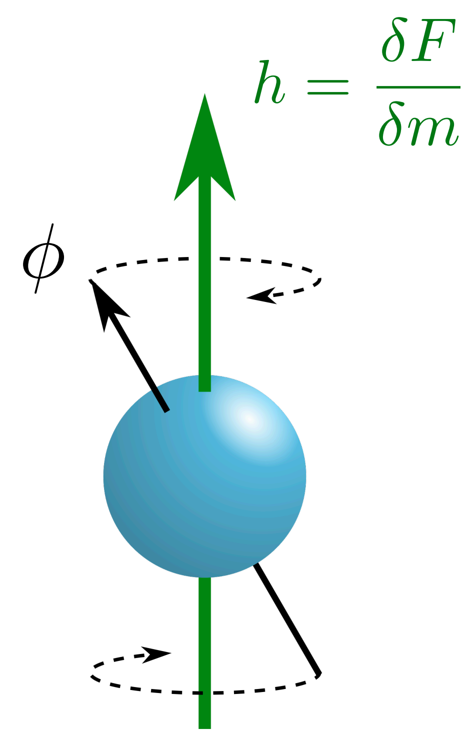


Larmor precession

Verify: eom's symmetric under displacement of m & corresponding precession of ϕ

$$m(t, \vec{x}) \to m(t, \vec{x}) + \delta m$$

$$\phi(t, \vec{x}) \to e^{i\omega t}\phi(t, \vec{x})$$

Larmor frequency: $\omega = \frac{g}{\chi}\delta m$

here $\phi = \phi_1 + i\phi_2$

exact symmetry! $\rightsquigarrow$ Ward identities

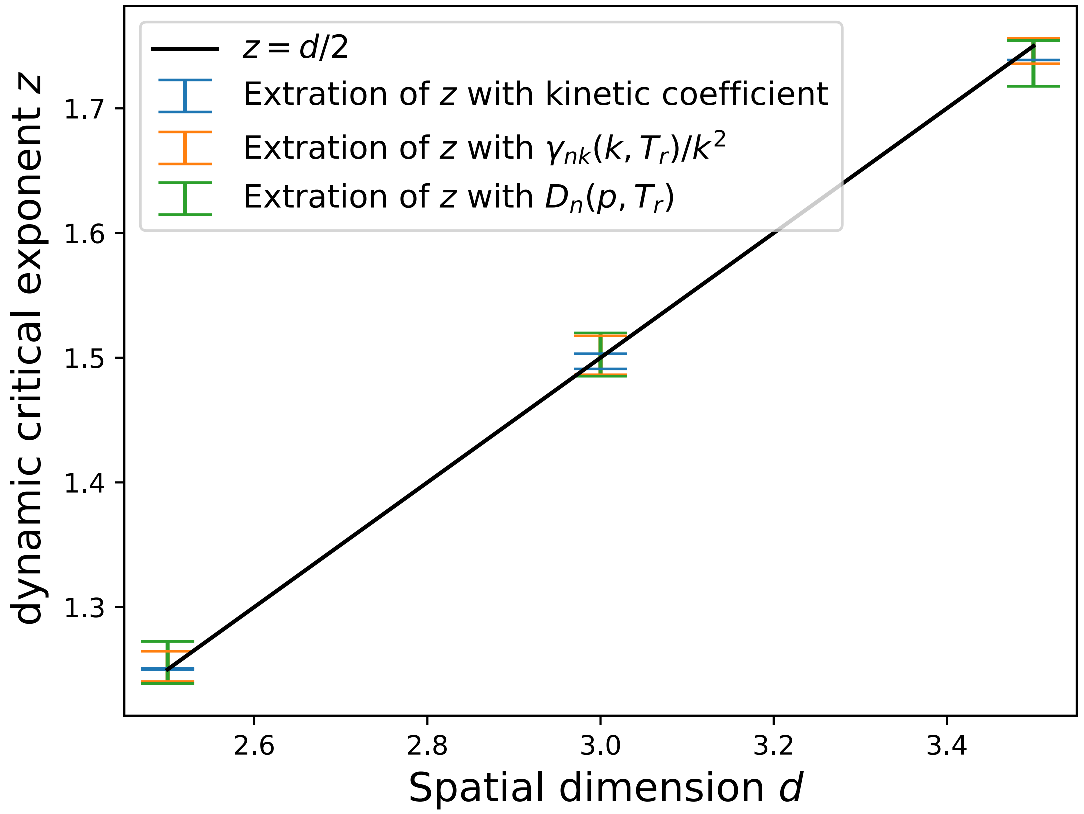
Results for dynamic scaling behavior

Model G
$z = d/2$

Dependence of dynamic critical exponent on spatial dimension:

$$z = \frac{d}{2}$$

Rajagopal and Wilczek,
Nucl. Phys. B **399** (1993) 395-425



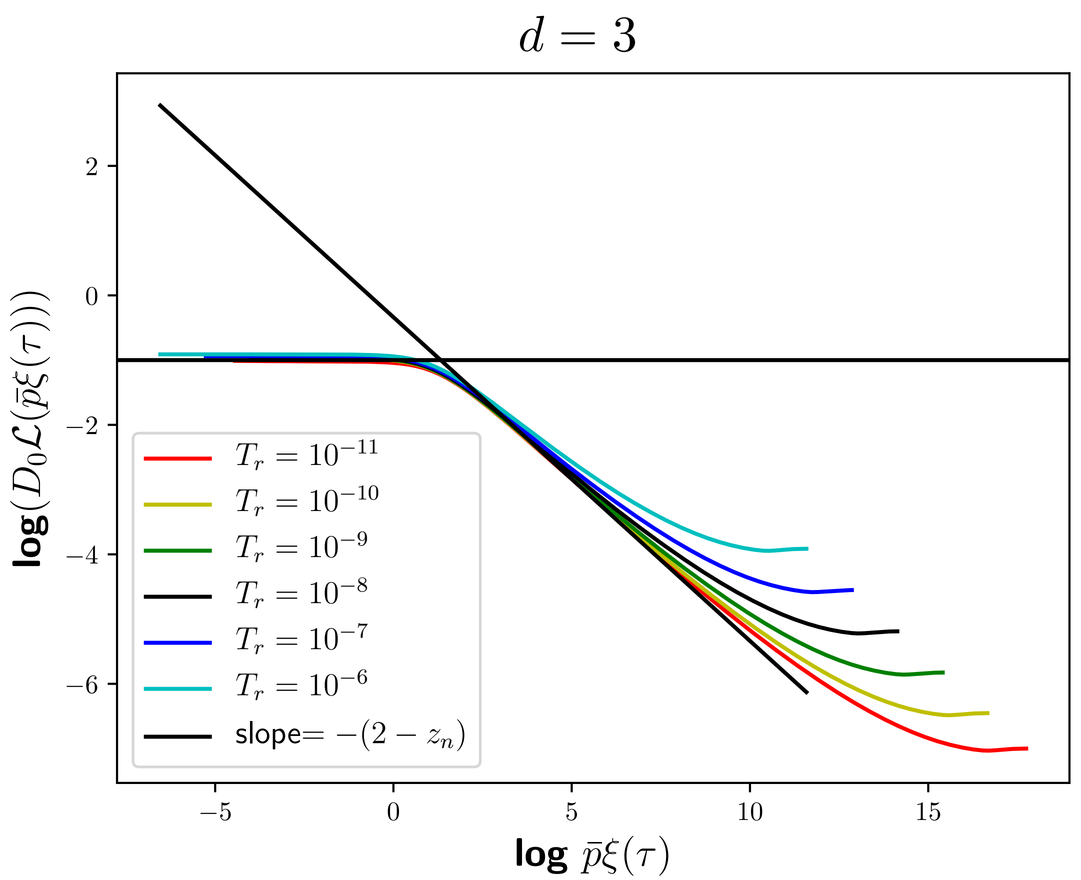
confirmed non-perturbatively!

Diffusion coefficient of isovector and isoaxial charges:

$$D_n(p, \tau) = s^{2-z} D_n(sp, s^{1/\nu}\tau)$$

$$\Longrightarrow \quad D_n(p, \tau) \sim \tau^{-\nu(2-z)} \mathcal{L}(\tau^{-\nu}\bar{p})$$

$$\bar{p} = f^+ p$$



described by **universal scaling function** for different reduced temperatures T_r

JR, S. Schlichting, L. von Smekal, Y. Ye, in preparation