Meson distribution functions and GFFs from data and DSEs

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FÍSICA
MATEMÁTICAS
COMPUTACIÓN

In collaboration with Yin-Zhen Xu, Khépani Raya, C.D. Roberts, ...

Complex QFT Workshop, Maynooth; June 14th - 17th, 2023.

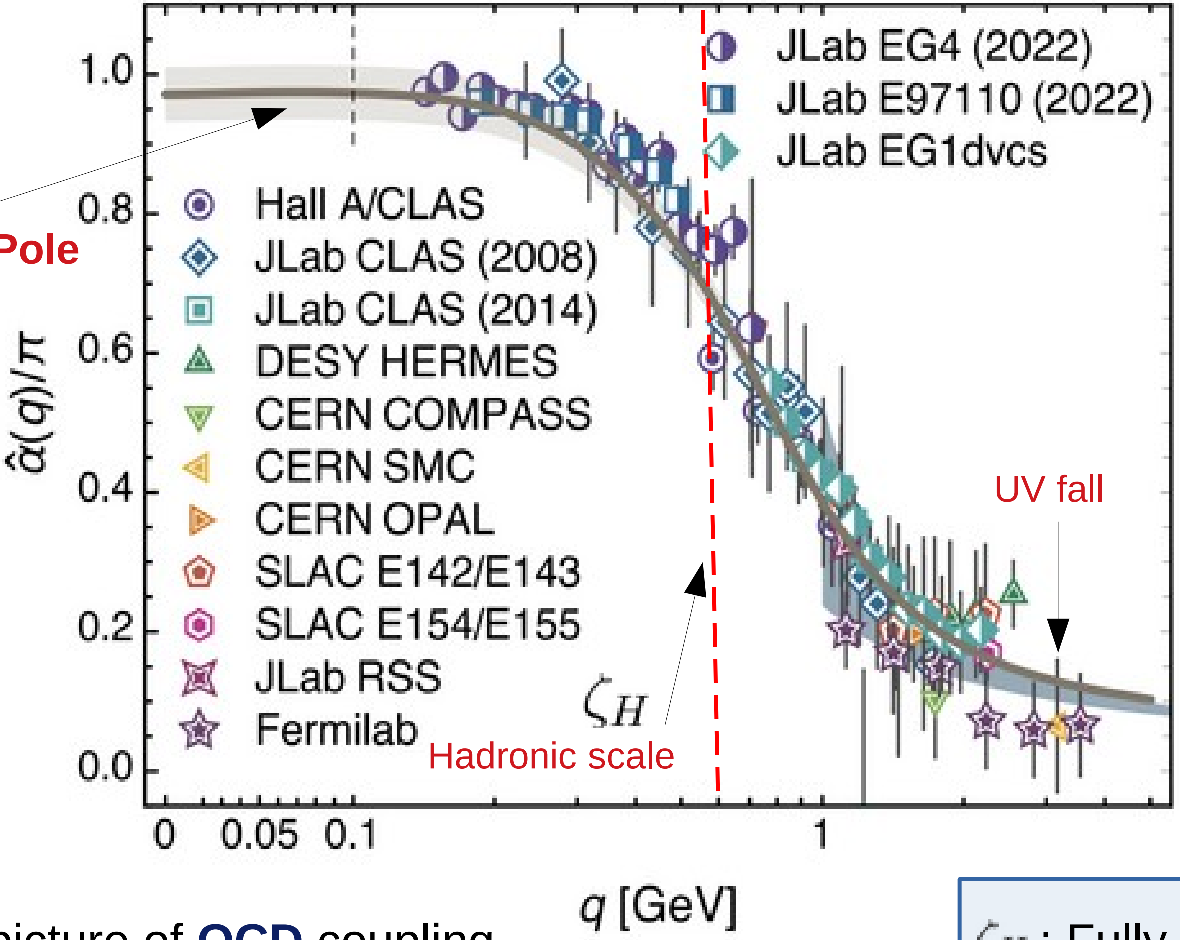
QCD: Basic Facts

- **Confinement** and the **EHM** are tightly connected with **QCD's running coupling**.

'Effective Charge' (figure: D. Binosi's courtesy!)

No Landau Pole

Enhancement



$$\mathcal{L}_{QCD} = \sum_{j=u,d,s,\ldots} \bar{q}_j[\gamma_\mu D_\mu + m_j]q_j + \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu},$$

$$D_\mu = \partial_\mu + ig\frac{1}{2}\lambda^a A^a_\mu,$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu + \partial_\nu A^a_\mu - \underline{g f^{abc} A^b_\mu A^c_\nu},$$

Modern picture of **QCD** coupling.

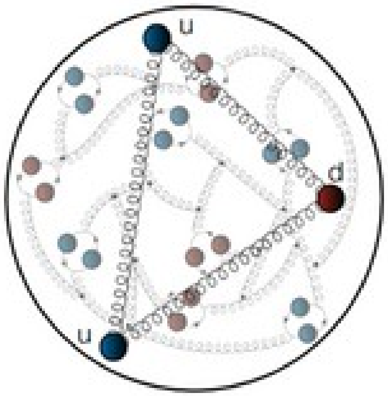
Combined continuum + QCD lattice analysis

ζ_H : Fully **dressed valence** quarks express all hadron's properties

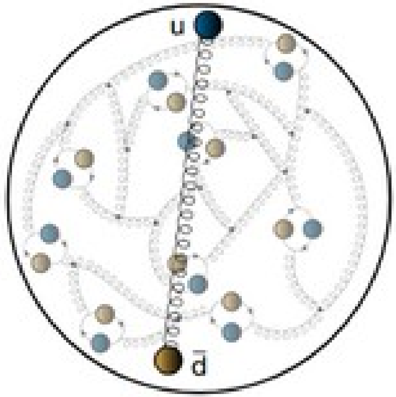
Why pions and kaons?: understanding EHM

- **Pions** and **kaons** emerge as (pseudo)-**Goldstone** bosons of **DCSB**.

(besides being 'simple' bound states)



$m_p \approx 0.940\ \text{GeV}$

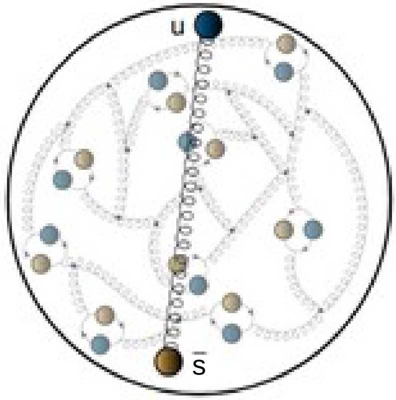


$m_\pi \approx 0.140\ \text{GeV}$

- Their study is crucial to understand the **EHM** and the ***hadron structure***:

  - Dominated by **QCD** dynamics

    Simultaneously explains the mass of the proton and the *masslessness* of the pion



$m_K \approx 0.490\ \text{GeV}$

  - Interplay between **Higgs** and **strong** mass generating mechanisms.

'Higgs' masses

$m_{u/d} \approx 0.004\ \text{GeV}$

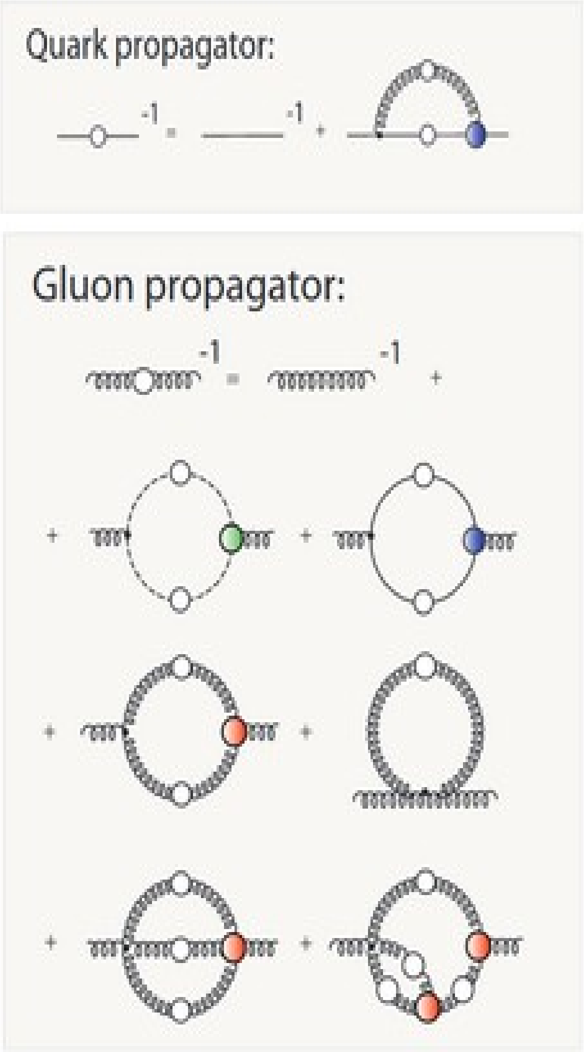
$m_s \approx 0.095\ \text{GeV}$

CSM: the DSE approach

- Equations of motion of a **quantum field theory**
- Relate Green functions with higher-order Green functions
  - **Infinite** tower of coupled equations.
  - ✗ Systematic **truncation** required
- ✓ **No assumptions** on the **coupling** for their derivation.
  - ✓ Capture both **perturbative** and **non-perturbative** facets of **QCD**
- ✓ **Not limited** to a certain domain of current **quark masses**
- ✓ Maintain a **traceable connection** to QCD.

C.D. Roberts and a A.G. Williams,
Prog.Part.Nucl.Phys. 33 (1994) 477-575

Example DSEs



Eichmann:2009zx

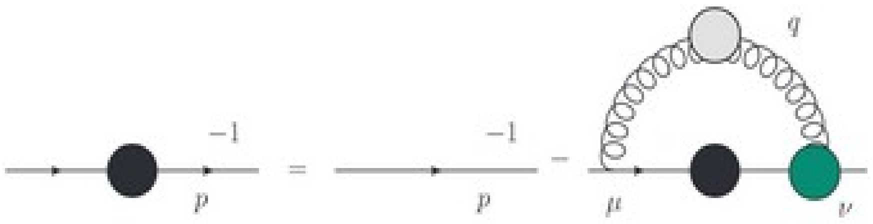
CSM: the DSE approach

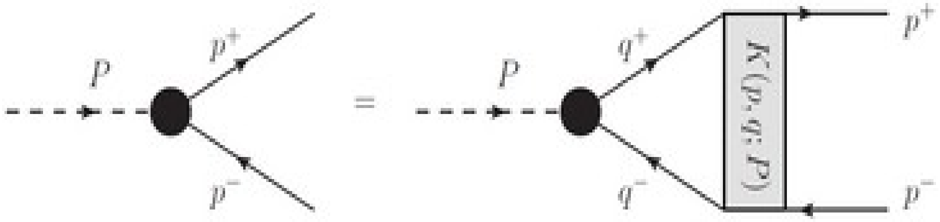
- **BSWF**: sandwich of the Bethe-Salpeter amplitude and quark propagators:

$$\chi_H(k^H;P_H) = S_q(k)\Gamma_H(k^H;P_H)S_{\bar{q}}(k-P_H)\ ,\ k^H = k - P_H/2\ .$$

$P^2 = -m_H^2$: **meson's mass**; Γ_H **BS amplitude**; $S_{q(\bar{q})}$ **quark (antiquark) propagator**

- Quark **propagator** and **BSA** should come from solutions of:





Quark DSE

Meson BSE

- Relates the quark propagator with **QGV** and **gluon propagator**.

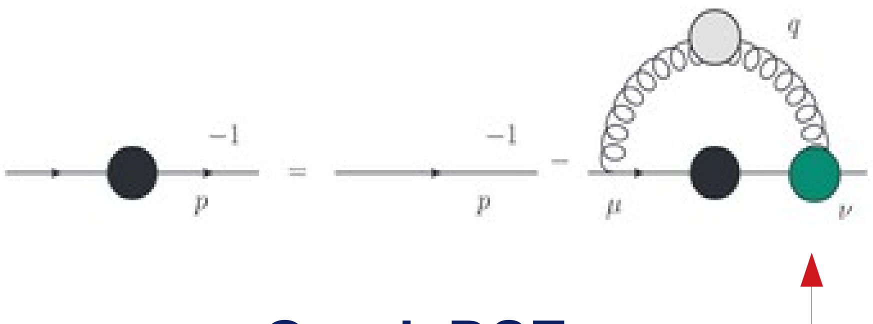
- Contains **all interactions** between the quark and antiquark

CSM: the DSE approach

- For the ground-state pseudoscalar and vector mesons, it is typical to employ the so called Rainbow-Ladder (**RL**) truncation:

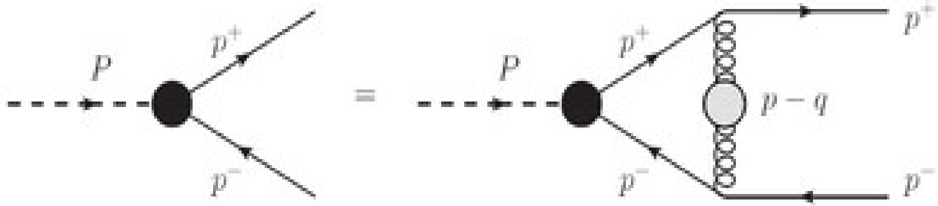
Y-Z Xu et al., PRD 100 (2019) 11, 114038.

K. Raya et al., PRD 101 (2020) 7, 074021.



Quark DSE

bare



Meson BSE

- It preserves the **Goldstone's Theorem**, whose most fundamental expression is captured in:

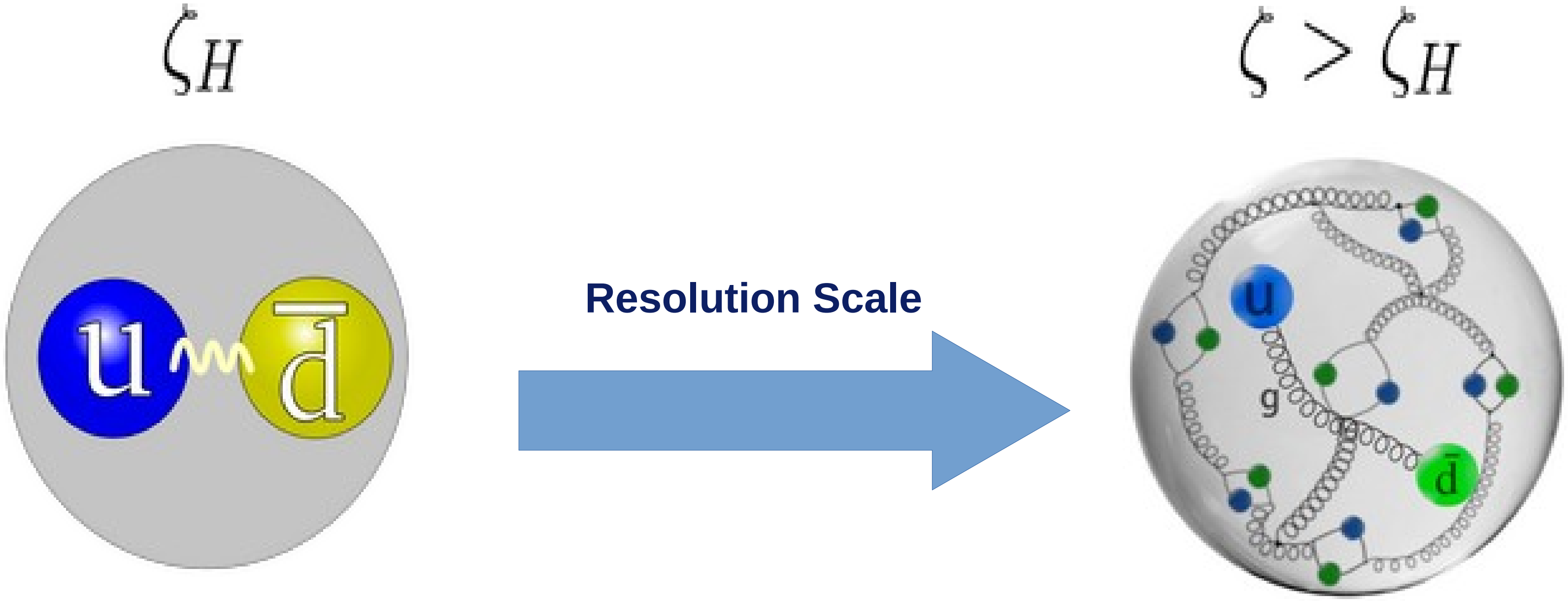
$$f_\pi E_\pi(k;P=0) = B(k^2)$$

Leading BSA

"***Mass*** Function"

"**Pions** exists, if and only if, **DCSB** occurs."

Parton distributions: energy scales



ζ_H

Resolution Scale

$\zeta > \zeta_H$

- **Fully-dressed valence quarks**

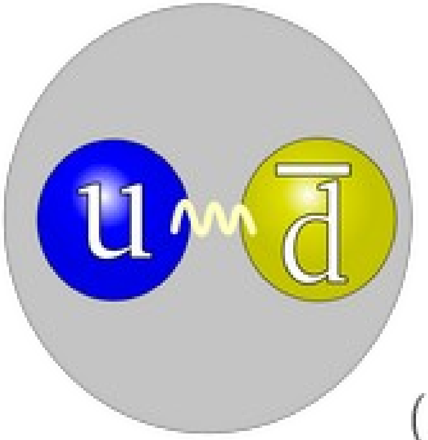
(quasiparticles)

- **Unveiling of glue and sea d.o.f.**

(partons)

Parton distributions: energy scales

ζ_H



$(M_u = M_d)$

- **Fully-dressed valence quarks**
  - At this scale, **all properties** of the hadron are contained within their valence quarks.
  - **QCD constraints** are defined from here (e.g. large-x behavior of the PDF)

$$u^{\pi}(x;\zeta) \overset{x\simeq 1}{\approx} (1-x)^{\beta = 2+\gamma(\zeta)}$$

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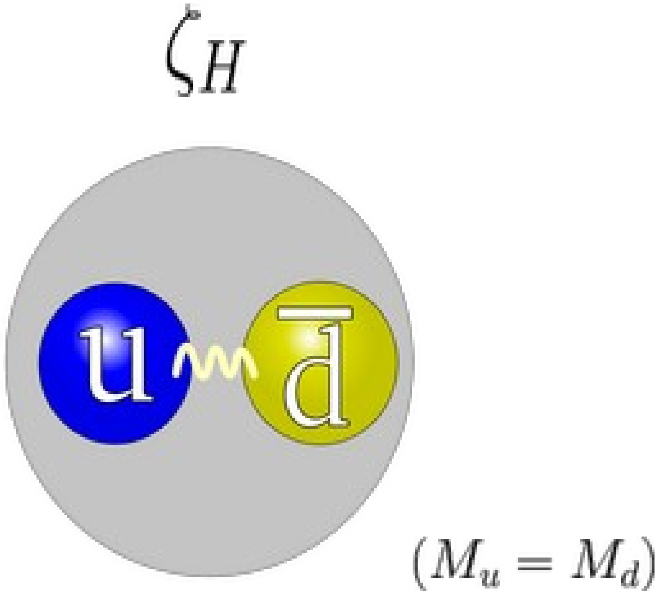

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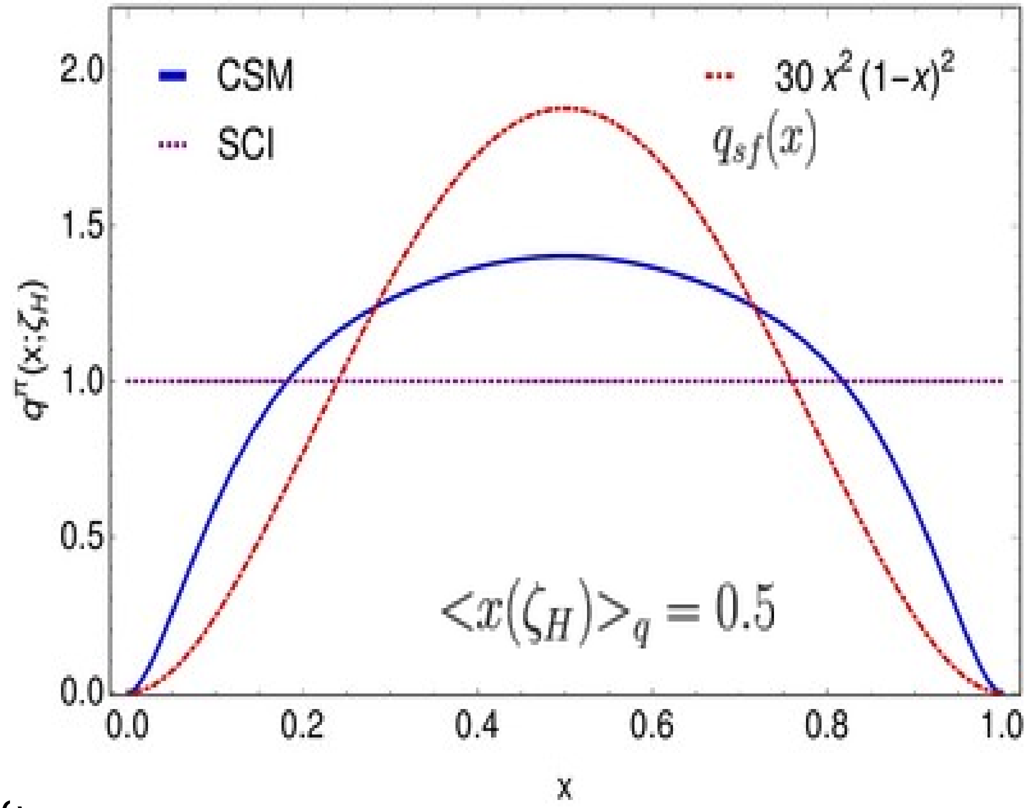
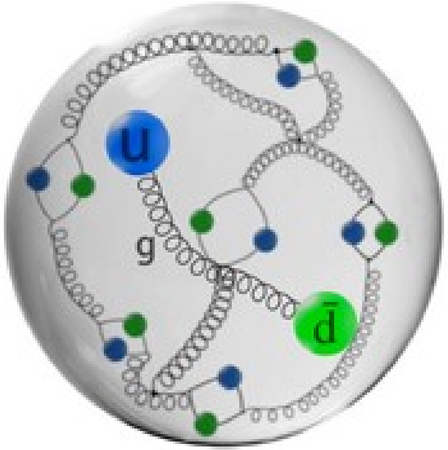


- **CSM** results produce:
  - **EHM-induced** dilated distributions
  - Soft end-point behavior

`Cui:2020tdf`

Parton distributions: energy scales

$$\zeta > \zeta_H$$

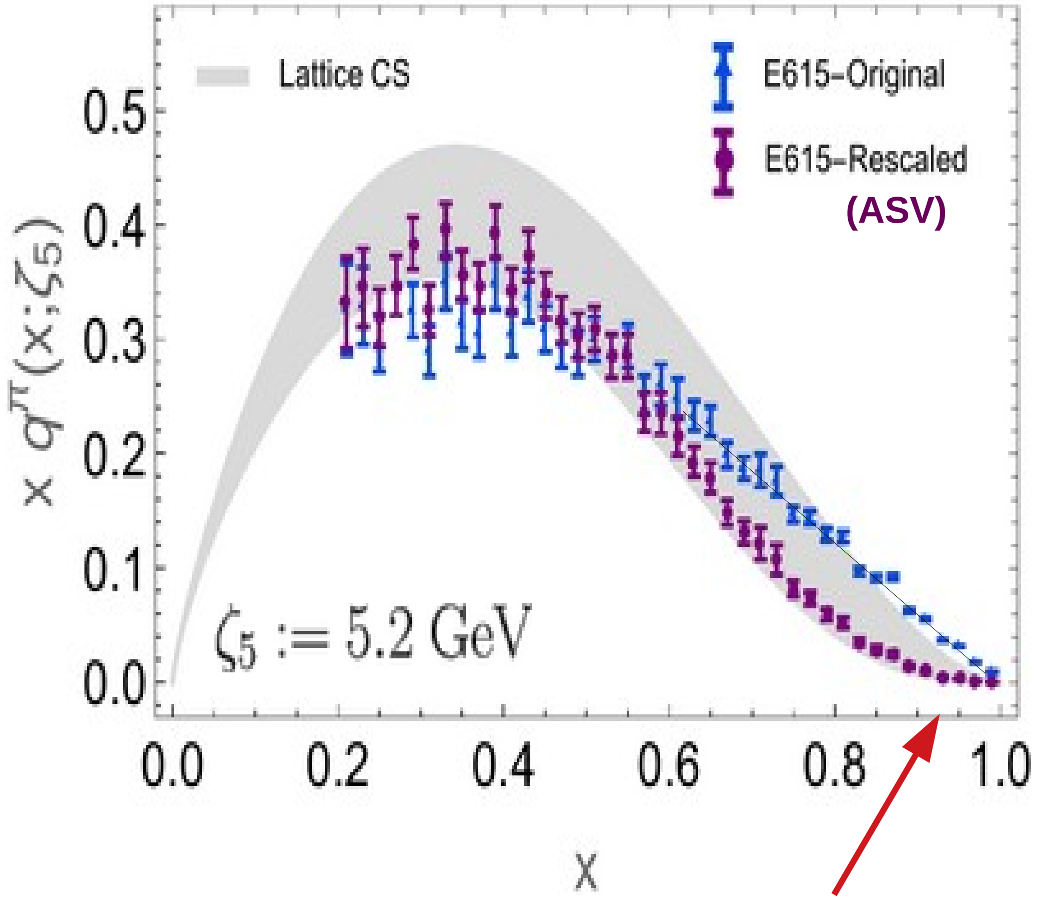


- **Unveiling of glue and sea d.o.f.**

- **Experimental** data is given **here**.
- The interpretation of parton distributions from cross sections demands **special care**.
- In addition, the synergy with **lattice QCD** and phenomenological approaches is welcome.

Parton distributions: energy scales

`Conway:1989fs` `Aicher:2010cb` `Sufian:2020vzb`



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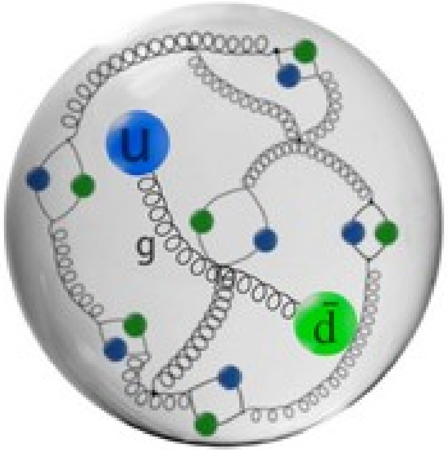

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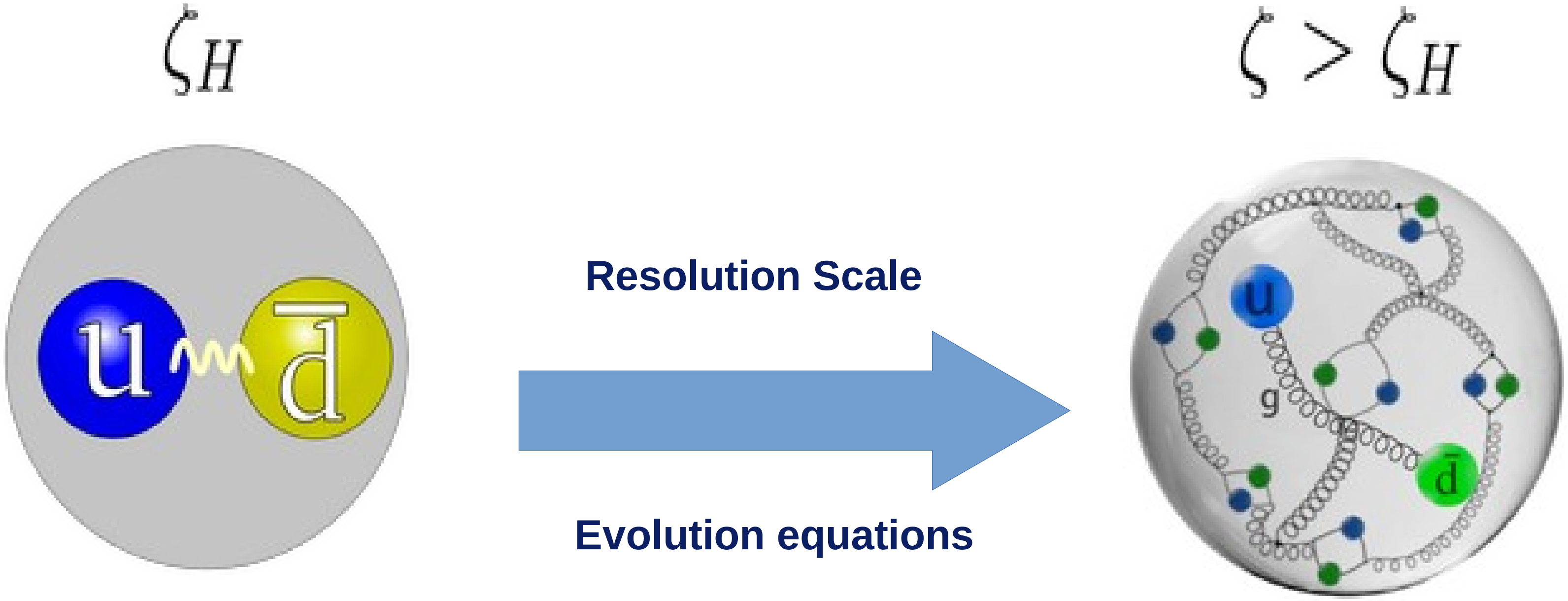

$$\zeta_H \qquad\qquad \zeta > \zeta_H$$

Resolution Scale

Evolution equations

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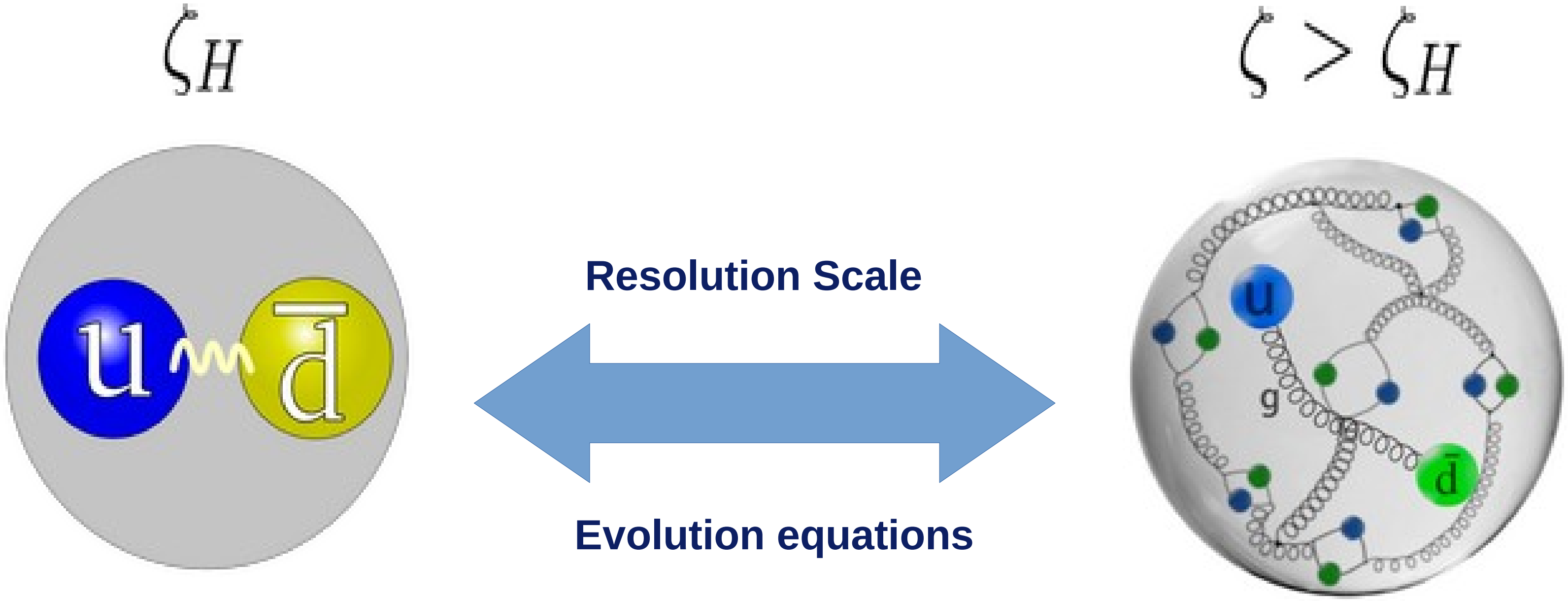

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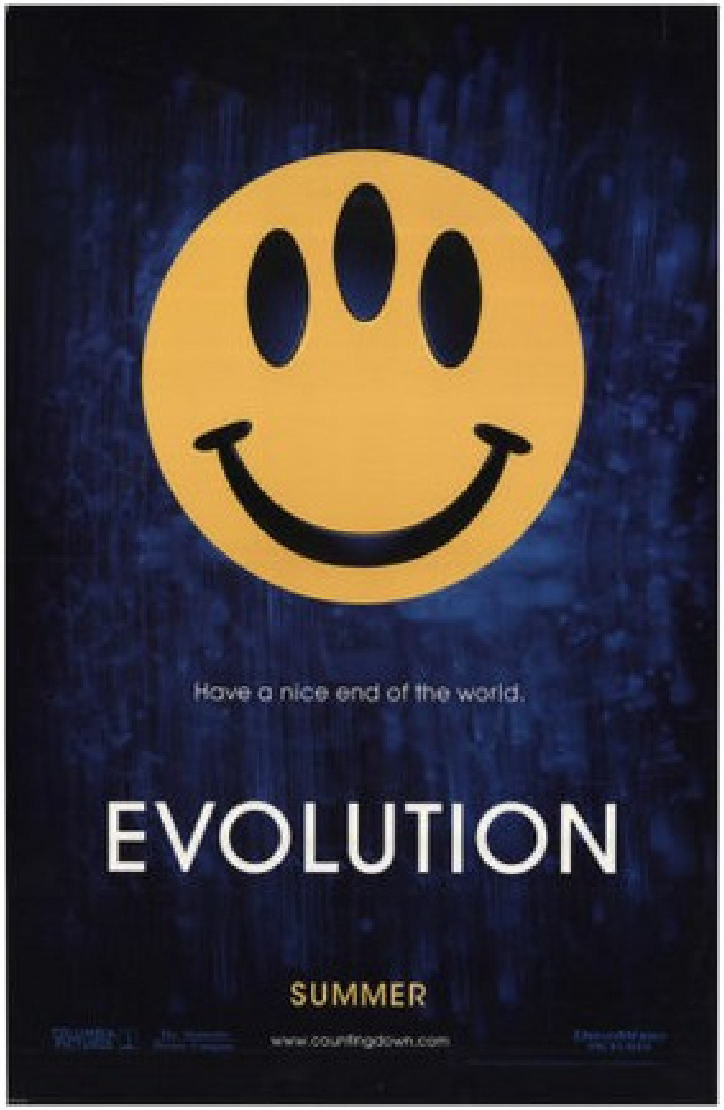
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www.countingdown.com

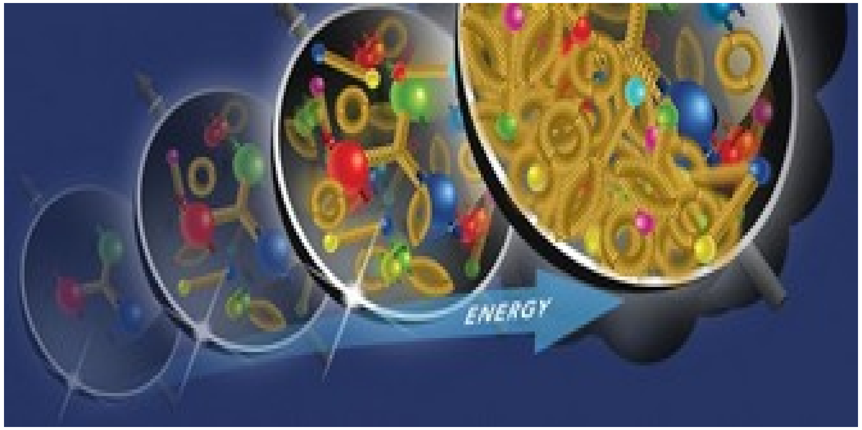
DGLAP: All orders evolution

Raya:2021zrz

Cui:2020tdf

$$\left\{\zeta^2\frac{d}{d\zeta^2}\int_0^1 dy\delta(y-x) - \frac{\alpha(\zeta^2)}{4\pi}\int_x^1\frac{dy}{y}\begin{pmatrix} P_{qq}^{\mathrm{NS}}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{\mathrm{S}}\left(\frac{\mathbf{x}}{\mathbf{y}}\right)\end{pmatrix}\right\}\begin{pmatrix} H_\pi^{\mathrm{NS},+}(y,t;\zeta)\\ \mathbf{H}_\pi^{\mathrm{S}}(y,t;\zeta)\end{pmatrix} = 0$$

DGLAP leading-order evolution equations



DGLAP: All orders evolution

Assumption: define an **effective** charge such that

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Starting from fully-dressed **quasiparticles**, at ζ_H → **Sea** and **Gluon** content unveils, as prescribed by **QCD**

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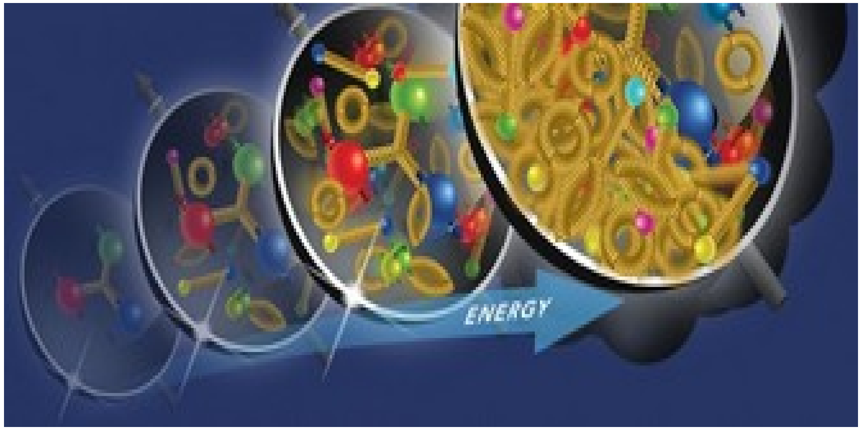
DGLAP ~~leading order~~ evolution equations

- **Not** the **LO** QCD coupling but an **effective** one.
- Making this equation **exact**.
- Connecting with the **hadron scale**, at which the **fully-dressed** valence-**quarks** express **all** of the hadron's properties.

(thus carrying all the momentum)



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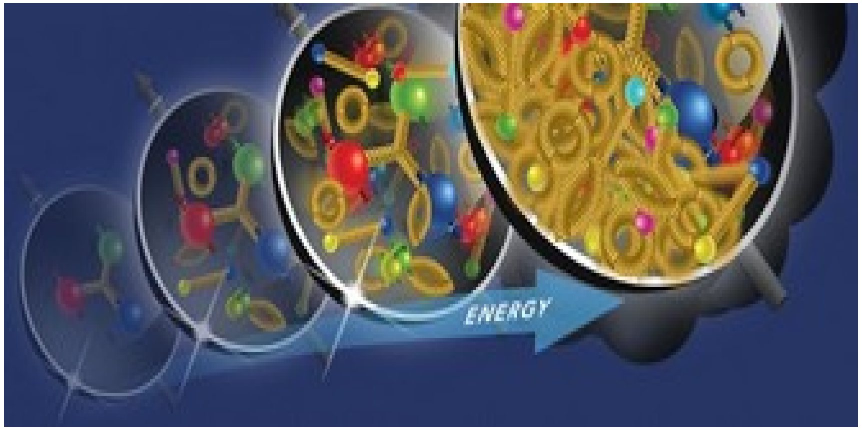
$$\left\{\zeta^2\frac{d}{d\zeta^2}\mathbb{1}+\boxed{\frac{\alpha(\zeta^2)}{4\pi}}\begin{pmatrix}\gamma_{qq}^{(n)} & 0 & 0\\ 0 & \gamma_{qq}^{(n)} & 2n_f\gamma_{qg}^{(n)}\\ 0 & \gamma_{gq}^{(n)} & \gamma_{gg}^{(n)}\end{pmatrix}\right\}\begin{pmatrix}\langle x^n\rangle_{\rm NS}(\zeta)\\ \langle x^n\rangle_{\rm S}(\zeta)\\ \langle x^n\rangle_g(\zeta)\end{pmatrix}=0$$

DGLAP ~~leading order~~ evolution equations

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$



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DGLAP: All orders evolution

Cui:2020tdf

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\mathsf{H}}(x) = q_{\mathsf{H}}(x) + \bar{q}_{\mathsf{H}}(x)$$

singlet combination

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Valence-quark PDF in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_H} \;=\; -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_H}$$

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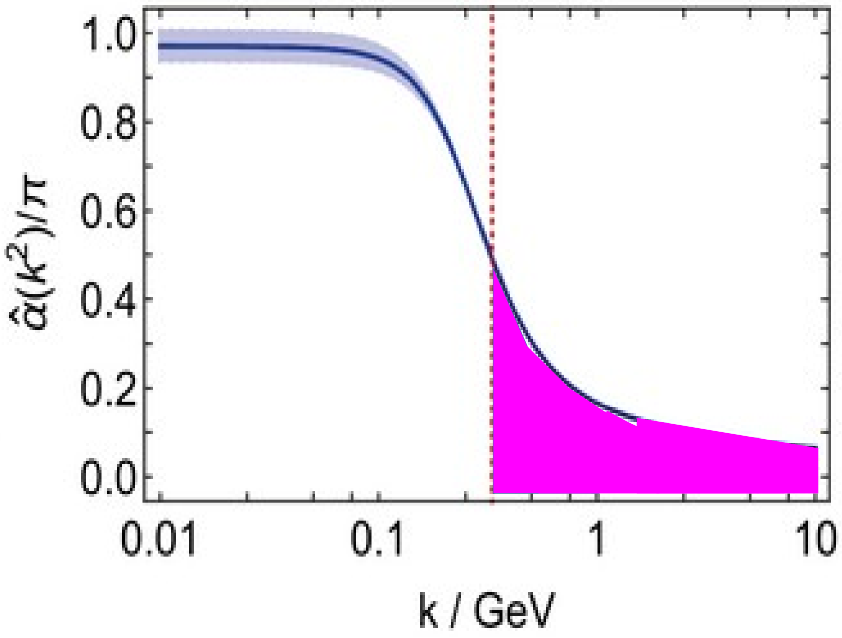

Moments' evolution is controlled by the integrated "strength" of the coupling beyond the hadron scale

DGLAP: All orders evolution

Cui:2020tdf

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$$\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

The ratio of lightcone momentum fractions encodes the required information of the charge

DGLAP: All orders evolution

Cui:2020tdf

Implication 1: valence-quark PDF

$$\langle x^n \rangle^{\zeta}_{q_H} = \langle x^n \rangle^{\zeta_H}_{q_H} \exp\left(-\frac{\gamma^n_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right) = \langle x^n \rangle^{\zeta_H}_{q_H} \left[\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} \right]^{\gamma^n_{qq}/\gamma_{qq}}$$

This ratio encodes the information of the charge

Direct connection bridging from hadron to experimental scale: only one input is needed to evolve “all” the Mellin moments up and reconstruct the PDF.

DGLAP: All orders evolution

Cui:2020tdf

Implication 1: valence-quark PDF

$$\langle x^n\rangle^{\zeta}_{q_\pi} = \langle x^n\rangle^{\zeta_H}_{q_\pi}\ \exp\left(-\frac{\gamma^n_{qq}}{2\pi}\int_{\zeta_H}^{\zeta}\frac{dz}{z}\alpha(z^2)\right) = \langle x^n\rangle^{\zeta_H}_{q_\pi}\left[\langle 2x\rangle^{\zeta}_{q_\pi}\right]^{\gamma^n_{qq}/\gamma_{qq}}$$

Direct connection bridging from hadron to experimental scale: only one input is needed to evolve “all” the Mellin moments up and reconstruct the PDF.

This ratio encodes the information of the charge and use isospin symmetry (pion case)

$$\langle x\rangle^{\zeta_H}_{u_\pi} = \langle x\rangle^{\zeta_H}_{d_\pi} = \frac{1}{2}$$

DGLAP: All orders evolution

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Capitalizing on the Mellin moments of asymptotically large order:

$$q(x;\zeta) \underset{x\to 1}{\sim} (1-x)^{\beta(\zeta)}\,(1+\mathcal{O}(1-x))$$

$$\beta(\zeta) = \beta(\zeta_H) + \frac{3}{2}\ln\frac{\langle x(\zeta_H)\rangle}{\langle x(\zeta)\rangle}$$

DGLAP: All orders evolution

Cui:2020tdf

Implication 1: valence-quark PDF

$$\langle x^n\rangle_{q_\pi}^{\zeta} = \langle x^n\rangle_{q_\pi}^{\zeta_H} \exp\left(-\frac{\gamma_{qq}^n}{2\pi}\int_{\zeta_H}^{\zeta}\frac{dz}{z}\alpha(z^2)\right) = \langle x^n\rangle_{q_\pi}^{\zeta_H}\left[\langle 2x\rangle_{q_\pi}^{\zeta}\right]^{\gamma_{qq}^n/\gamma_{qq}}$$

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Under a sensible assumption at large momentum scale:

$$q(x;\zeta) \underset{x\to 0}{\sim} x^{\alpha(\zeta)}\,(1+\mathcal{O}(x))$$

$$1+\alpha(\zeta) = \frac{3}{2}\langle x(\zeta)\rangle\ln\frac{\langle x(\zeta_H)\rangle}{\langle x(\zeta)\rangle} + \beta(\zeta_H)\langle x(\zeta)\rangle + \mathcal{O}\left(\frac{\langle x(\zeta)\rangle}{\ln\frac{\langle x(\zeta_H)\rangle}{\langle x(\zeta)\rangle}}\right)$$

DGLAP: All orders evolution

Cui:2020tdf

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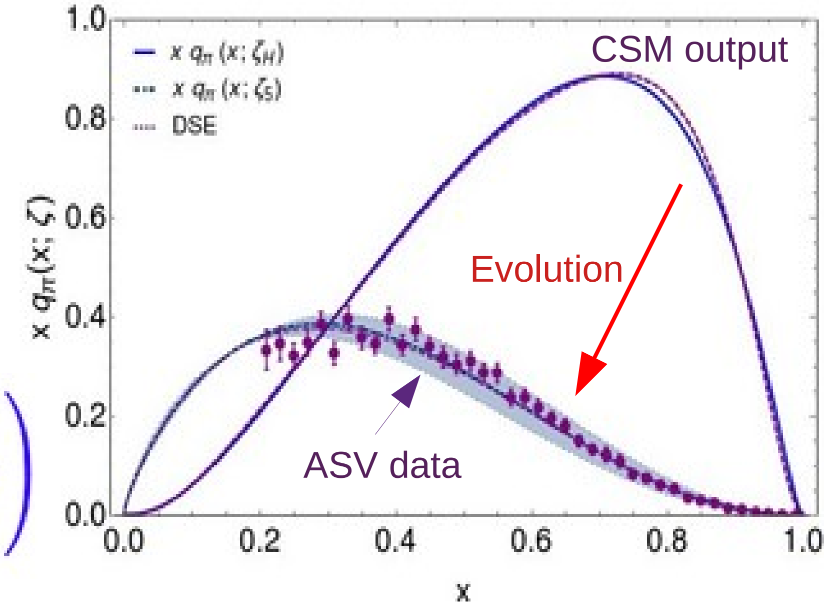
Reconstruction after evolving:



DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta_H}_{u_\pi} = \frac{1}{2(n+1)}$$

$$\times \sum_{j=0,1,\ldots}^{2n} (-)^j \begin{pmatrix} 2(n+1) \\ j \end{pmatrix} \langle x^j\rangle^{\zeta_H}_{u_\pi}$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments **(pion case)**

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{(\langle 2x\rangle^{\zeta}_{u_\pi})^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \begin{pmatrix} 2(n+1) \\ j \end{pmatrix} \langle x^j\rangle^{\zeta}_{u_\pi} (\langle 2x\rangle^{\zeta}_{u_\pi})^{-\gamma_0^j/\gamma_0^1} .$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

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DGLAP: All orders evolution

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$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{(\langle 2x\rangle^{\zeta}_{u_\pi})^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)}$$

$$\times \sum_{j=0,1,\ldots}^{2n} (-)^j \left(\begin{array}{c} 2(n+1) \\ j \end{array}\right) \langle x^j\rangle^{\zeta}_{u_\pi} (\langle 2x\rangle^{\zeta}_{u_\pi})^{-\gamma_0^j/\gamma_0^1} .$$

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- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

DGLAP: All orders evolution

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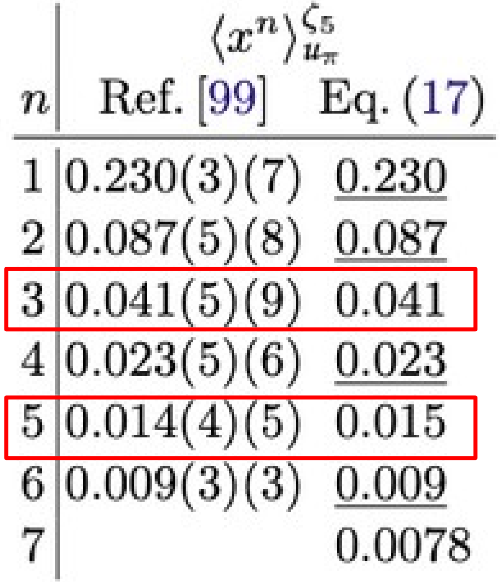
Reported lattice moments agree very well with the recursion formula



n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7		0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{(\langle 2x\rangle^{\zeta}_{u_\pi})^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \begin{pmatrix} 2(n+1) \\ j \end{pmatrix} \langle x^j\rangle^{\zeta}_{u_\pi} (\langle 2x\rangle^{\zeta}_{u_\pi})^{-\gamma_0^j/\gamma_0^1} .$$

- Since **isospin symmetry** limit implies:

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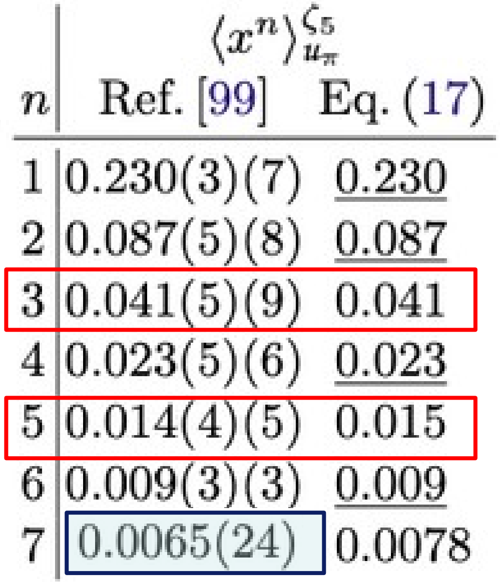
Reported lattice moments agree very well with the recursion formula and so also does and estimate for the 7-th moment from lattice reconstruction.



n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
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3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7	0.0065(24)	0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{\left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \begin{pmatrix} 2(n+1) \\ j \end{pmatrix} \langle x^j\rangle^{\zeta}_{u_\pi} \left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^j/\gamma_0^1} .$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

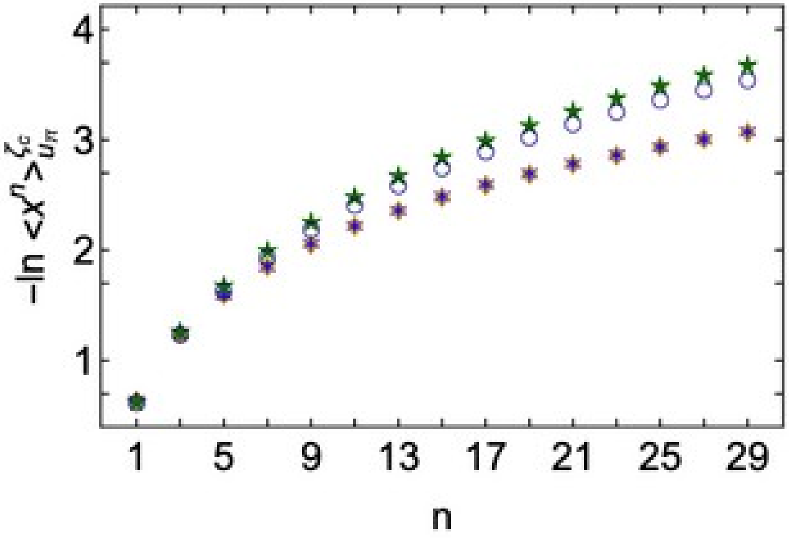
Reported lattice moments agree very well with the recursion formula and so also does and estimate for the 7-th moment from lattice reconstruction.

Moments from global fits can be also compared to the estimated from recursion !

	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$	
n	Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7	0.0065(24)	0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

Moments computed from: P. Barry et al., PRL127(2021)232001



DGLAP: All orders evolution

Implication 3: physical bounds (pion case).

$$\langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1}$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1}$$

$$\uparrow$$

$$q(x;\zeta_H) = \delta(x - 1/2)$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Lower bound** is imposed by considering the limit of a system of two strongly massive and maximally correlated) partons: both carry half of the momentum.

DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x;\zeta_H) = \delta(x - 1/2) \qquad q(x;\zeta_H) = 1$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

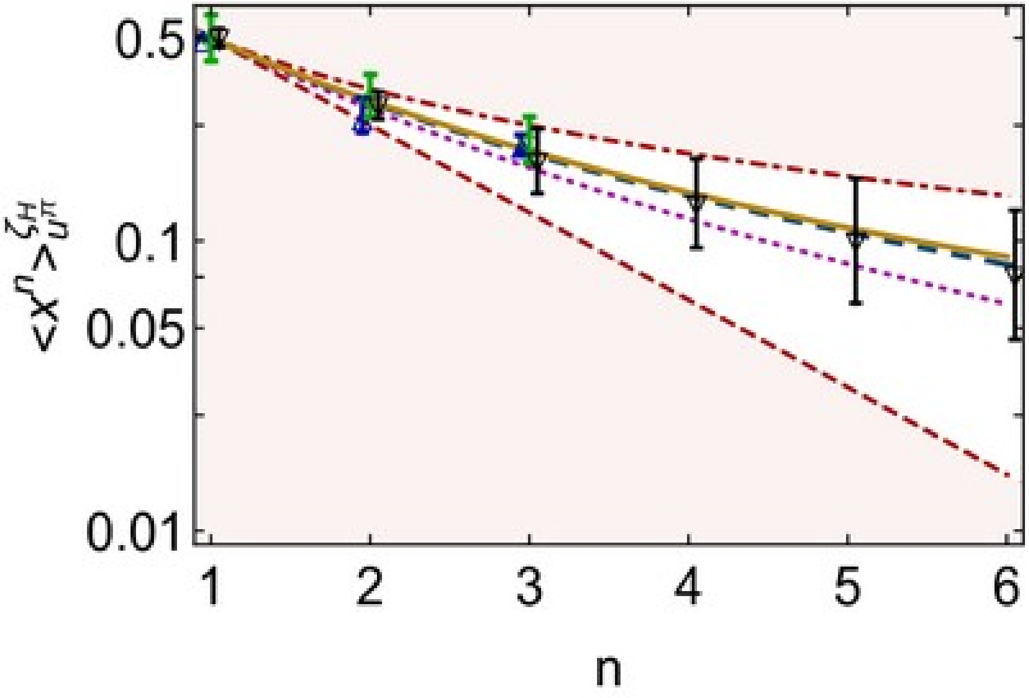
- **Lower bound** is imposed by considering the limit of a system of two strongly massive and maximally correlated) partons: both carry half of the momentum.
- **Upper bound** comes out from considering the opposite limit of a weekly interacting system of two (then fully decorrelated) partons: all the momentum fractions are equally probable.

DGLAP: All orders evolution

Implication 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n\rangle^{\zeta}_{u_\pi} \left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x;\zeta_H) = \delta(x-1/2) \qquad q(x;\zeta_H) = 1$$



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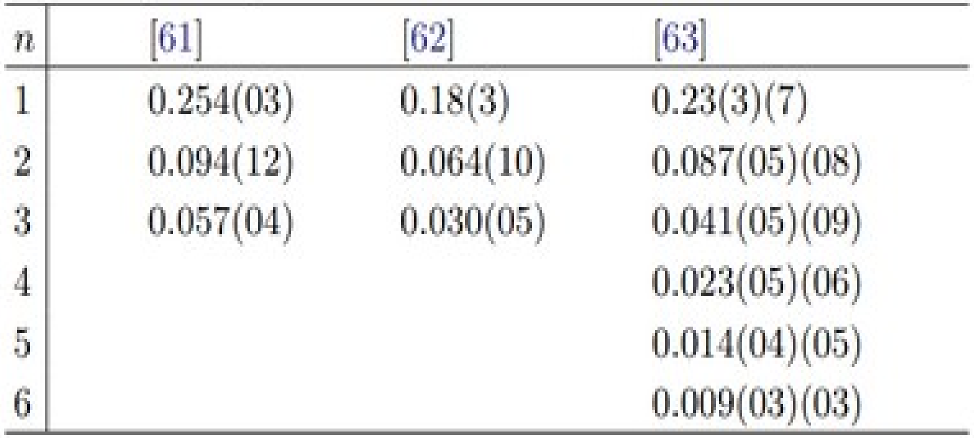
`Joo:2019bzr` `Sufian:2019bol` `Alexandrou:2021mmi`



n	[61]	[62]	[63]
1	0.254(03)	0.18(3)	0.23(3)(7)
2	0.094(12)	0.064(10)	0.087(05)(08)
3	0.057(04)	0.030(05)	0.041(05)(09)
4			0.023(05)(06)
5			0.014(04)(05)
6			0.009(03)(03)

Lattice moments verifying the **recurrence relation** too.

DGLAP: All orders evolution

Cui:2020tdf

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\mathsf{H}}(x) = q_{\mathsf{H}}(x) + \bar{q}_{\mathsf{H}}(x)$$

singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\mathsf{H}}(y)$$

$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + 2P^{\zeta}_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

Hard-wall threshold

$$\mathcal{P}^{\zeta}_{q} = \theta(\zeta - M_q)$$

Quark singlet and glue PDFs in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + 2\mathcal{P}^{\zeta}_{q} \gamma^n_{qg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

DGLAP: All orders evolution

`Cui:2020tdf`

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$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

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$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

Sea-quark PDF

$$\langle x^n \rangle^{\zeta}_{S^q_H} = \langle x^n \rangle^{\zeta}_{\Sigma^q_H} - \langle x^n \rangle^{\zeta}_{q_H}$$

DGLAP: All orders evolution

Cui:2020tdf

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$$\Sigma^q_{\rm H}(x) = q_{\rm H}(x) + \bar{q}_{\rm H}(x)$$

singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\rm H}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\rm H}(y)$$

$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\rm H}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\rm H}(y) + 2P^{\zeta}_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\rm H}(y) \right\}$$

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Hard-wall threshold

$$\mathcal{P}^{\zeta}_{q} = \theta(\zeta - M_q)$$

Quark singlet and glue PDFs in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^{\zeta}_{\Sigma_H} + 2 n_f \mathcal{P}^{\zeta}_{q} \gamma^n_{ug} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gu} \langle x^n \rangle^{\zeta}_{\Sigma_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

Sea-quark PDF

$$\langle x^n \rangle^{\zeta}_{S^q_H} = \langle x^n \rangle^{\zeta}_{\Sigma^q_H} - \langle x^n \rangle^{\zeta}_{q_H}$$

Full singlet and sea

$$\langle x^n \rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x^n \rangle^{\zeta}_{\Sigma^q_H}, \; \langle x^n \rangle^{\zeta}_{S_H} = \sum_q \langle x^n \rangle^{\zeta}_{S^q_H}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H\,,\ \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2}\left(\begin{array}{c}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{array}\right) = \left(\begin{array}{cc}\gamma^n_{uu} & 2n_f\gamma^n_{ug}\\ \gamma^n_{gu} & \gamma^n_{gg}\end{array}\right)\left(\begin{array}{c}\langle x^n\rangle^{\zeta}_{\Sigma_H}\\ \langle x^n\rangle^{\zeta}_{g_H}\end{array}\right)$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

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$$\zeta^2\frac{d}{d\zeta^2}\left(\begin{array}{c}\langle x^n\rangle^{\zeta}_{\Sigma H}\\ \langle x^n\rangle^{\zeta}_{gH}\end{array}\right) = \left(\begin{array}{cc}\gamma^n_{uu} & 2n_f\gamma^n_{ug}\\ \gamma^n_{gu} & \gamma^n_{gg}\end{array}\right)\left(\begin{array}{c}\langle x^n\rangle^{\zeta}_{\Sigma H}\\ \langle x^n\rangle^{\zeta}_{gH}\end{array}\right)$$

$$\left(\begin{array}{c}\langle x^n\rangle^{\zeta}_{\Sigma H}\\ \langle x^n\rangle^{\zeta}_{gH}\end{array}\right) = \left(\begin{array}{cc}\alpha^n_+S^n_- + \alpha^n_-S^n_+ & \beta^n_{\Sigma g}\left(S^n_- - S^n_+\right)\\ \beta^n_{g\Sigma}\left(S^n_- - S^n_+\right) & \alpha^n_-S^n_- + \alpha^n_+S^n_+\end{array}\right)\left(\begin{array}{c}\langle x^n\rangle^{\zeta_H}_{\Sigma H}\\ \langle x^n\rangle^{\zeta_H}_{gH}\end{array}\right)$$

$$\alpha^n_\pm = \pm\frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = -\frac{2n_f\gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = \left[S(\zeta_H,\zeta)\right]^{\lambda^n_\pm/\gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f\gamma^n_{ug}(\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H \, , \; \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2 n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = - \frac{2 n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr} \left(\Gamma^n \right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2 \left(\Gamma^n \right) - \mathrm{Det} \left(\Gamma^n \right)}$$

$$S^n_\pm = [S(\zeta_H, \zeta)]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2 n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

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In terms of the moments for the sum of all valence-quark distributions at hadronic scale

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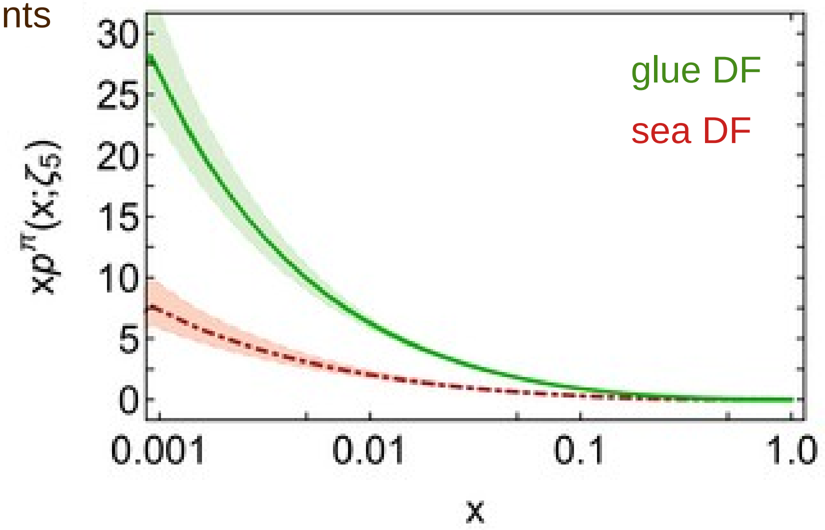
Compute all the moments and reconstruct:



DGLAP: All orders evolution

Implication 4: glue and sea from valence

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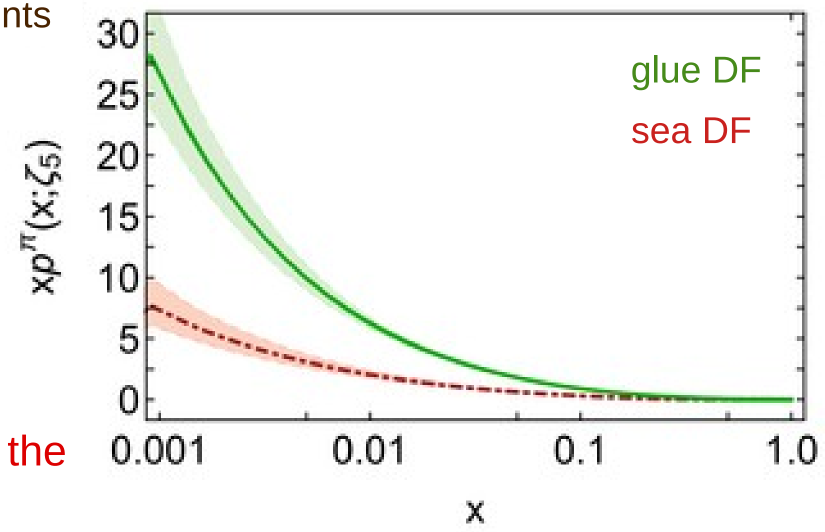
Compute all the moments and reconstruct:

$$S^n_\pm = \left[S(\zeta_H,\zeta)\right]^{\lambda^n_\pm/\gamma_{uu}} \longrightarrow = \left[\frac{\langle x\rangle^{\zeta}_{q_H}}{\langle x\rangle^{\zeta_H}_{q_H}}\right]^{\lambda^n_\pm/\gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f\gamma^n_{ug}(\lambda^n_+ - \lambda^n_-)}$$



★ The only required input is the the momentum fraction at the probed empirical scale!!

DGLAP: All orders evolution

Implication 4: glue and sea from valence

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All quarks active

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n=1 case

$n_f = 4$

$$\langle x\rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x\rangle^{\zeta}_{q_H} + \langle x\rangle^{\zeta}_{S_H} = \frac{3}{7} + \frac{4}{7}[S(\zeta_H,\zeta)]^{7/4}$$

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DGLAP: All orders evolution

Implication 4: glue and sea from valence

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$$\alpha^n_\pm = \pm\frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

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ζ_5	$\langle 2x\rangle^\pi_q$	$\langle x\rangle^\pi_g$	$\langle x\rangle^\pi_{\text{sea}}$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

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Z-F. Cui et al., arXiv:2006.1465

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R.S. Sufian et al., arXiv:2001.04960

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Asymptotic limit: G. Altarelli, Phys. Rep. 81, 1 (1982)

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owing to $\lambda^n_\pm > 0$

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In terms of the moments for the sum of all valence-quark distributions at hadronic scale

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = - \frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr} \left(\Gamma^n \right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2 \left(\Gamma^n \right) - \mathrm{Det} \left(\Gamma^n \right)}$$

n=1 case

$n_f = 4$

$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}} \longrightarrow = \left[\langle 2x \rangle^{\zeta}_{q_\pi} \right]^{\gamma^n_{qq} / \gamma_{qq}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

$$\Sigma_H(x) \underset{\zeta^2 \to \infty}{=} \frac{3}{7} \frac{\delta(x)}{x}$$

$$g_H(x) \underset{\zeta^2 \to \infty}{=} \frac{4}{7} \frac{\delta(x)}{x}$$

★ The only required input is the the pion momentum fraction at the probed empirical scale (assuming charge universality)!!

DGLAP: All orders evolution

Implication 5: correlating glue and sea

$$M_q = \zeta_H \,,\ \forall q$$

All quarks active

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

DGLAP: All orders evolution

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$$\begin{pmatrix} \alpha_+^n \left[S_-^n\right]^{-1} + \alpha_-^n \left[S_+^n\right]^{-1} & \beta_{\Sigma g}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) \\ \beta_{g\Sigma}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) & \alpha_-^n \left[S_-^n\right]^{-1} + \alpha_+^n \left[S_+^n\right]^{-1} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta_H} \\ \langle x^n \rangle_{g_H}^{\zeta_H} \end{pmatrix}$$

The equation can be easily inverted

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The equation can be easily inverted and, relying on the hadronic scale definition, delivers a constraint for all Mellin moments of glue and sea at any experimental scale:

$$\frac{\langle x^n \rangle_{\Sigma_\pi}^{\zeta}}{\langle x^n \rangle_{g_\pi}^{\zeta}} = \frac{\langle x^n \rangle_{\mathcal{S}_\pi}^{\zeta} + \langle 2x^n \rangle_{u_\pi}^{\zeta}}{\langle x^n \rangle_{g_\pi}^{\zeta}} = \frac{\alpha_-^n S_+^n + \alpha_+^n S_-^n}{\beta_{g\Sigma}^n \left(S_-^n - S_+^n\right)}$$

DGLAP: All orders evolution

Implication 5: correlating glue and sea

$$M_q = \zeta_H \,, \; \forall q$$

All quarks active

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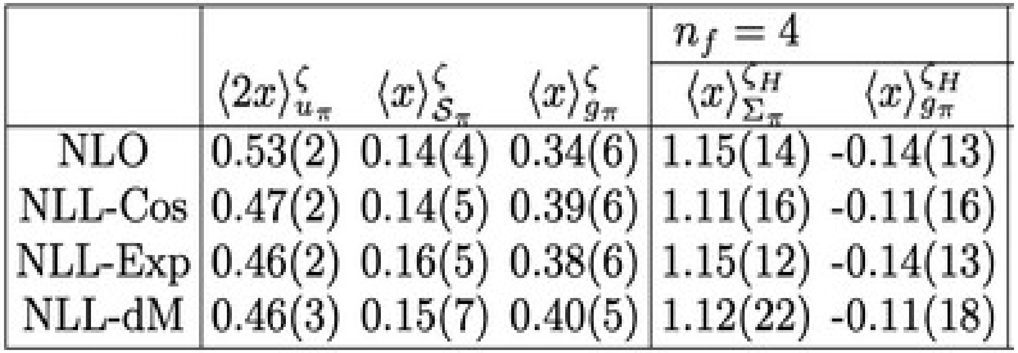
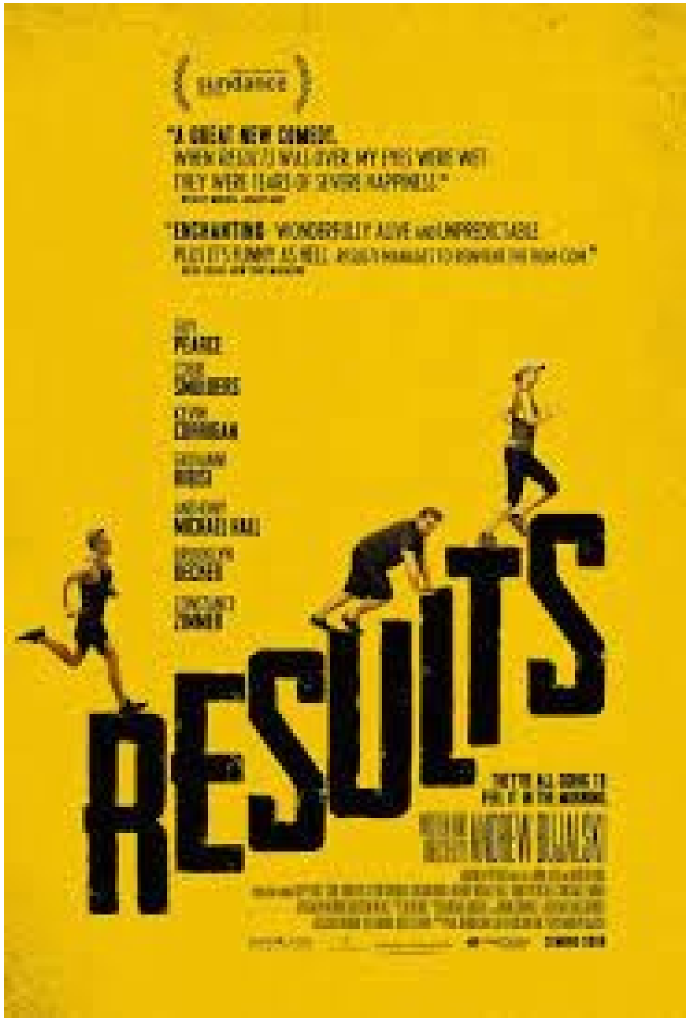
$$\frac{\langle x \rangle_{\Sigma_\pi}^{\zeta}}{\langle x \rangle_{g_\pi}^{\zeta}} = \frac{\langle x \rangle_{\mathcal{S}_\pi}^{\zeta} + \langle 2x \rangle_{u_\pi}^{\zeta}}{\langle x \rangle_{g_\pi}^{\zeta}} = \frac{\frac{3}{4} + \left(\langle 2x \rangle_{u_\pi}^{\zeta}\right)^{7/4}}{1 - \left(\langle 2x \rangle_{u_\pi}^{\zeta}\right)^{7/4}}$$



	$\langle 2x \rangle_{u_\pi}^{\zeta}$	$\langle x \rangle_{\mathcal{S}_\pi}^{\zeta}$	$\langle x \rangle_{g_\pi}^{\zeta}$	$n_f = 4$ $\langle x \rangle_{\Sigma_\pi}^{\zeta_H}$	$\langle x \rangle_{g_\pi}^{\zeta_H}$
NLO	0.53(2)	0.14(4)	0.34(6)	1.15(14)	-0.14(13)
NLL-Cos	0.47(2)	0.14(5)	0.39(6)	1.11(16)	-0.11(16)
NLL-Exp	0.46(2)	0.16(5)	0.38(6)	1.15(12)	-0.14(13)
NLL-dM	0.46(3)	0.15(7)	0.40(5)	1.12(22)	-0.11(18)


sundance
"A GREAT NEW COMEDY.
WHEN RESULTS WAS OVER, MY EYES WERE WET.
THEY WERE TEARS OF SEVERE HAPPINESS."
"ENCHANTING. WONDERFULLY ALIVE AND UNPREDICTABLE.
PLUS IT'S FUNNY AS HELL. RESULTS MANAGES TO REINVENT THE ROM-COM."
GUY PEARCE
COBIE SMULDERS
KEVIN CORRIGAN
GIOVANNI RIBISI
ANTHONY MICHAEL HALL
BROOKLYN DECKER
CONSTANCE ZIMMER
RESULTS
THEY'RE ALL GOING TO
WRITTEN AND DIRECTED BY ANDREW BUJALSKI

Pion PDF: from CSM (DSEs) to the experiment

Symmetry-preserving DSE computation of the valence-quark PDF:

[L. Chang et al., Phys.Lett.B737(2014)23]
[M. Ding et al., Phys.Rev.D101(2020)054014

$$q^{\pi}(x;\zeta) = N_c \mathrm{tr} \int_{dk} \delta_n^x(k_\eta) \Gamma_\pi^P(k_{\bar{\eta}\eta};\zeta)\, S(k_{\bar{\eta}};\zeta) \times \{n \cdot \frac{\partial}{\partial k_\eta} [\Gamma_\pi^{-P}(k_{\eta\bar{\eta}};\zeta) S(k_\eta;\zeta)]\} .$$

$$q_{\mathrm{O}}^{\pi}(x;\zeta_H) = 213.32\, x^2(1-x)^2 \times [1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)]$$

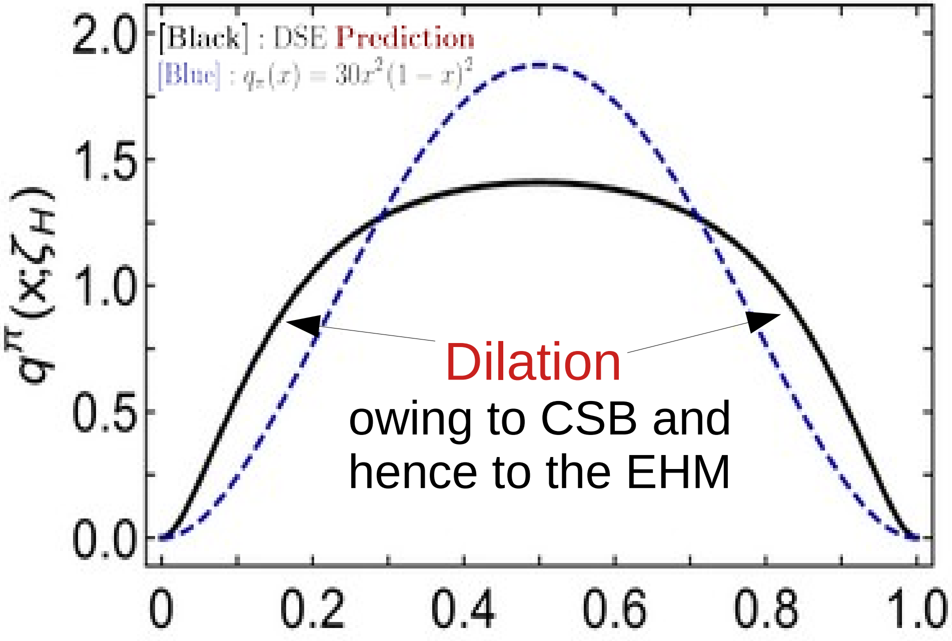
$$q(x;\zeta) \underset{x\to 1}{\sim} (1-x)^{\beta(\zeta)}\, (1+\mathcal{O}(1-x))$$

$$\beta(\zeta_H) = 2$$

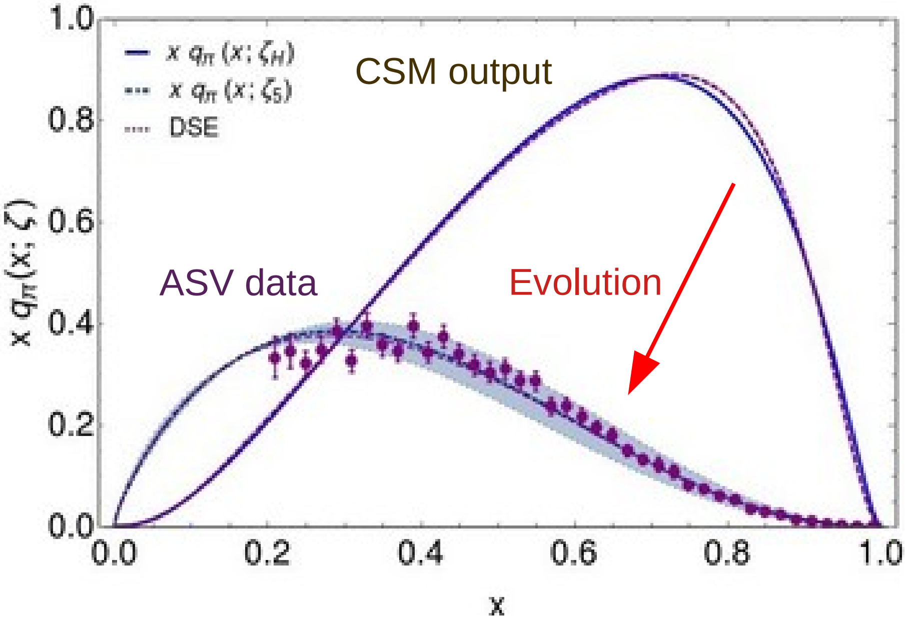
Farrar, Jackson, Phys.Rev.Lett 35(1975)1416
Berger, Brodsky, Phys.Rev.Lett 42(1979)940

- The EHM-triggered broadening shortens the extent of the domain of convexity lying on the neighborhood of the endpoints, induced too by the QCD dynamics

- It cannot however spoil the asymptotic QCD behaviour at large-x (and, owing to isospin symmetry, at low-x)



Dilation
owing to CSB and
hence to the EHM



CSM output

ASV data

Evolution

Pion PDF: from CSM (DSEs) to the experiment

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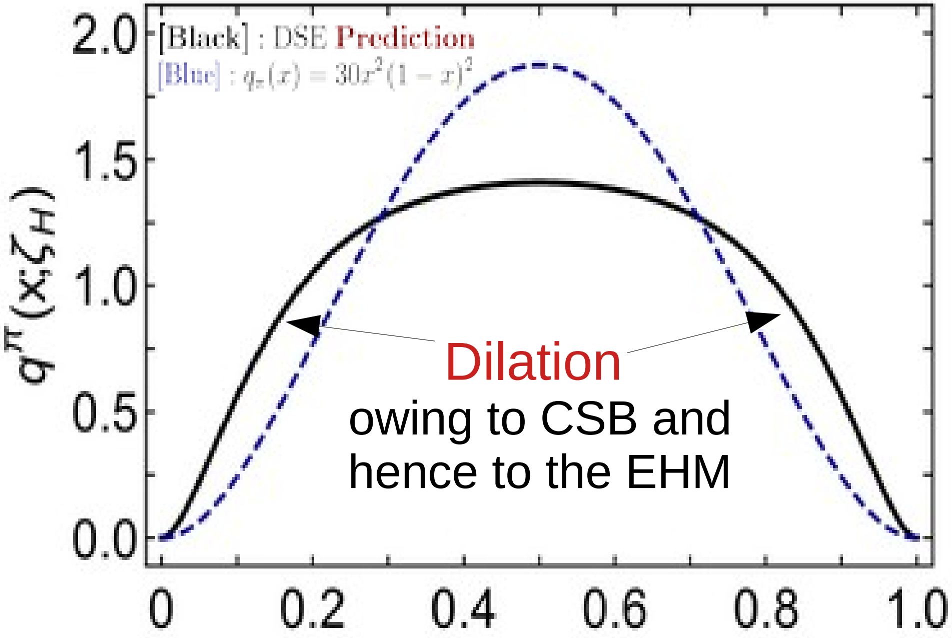
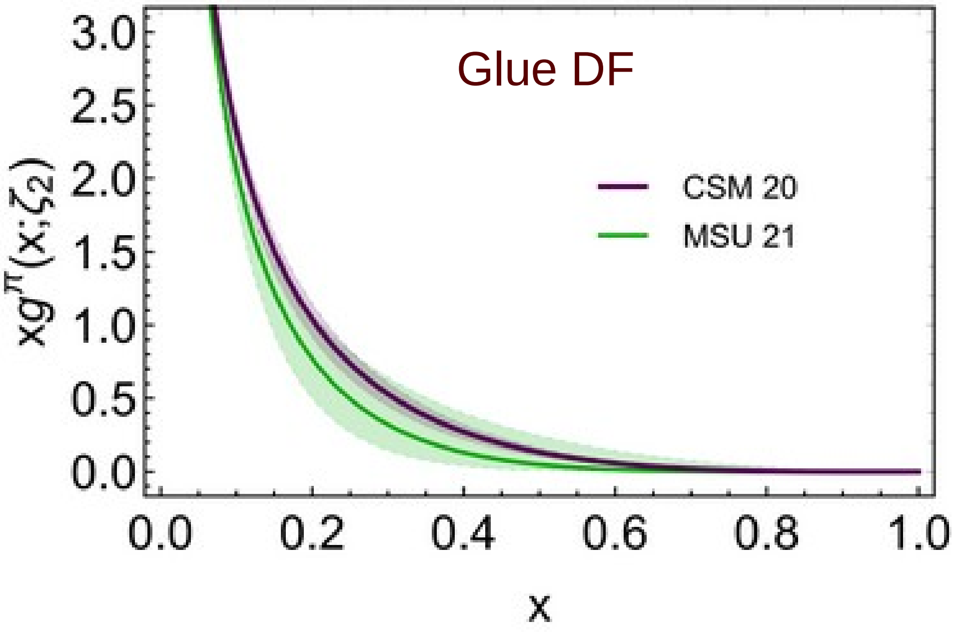


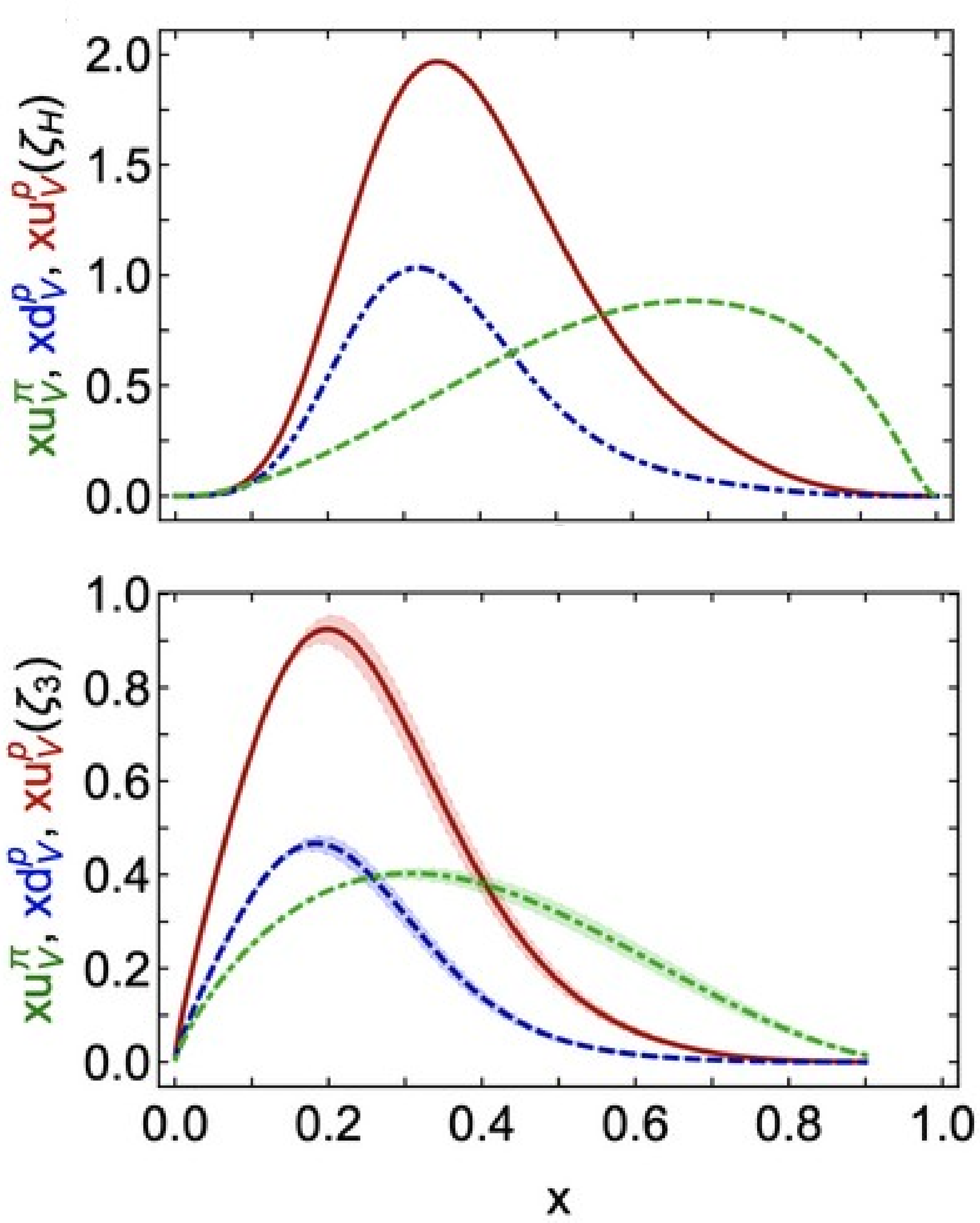
Dilation
owing to CSB and hence to the EHM



Proton PDF: from CSM (DSEs) to the experiment

An analogous symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach:

[L. Chang et al., Phys.Lett.B, arXiv:2201.07870]



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An analogous symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach:
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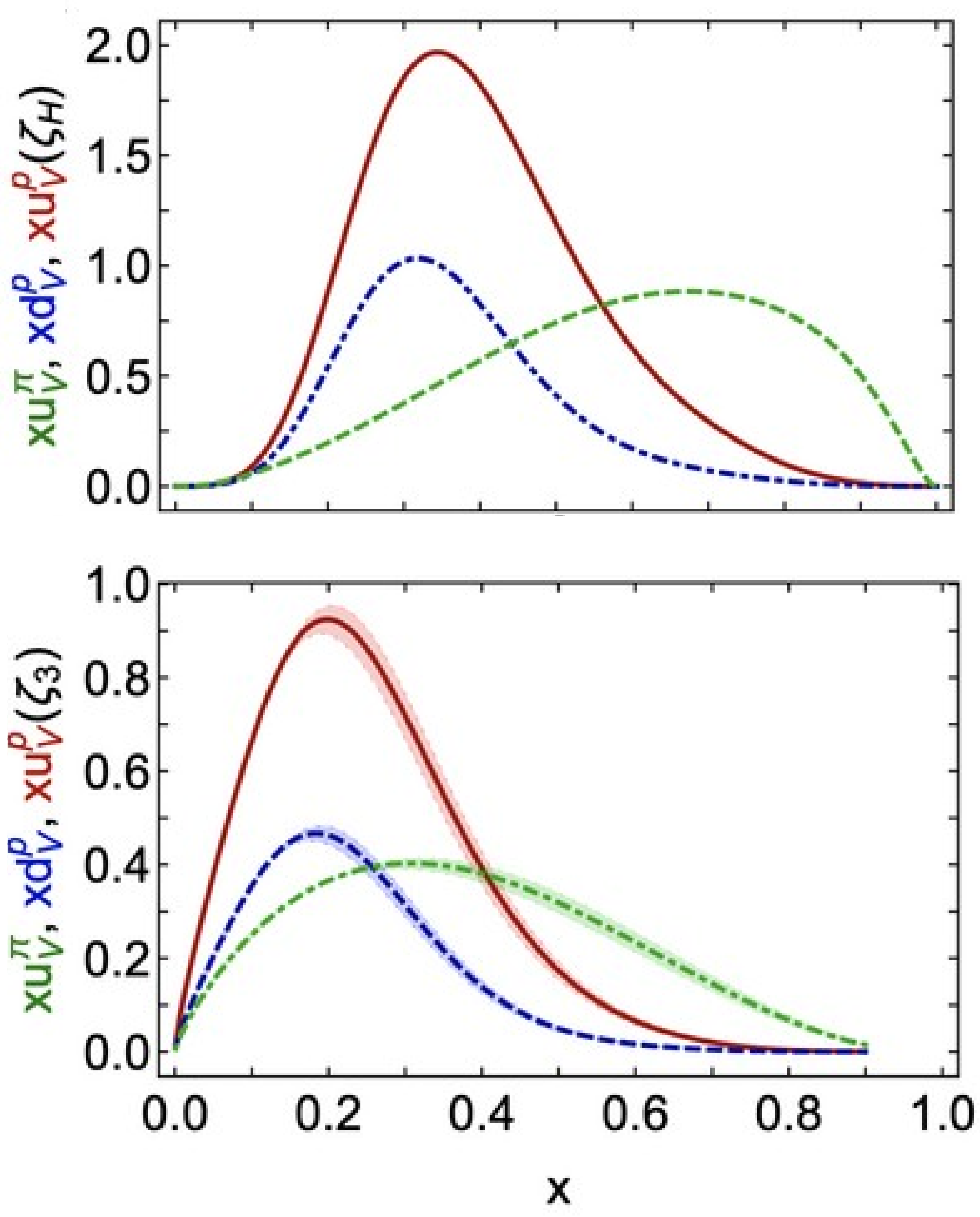
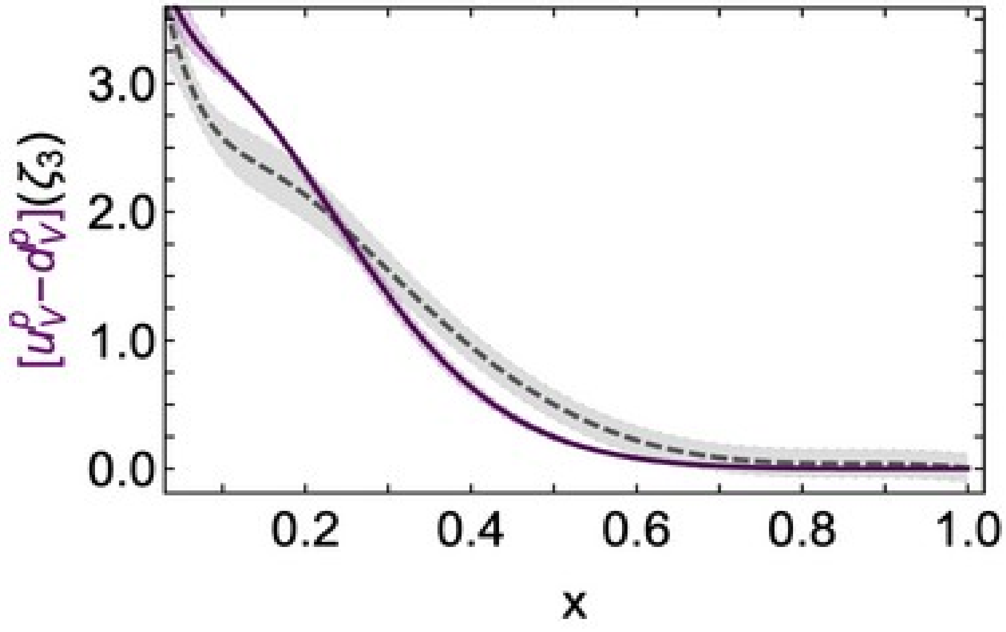
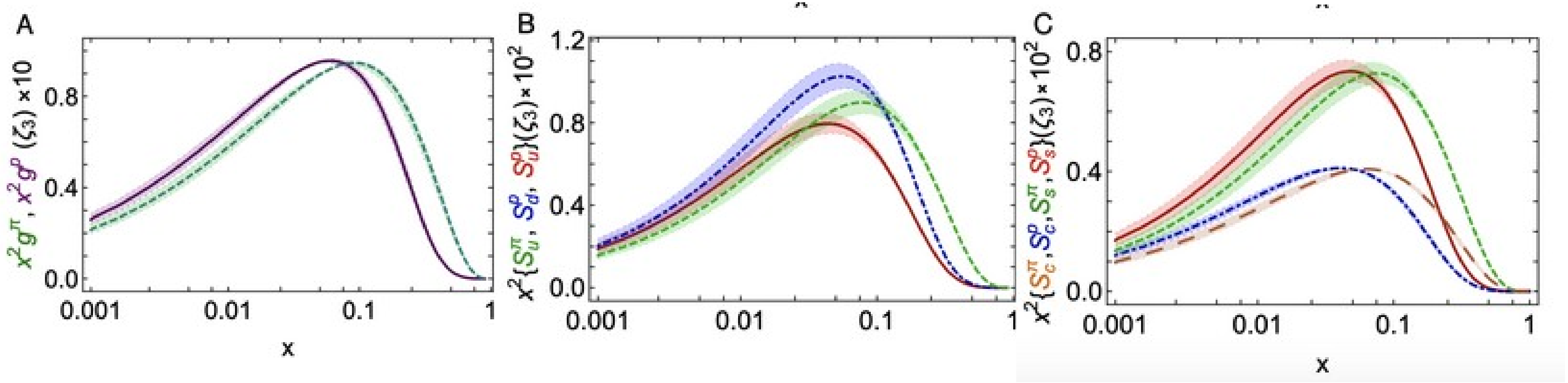
Producing an isovector distribution in fair agreement with lattice results
[H-W. Lin et al., arXiv:2011.14791]





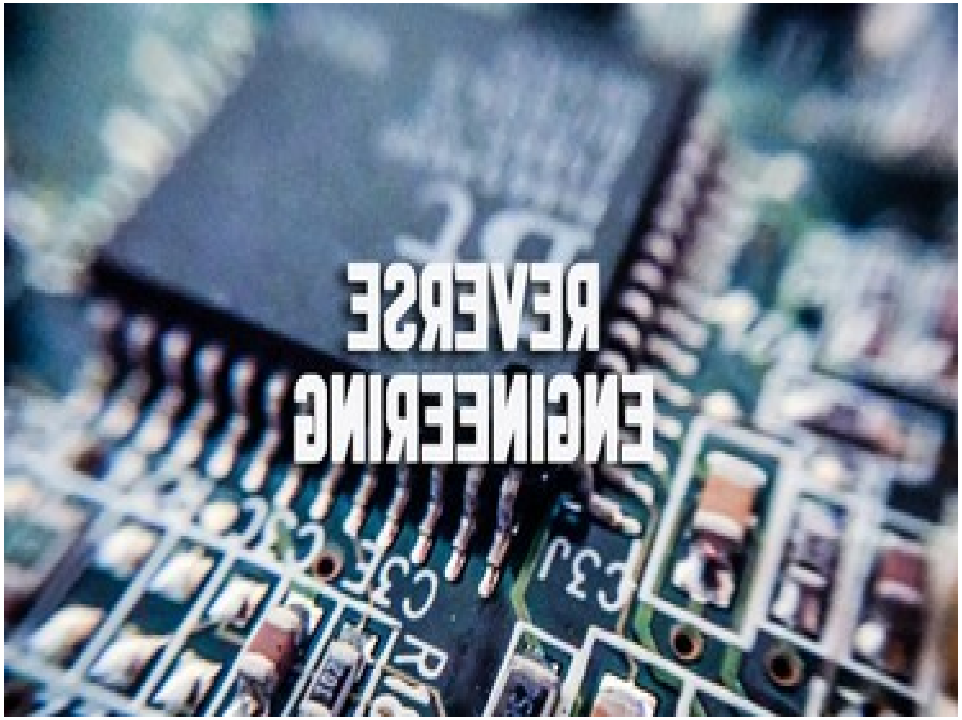
Proton PDF: pion and proton in counterpoint



pion	u^π	$\bar{d}^\pi$	g^π	S_π^u	$S_\pi^{\bar{d}}$	S_π^s	S_π^c
$\langle x\rangle^{\zeta_2}$	24.0(1.1)	24.0(1.1)	41.0(1.2)	3.3(3)	3.3(3)	2.65(22)	1.33(5)
$\langle x^2\rangle^{\zeta_2}$	9.5(7)	9.5(7)	3.7(1)	0.27(1)	0.27(1)	0.21(1)	0.092(2)
$\langle x^3\rangle^{\zeta_2}$	4.7(4)	4.7(4)	0.92(6)	0.057(1)	0.057(1)	0.044(0)	0.018(1)
$\langle x\rangle^{\zeta_3}$	22.1(1.0)	22.1(1.0)	42.9(1.0)	3.7(3)	3.7(3)	3.0(2)	1.83(6)
$\langle x^2\rangle^{\zeta_3}$	8.4(6)	8.4(6)	3.5(1)	0.27(1)	0.27(1)	0.22(1)	0.120(3)
$\langle x^3\rangle^{\zeta_3}$	4.0(3)	4.0(3)	0.82(5)	0.056(0)	0.056(0)	0.044(0)	0.022(1)

proton	u^p	d^p	g^p	S_p^u	S_p^d	S_p^s	S_p^c
$\langle x\rangle^{\zeta_2}$	32.9(1.4)	15.0(0.7)	40.9(1.1)	2.9(2)	3.7(3)	2.64(22)	1.32(5)
$\langle x^2\rangle^{\zeta_2}$	8.7(6)	3.6(2)	2.4(1)	0.14(1)	0.21(1)	0.13(0)	0.059(2)
$\langle x^3\rangle^{\zeta_2}$	2.9(3)	1.1(1)	0.39(2)	0.019(0)	0.030(1)	0.019(0)	0.008(0)
$\langle x\rangle^{\zeta_3}$	30.4(1.3)	13.8(0.6)	42.8(1.0)	3.3(3)	4.1(3)	3.0(2)	1.82(6)
$\langle x^2\rangle^{\zeta_3}$	7.7(5)	3.2(2)	2.2(1)	0.15(1)	0.21(1)	0.14(0)	0.075(2)
$\langle x^3\rangle^{\zeta_3}$	2.5(2)	0.9(1)	0.35(2)	0.019(0)	0.028(0)	0.019(0)	0.010(1)

Reverse engineering the PDF data



Pion **PDF**

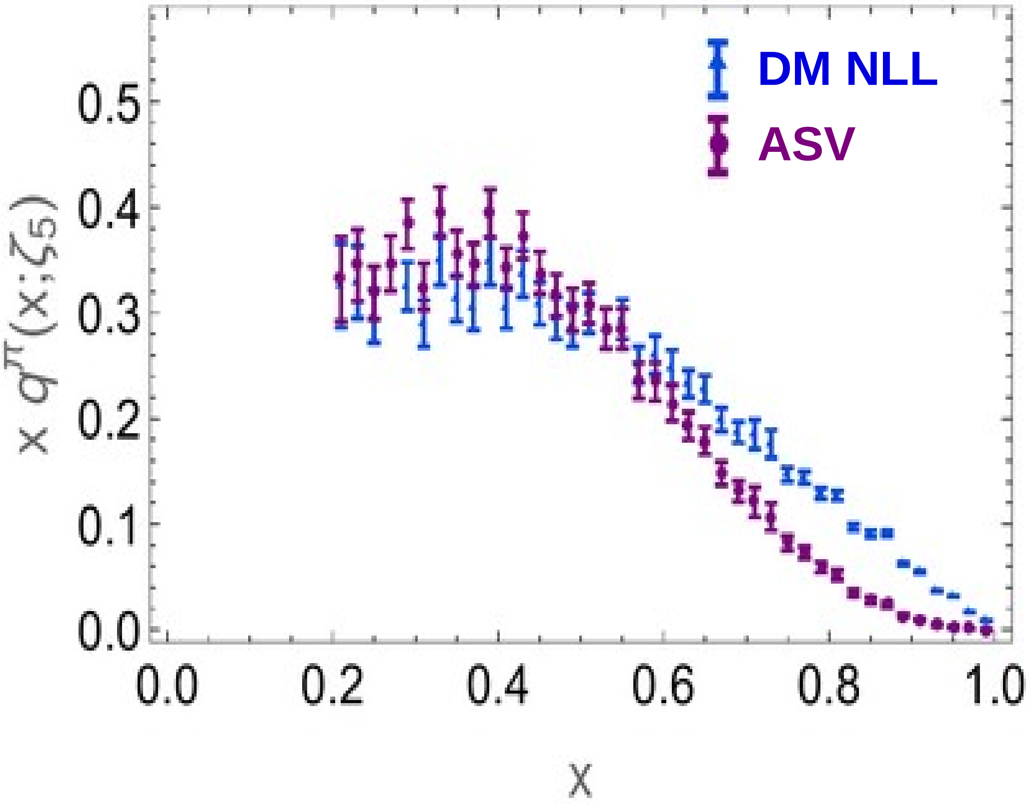
- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^\pi(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

$$\{\alpha_i^\zeta | i = 1,2,3\}$$

Normalization

Free parameters



- Then, we proceed as follows:

1) Determine the **best values α_i** via least-squares fit to the data.

2) Generate new **values α_i,** distributed randomly around the best fit.

3) Using the latter set, evaluate:

$$\chi^2 = \sum_{l=1}^{N} \frac{(u^\pi(x_l;[\alpha_i];\zeta_5) - u_j)^2}{\delta_l^2}$$

Data point with error

4) Accept a replica with probability:

$$\mathcal{P} = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}\,, \quad P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

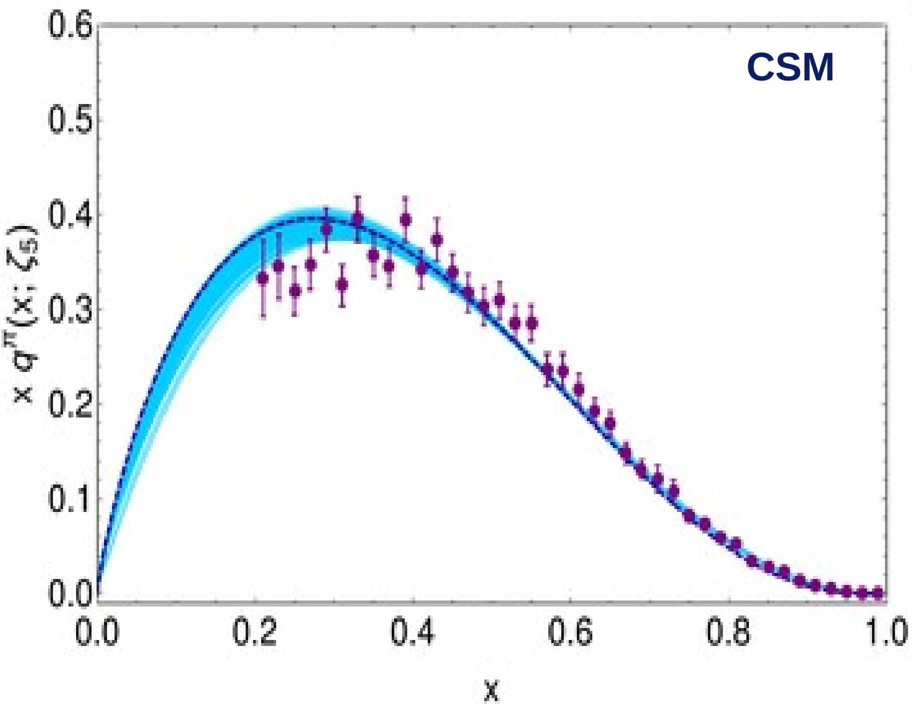
5) Evolve back to ζ_H

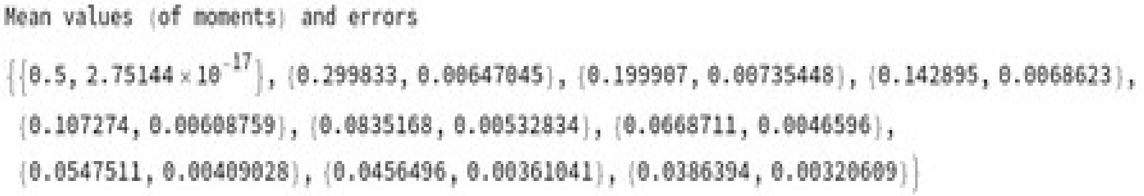
Repeat (2-5).

Pion PDF: ASV analysis of E615 data

- Applying this algorithm to the **ASV** **data** yields:

(average)





```
Mean values (of moments) and errors
{{0.5, 2.75144×10^-17}, {0.299833, 0.00647045}, {0.199907, 0.00735448}, {0.142895, 0.0068623},
 {0.107274, 0.00608759}, {0.0835168, 0.00532834}, {0.0668711, 0.0046596},
 {0.0547511, 0.00409028}, {0.0456496, 0.00361041}, {0.0386394, 0.00320609}}
```

- ✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.
- ✔ It seems it favors a **soft end-point** behavior just like the **CSM result**.

Pion PDF: ASV analysis of E615 data

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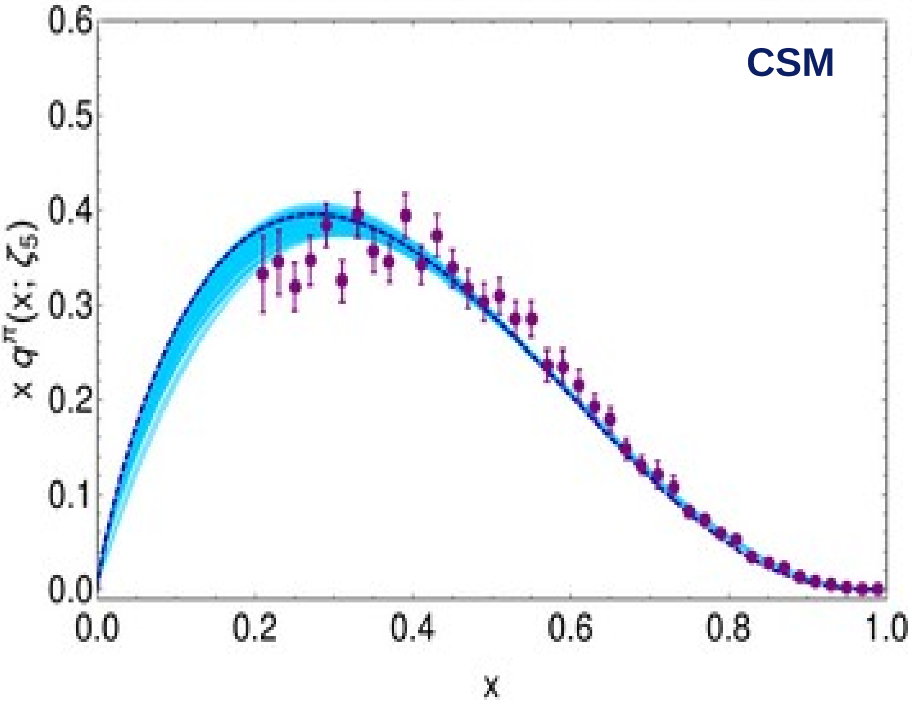

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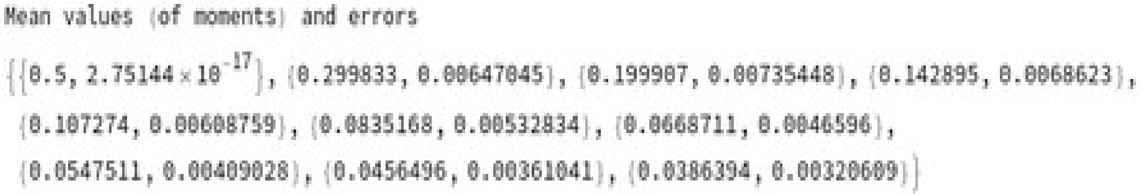

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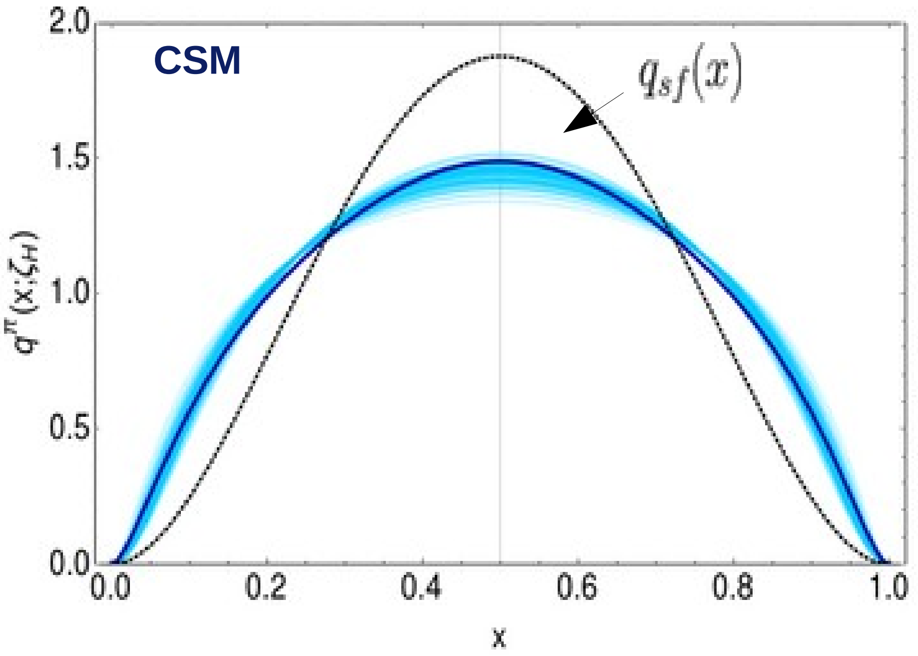
 {0.0547511, 0.00409028}, {0.0456496, 0.00361041}, {0.0386394, 0.00320609}}
```

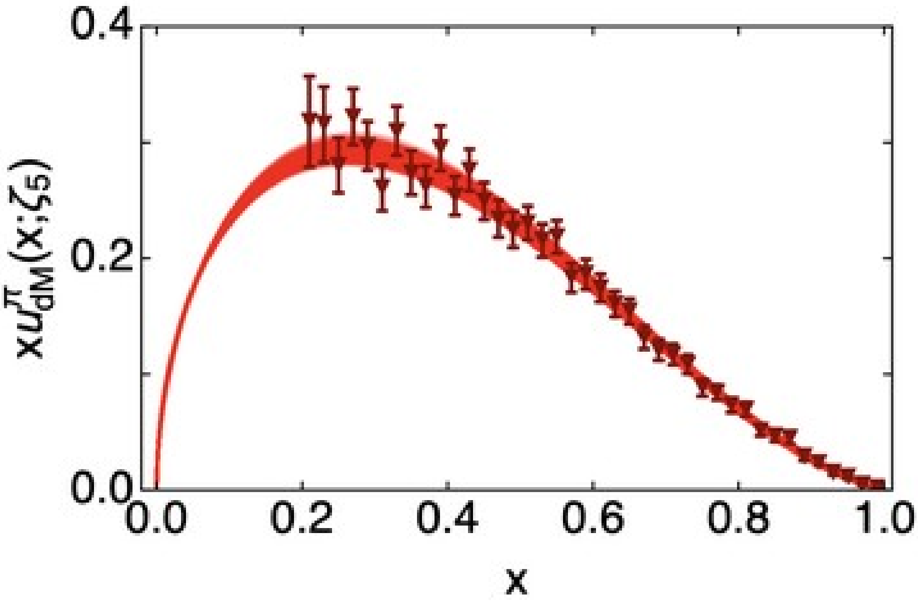
✔ Then, we can **reconstruct** the moments produced by each replica, using the **single-parameter Ansatz**:

$$u^{\pi}(x;\zeta_{\mathcal{H}}) = n_0 \ln(1 + x^2(1-x)^2/\rho^2)$$

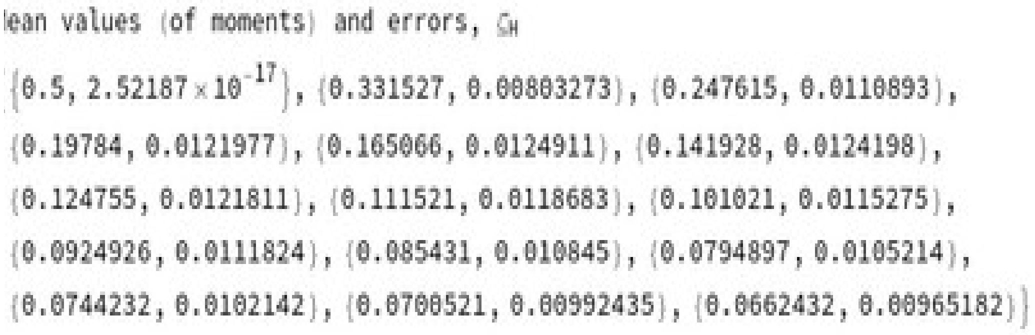


Pion PDF: dM NLL analysis of E615 data

- Applying this algorithm to the original data yields:

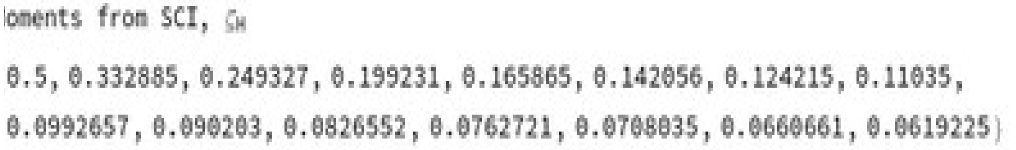


(average)

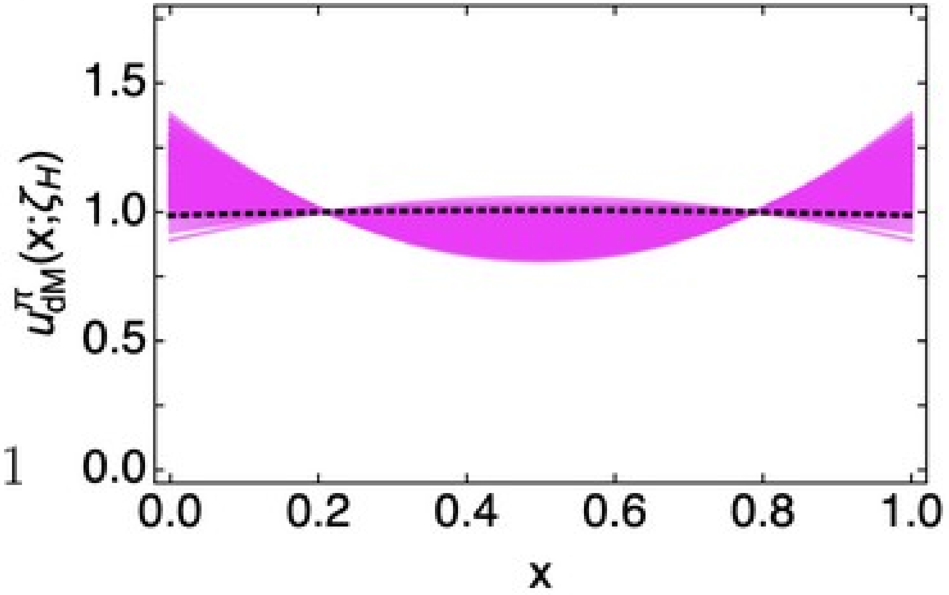


```
lean values (of moments) and errors, ζH
{0.5, 2.52187 × 10^-17}, {0.331527, 0.00803273}, {0.247615, 0.0110893},
{0.19784, 0.0121977}, {0.165066, 0.0124911}, {0.141928, 0.0124198},
{0.124755, 0.0121811}, {0.111521, 0.0118683}, {0.101021, 0.0115275},
{0.0924926, 0.0111824}, {0.085431, 0.010845}, {0.0794897, 0.0105214},
{0.0744232, 0.0102142}, {0.0700521, 0.00992435}, {0.0662432, 0.00965182}}
```

(SCI)



```
loments from SCI, ζH
0.5, 0.332885, 0.249327, 0.199231, 0.165865, 0.142056, 0.124215, 0.11035,
0.0992657, 0.090203, 0.0826552, 0.0762721, 0.0708035, 0.0660661, 0.0619225}
```



✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.

✗ But also exhibit agreement with the **SCI results**.

$$q_{\mathrm{SCI}}(x;\zeta_H) \approx 1$$

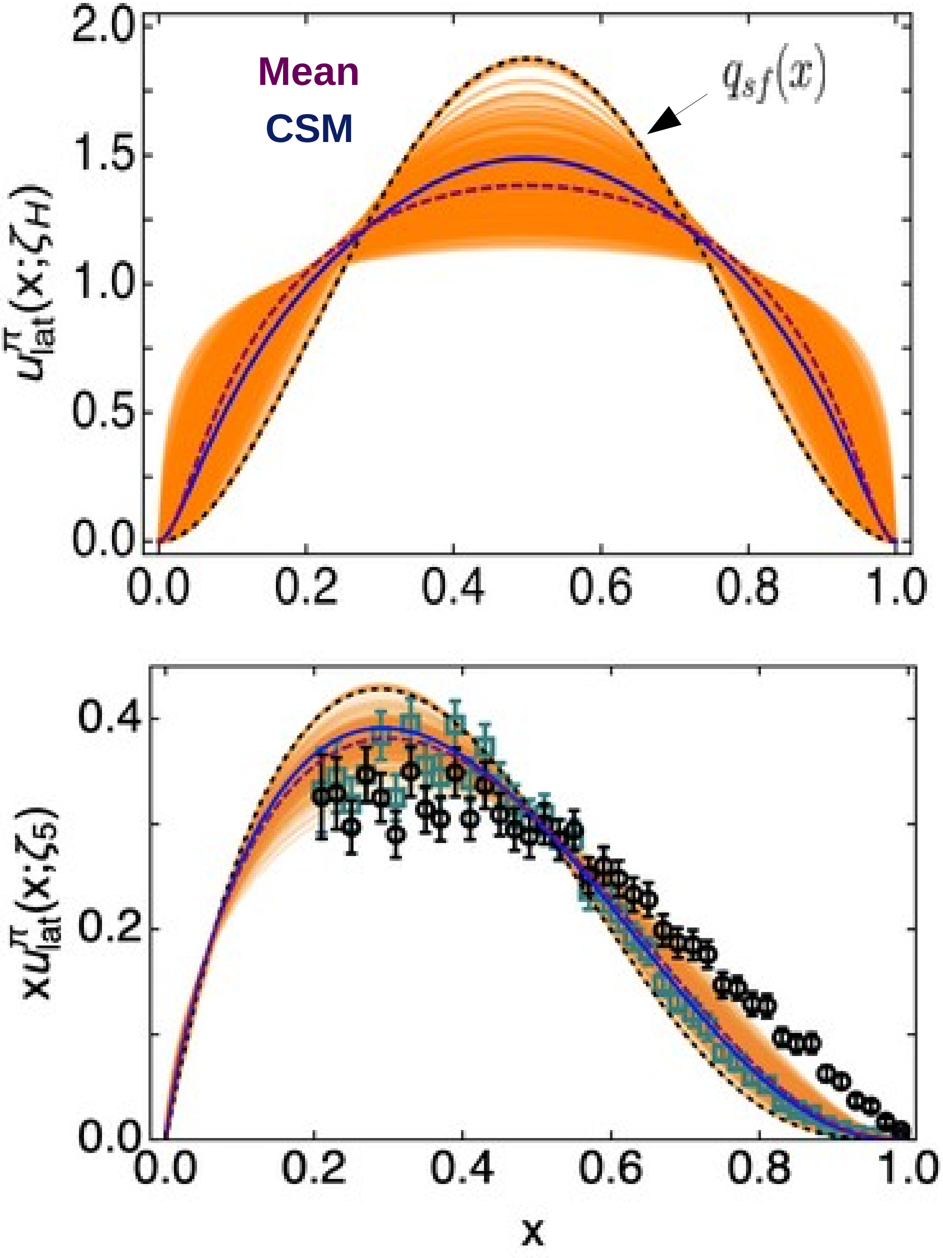
Pion PDF: lattice data

- An analagous procedure, similarly based on the all-orders evolution, can be applied to the **lattice data** for Mellin moments. Here, the moments obtained at the lattice scale are evolved down to the hadronic scale and up to the **experimental** one.

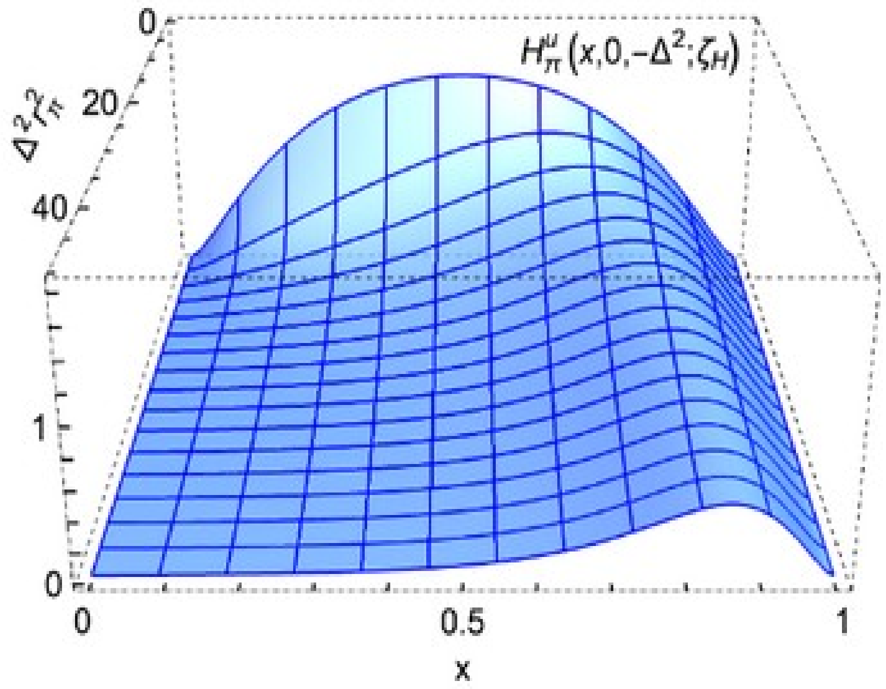
- Both (**ASV**) **experimental** and **lattice** data yield hadronic scale **PDFs** exhibiting soft end-point behavior and **EHM-induced broadening**.

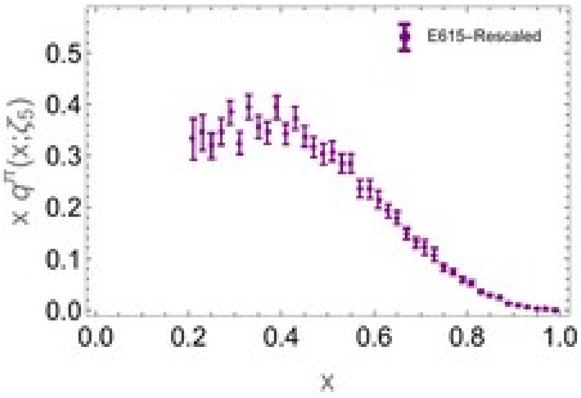
- The results are **compatible**, although current precision of the lattice moments still leaves us with a somewhat **wide band** of **uncertainty**.

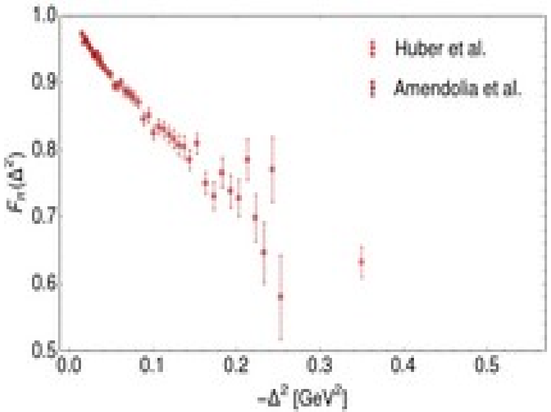
- **Lattice** results, analyzed on the basis of all-orders evolution, are clearly **inconsistent** with those resulting from the **dM NLL analysis** of E615 data.



GPDs from PDFs and form factors







Light-front wave functions

- Many **distributions** are related via the leading-twist light-front wave function (**LFWF**), e.g.:

Distribution amplitudes

$$f_{\mathsf{P}}\varphi_{\mathsf{P}}^{u}(x,\zeta_{\mathcal{H}}) = \int \frac{dk_{\perp}^{2}}{16\pi^{3}} \psi_{\mathsf{P}}^{u}\left(x,k_{\perp}^{2};\zeta_{\mathcal{H}}\right)$$

Distribution functions

$$u^{\mathsf{P}}\left(x;\zeta_{\mathcal{H}}\right) = \int \frac{d^{2}k_{\perp}}{16\pi^{3}} \left|\psi_{\mathsf{P}}^{u}\left(x,k_{\perp}^{2};\zeta_{\mathcal{H}}\right)\right|^{2}$$

- In the **DGLAP** kinematic domain, this is also the case of the valence-quark **GPD**:

$$H_{\mathsf{P}}^{u}(x,\xi,t;\zeta_{\mathcal{H}}) = \int \frac{d^{2}k_{\perp}}{16\pi^{3}} \psi_{\mathsf{P}}^{u*}\left(x_{-},k_{\perp -}^{2};\zeta_{\mathcal{H}}\right) \psi_{\mathsf{P}}^{u}\left(x_{+},k_{\perp +}^{2};\zeta_{\mathcal{H}}\right)$$

$$x_{\mp} = (x \mp \xi)/(1 \mp \xi)\,,$$
$$k_{\perp\mp} = k_{\perp} \pm (\Delta_{\perp}/2)(1-x)/(1 \mp \xi)$$

$$\psi_{\mathsf{P}}^{u}(x,k_{\perp}^{2};\zeta)$$

“One ring to rule them all”

LFWF: Factorized models

Raya:2021zrz

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$$x_\mp = (x \mp \xi)/(1 \mp \xi)\,,$$

$$k_{\perp\mp} = k_\perp \pm (\Delta_\perp/2)(1-x)/(1\mp\xi)$$

- If the **x-k** dependence is factorized, then:

$$\psi^{\uparrow\downarrow}_{\mathbf{P}_u}(x,k_\perp^2;\zeta_H) = \tilde{\psi}^{\mathbf{u}}_{\mathbf{P}_u}(k_\perp^2)\,[u^{\mathbf{P}}(x;\zeta_H)]^{1/2}$$

➔The **x-dependence** of the LFWF lies within the **PDF** or, equivalently, the **PDA**:

$$u^{\mathbf{P}}(x;\zeta_H) = [\varphi^u_{\mathbf{P}}(x;\zeta_H)]^2 / \int_0^1 dx[\varphi^u_{\mathbf{P}}(x;\zeta_H)]^2$$

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- Our experience with **CSM** have revealed correlations proportional to

$$M_{\mathbf{P}}^2,\ M_h^2 - M_q^2$$

- So it should be a very good **Ansatz** for the **pion**, and fairly good for the **kaon**.

LFWF: **Factorized models**

Raya:2021zrz

- Starting with a **factorized** **LFWF**, $\psi^{\uparrow\downarrow}_{\mathbf{P}_u}(x,k_\perp^2;\zeta_H) = \tilde{\psi}^{\mathbf{u}}_{\mathbf{P}_u}(k_\perp^2)\,[u^{\mathbf{P}}(x;\zeta_H)]^{1/2}$

This one shall be obtained as in the first part of the talk

- The overlap representation for the **GPD** entails:

$$H^u_{\mathsf{P}}(x,\xi,t;\zeta_{\mathcal{H}}) = \int \frac{d^2k_\perp}{16\pi^3}\psi^{u*}_{\mathsf{P}}\left(x_-,k^2_{\perp-};\zeta_{\mathcal{H}}\right)\psi^u_{\mathsf{P}}\left(x_+,k^2_{\perp+};\zeta_{\mathcal{H}}\right)$$

$$= \Theta(x_-)\sqrt{u^{\mathbf{P}}(x_-;\zeta_H)u^{\mathbf{P}}(x_+;\zeta_H)}\Phi_{\mathbf{P}}(z;\zeta_H)$$

Heaviside Theta

This dictates the off-forward behavior of the GPD

- Where $z = s_\perp^2 = -t(1-x)^2/(1-\xi^2)^2$ and:

… will be driven by the electromagnetic form factor

$$\boxed{\Phi^u_{\mathbf{P}}(z;\zeta_H) = \int\frac{d^2\mathbf{k}_\perp}{16\pi^3}\tilde{\psi}^{u*}_{\mathbf{P}}\left(\mathbf{k}_\perp^2;\zeta_H\right)\tilde{\psi}^u_{\mathbf{P}}\left(\left(\mathbf{k}_\perp-\mathbf{s}_\perp\right)^2;\zeta_H\right)}$$

The GPD **model**

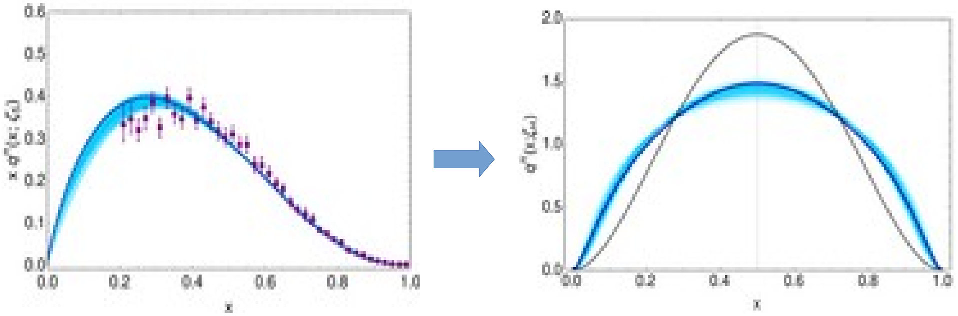
Raya:2021zrz

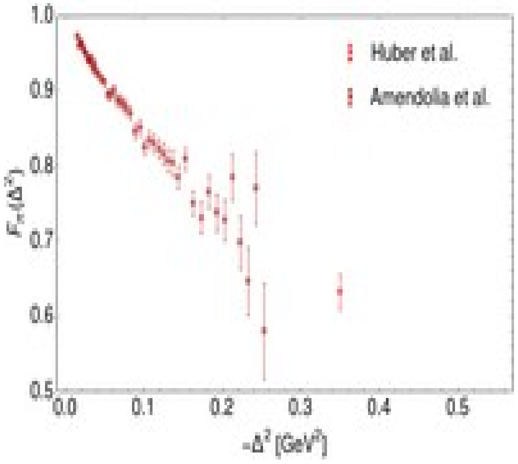
- The factorized **LFWF** motivates the following **GPD** model:

$$H_{\mathrm{P}}^{u}(x,\xi,t;\zeta_{\mathcal{H}}) = \Theta(x_-)\sqrt{u^{\mathrm{P}}(x_-;\zeta_H)u^{\mathrm{P}}(x_+;\zeta_H)}\Phi_{\mathrm{P}}(z;\zeta_H)$$

- The **PDF** might be inferred from **data**, as described before.
- Thus, **parameterized** by:

$$u^{\pi}(x;\zeta_{\mathcal{H}}) = n_0 \ln(1 + x^2(1-x)^2/\rho^2)$$





- The **GPD** connects **Φ(z)** with the **EFF** via**:**

$$F_\pi(t) = \int_0^1 dx\, u^{\pi}(x;\zeta_H)\Phi_\pi(z;\zeta_H)$$

- A useful **parametrization** is:

$$\Phi_\pi(z;\zeta_H) = \frac{1+(b_1-1)r_\pi^2/(6<x^2>)z}{1+b_1 r_\pi^2/(6<x^2>)z + b_2 z^2}$$

- Where $\mathbf{r_\pi}$ is taken from **PDG** and $\mathbf{b_{1,2}}$ are parameters to be fitted to the experimental data.

The GPD model

Raya:2021zrz

- We have a **3-parameter** model for the **GPD**: $\{\rho, b_1, b_2\}$

$$H^u_{\mathrm{P}}(x,\xi,t;\zeta_{\mathcal{H}}) = \Theta(x_-)\sqrt{u^{\mathrm{P}}(x_-;\zeta_H)u^{\mathrm{P}}(x_+;\zeta_H)}\Phi_{\mathrm{P}}(z;\zeta_H)$$

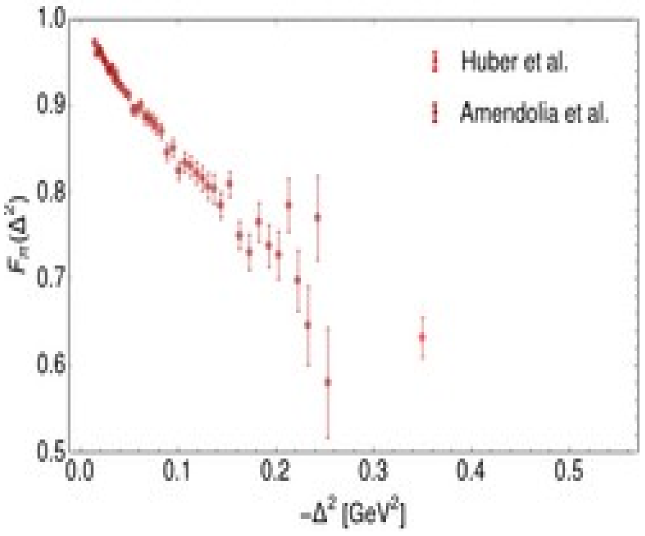
$$u^\pi(x;\zeta_{\mathcal{H}}) = n_0 \ln(1+x^2(1-x)^2/\rho^2) \qquad \Phi_\pi(z;\zeta_H) = \frac{1+(b_1-1)r_\pi^2/(6<x^2>)z}{1+b_1 r_\pi^2/(6<x^2>)z+b_2 z^2}$$

- The **strategy** is as follows:

1) Following the described procedure for the **PDF**, generate a replica "*i*", storing the value ρ_i, and its probability of acceptance $P(\rho_i)$.

2) Using such **replica**, integrate the **GPD** (for ξ=0) using random values of $b_{1,2}$ and varying randomly r_π within the range 0.659 +/- 0.005 fm (in agreement with its **PDG** value).

3) Compute the χ^2_i by comparing with the **EFF** experimental data [Amendolia:1984nz, JeffersonLab:2008jve].



The GPD **model**

Raya:2021zrz

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$$\{\rho, b_1, b_2\}$$

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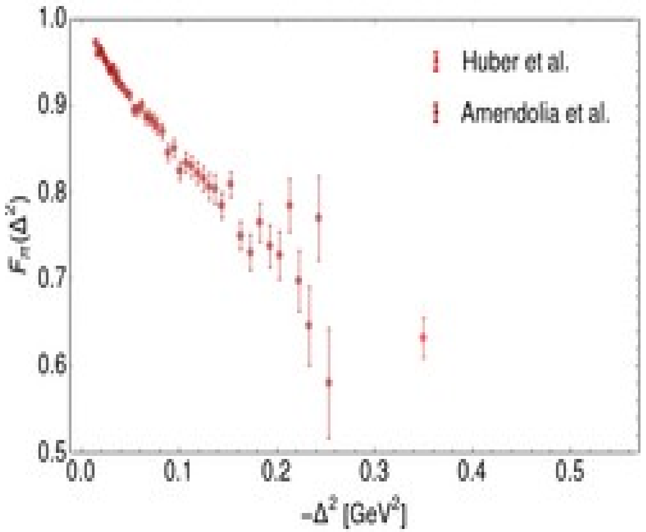
$$u^{\pi}(x;\zeta_{\mathcal{H}}) = n_0 \ln(1+x^2(1-x)^2/\rho^2) \qquad \Phi_{\pi}(z;\zeta_H) = \frac{1+(b_1-1)r_{\pi}^2/(6<x^2>)z}{1+b_1 r_{\pi}^2/(6<x^2>)z+b_2 z^2}$$

- The **strategy** is as follows:

4) Use χ^2_i to **calculate** $P(\{b_1^i, b_2^i\}|\rho_i)$

Subsequently, accept the set of parameters with probability:

$$P(\{\rho_i, b_1^{(i)}, b_2^{(i)}\}) = P(\{b_1^{(i)}, b_2^{(i)}\}|\rho_i)P(\rho_i)$$



Repeat.

The GPD from

$$H^u_{\mathbf{P}}(x,\xi,t;\zeta_{\mathcal{H}}) = \Theta(x_-)\sqrt{u^{\mathbf{P}}(x_-;\zeta_H)u^{\mathbf{P}}(x_+;\zeta_H)}\,\Phi_{\mathbf{P}}(z;\zeta_H)$$

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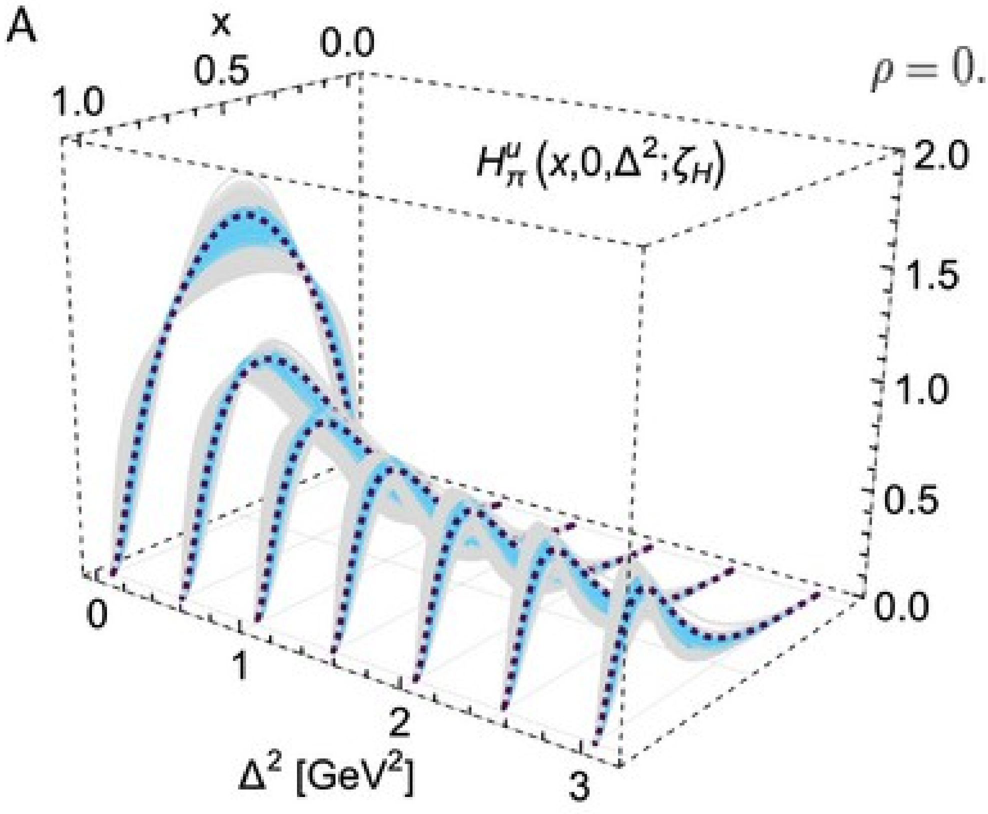
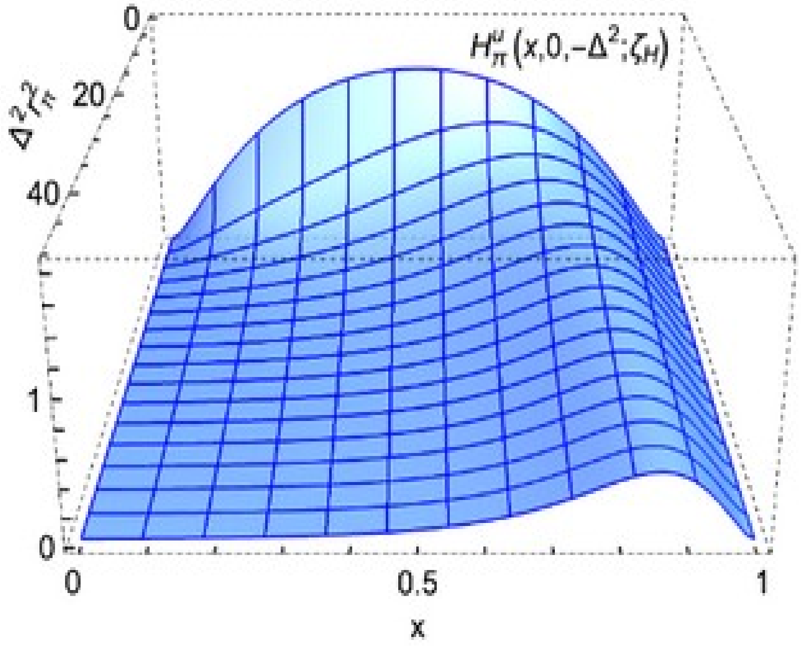
- Combining **pion PDF** data (**ASV**) and **pion EFF** data, one arrives at:



$$\rho = 0.07\pm0.03,\ b_1 = 0.46\pm0.40,\ b_2 = 18.67\pm4.38$$

(with proper mass units)



Numerical **Results**

$$H^u_{\mathbf{P}}(x,\xi,t;\zeta_{\mathcal{H}}) = \Theta(x_-)\sqrt{u^{\mathbf{P}}(x_-;\zeta_H)u^{\mathbf{P}}(x_+;\zeta_H)}\Phi_{\mathbf{P}}(z;\zeta_H)$$

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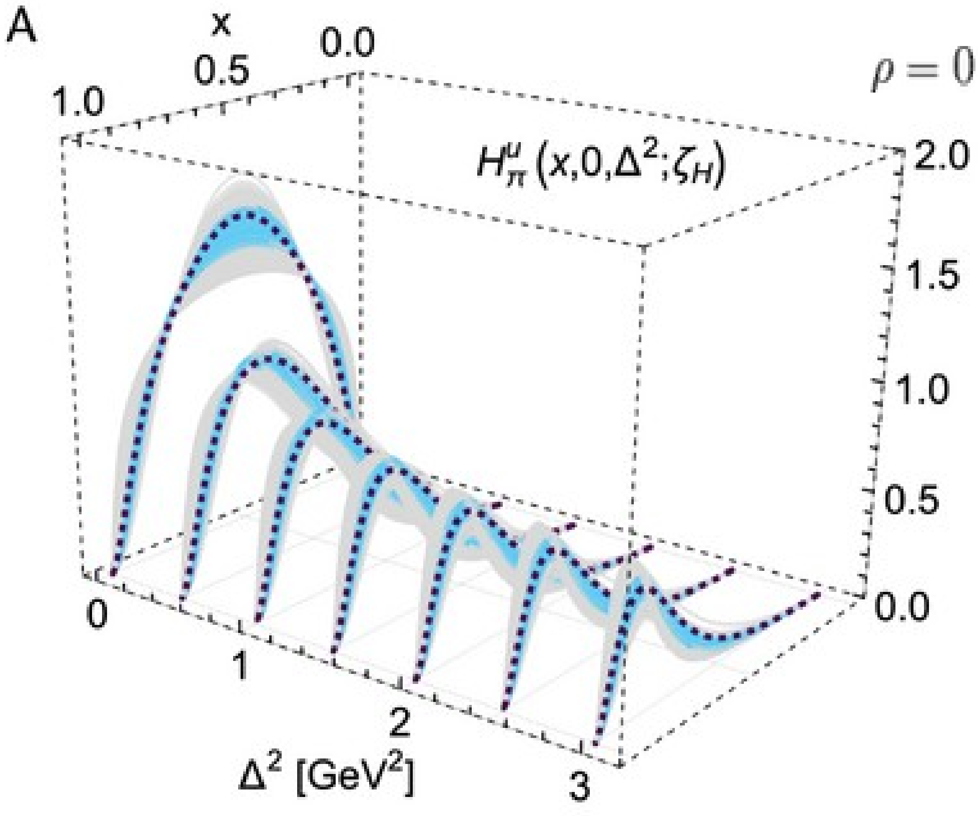

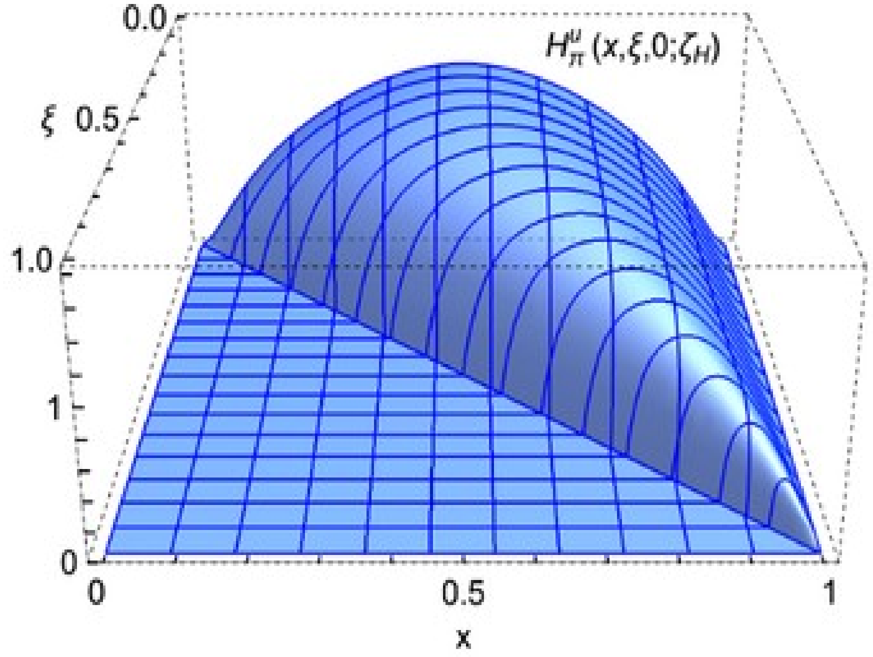

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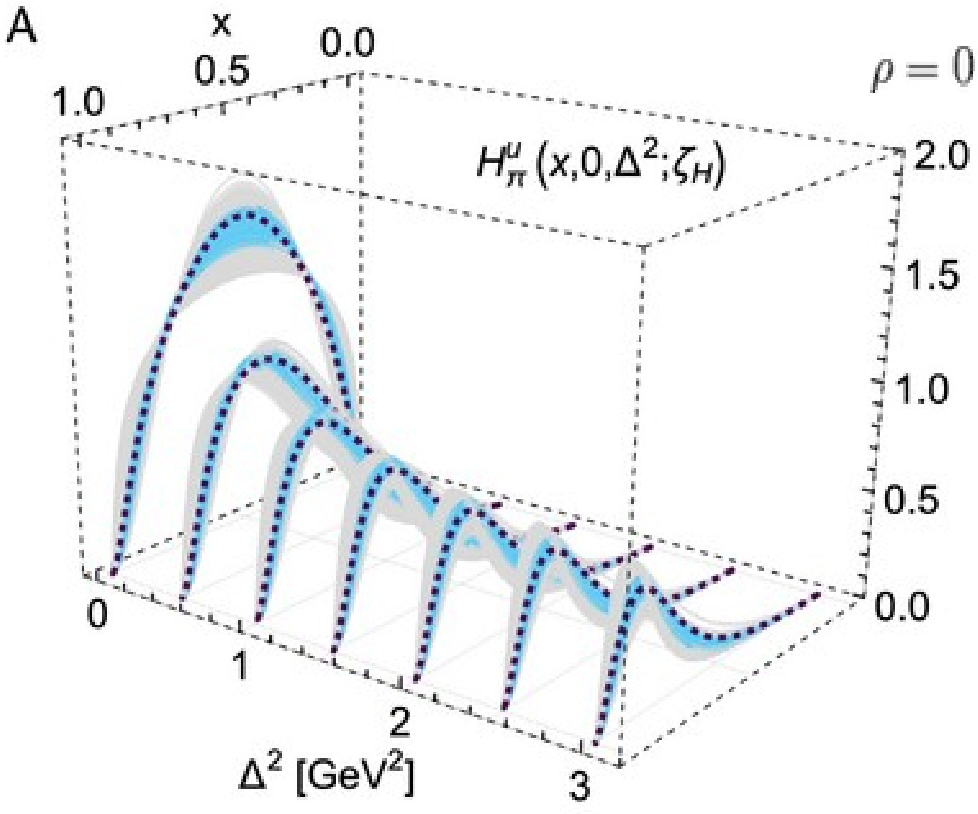

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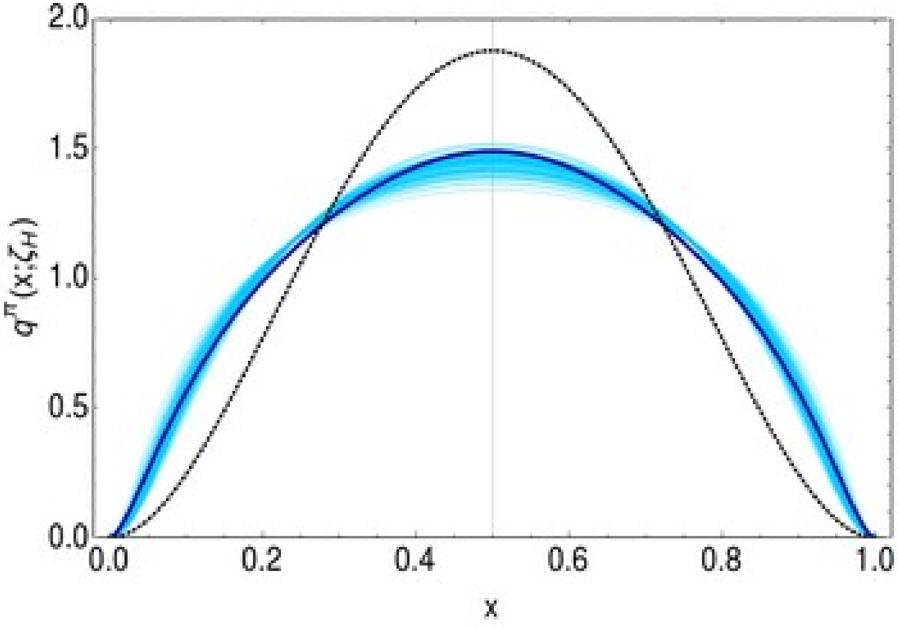

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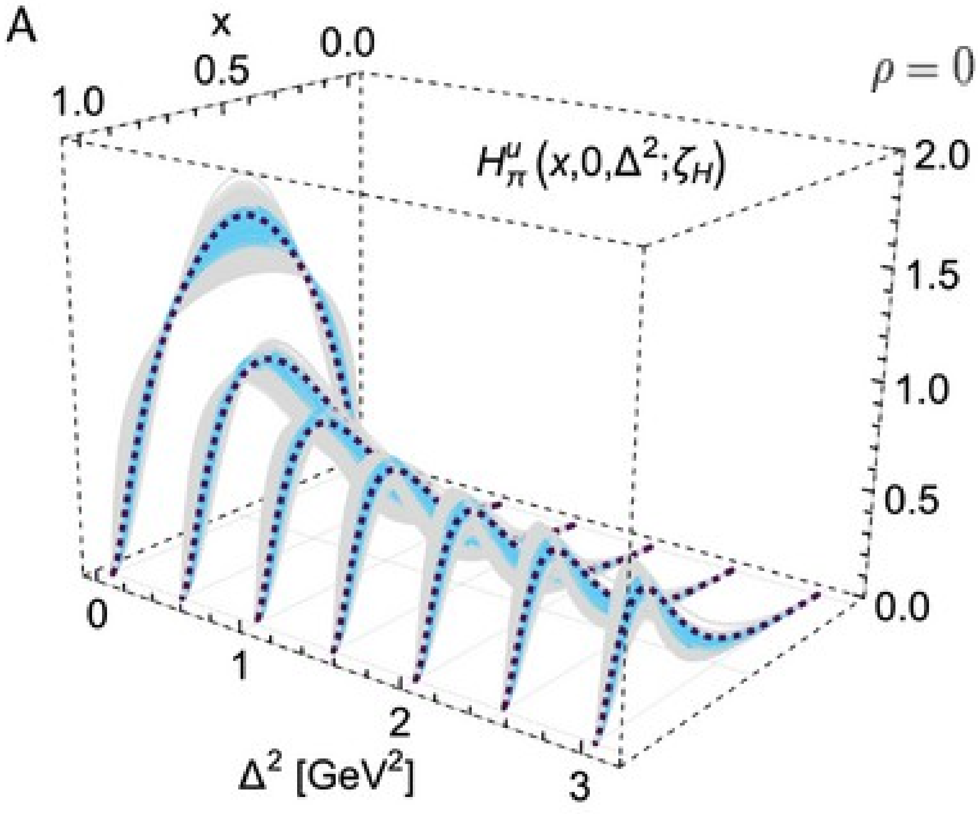

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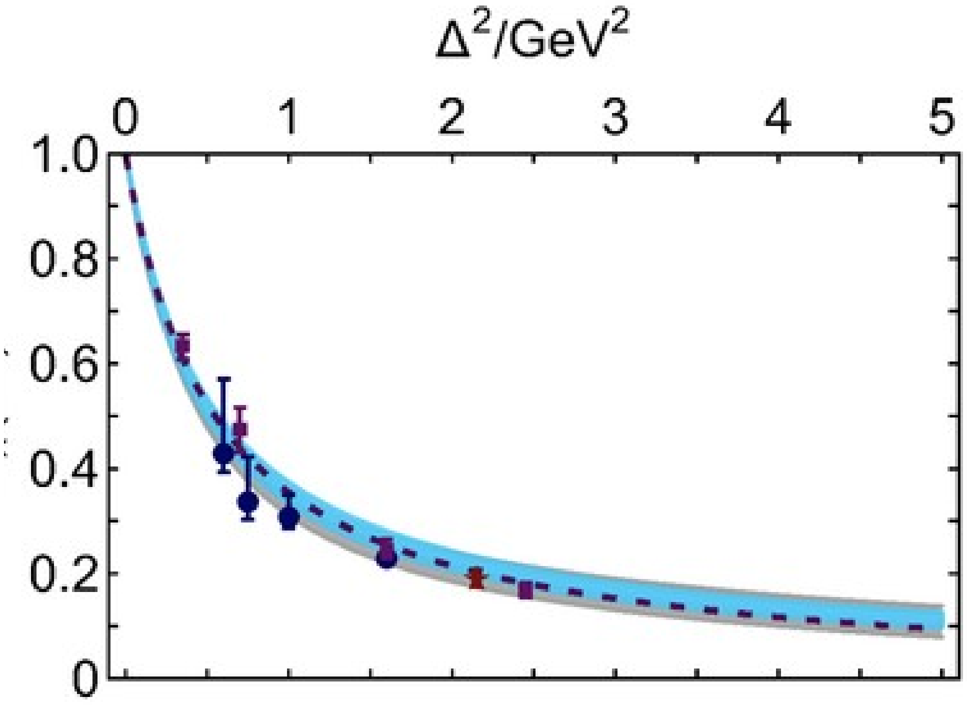

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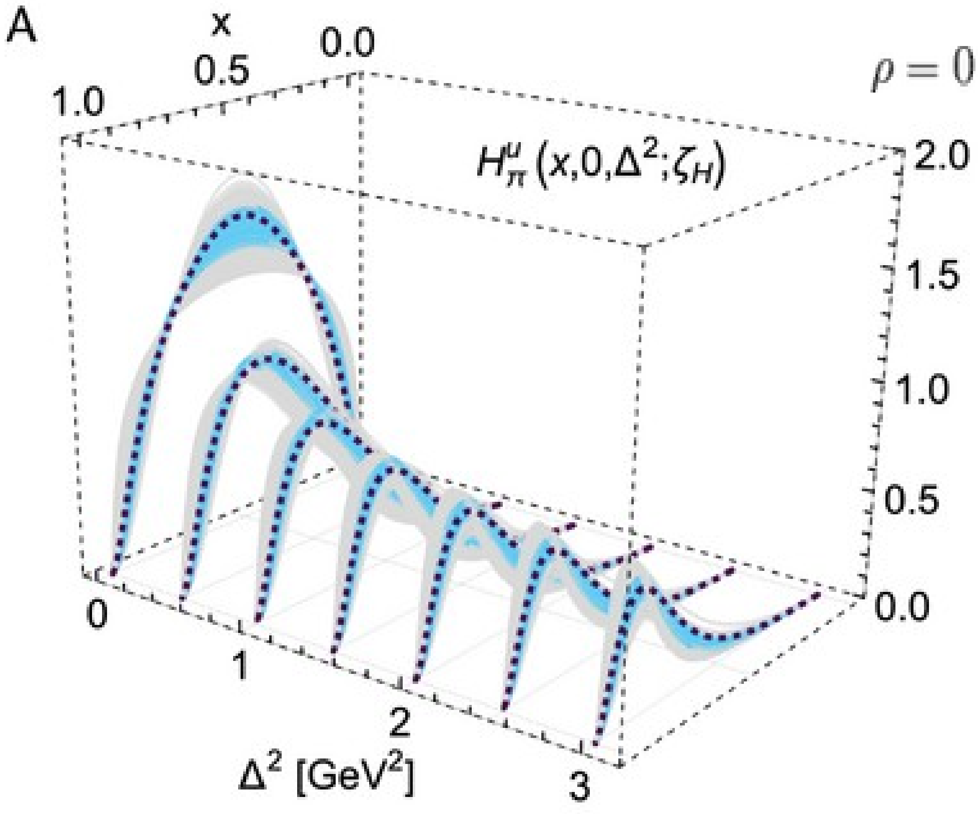

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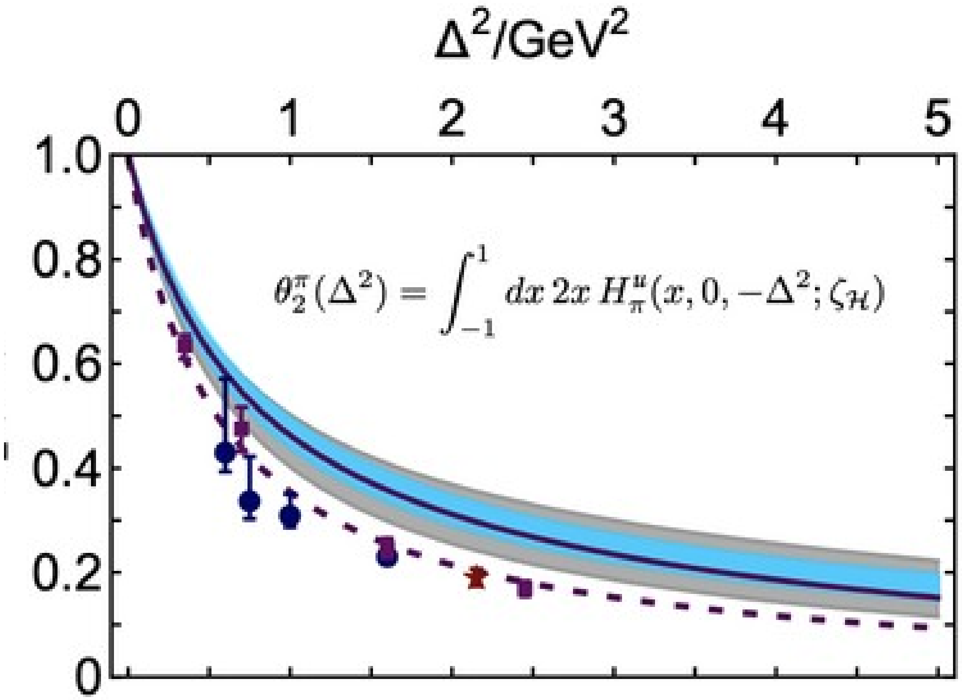
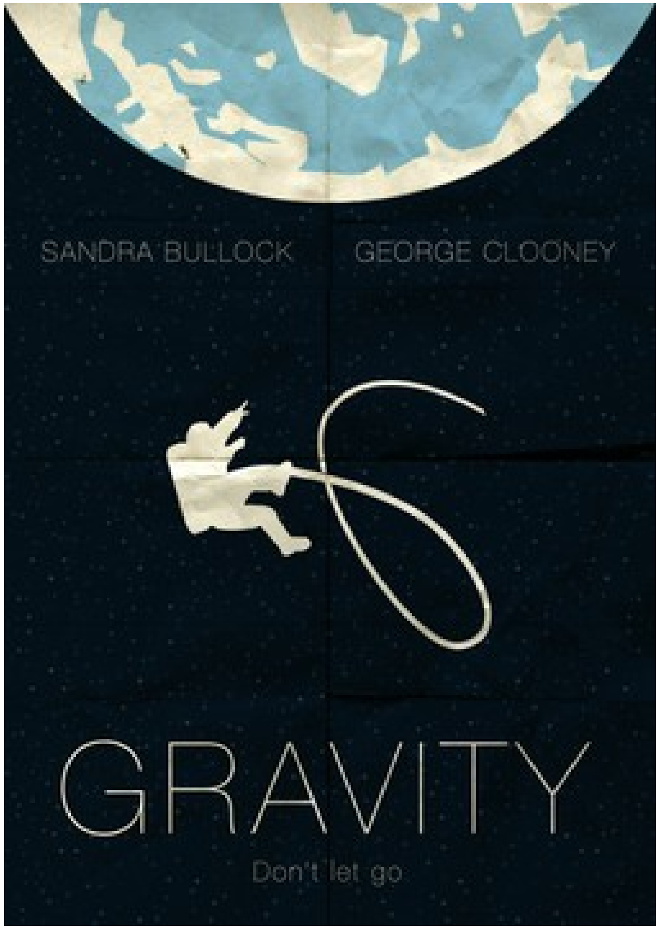


Gravitational form factors



Gravitational form factors

- For a given parton class, the **spin-0** energy-momentum tensor (**EMT**) can take the following form:

$$\underbrace{\Lambda^{a}_{\mu\nu}(P,Q)}_{\langle P_f|T_{\mu\nu}(0)|P_i\rangle} = 2P_\mu P_\nu \theta^{a}_2(Q^2) + \frac{1}{2}\left(Q^2 g_{\mu\nu} - Q_\mu Q_\nu\right)\theta^{a}_1(Q^2) + 2m_\pi^2 g_{\mu\nu}\bar{c}^{a}(Q^2)$$

With: $P = [P_f + P_i]/2 \text{ and } Q = P_f - P_i$

- Such that $\theta_{1,2}(Q^2),\ \bar{c}(Q^2)$ define the so called **gravitational form factors** (**GFFs**).

They can be extracted with the appropriate projections.
Particularly:

$$\theta_{1,2}(Q^2) = P^{\mu\nu}_{1,2}\Lambda_{\mu\nu}(P,Q)$$

With:

$$\boxed{\begin{aligned} P_2^{\mu\nu} &= \frac{3P^\mu P^\nu}{4P^4} + \frac{Q^\mu Q^\nu - Q^2 g^{\mu\nu}}{4P^2Q^2} \\ P_1^{\mu\nu} &= -\frac{P^\mu P^\nu}{P^2Q^2} - \frac{3\left(Q^\mu Q^\nu - Q^2 g^{\mu\nu}\right)}{Q^4} - \frac{2g^{\mu\nu}}{Q^2} \end{aligned}}$$

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p(r) : pressure
s(r) : shear forces

$$T^{ij}_q(\vec{r}) = p_q(r)\,\delta_{ij} + s_q(r)\left(\frac{r_i r_j}{r^2} - \frac{1}{3}\delta_{ij}\right) \Rightarrow \theta_1(Q^2)$$

Is connected with the **mechanical** properties of the hadron

$$\int d^3r\, T^{00}_q(\vec{r}) = m_\pi \Theta_{2,q}(0) \Rightarrow \theta_2(Q^2)$$

connected with the **mass** distribution inside the hadron

M. Polyakov, Phys.Lett. B555 (2003) 56-62
M. Polyakov, P. Schweitzer, Int. J. Mod. Phys. A 33 (2018) 1830025

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With: $P = [P_f + P_i]/2 \text{ and } Q = P_f - P_i$

- Such that $\theta_{1,2}(Q^2),\ \bar{c}(Q^2)$ define the so called **gravitational form factors** (**GFFs**).

- Energy-momentum **conservation** entails the following **sum rules**:

$$\sum_{q,g}\theta_2(0) = 1 \qquad \sum_{q,g}\bar{c}(t) = 0$$

- While, in the **chiral limit**, the **soft-pion theorem** constraints:

$$\sum_{q,g}\theta_1(0) = 1$$

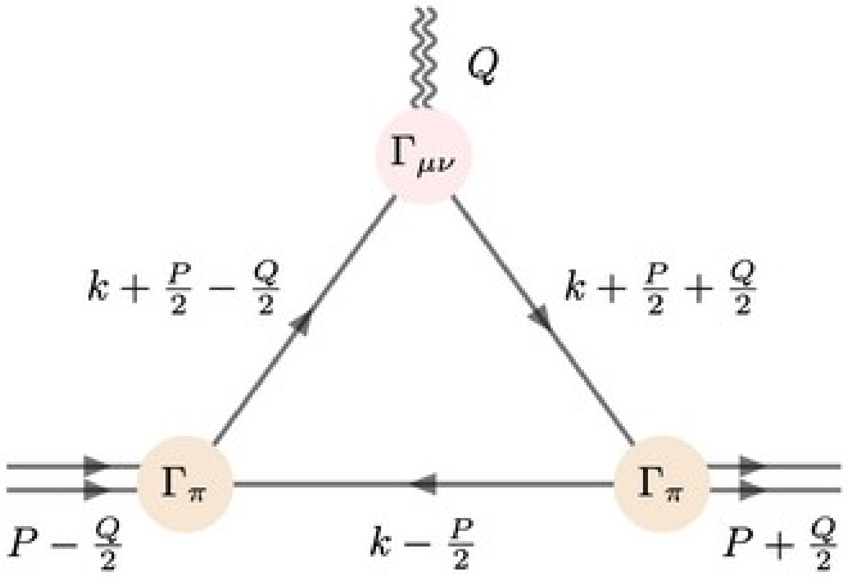
Gravitational form factors: **CSM**

- For a given parton class, the **spin-0** energy-momentum tensor (**EMT**) can take the following form:

$$\underbrace{\Lambda_{\mu\nu}(P,Q)}_{} = 2P_\mu P_\nu \theta_2(Q^2) + \frac{1}{2}\left(Q^2 g_{\mu\nu} - Q_\mu Q_\nu\right)\theta_1(Q^2) + 2m_\pi^2 g_{\mu\nu}\bar{c}(Q^2)$$

In pion's case, both u- and d-in-π valence-quark contributions are the same

$$\Lambda_{\mu\nu}(P,Q) = N_c \int_{dk} \mathrm{Tr}\left[\Gamma_\pi\left(k-\frac{Q}{4}, P-\frac{Q}{2}\right) S\left(k-\frac{P}{2}\right)\Gamma_\pi\left(k+\frac{Q}{4}, P+\frac{Q}{2}\right)\right.$$
$$\left. S\left(k+\frac{P}{2}+\frac{Q}{2}\right)\Gamma_{\mu\nu}\left(k+\frac{P}{2}, Q\right) S\left(k+\frac{P}{2}-\frac{Q}{2}\right)\right]$$



+ beyond **I.A.**

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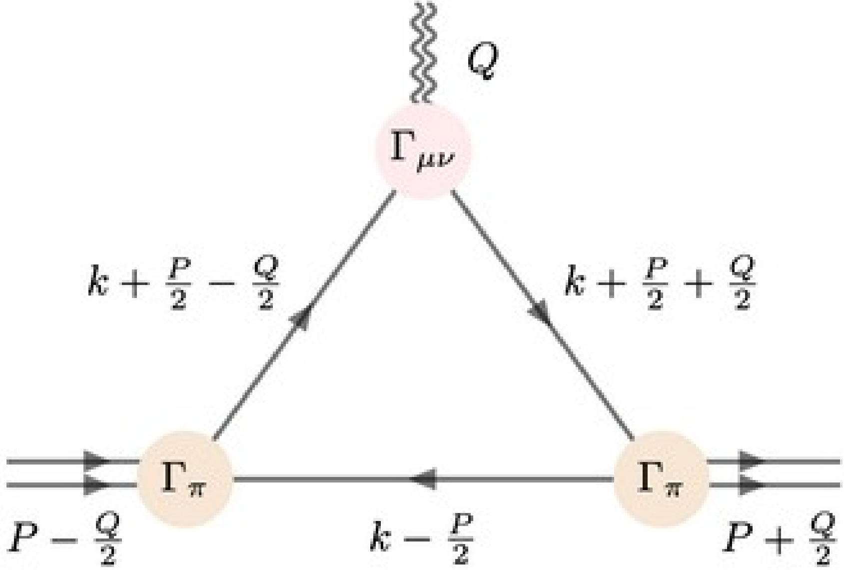
In pion's case, both u- and d-in-π valence-quark contributions are the same

$$\Lambda_{\mu\nu}(P,Q) = N_c \int_{dk} \mathrm{Tr}\left[\Gamma_\pi\left(k-\frac{Q}{4}, P-\frac{Q}{2}\right) S\left(k-\frac{P}{2}\right) \Gamma_\pi\left(k+\frac{Q}{4}, P+\frac{Q}{2}\right)\right.$$

$$\left. S\left(k+\frac{P}{2}+\frac{Q}{2}\right) \Gamma_{\mu\nu}\left(k+\frac{P}{2}, Q\right) S\left(k+\frac{P}{2}-\frac{Q}{2}\right)\right]$$



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EM conservation implies:

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Gravitational form factors: CSM

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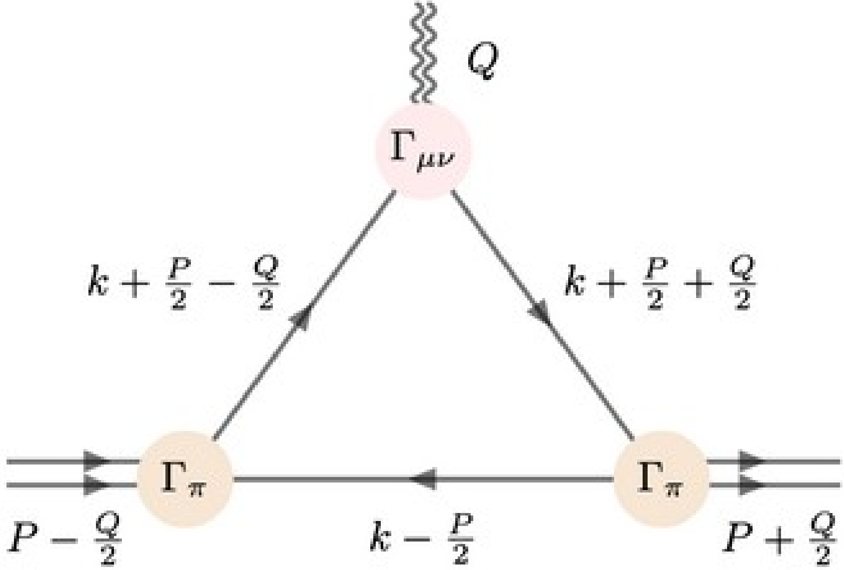

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Also note:

$$\boxed{Q_\mu \Lambda_{\mu\nu}(P,Q) \sim \bar{c}(Q^2)}$$

Thus restricting the structure of contributions beyond I.A.

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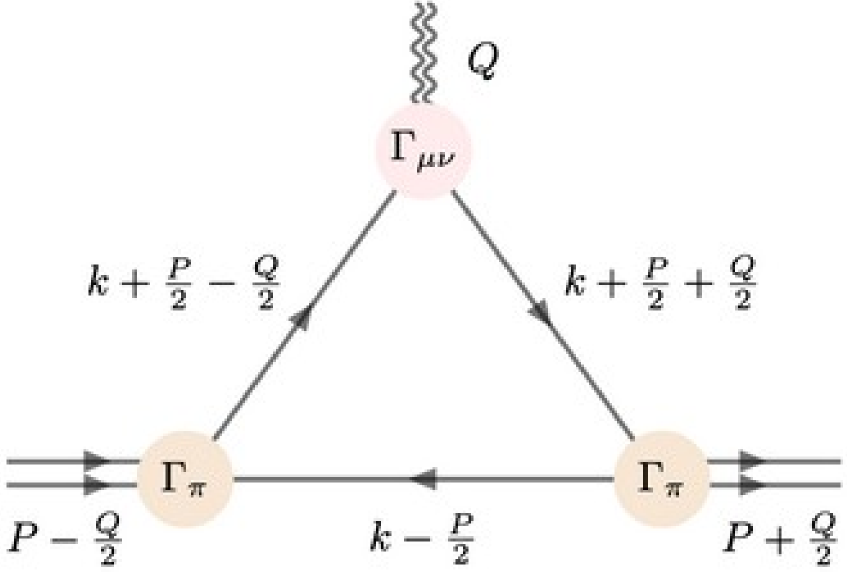

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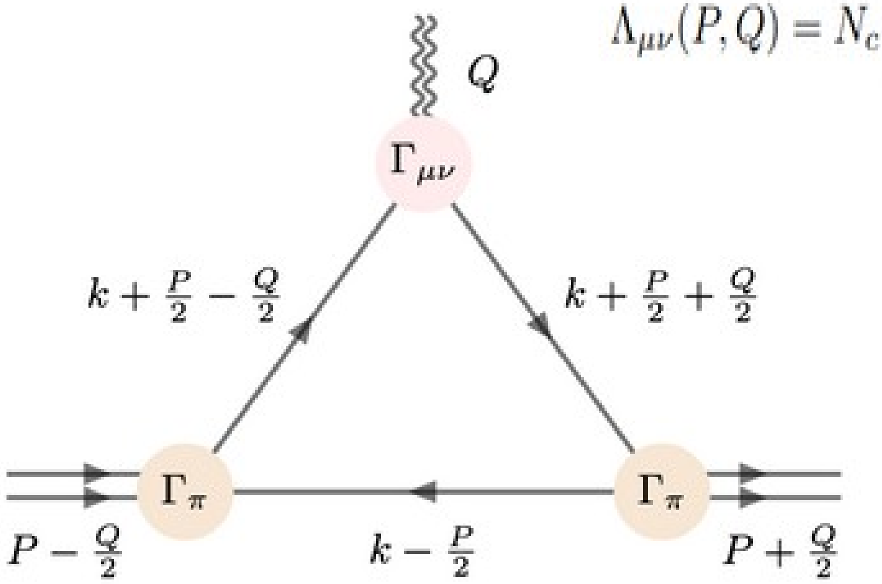
$$\boxed{Q_\mu \Lambda_{\mu\nu}(P,Q) \sim \bar{c}(Q^2)}$$

Thus restricting the structure of contributions beyond I.A.

Remark: the graviton-quark vertex obeys a tensor WGTI making it to rely on the quark propagators and such that $\bar{c}(Q^2)$ is irrespective of it.

$$\boxed{iQ_\mu \Gamma^{\mu\nu}(P,Q) = P_i^\nu S^{-1}(P_f) - P_f^\nu S^{-1}(P_i)}$$

Gravitational form factors: CSM ingredients



$$\Lambda_{\mu\nu}(P,Q) = N_c \int_{dk} \mathrm{Tr}\left[\Gamma_\pi\left(k-\frac{Q}{4}, P-\frac{Q}{2}\right) S\left(k-\frac{P}{2}\right) \Gamma_\pi\left(k+\frac{Q}{4}, P+\frac{Q}{2}\right)\right.$$
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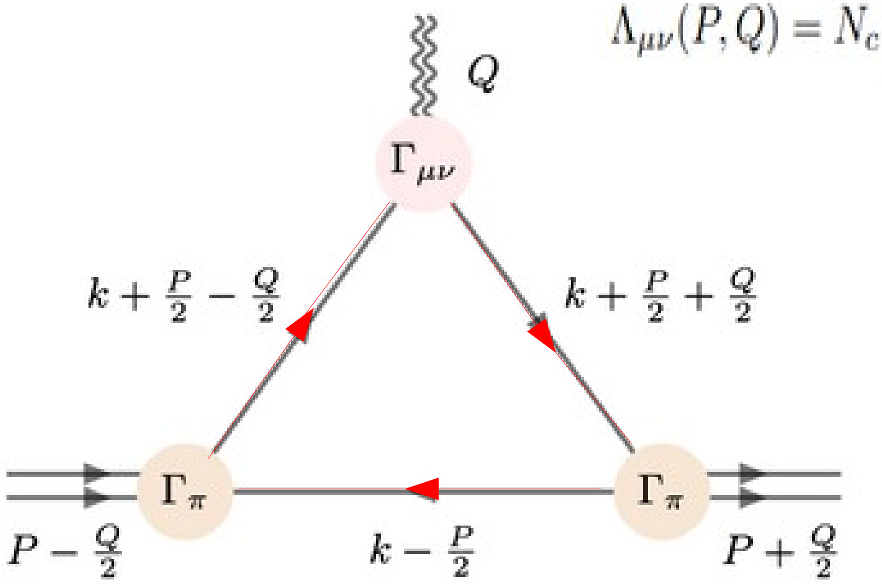
+ beyond **I.A.**

Gravitational form factors: CSM ingredients

$$\Lambda_{\mu\nu}(P,Q) = N_c \int_{dk} \mathrm{Tr}\left[\Gamma_\pi\left(k-\frac{Q}{4},P-\frac{Q}{2}\right)\boxed{S\left(k-\frac{P}{2}\right)}\Gamma_\pi\left(k+\frac{Q}{4},P+\frac{Q}{2}\right)\right.$$

$$\left.\boxed{S\left(k+\frac{P}{2}+\frac{Q}{2}\right)}\Gamma_{\mu\nu}\left(k+\frac{P}{2},Q\right)\boxed{S\left(k+\frac{P}{2}-\frac{Q}{2}\right)}\right]$$

+ beyond **I.A.**



Solutions of the quark gap equation in RL with a realistic quark-gluon interaction.

Gravitational form factors: CSM ingredients

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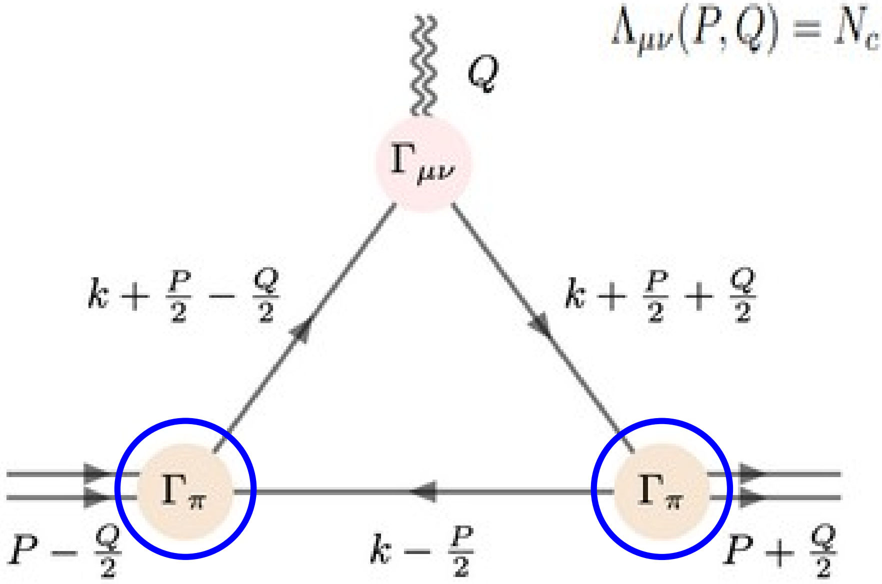
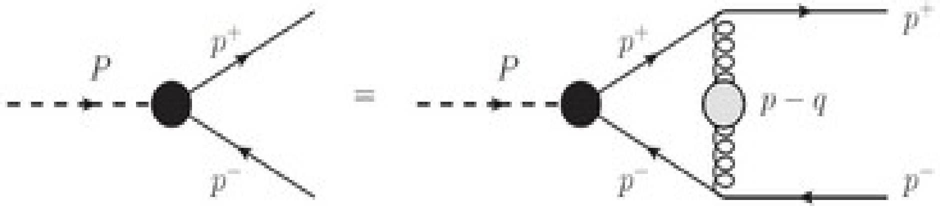


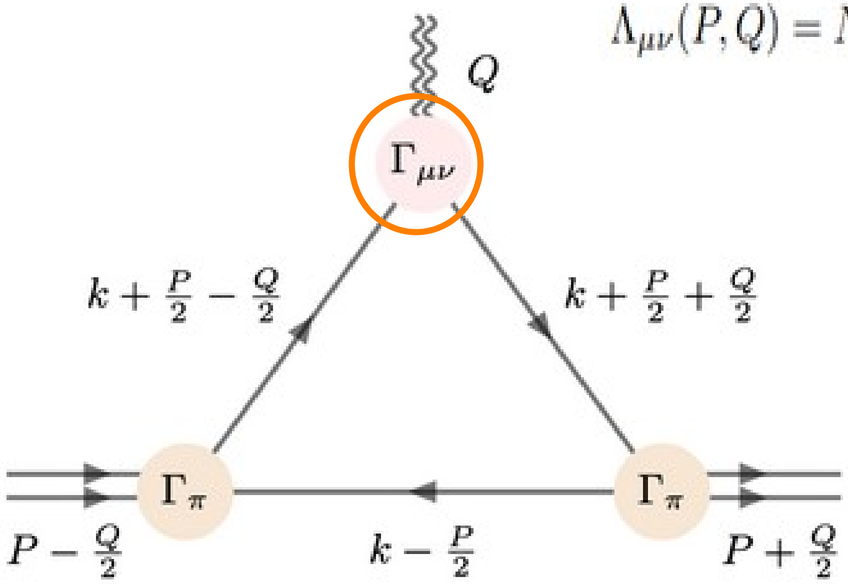
Solutions of the quark gap equation in RL with a realistic quark-gluon interaction.

Solutions of the Bethe-Salpeter equation with the corresponding RL kernel, derived from the realistic quark-gluon interaction.



The interaction parameters are properly fixed such that: $m_\pi = 0.14\,, m_K = 0.49\,, f_\pi = 0.095\,, f_K = 0.116\,[GeV]$ but one can also consistently compute with an effective interaction relying on the PI effective charge.

Quark-tensor vertex



$$\Lambda_{\mu\nu}(P,Q) = N_c \int_{dk} \mathrm{Tr}\left[\Gamma_\pi\left(k-\frac{Q}{4},P-\frac{Q}{2}\right) S\left(k-\frac{P}{2}\right)\Gamma_\pi\left(k+\frac{Q}{4},P+\frac{Q}{2}\right)\right.$$
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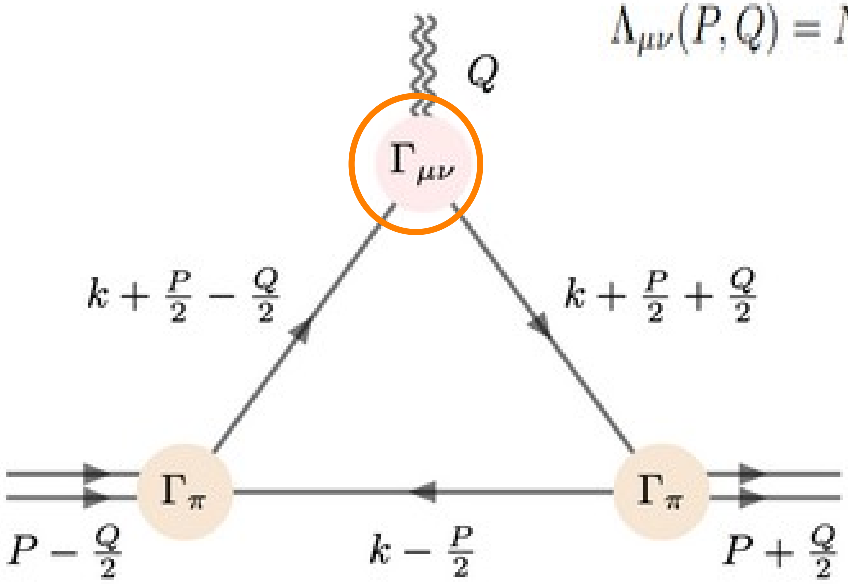
Quark-tensor vertex (QTV): The brand-new ingredient encoding the interaction of a quark and a spin-2 probe. To be determined by its **DSE**, but also its Ward-Green-Takahashi identity (**WGTI**) and other **symmetry properties**.

As the quark-photon vertex (QPV), QTV obeys its own DSE:

$$i\Gamma^{\mu\nu}(P,Q) = \underbrace{i\Gamma_0^{\mu\nu}(P,Q)}_{\text{Tree-level}} + \int \underbrace{K^{(2)}(P,Q|P',Q')}_{\text{IA Kernel}}\, i\Gamma^{\mu\nu}(P',Q') + \underbrace{\Delta^{\mu\nu}(P,Q)}_{\text{Symmetry-restoring term}}$$

$$i\Gamma_0^{\mu\nu}(P,Q) = i\gamma^\mu P_i^\nu - g^{\mu\nu} S_0^{-1}(P_i)$$

Quark-tensor vertex



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$$i\Gamma_0^{\mu\nu}(P,Q) = i\gamma^\mu P_i^\nu - g^{\mu\nu} S_0^{-1}(P_i)$$

...and its own WGTI, constraining its structure from symmetry principles:

$$\boxed{iQ_\mu\Gamma^{\mu\nu}(P,Q) = P_i^\nu S^{-1}(P_f) - P_f^\nu S^{-1}(P_i)}$$

Quark-tensor **vertex**

A symmetry preserving quark-tensor vertex can be minimally built as:

$$i\Gamma^{\mu\nu}(P,Q) = i\Gamma^{\mu}_{L}(P,Q)P_i^{\nu} - g^{\mu\nu}S^{-1}(P_i) + i\Gamma^{\mu}_{T}(P,Q)P_i^{\nu} + i\Gamma^{\mu\nu}_{T}(P,Q)$$

$$Q_\mu \Gamma_T^{\mu\nu} = 0$$

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$$Q_\mu\Gamma^{\mu\nu}_{T} = 0$$

This part being fully determined by the **quark-propagator** and **QPV**,

$$\Gamma^\mu = \Gamma^\mu_L + \Gamma^\mu_T\,, \qquad Q_\mu\Gamma^\mu_T = 0$$

Obeying its vector WGTI: (the transverse part resulting from the QV IBSE.)

$$\boxed{iQ_\mu\Gamma^\mu_L(P,Q) = S^{-1}(P_f) - S^{-1}(P_i)}$$

$$i\Gamma^{\mu\nu}_{L}(P,Q) = \sum_{i=1}^{14} F_i(P^2,Q^2,P\cdot Q)\,\tau_i^{\mu\nu}(P,Q)$$

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Capitalizing on the latter, we propose the following **minimal representation**:

$$i\Gamma^{\mu\nu}_{L}(P,Q) = \sum_{i=1}^{14} F_i(P^2,Q^2,P\cdot Q)\,\tau^{\mu\nu}_{i}(P,Q)$$

$$i\Gamma^{\mu\nu}_{T}(P,Q) \;=\; F_{15}(P^2,Q^2,P\cdot Q)\,\tau^{\mu\nu}_{15}(P,Q) = \boxed{i\mathbb{1}\left(Q^2 g^{\mu\nu} - Q^\mu Q^\nu\right)} F_{15}(P^2,Q^2,P\cdot Q)$$

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This part being fully determined by the **quark-propagator** and **QPV**,

$$\Gamma^{\mu} = \Gamma_L^{\mu} + \Gamma_T^{\mu}\ , \qquad Q_\mu \Gamma_T^{\mu} = 0$$

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Then we proceed to solve the **QTV IBSE.**

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Setting $\Gamma^{\mu\nu} \equiv \Gamma^{\mu\nu}_L$ is sufficient to produce a sensible result for $\theta_2(Q^2)$, it is convenient do not spoil this outcome.

$$i\Gamma^{\mu\nu}_L(P,Q) = \sum_{i=1}^{14} F_i(P^2,Q^2,P\cdot Q)\, \tau^{\mu\nu}_i(P,Q)$$

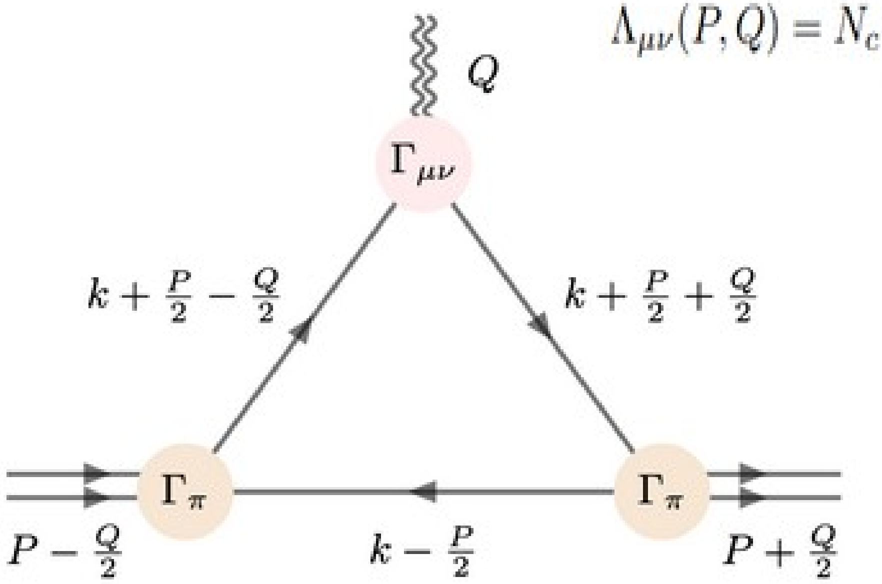
Capitalizing on the latter, we propose the following **minimal representation**:

$$i\Gamma^{\mu\nu}_T(P,Q) \;=\; F_{15}(P^2,Q^2,P\cdot Q)\,\tau^{\mu\nu}_{15}(P,Q) = \boxed{i\mathbb{1}\left(Q^2 g^{\mu\nu} - Q^\mu Q^\nu\right)} F_{15}(P^2,Q^2,P\cdot Q)$$

$$\boxed{i\left(Q^2 g^{\mu\nu} - Q^\mu Q^\nu\right) \otimes \{1,\, \gamma\cdot Q,\, \gamma\cdot K,\, \sigma_{\alpha\beta}K^\alpha Q^\beta\}}$$

Then we proceed to solve the **QTV IBSE.**
Can also consider more general Dirac structures with similar results

Gravitational form factors: Algebraic Model



$$\Lambda_{\mu\nu}(P,Q) = N_c \int_{dk} \mathrm{Tr}\left[\Gamma_\pi\left(k-\frac{Q}{4},P-\frac{Q}{2}\right) S\left(k-\frac{P}{2}\right)\Gamma_\pi\left(k+\frac{Q}{4},P+\frac{Q}{2}\right)\right.$$
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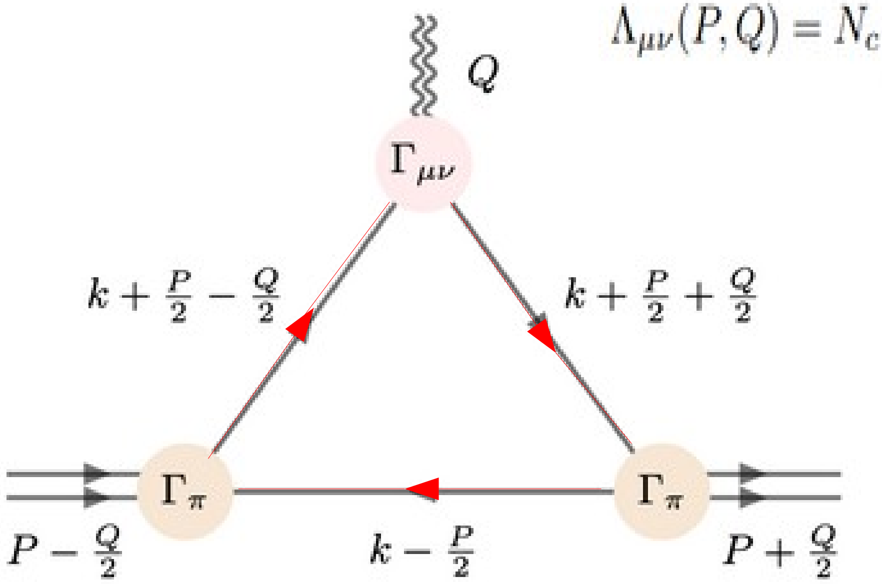
+ beyond **I.A.**

Gravitational form factors: Algebraic Model



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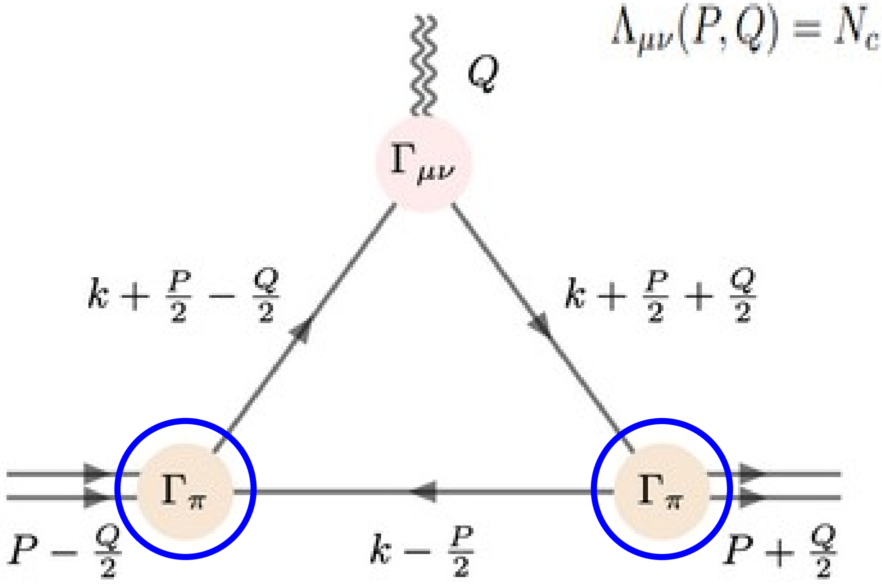
+ beyond **I.A.**

$$S(p) = (-i\gamma\cdot p + M)\Delta_M(p^2)\,,\ \Delta_M(p^2) = (p^2+M^2)^{-1}$$

Gravitational form factors: Algebraic Model



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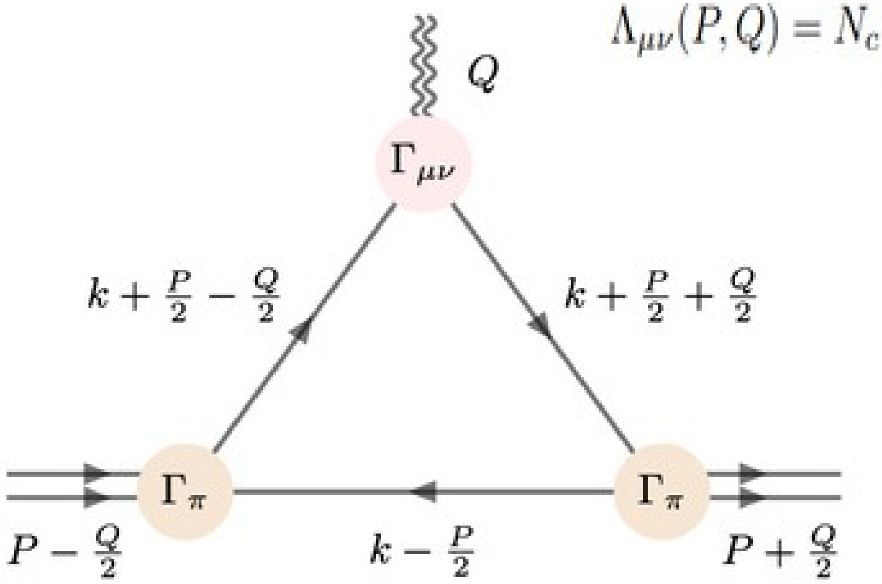
$$\Gamma_\pi(k;P) = i\gamma_5 \int_{-1}^{1} d\omega\, \rho(\omega)\hat{\Delta}_M(k_\omega^2)\,,\ \begin{cases}\hat{\Delta}_M(s) = M^2\Delta_M(s)\\ k_\omega = k + (\omega/2)P\end{cases}$$

Gravitational form factors: Algebraic Model



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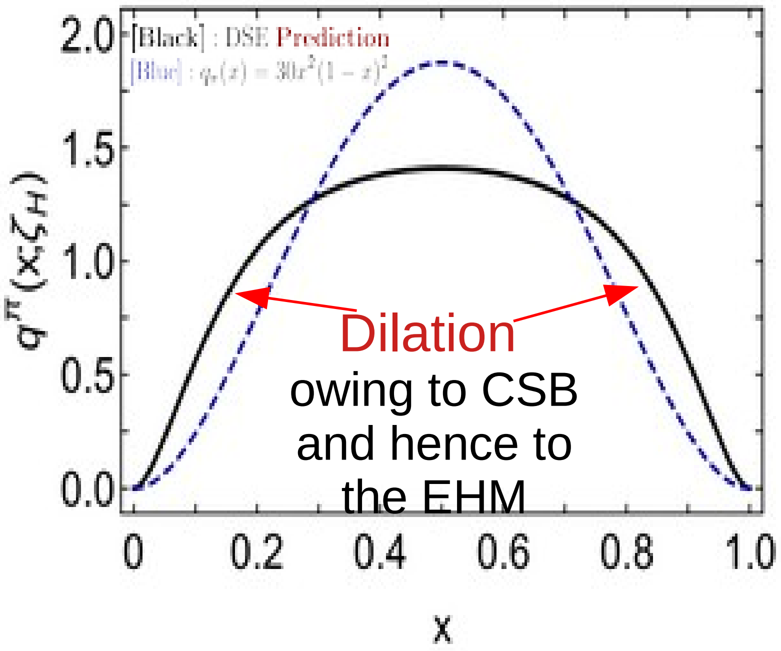
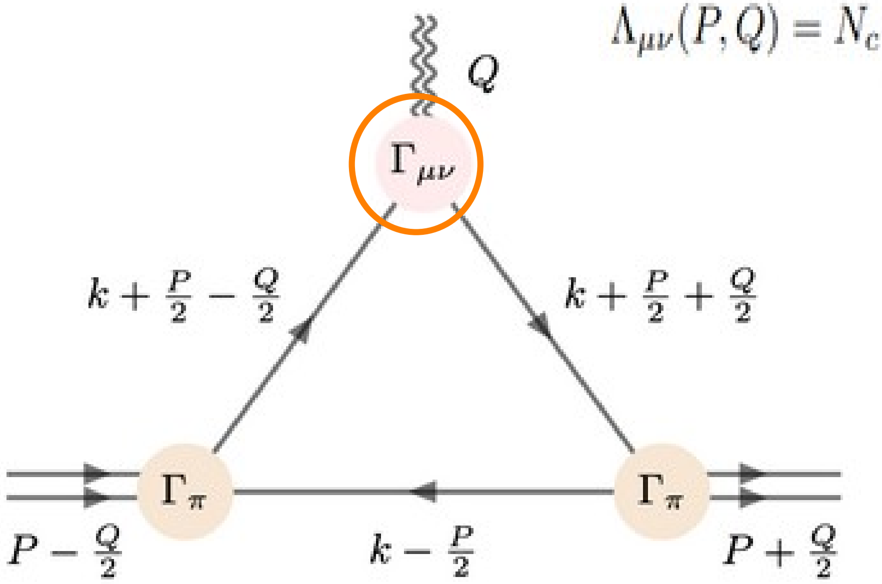


Dilation
owing to CSB
and hence to
the EHM

Gravitational form factors: Algebraic Model



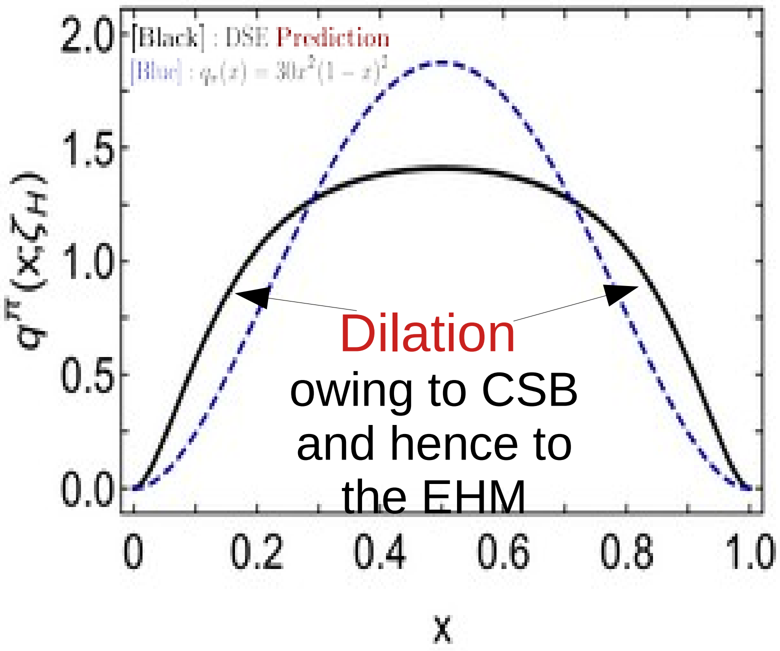
$$\Lambda_{\mu\nu}(P,Q) = N_c \int_{dk} \mathrm{Tr}\left[\Gamma_\pi\left(k-\frac{Q}{4},P-\frac{Q}{2}\right)S\left(k-\frac{P}{2}\right)\Gamma_\pi\left(k+\frac{Q}{4},P+\frac{Q}{2}\right)\right.$$
$$\left. S\left(k+\frac{P}{2}+\frac{Q}{2}\right)\boxed{\Gamma_{\mu\nu}\left(k+\frac{P}{2},Q\right)}S\left(k+\frac{P}{2}-\frac{Q}{2}\right)\right]$$

+ beyond **I.A.**

$$S(p) = (-i\gamma\cdot p + M)\Delta_M(p^2)\,,\ \Delta_M(p^2) = (p^2+M^2)^{-1}$$

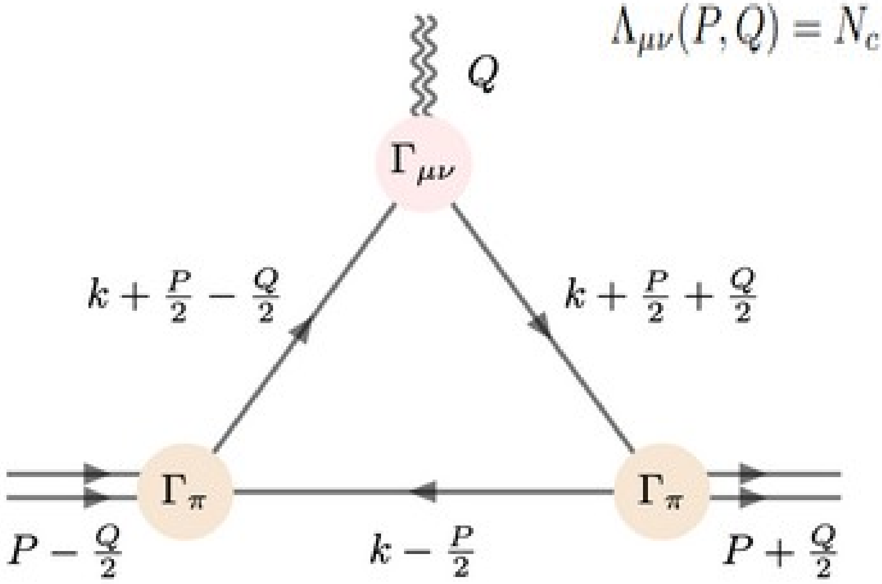
$$\Gamma_\pi(k;P) = i\gamma_5\int_{-1}^{1} d\omega\,\rho(\omega)\hat{\Delta}_M(k_\omega^2)\,,\ \begin{cases}\hat{\Delta}_M(s) = M^2\Delta_M(s)\\ k_\omega = k+(\omega/2)P\end{cases}$$

$$i\Gamma^{\mu\nu} = [i\gamma^\mu p^\nu - g^{\mu\nu}S^{-1}(p)] + i(Q^\mu Q^\nu - Q^2 g^{\mu\nu})F_{15}(k,p)$$



Dilation
owing to CSB
and hence to
the EHM

Gravitational form factors: Algebraic Model



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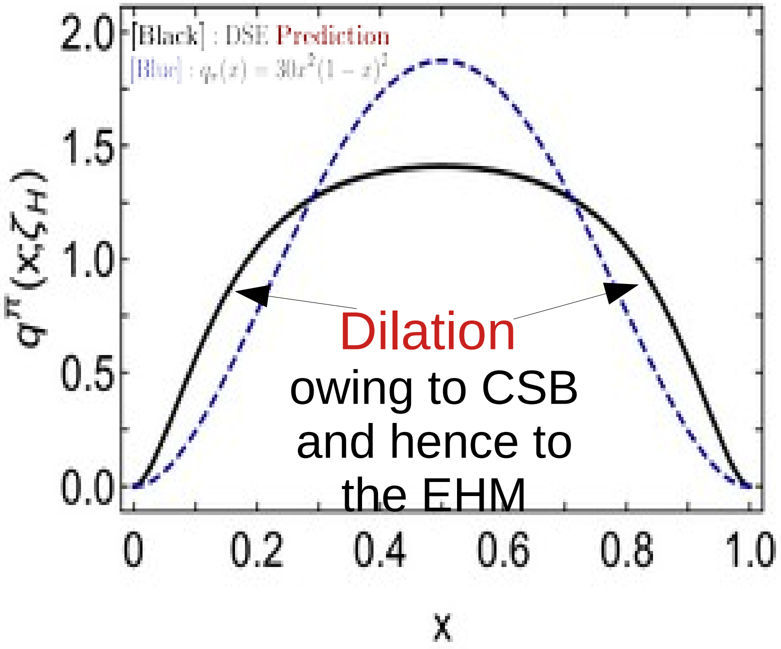
$$F_{15}(k,p) \to F_{15}(Q^2) := \frac{\eta_{15}}{1+Q^2/m_\sigma^2}$$

Inspired by a SCI analysis



Dilation
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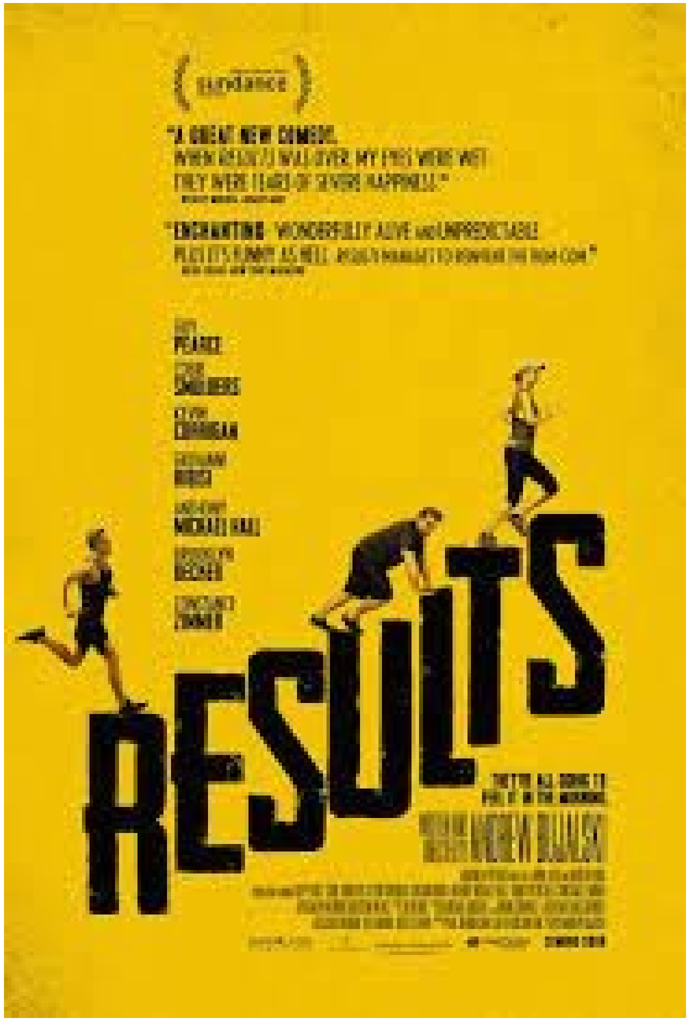
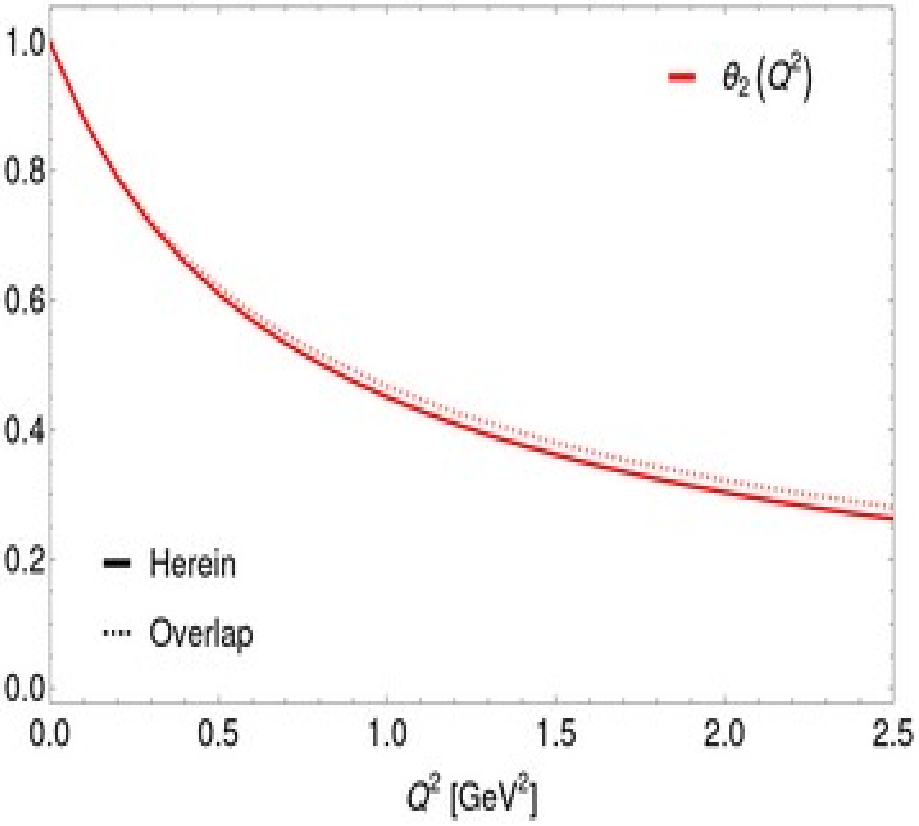

sundance
"A GREAT NEW COMEDY.
WHEN RESULTS WAS OVER, MY EYES WERE WET.
THEY WERE TEARS OF SEVERE HAPPINESS."
"ENCHANTING. WONDERFULLY ALIVE AND UNPREDICTABLE.
PLUS IT'S FUNNY AS HELL. RESULTS MANAGES TO REINVENT THE ROM-COM."
GUY PEARCE
COBIE SMULDERS
KEVIN CORRIGAN
GIOVANNI RIBISI
ANTHONY MICHAEL HALL
BROOKLYN DECKER
CONSTANCE ZIMMER
RESULTS
THEY'RE ALL GOING TO WORK IT IN THE MORNING.
WRITTEN AND DIRECTED BY ANDREW BUJALSKI

Results: Pion’s GFFs

- Recall the **GFFs** are extracted from: $\Lambda_{\mu\nu}(P,Q) = 2P_\mu P_\nu \theta_2(Q^2) + \frac{1}{2}\left(Q^2 g_{\mu\nu} - Q_\mu Q_\nu\right)\theta_1(Q^2) + 2m_\pi^2 g_{\mu\nu}\bar{c}(Q^2)$ 0
- $\theta_2(Q^2)$ Is well described by the part of the **QTV** that satisfies its **WGTI** alone:

$$iQ_\mu \Gamma^{\mu\nu}(P,Q) = P_i^\nu S^{-1}(P_f) - P_f^\nu S^{-1}(P_i)$$

Which is fully determined by the **QPV** and the **quark propagator**



$$i\Gamma^{\mu\nu}(P,Q) = \underbrace{i\Gamma_L^{\mu}(P,Q)P_i^\nu - g^{\mu\nu}S^{-1}(P_i)}_{i\Gamma_L^{\mu\nu}(P,Q)} + i\Gamma_T^{\mu}(P,Q)P_i^\nu$$

Overlap: Result obtained via the computation of the pion **LFWF** and **GPD**

$$\int_{-1}^{1} dx\, x\, H_{\mathrm{P}}^{q}(x,\xi,-\Delta^2;\zeta_{\mathcal{H}}) = \theta_2^{\mathrm{P}}(\Delta^2) - \xi^2\theta_1^{\mathrm{P}}(\Delta^2)$$

Raya:2021zrz

Results: Pion’s GFFs

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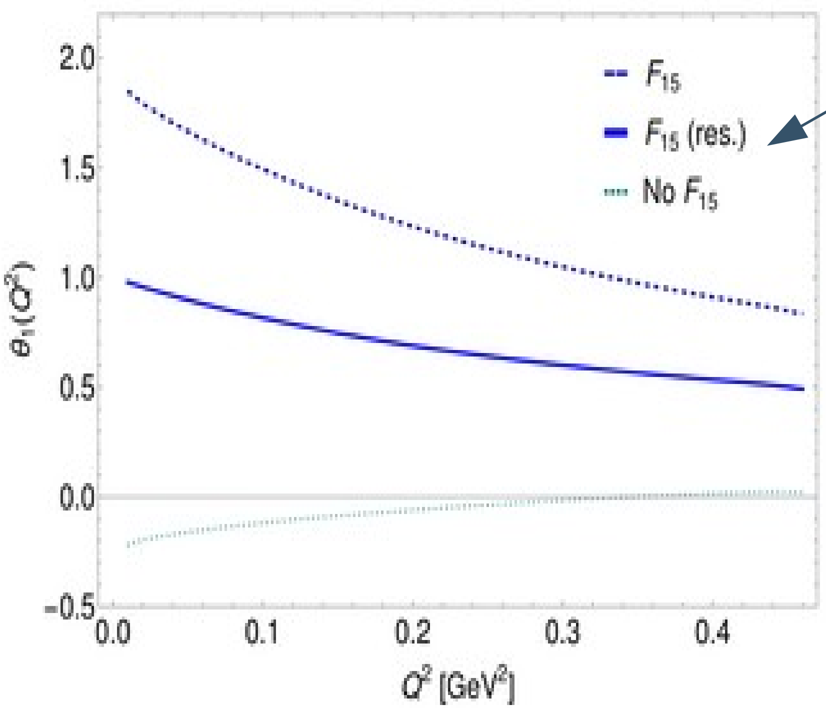
0

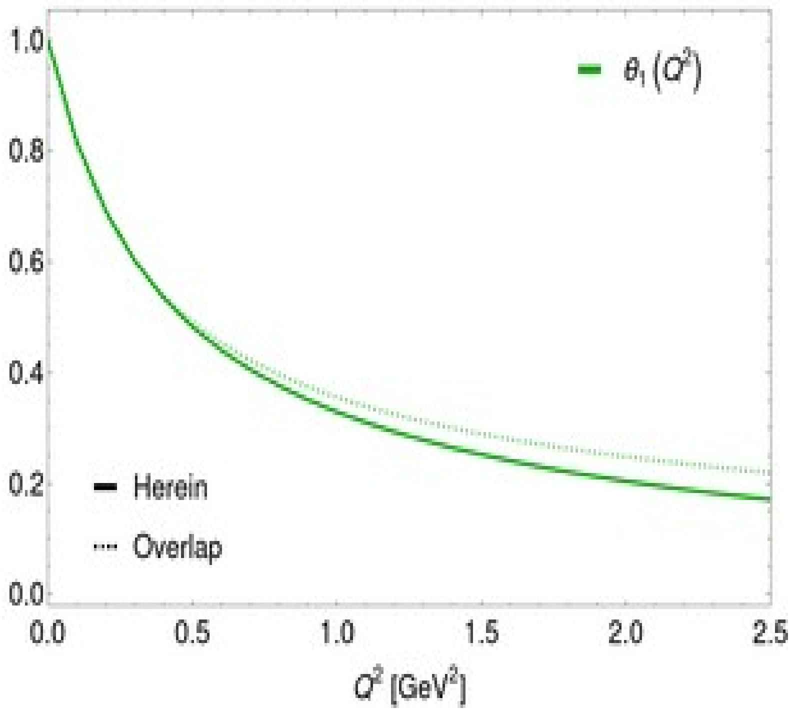
- $\theta_1(Q^2)$ Requires the inclusion of fully transverse pieces in the **QTV;** our *minimal* extension:

$$i\Gamma_T^{\mu\nu}(P,Q) \;=\; F_{15}(P^2,Q^2,P\cdot Q)\,\tau_{15}^{\mu\nu}(P,Q) = i\mathbb{1}\left(Q^2 g^{\mu\nu} - Q^\mu Q^\nu\right) F_{15}(P^2,Q^2,P\cdot Q)$$



Rescaled to account for soft-pion theorem: $\sum_{q,g}\theta_1(0) = 1$

- The **complete** result:



Results: Pion’s GFFs

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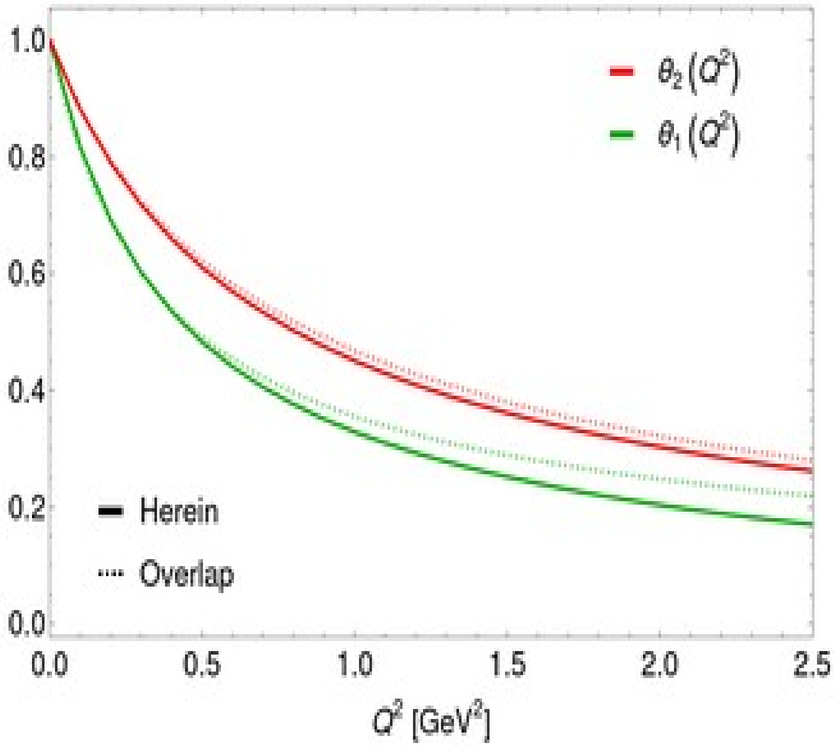
0

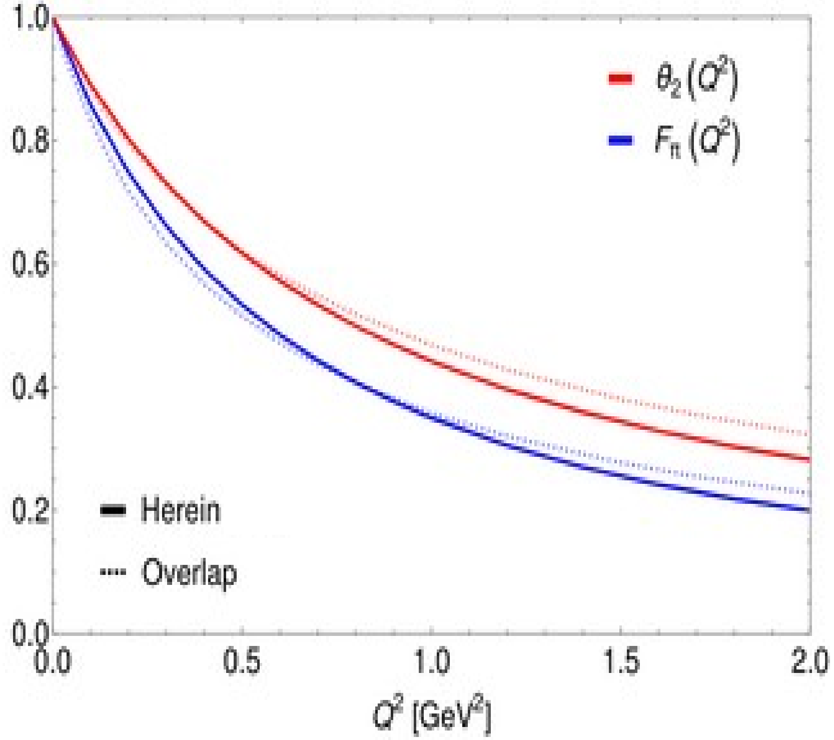
- $\theta_2(Q^2)$ Is harder than $\theta_1(Q^2)$ (and than the pion electromagnetic form factor):



- In fact, one finds: $r_{\theta_2} \approx 0.8 r_\pi$, $r_{\theta_2} \langle r_\pi \langle r_{\theta_1}$

Not an accident! Can be proven via GPD

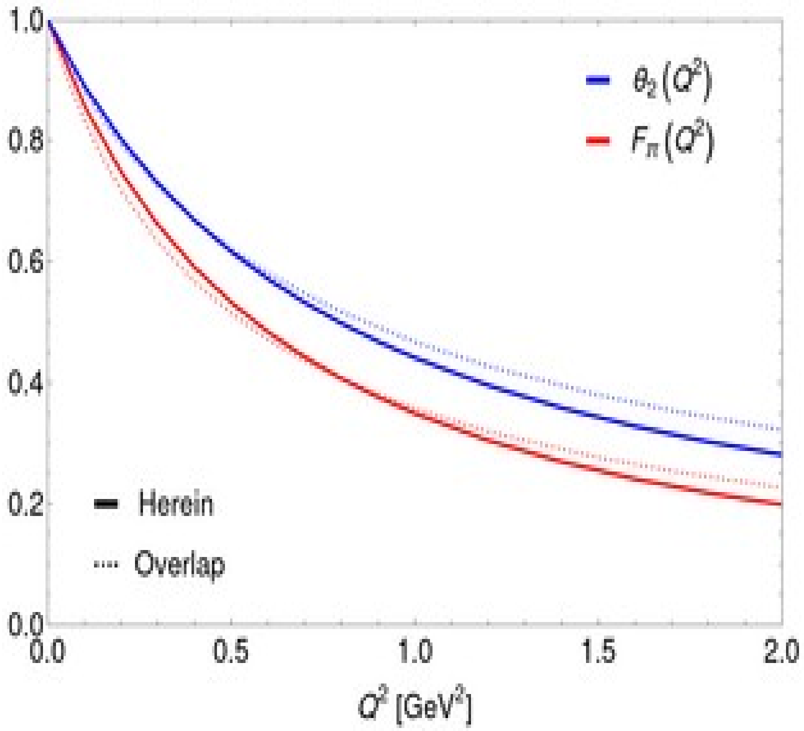
Raya:2021zrz



Overlap: Result obtained via the computation of the pion **LFWF** and **GPD**

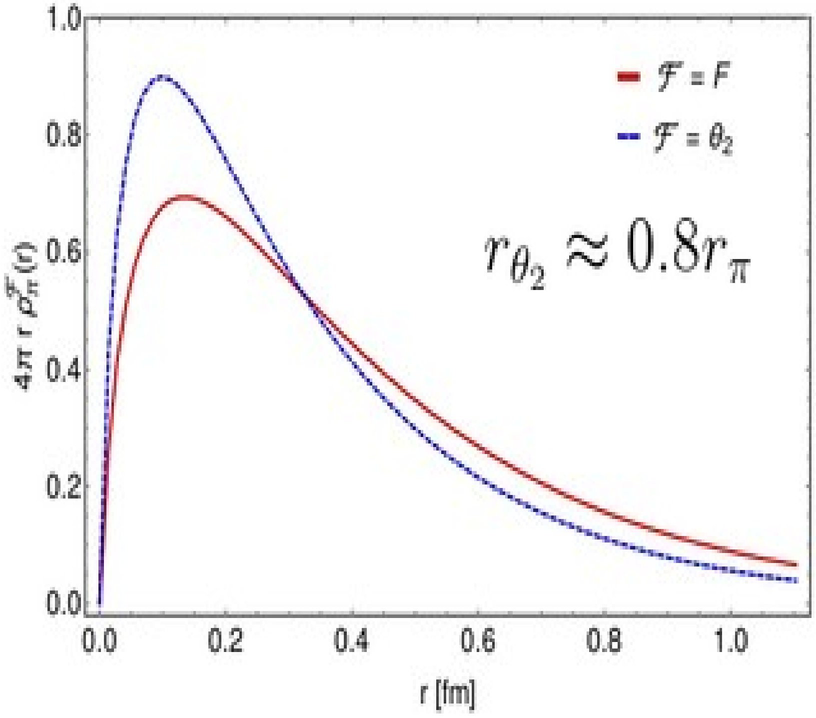
Results: Mass distribution

- Recall the **GFFs** are extracted from: $\Lambda_{\mu\nu}(P,Q) = 2P_\mu P_\nu \theta_2(Q^2) + \frac{1}{2}\left(Q^2 g_{\mu\nu} - Q_\mu Q_\nu\right)\theta_1(Q^2) + 2m_\pi^2 g_{\mu\nu}\cancelto{0}{\bar{c}(Q^2)}$
- $\theta_2(Q^2)$ Is harder than $\theta_1(Q^2)$ (and than the pion electromagnetic form factor):



→ The **charge** and **mass** distributions:

$$\rho_{\mathbf{P}}^{\mathfrak{F}}(r) = \frac{1}{2\pi}\int_0^\infty d\Delta \Delta J_0(\Delta r)\mathfrak{F}_{\mathbf{P}}(\Delta^2)$$



$$r_{\theta_2} \approx 0.8 r_\pi$$

Overlap: Result obtained via the computation of the pion **LFWF** and **GPD**

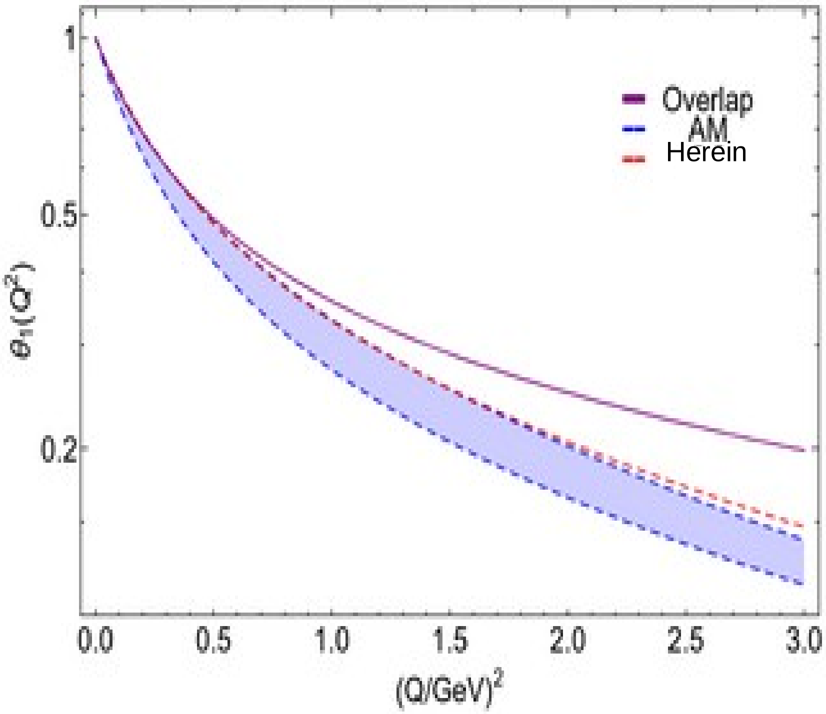
`Raya:2021zrz`

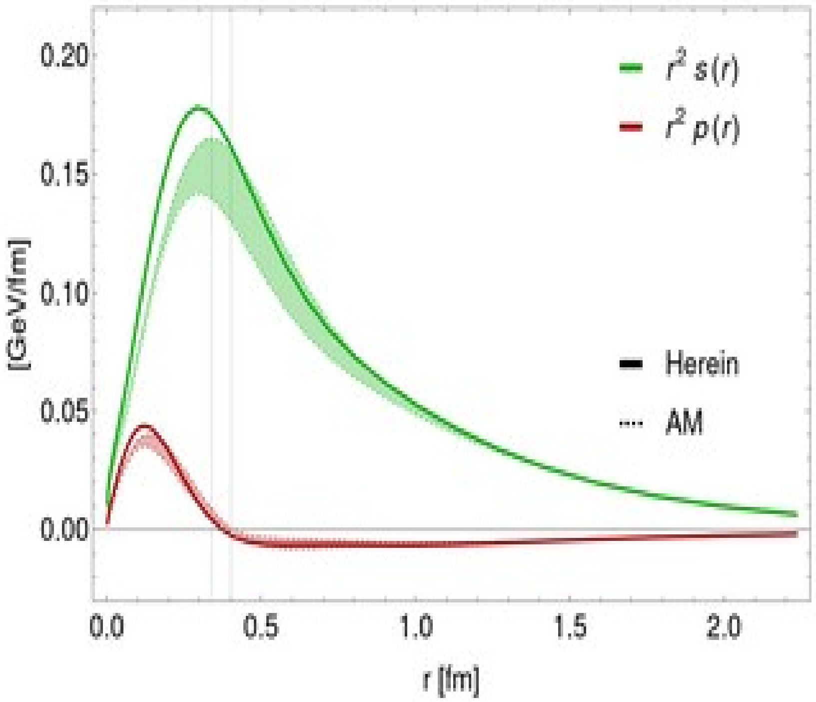
Results: Pressure profiles

- **Pressure** and **shear** forces are obtained from $\theta_1(Q^2)$:

$$p_{\mathbf{P}}(r) = \frac{1}{6\pi^2 r}\int_0^\infty d\Delta \, \frac{\Delta}{2E(\Delta)} \sin(\Delta r)[\Delta^2\theta_1^{\mathbf{P}}(\Delta^2)]$$

$$s_{\mathbf{P}}(r) = \frac{3}{8\pi^2}\int_0^\infty d\Delta \, \frac{\Delta^2}{2E(\Delta)} j_2(\Delta r)\,[\Delta^2\theta_1^{\mathbf{P}}(\Delta^2)]$$



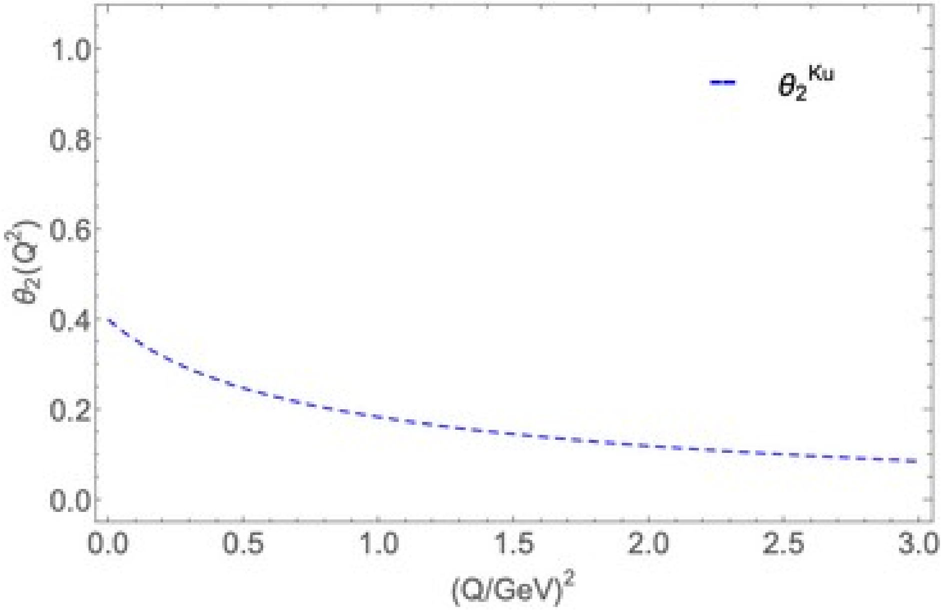


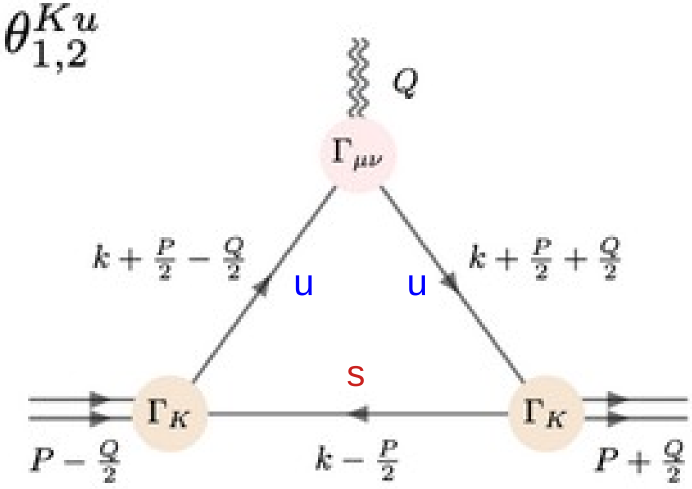
AM: Ingredients from **Overlap** but using the diagrammatic approach discussed herein.

`Raya:2021zrz`

Shear forces are maximal where the **pressure** shifts sign, i.e. where **confinement** forces become dominant.

Results: Kaon’s GFFs and profiles





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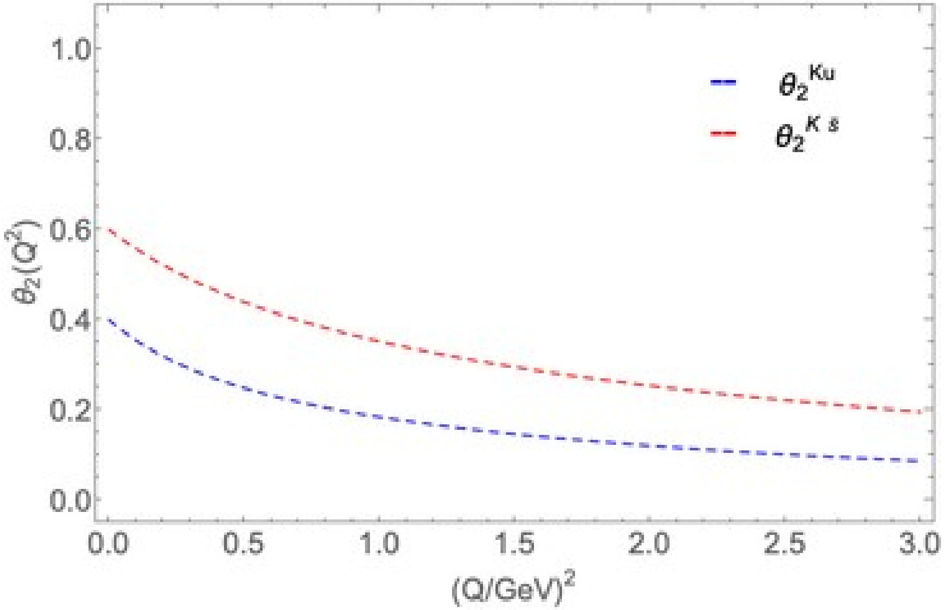

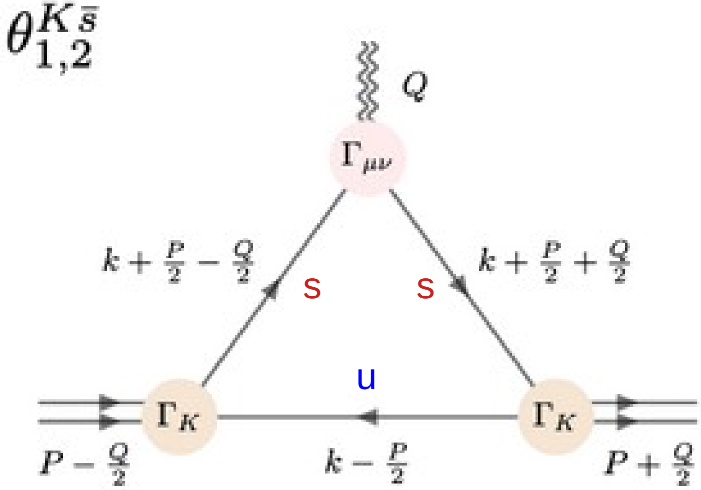

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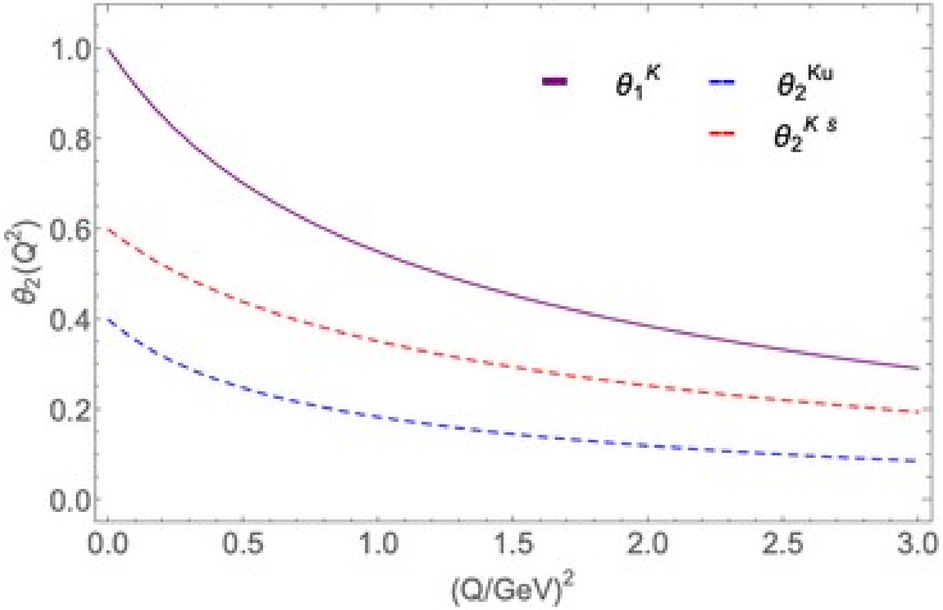

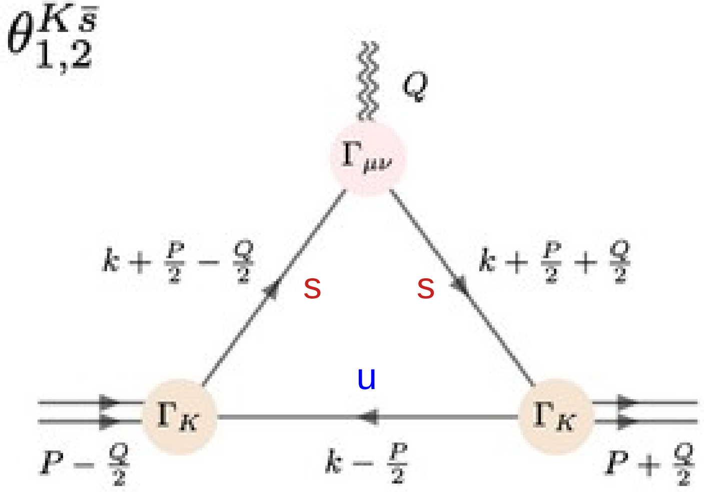

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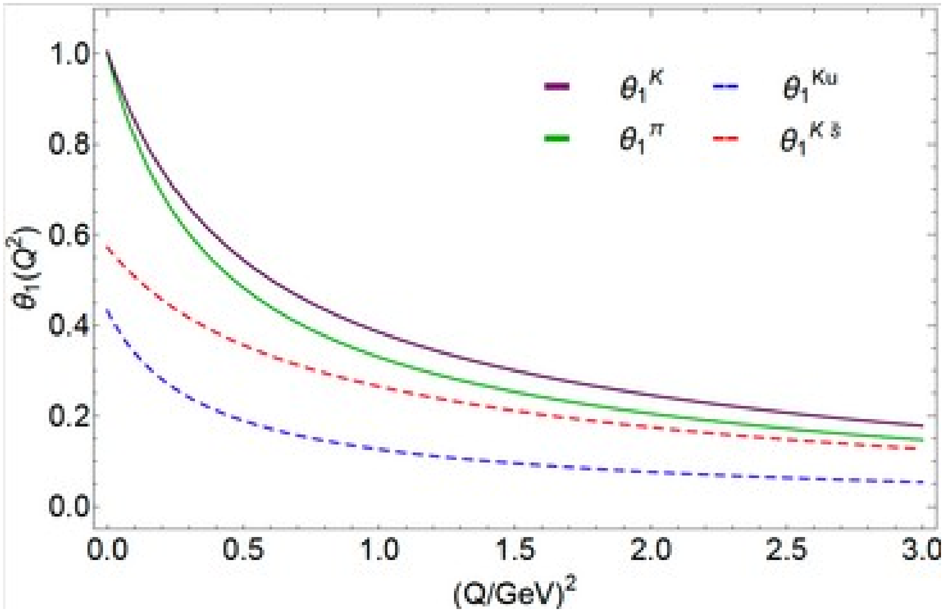
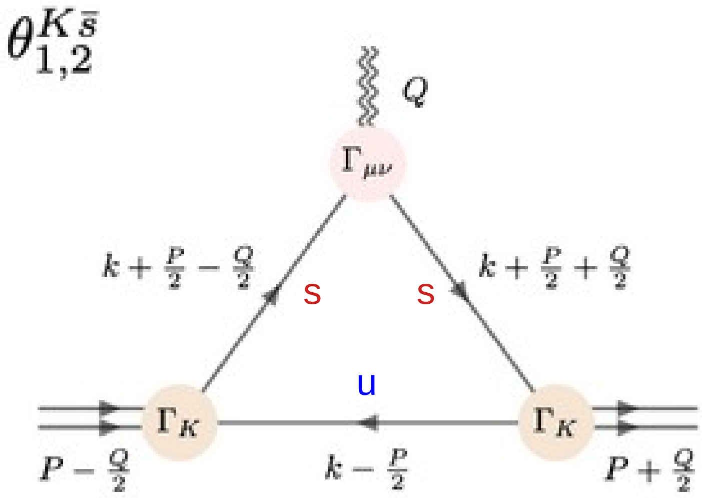
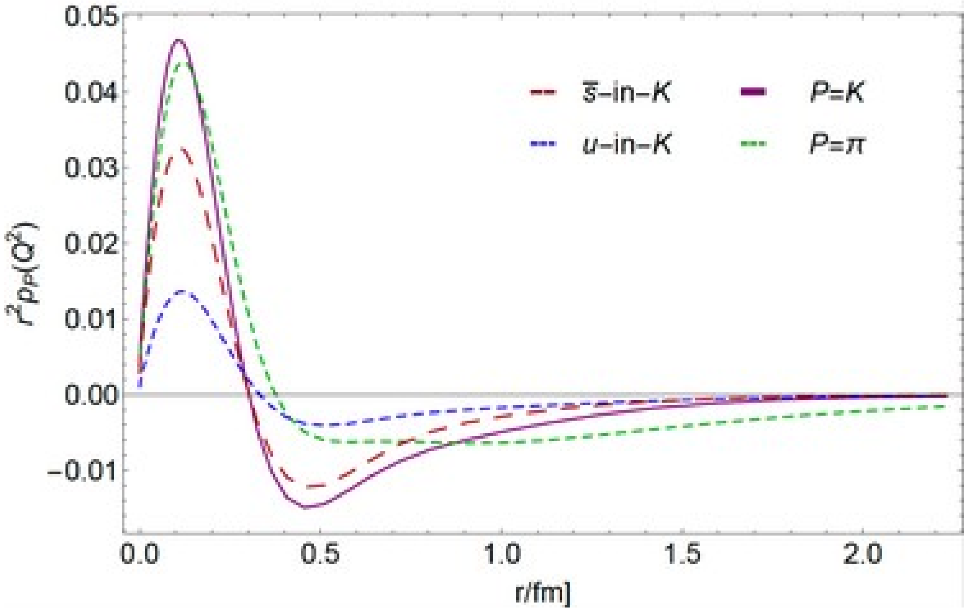
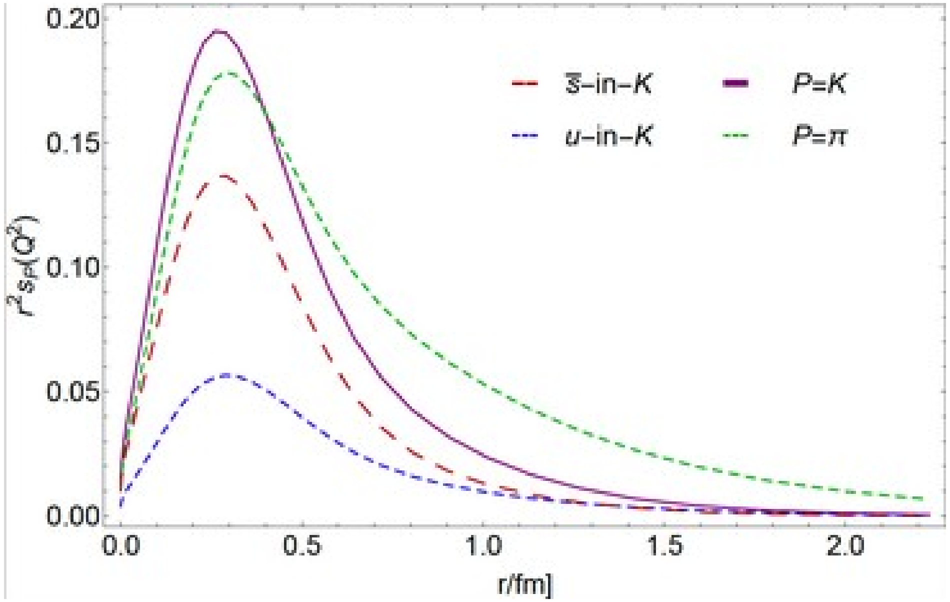
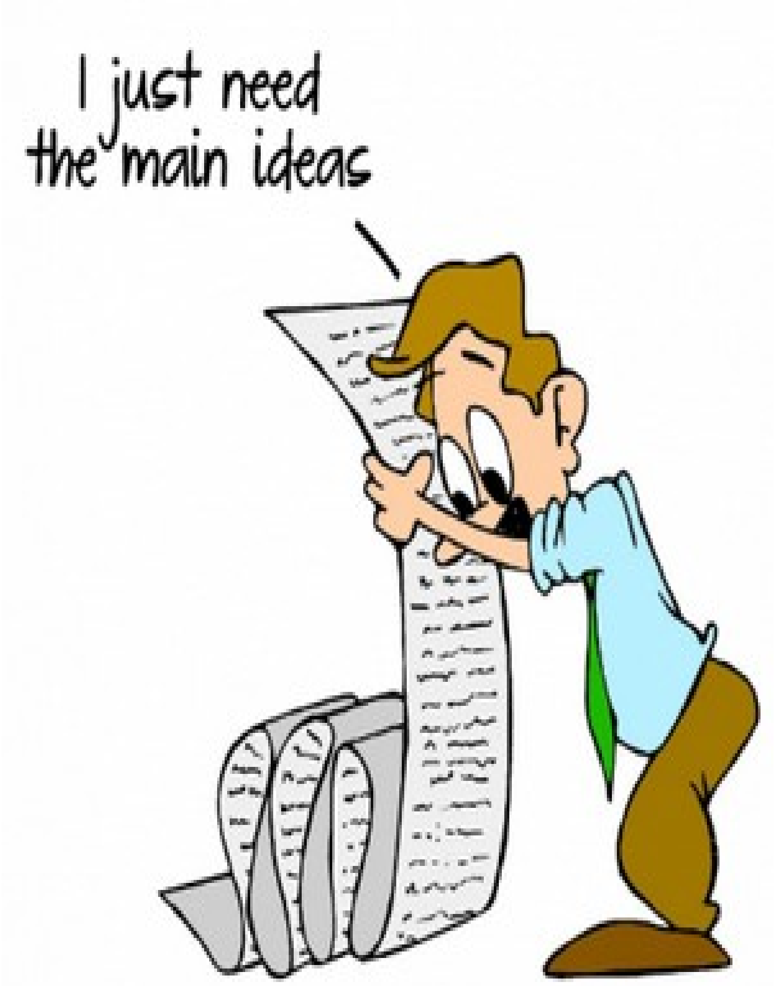








Summary and scopes



Summary and scopes

- The **EHM** is argued to be intimately connected to a **PI effective** charge which enters a conformal regime, below a given momentum scale, where gluons acquiring a dynamical mass decouple from interaction.

- Capitalizing on the latter, two main ideas emerge: (I) the identification of that decoupling with a **hadronic scale** at which the structure of hadrons can be expressed only in terms of valence dressed partons; and (ii) the reliability of an **all-orders** evolution scheme to describe the splitting of valence into more partons, generating thus the glue and sea, when the resolution scale decreases.

- Key implications stemming from both ideas have been derived and tested for the pion PDFs. Grounding on them, **Lattice QCD** and experimental data have been shown to confirm **CSM** results.

- The robustness of the approach based on **all-orders** evolution from **hadronic** to experimental scale has been proved with its application to the pion and proton case; and used to produce, via reverse engineering, results for pion GPDs and mass density distributions from data.

Summary and scopes

- We have described a **CSM based** computation of the pion **GFFs,** the new-brand ingredient for which is the **QTV** entering the **game**.

- The obtained results expose the **robustness** of the framework and the importance of **symmetries**:
  - Both **QPV** and **QTV** obey their own **WGTI**
  - This is sufficient to produce a sensible result for $\theta_2(Q^2)$
  - **Beyond I.A.**, additional diagrams are crucial to ensure $\sum_{q,g} \bar{c}(t) = 0$, but not needed for the two other form factors.

- **Physically** meaningful pictures are drawn:
  - **Charge** effects span over a larger domain than **mass** effects
  - **Shear** forces are maximal where **confinement** forces become dominant

- Other hadrons are **within reach:**
  - we can **analogously** proceed with **heavy quarkonia**
  - and, capitalizing on **Faddeev amplitudes**, compute **proton GFFs**

To be continued...