Introduction to the standard model

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I'll assume you all know that ...

- **The relativistic version of Quantum Mechanics is Quantum Field Theory (non-conservation of the particle number in reactions)**
- **QFT's are completely determined by symmetry properties (hinted by experiments)**
- **Gauge invariance plays a special role.**

... but I'll be happy to answer questions!

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Lay-out:

1. **The standard model:** *A model of leptons*[a] **(and hadrons)**
2. **Spontaneous breaking of the gauge symmetry**
3. **Explicit breaking of accidental symmetries**
4. **Are we happy? Experimental tests and future prospects**

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[a] The title of S. Weinberg's paper, PRL **19** (1967) 1264.

1. Construction of the standard model

- **Phenomenological input: β and μ decays, parity violation, production and decays of strange particles.**
- **Theoretical constraints: unitarity and perturbative renormalizability.**

Our attitude towards renormalizability has changed in time ...

The idea of interpreting the Fermi four-fermion interaction vertex as originated by vector boson exchange dates back to Fermi himself.

There is only one way to build a unitary and renormalizable field theory of vectors: a gauge theory.

An endless list of experimental confirmations of this fact. The most striking one: universality of couplings.

The gauge symmetry suggested by early data (and later confirmed) is based on the invariance group

$$SU(2)_L \otimes U(1)_Y$$

which requires four gauge vector bosons:

$$W^a_\mu, a = 1, 2, 3 \qquad \text{for } \; SU(2)_L$$
$$B_\mu \qquad \text{for } \; U(1)_Y$$

Vector boson dynamics is governed by the usual Yang-Mills action

$$\mathcal{L}_{\mathrm{Yang-Mills}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

which contains cubic and quartic interaction vertices.

Next, one must associate fermion matter fields to representations of the gauge group.

Six flavours of quarks:

$$u \quad d \quad s \quad c \quad b \quad t$$

Three charged leptons:

$$e \quad \mu \quad \tau$$

and three neutrinos:

$$\nu_e \quad \nu_\mu \quad \nu_\tau$$

(I'm making a long story VERY short!)

Data are consistent with the following scheme:

$$\begin{array}{ccccc} Q^i_L = \begin{pmatrix} u^i_L \\ d^i_L \end{pmatrix} & u^i_R & d^i_R & L^i_L = \begin{pmatrix} \nu^i_L \\ \ell^i_L \end{pmatrix} & \ell^i_R \\ \psi^i_1 & \psi^i_2 & \psi^i_3 & \psi^i_4 & \psi^i_5 \end{array}$$

A family structure emerges:

$$\psi_r{}^i$$

The index i labels fermion generations: $i = 1, \ldots, 3$ (as far as we know).

The index r labels group representations

Comments:

- **Fermion fields with different chiralities transform differently:**

$$\psi_R = \frac{1+\gamma_5}{2}\psi \qquad \psi_L = \frac{1-\gamma_5}{2}\psi$$

  Parity is not conserved by weak interacions.

- **Left-handed quarks Q_L and leptons L_L transform as $SU(2)$ doublets ($r=1$ and $r=4$), right-handed fermions as $SU(2)$ singlets ($r=2,3,5$). No right-handed fermion participate in charged-current interactions.**

- **Different representations have different values of the hypercharge quantum number (more on this later).**

- **neutrinos are massless: no right-handed neutrinos around. Much more on this later.**

A unique *gauge-invariant lagrangian density* **can now be written:**

$$\mathcal{L}_{SM} = \mathcal{L}_{\mathrm{Yang-Mills}} + \sum_{i=1}^{N}\sum_{r=1}^{5} \bar{\psi}_r^i \, i \not{D}_r \, \psi_r^i$$

with

$$D_r^\mu = \partial^\mu - igT_r^a W_a^\mu - ig'\frac{Y_r}{2}B^\mu$$

$$T_r^a = \frac{\tau^a}{2} \quad \text{for SU(2) doublets} \quad (r = 1, 4)$$

$$T_r^a = 0 \quad \text{for SU(2) singlets} \quad (r = 2, 3, 5)$$

- **Hypercharge values undetermined so far**
- **Axial anomaly cancelled if $n_q = n_\ell = N$ (a prediction of the standard model.)**

The interaction lagrangian density includes a charged-current term which can be written as

$$\mathcal{L}_{cc} \quad = \quad \frac{g}{\sqrt{2}}\left[\overline{L}_L \gamma^\mu \tau^+ L_L \, W_\mu^+ + \overline{L}_L \gamma^\mu \tau^- L_L \, W_\mu^- \right.$$
$$\left. + \, \overline{Q}_L \gamma^\mu \tau^+ Q_L \, W_\mu^+ + \overline{Q}_L \gamma^\mu \tau^- Q_L \, W_\mu^- \right]$$

with the definitions

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2); \qquad \tau^\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$$

This interaction term accounts for all processes described by the Fermi theory. For example

$$\mathcal{L}_{cc} = \frac{g}{2\sqrt{2}} \bar{u}\gamma^\mu(1-\gamma_5) d \, W_\mu^- + \frac{g}{2\sqrt{2}} \bar{e}\gamma^\mu(1-\gamma_5)\nu_e \, W_\mu^- + \ldots$$

Electro-weak unification

The neutral-current interaction term is

$$\mathcal{L}_{nc} = g\,\bar{\psi}\,\gamma_\mu\,T_3\,\psi\,W_3^\mu + g'\,\bar{\psi}\,\gamma_\mu\,\frac{Y}{2}\,\psi\,B^\mu$$

where

$$\psi = \psi_r^i, \quad r = 1, \ldots, 5$$

$$T_3 = (T_3)_r$$

$$(T_3)_r = \frac{\tau_3}{2} \qquad \text{for doublets } \ (r = 1, 4)$$

$$(T_3)_r = 0 \qquad \text{for singlets } \ (r = 2, 3, 5)$$

$$Y = Y_r \ \ (r = 1, \ldots, 5)$$

Reparametrization of the neutral sector:

$$W_3^\mu = A^\mu \sin\theta_W + Z^\mu \cos\theta_W$$
$$B^\mu = A^\mu \cos\theta_W - Z^\mu \sin\theta_W$$

(an orthogonal transformation, in order to keep kinetic terms diagonal in the vector fields).

$$\begin{aligned}\mathcal{L}_{nc} &= \bar\psi\,\gamma_\mu \left[g \sin\theta_W\, T_3 + g' \cos\theta_W \frac{Y}{2}\right] \psi\, A^\mu \\ &+ \bar\psi\,\gamma_\mu \left[g \cos\theta_W\, T_3 - g' \sin\theta_W \frac{Y}{2}\right] \psi\, Z^\mu\end{aligned}$$

We may identify A_μ with the photon field provided

$$eQ = g \sin\theta_W\, T_3 + g' \cos\theta_W \frac{Y}{2}$$

Choosing g and g' so that

$$g \sin\theta_W = g' \cos\theta_W = e$$

we obtain

$$Q = T_3 + \frac{Y}{2}$$

for all fermions:

$$Y(u_L) = 2\left(\frac{2}{3} - \frac{1}{2}\right) = \frac{1}{3}$$

$$Y(d_L) = 2\left(-\frac{1}{3} + \frac{1}{2}\right) = \frac{1}{3}$$

$$Y(e_L) = 2\left(-1 + \frac{1}{2}\right) = -1$$

$$\ldots$$

[Alternatively: choose e.g. $Y = -1$ **for the lepton doublets, and solve**

$$+\tfrac{1}{2} g \sin\theta_W - \tfrac{1}{2} g' \cos\theta_W = 0$$
$$-\tfrac{1}{2} g \sin\theta_W - \tfrac{1}{2} g' \cos\theta_W = -e$$

with respect to $g\sin\theta_W, g'\cos\theta_W$**. You get** $g\sin\theta_W = g'\cos\theta_W = e$**, and** $Q = T_3 + Y/2$ **follows.]**

The value of $\sin\theta_W$ can only be extracted from the observation of weak neutral-current phenomena, induced by interactions with the Z^0 boson:

$$\begin{aligned}\mathcal{L}_{nc} &= e\bar{\psi}\,\gamma_\mu\,Q\,\psi\,A^\mu \\ &+ e\bar{\psi}\,\gamma_\mu\left[\frac{\cos\theta_W}{\sin\theta_W}\,T_3 - \frac{\sin\theta_W}{\cos\theta_W}\,\frac{Y}{2}\right]\psi\,Z^\mu\end{aligned}$$

Historical example: neutral-current deep inelastic scattering.

$$\begin{aligned}\mathrm{NC}: &\quad \nu_\mu + H \to \nu_\mu + X \\ \mathrm{CC}: &\quad \nu_\mu + H \to \mu^- + X\end{aligned}$$

$$R = \frac{\sigma_{\bar{\nu}}^{(\mathrm{NC})} - \sigma_{\nu}^{(\mathrm{NC})}}{\sigma_{\bar{\nu}}^{(\mathrm{CC})} - \sigma_{\nu}^{(\mathrm{CC})}} \simeq \frac{1 - 2\sin^2\theta_W}{2}$$

The most precise determinations of $\sin\theta_W$ come from forward-backward asymmetries measured in e^+e^- collisions:

$$A_{FB}(f) = \frac{\int_{\cos\theta>0} d\sigma(e^+e^- \to f\bar{f}) - \int_{\cos\theta<0} d\sigma(e^+e^- \to f\bar{f})}{\sigma(e^+e^- \to f\bar{f})}$$

where f is any charged fermion. Present result:

$$\sin^2\theta_W = 0.23116(13)$$

It follows that g, g' are of the same order of magnitude as e.

Not yet a realistic theory

- **The gauge symmetry must be (spontaneously) broken,**

$$SU(2)_L \otimes U(1)_Y \to U(1)_{\mathrm{em}}$$

  because weak vector bosons are observed to be massive (short range of weak interactions).

- **The fermionic sector has a large global symmetry which is not observed. An explicit breaking**

$$[U(N)]^5 \to U(1)_B \otimes U(1)_e \otimes U(1)_\mu \otimes U(1)_\tau$$

  or

$$[U(N)]^5 \to U(1)_B \otimes U(1)_L$$

  is needed.

Addendum n. 1

$$\mathcal{L}_{\rm Yang-Mills} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W^a_{\mu\nu} W^{\mu\nu}_a$$

$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$$

$$W^{\mu\nu}_i = \partial^\mu W^\nu_i - \partial^\nu W^\mu_i + g \epsilon_{ijk} W^\mu_j W^\nu_k$$

The corresponding expressions in terms of $W^\pm_\mu$, Z_μ and A_μ can be easily worked out:

$$W^1_\mu = \frac{1}{\sqrt{2}} (W^+_\mu + W^-_\mu)$$

$$W^2_\mu = \frac{i}{\sqrt{2}} (W^+_\mu - W^-_\mu)$$

$$W^3_\mu = A_\mu \sin\theta_W + Z_\mu \cos\theta_W$$

$$B_\mu = A_\mu \cos\theta_W - Z_\mu \sin\theta_W$$

We get

$$
\begin{aligned}
W^1_{\mu\nu} &= \frac{1}{\sqrt{2}}\left[W^+_{\mu\nu} + ig\,\sin\theta_W(W^+_\mu A_\nu - W^+_\nu A_\mu)\right.\\
&\qquad\qquad \left. + \,ig\,\cos\theta_W(W^+_\mu Z_\nu - W^+_\nu Z_\mu)\right] + \text{h.c.}\\
W^2_{\mu\nu} &= \frac{i}{\sqrt{2}}\left[W^+_{\mu\nu} + ig\,\sin\theta_W(W^+_\mu A_\nu - W^+_\nu A_\mu)\right.\\
&\qquad\qquad \left. + \,ig\,\cos\theta_W(W^+_\mu Z_\nu - W^+_\nu Z_\mu)\right] + \text{h.c.}\\
W^3_{\mu\nu} &= F_{\mu\nu}\sin\theta_W + Z_{\mu\nu}\cos\theta_W - ig(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu)\\
B_{\mu\nu} &= F_{\mu\nu}\cos\theta_W - Z_{\mu\nu}\sin\theta_W
\end{aligned}
$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$Z^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu$$

$$W^{\mu\nu}_\pm = \partial^\mu W^\nu_\pm - \partial^\nu W^\mu_\pm$$

It follows that

$$
\begin{aligned}
\mathcal{L}_{\text{Yang-Mills}} \quad = \quad & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}W^{+}_{\mu\nu}W^{\mu\nu}_{-} \\
& + ig\sin\theta_W \left(W^{+}_{\mu\nu}W^{\mu}_{-}A^{\nu} - W^{-}_{\mu\nu}W^{\mu}_{+}A^{\nu} + F_{\mu\nu}W^{\mu}_{+}W^{\nu}_{-}\right) \\
& + ig\cos\theta_W \left(W^{+}_{\mu\nu}W^{\mu}_{-}Z^{\nu} - W^{-}_{\mu\nu}W^{\mu}_{+}Z^{\nu} + Z_{\mu\nu}W^{\mu}_{+}W^{\nu}_{-}\right) \\
& + \frac{g^2}{2}\left(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)\Bigg[\frac{1}{2}W^{+}_{\mu}W^{+}_{\nu}W^{-}_{\rho}W^{-}_{\sigma} \\
& - W^{+}_{\mu}W^{-}_{\nu}\left(A_\rho A_\sigma \sin^2\theta_W + Z_\rho Z_\sigma \cos^2\theta_W + 2A_\rho Z_\sigma \sin\theta_W\cos\theta_W\right)\Bigg]
\end{aligned}
$$

Addendum n. 2

Why did I say that

$$\mathcal{L}_{SM} = \mathcal{L}_{\text{Yang-Mills}} + \sum_{i=1}^{N} \sum_{r=1}^{5} \bar{\psi}_r^i \, i \not{D}_r \, \psi_r^i$$

has a global $[U(N)]^5$ invariance?

For each $r = 1, \ldots, 5$ consider the transformation

$$\psi_r^{i\,\prime} = \sum_{j=1}^{N} U_r^{ij} \psi_r^j$$

where U^r is a constant unitary $N \times N$ matrix. This transformation leaves $\mathcal{L}_{SM}$ unchanged. There are 5 symmetries of this kind, one for each representation r. The full global symmetry is therefore $[U(N)]^5$, as announced.

2. Spontaneous breaking of the gauge symmetry

A simple argument shows that the W boson must be massive. The amplitude for β decay in the Fermi theory is given by

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}}\overline{u}\gamma^\mu(1-\gamma_5)d\,\overline{e}\gamma_\mu(1-\gamma_5)\nu_e$$

In the standard model, the same process is induced by the exchange of a W boson:

$$\mathcal{M}^{\mathrm{SM}} = \left(\frac{g}{\sqrt{2}}\overline{u}_L\gamma^\mu d_L\right)\frac{1}{q^2-m_W^2}\left(\frac{g}{\sqrt{2}}\overline{e}_L\gamma_\mu\nu_{e\,L}\right)$$

We have

$$q^2 \leq (m_N - m_P)^2 \sim (1.3\ \mathrm{MeV})^2$$

Hence, the two amplitudes coincide in the limit $m_W^2 \gg q^2$ **if**

$$\frac{G_F}{\sqrt{2}} = \left(\frac{g}{2\sqrt{2}}\right)^2 \frac{1}{m_W^2}.$$

A lower bound on the W **mass can be set: since**

$$g = \frac{e}{\sin\theta_W}$$

we obtain

$$m_W^2 = \frac{\sqrt{2}}{G_F}\frac{g^2}{8} \geq \frac{\sqrt{2}}{G_F}\frac{e^2}{8} \sim (40\ \mathrm{GeV})^2$$

quite a large value, compared to the nucleon mass, and an enormous number, compared to the present upper bound on the photon mass

$$m_\gamma \leq 2 \cdot 10^{-16}\ \mathrm{eV}.$$

Breaking gauge invariance explicitly with a mass term

$$m_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2} m_Z^2 Z_\mu Z^\mu$$

leads to a non-renormalizable and non-unitary theory.

The gauge symmetry of the standard model must be spontaneously broken, in order to introduce masses for the W and Z vector bosons without spoiling unitarity and renormalizability.

A flavour of the argument: a mass term inserted by hand (explicit breaking) leads to a massive gauge boson propagator

$$\Delta^{\mu\nu}(k) = \frac{i}{k^2 - m^2}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right)$$

For large k, the term proportional to $k^\mu k^\nu$ dominates, and $\Delta(k) \sim k^0$ rather than k^{-2}: the behaviour of this propagator at large k is much worse than that of the scalar propagator. This suggests a worse UV behaviour of the Feynman integrals, which leads to a non-renormalizable theory.

A related problem: unitarity of the scattering matrix. The amplitude for a generic physical process with the emission or the absorption of a vector boson with four-momentum k and polarization vector $\epsilon(k)$ has the form

$$\mathcal{M} = \mathcal{M}^{\mu}\epsilon_{\mu}(k).$$

A massive vector (contrary to a massless one) may be polarized longitudinally. In this case, choosing the z axis along the direction of the 3-momentum of the vector boson, the polarization is given by

$$\epsilon_L = \left(\frac{|\vec{k}|}{m}, 0, 0, \frac{E}{m} \right) = \frac{k}{m} + \mathcal{O}\left(\frac{m^2}{E^2} \right),$$

(recall that the transversity condition $k \cdot \epsilon = 0$ and the normalization $\epsilon^2 = -1$.)

The amplitude $\mathcal{M}$ grows indefinitely with the energy E, and eventually violates the unitarity bound.

Both sources of power-counting violation are rendered harmless if the vector particles are coupled to conserved currents, so that

$$k^\mu \mathcal{M}_\mu = 0$$

Gauge invariance provides such conservation relations.

Spontaneous symmetry breaking is not really a way of breaking a symmetry: rather, it is a different realization of the symmetry itself.

More precisely, SSB takes place whenever the ground state is not invariant under symmetry transformations. As a consequence, the lagrangian density is symmetric, but the spectrum of physical states is not.

The prototype: ferromagnetism.

In quantum field theory, SSB takes place when some operator with non-trivial transformation properties under the gauge group has non-vanishing vacuum expectation value:

$$\langle 0|\phi_j|0\rangle = v_j \neq 0$$

Easy to prove: after an infinitesimal transformation

$$\phi_i \rightarrow \phi_i + i\alpha^a \, t^a_{ij}\phi_j = \phi_i + i\alpha^a \, [Q^a, \phi_i]$$

$$t^a_{ij}\langle 0|\phi_j|0\rangle = \langle 0| \, [Q^a, \phi_i] \, |0\rangle \neq 0 \; \Leftrightarrow \; Q^a|0\rangle \neq 0$$

which is the condition for spontaneous symmetry breaking, i.e. non-invariance of the vacuum state.

Observations:

- $\langle 0|\phi_i|0\rangle$ **is a constant if the vacuum is invariant under translations:**

$$\langle 0|\phi_i(x)|0\rangle = \langle 0|e^{iPx}\phi_i(0)e^{-iPx}|0\rangle = \langle 0|\phi_i(0)|0\rangle$$

- ϕ **must be a scalar, otherwise its vacuum expectation value is frame-dependent.**
- ϕ **is not necessarily an elementary field**

The simplest realization: the Higgs mechanism

$$\mathcal{L}_{SM} = \mathcal{L}_{\text{Yang-Mills}} + \sum_{i=1}^{N} \sum_{r=1}^{5} \bar{\psi}_r^i \, i \not{D}_r \, \psi_r^i + \mathcal{L}_{\text{Higgs}}$$

$$\mathcal{L}_{\text{Higgs}} = (D_\mu \phi)^\dagger D^\mu \phi - m^2 \, |\phi|^2 - \lambda \, |\phi|^4$$

The simplest among simplest: ϕ is an $SU(2)_L$ doublet:

$$D_\mu \phi = \partial_\mu \phi - \frac{ig}{2} W_\mu^a \tau^a \phi - \frac{ig'}{2} Y_\phi B_\mu \phi$$

If $m^2 < 0$ the scalar potential has a minimum at

$$\langle 0|\phi|0\rangle = \frac{V}{\sqrt{2}}; \qquad |V|^2 = -\frac{m^2}{\lambda} \equiv v^2.$$

The value of the hypercharge Y_ϕ is dictated by the requirement that $U_{\rm em}(1)$ remains unbroken:

$$e^{ieQ}V = V$$

or

$$Q\,V = \frac{1}{2}\left(\tau^3 + Y\right)\,V = 0 \quad \Leftrightarrow \quad (Y_\phi + 1)(Y_\phi - 1) = 0$$

where $Y = Y_\phi\,\mathbb{I}_2$. Two solutions:

$$Y_\phi = 1, \quad V = \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad Y_\phi = -1, \quad V = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

We choose $Y_\phi = 1$, so that

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

The $|D\phi|^2$ term contains a term

$$
\begin{aligned}
\mathcal{L}_{\phi\phi VV} &= \frac{1}{4}(g^2 W_a^\mu W_\mu^a + {g'}^2 B^\mu B_\mu)\phi^\dagger\phi + \frac{1}{2} g g' B^\mu W_\mu^i \phi^\dagger \tau^i \phi \\
&= \frac{1}{4} g^2 v^2 (W_1^\mu W_\mu^1 + W_2^\mu W_\mu^2) + \frac{1}{4} v^2 (W_3^\mu \;\; B^\mu) \begin{bmatrix} g^2 & -gg' \\ -gg' & {g'}^2 \end{bmatrix} \begin{pmatrix} W_{3\mu} \\ B_\mu \end{pmatrix} \\
&+ \; \ldots
\end{aligned}
$$

Mass terms for the W and the Z appear:

$$
m_W^2 = \frac{1}{4}\, g^2\, v^2 \qquad m_Z^2 = \frac{1}{4}\,(g^2 + {g'}^2)\, v^2 \qquad m_\gamma^2 = 0
$$

The value of the order parameter v^2 is obtained from matching with the Fermi theory of β decay: from

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}; \qquad m_W^2 = \frac{1}{4}g^2v^2$$

we get

$$v = (\sqrt{2}G_F)^{-1/2} \sim (247 \text{ GeV})^2$$

where we have used the measured value $G_F \sim 1.1 \times 10^{-5}$ GeV^{-2}.

Weak interactions have a characteristic energy scale of the order of a few hundred GeV.

Three of the four scalar degrees of freedom in ϕ are unphysical: they can be eliminated from the spectrum by a gauge choice.

An easy (but slightly deceptive) way to see it: parametrize ϕ by

$$\phi = \frac{1}{\sqrt{2}} e^{\frac{i\tau^i \theta^i(x)}{v}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}$$

after a suitable gauge transformation. The massive gauge boson propagators take the form

$$\Delta^{\mu\nu}(k) = \frac{i}{k^2 - m^2} \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right)$$

It looks like we are in troubles again with renormalization!

This is not true: we are working with a renormalizable theory, so renormalizability must arise in calculations, even though it is not manifest (the propagator does not respect the usual power-counting rule).

This is called the unitary gauge: unitarity is manifest, in the sense that unphysical degrees of freedom are removed from the spectrum, but manifest renormalizability is lost.

Useful in tree-level calculation.

When loop corrections become relevant, it is advisable to adopt a renormalizable gauge. The starting point is a linear parametrization of the scalar field:

$$\phi = \phi_1 + \phi_2,$$

$$\phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad \phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} G_1(x) + iG_2(x) \\ H(x) + iG_3(x) \end{pmatrix}$$

A convenient gauge-fixing term (suggested by 't Hooft) is

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} \left[\partial^\mu W_\mu^i - \xi f^i(\phi)\right]^2 - \frac{1}{2\xi} \left[\partial^\mu B_\mu - \xi f(\phi)\right]^2$$

with

$$f^i(\phi) = \frac{ig}{2}(\phi_1^\dagger \tau^i \phi_2 - \phi_2^\dagger \tau^i \phi_1) \qquad f(\phi) = \frac{ig'}{2}(\phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_1)$$

Two main advantages:

- **No mixing between vector fields and derivative of the scalar field**
- **Manifest renormalizability:**

$$\Delta_\xi^{\mu\nu}(k) = \frac{i}{k^2 - m^2}\left[-g^{\mu\nu} + \frac{(1-\xi)k^\mu k^\nu}{k^2 - \xi m^2}\right]$$

(the unitary gauge is recovered in the limit $\xi \to \infty$).

Draw-back: the unphysical scalars G_1, G_2, G_3 are in the game. They cannot appear as asymptotic states (external lines in Feynman diagrams), and their contributions as internal lines is cancelled by the unphysical singularity in the vector boson propagators (not easy to prove).

Physical interpretation: massless vector bosons have two physical degrees of freedom: the two helicity states (no longitudinal polarization).

After SSB, vector bosons become massive: the longitudinal modes are provided by the three would-be Goldstone bosons, which disappear from the spectrum.

The existence of longitudinally polarized W and Z is the most striking evidence of spontaneous gauge symmetry breaking.

The scalar potential simplifies considerably in the unitary gauge:

$$
\begin{aligned}
V(\phi) &= m^2 |\phi|^2 + \lambda |\phi|^4 \\
&= \frac{m^2}{2}(v+H)^2 + \frac{\lambda}{4}(v+H)^4 \\
&= H(m^2 v + \lambda v^3) + \frac{1}{2}H^2(m^2 + 3\lambda v^2) + \lambda v H^3 + \frac{\lambda}{4}H^4
\end{aligned}
$$

Since $m^2 = -\lambda v^2$**, the linear term vanishes, and the quadratic term has a coefficient**

$$\frac{1}{2}2\lambda v^2$$

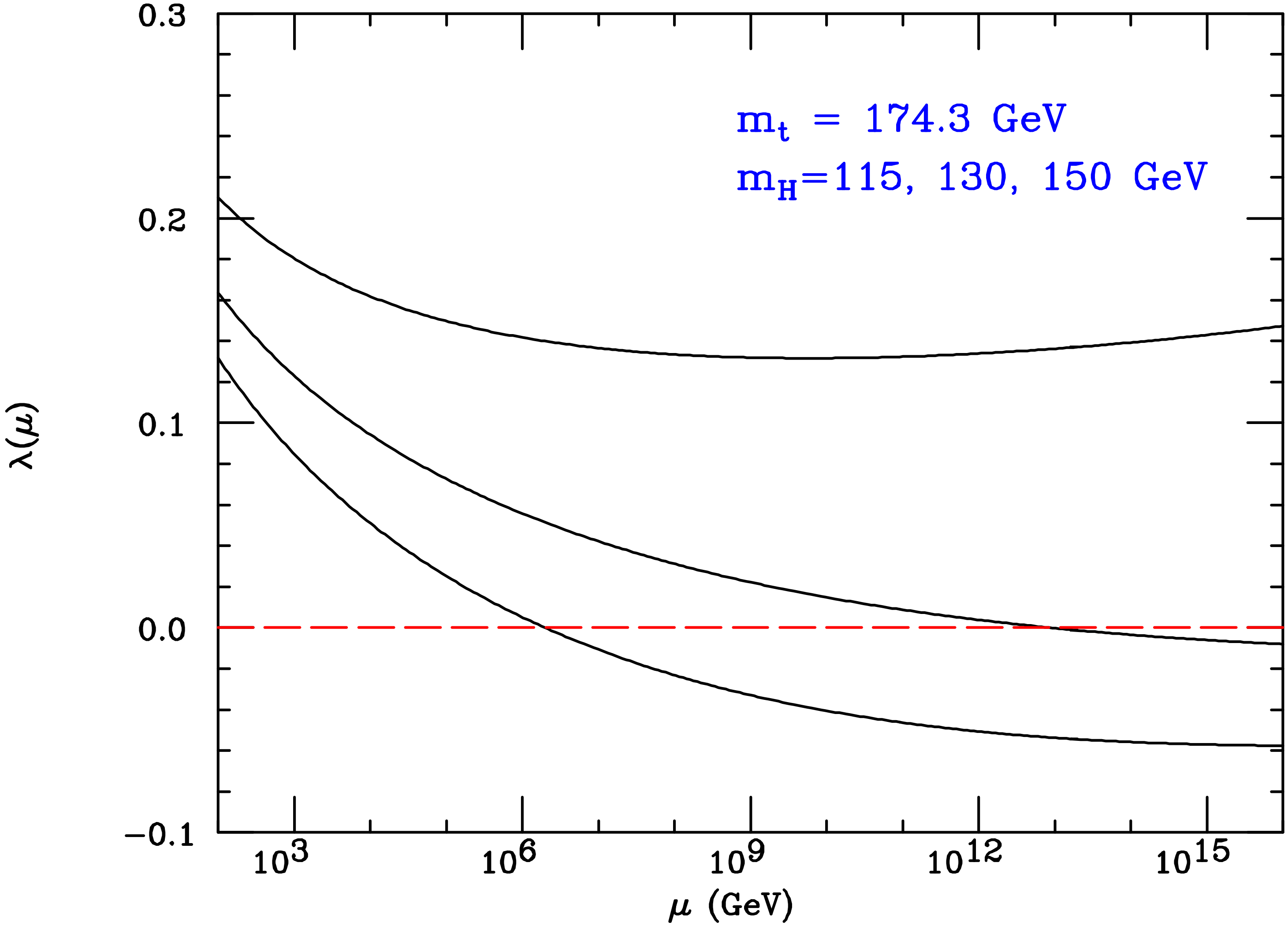
and can be interpreted as a true mass term for the scalar field H**.**

What do we know about the Higgs boson mass?

- **At tree level, $m_H^2 = 2\lambda v^2$. We do not know the value of λ, but v^2 is fixed, so if λ is in the perturbative domain, m_H must be something similar to m_W, m_Z. In any case, it grows with λ.**

- **An upper bound of about ~ 1 TeV on m_H comes from unitarity considerations (perturbative unitarity of $W_L W_L \to W_L W_L$, similar to the upper bound on m_W in the Fermi theory).**

The quartic running coupling $\lambda(\mu)$ becomes negative for μ large enough; then, at some large values of μ it has a Landau pole:

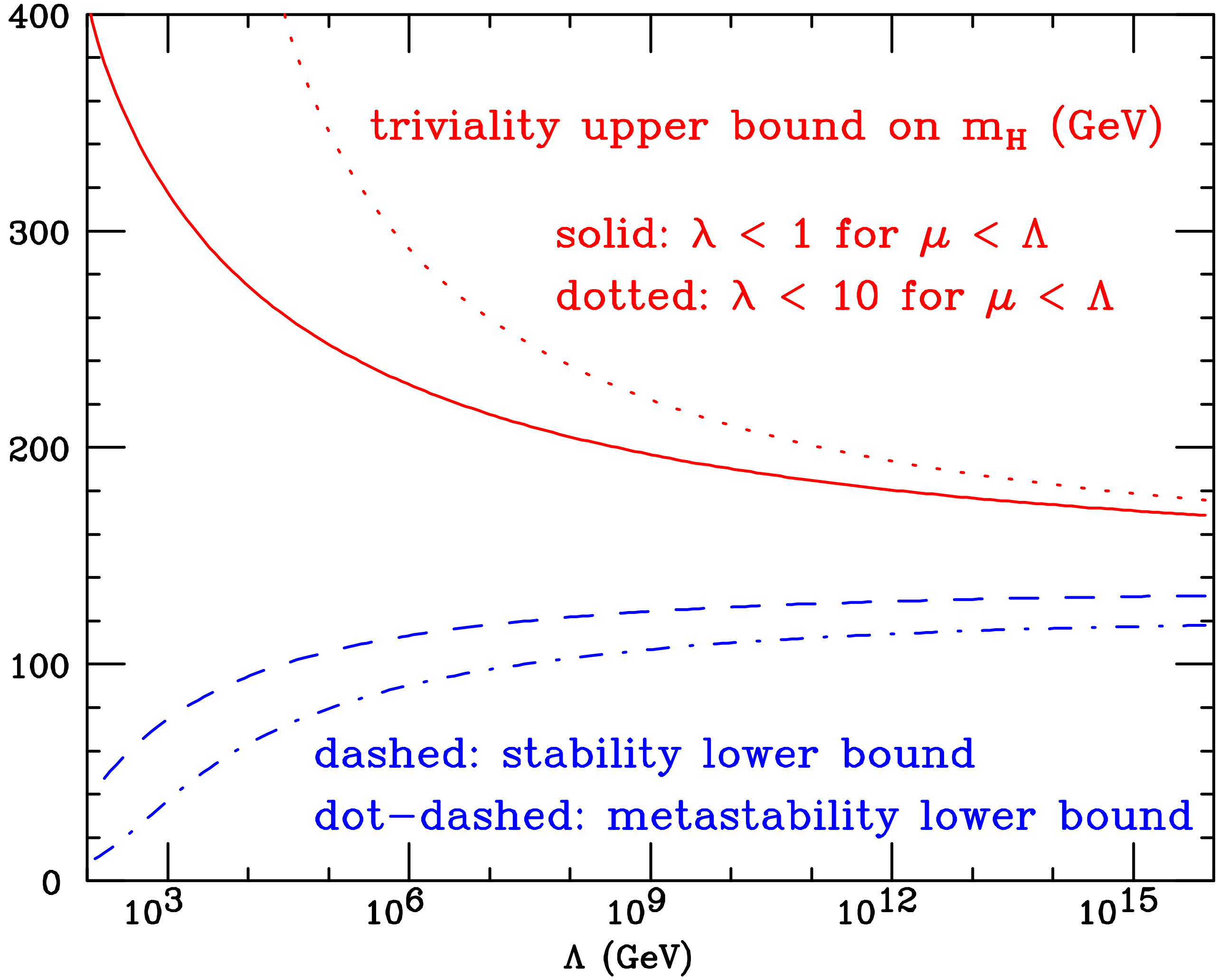


One lesson:

The standard model can not be valid up to arbitrarily large energy scales: it can only work up to some energy scale Λ

and two consequences:

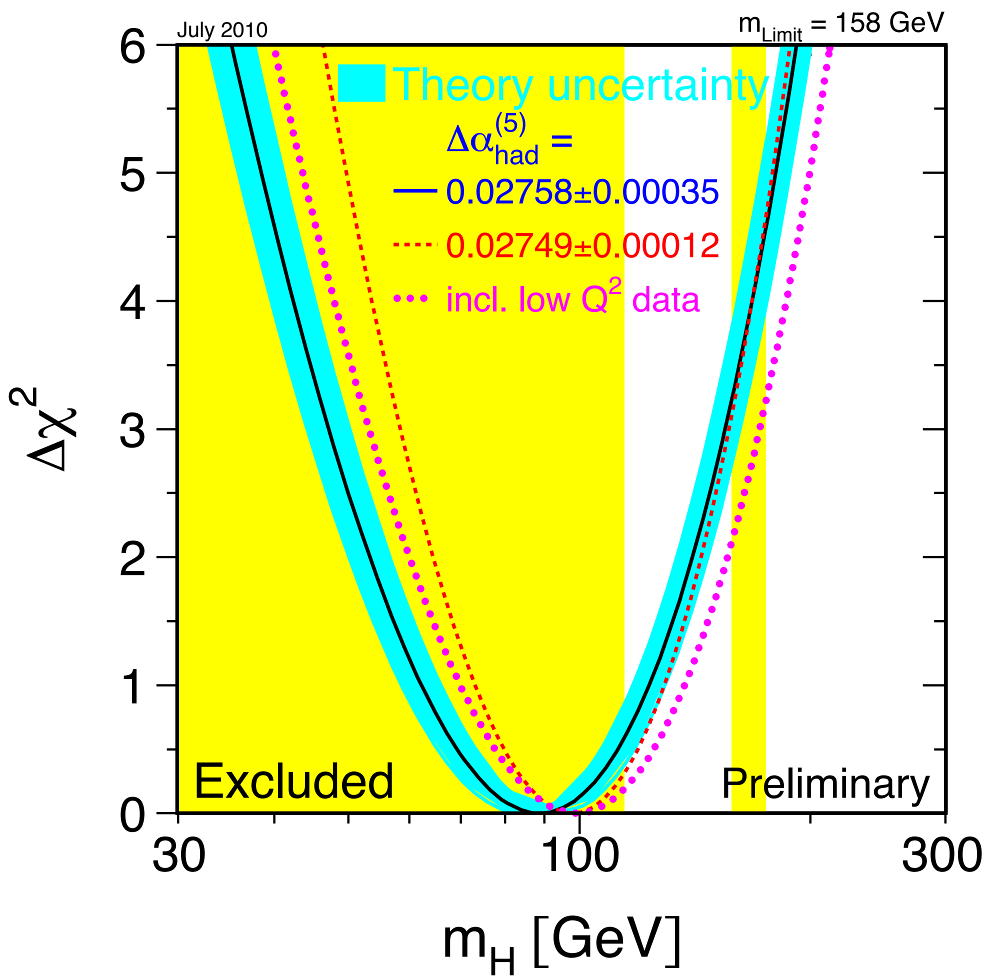
- **since $m_H^2 \simeq -2m^2 \simeq 2\lambda(v)v^2$, the smaller m_H, the smaller Λ.**
- **On the other hand, the larger m_H the larger $\lambda(v)$: if we insist on λ being in the perturbative domain, an upper bound on m_H is generated.**



The lower stability bound increases with $m_{\rm top}$.

Global fit to precision data.
The minimum shifts to larger values of m_H as $m_{\rm top}$ becomes larger.

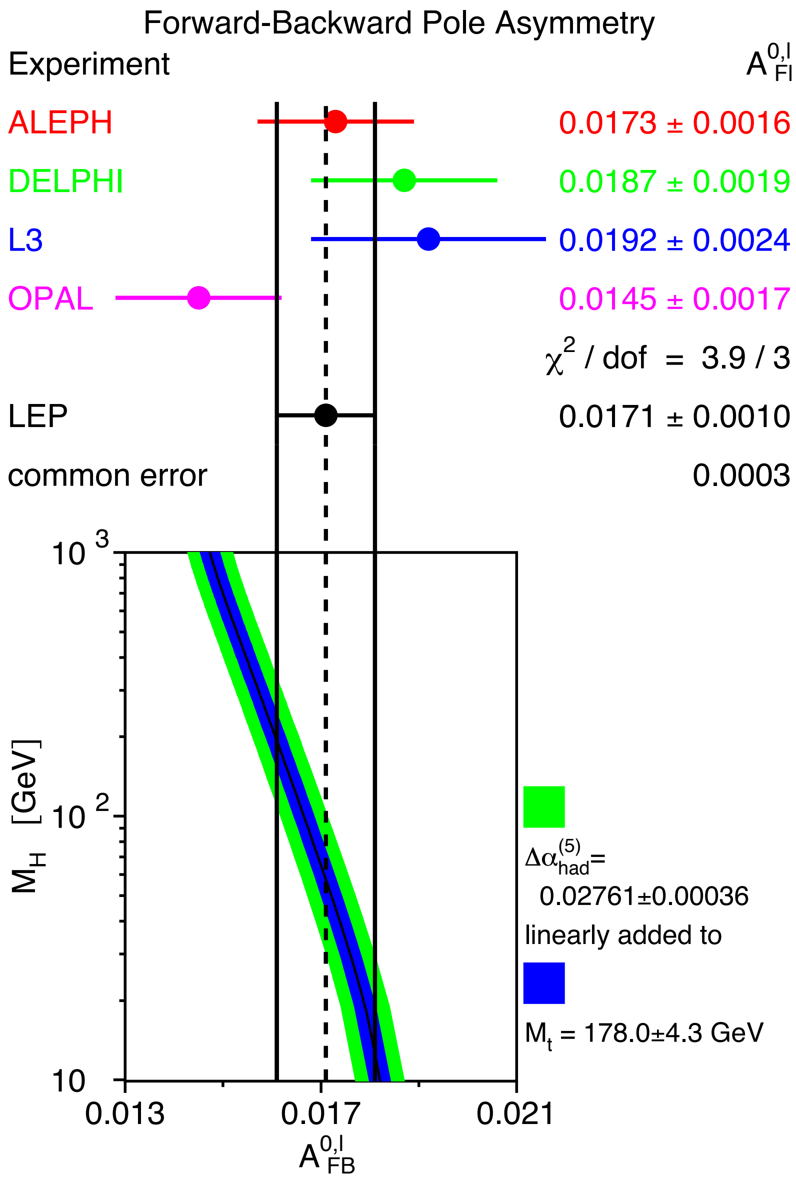
A precise determination of $m_{\rm top}$ is very important in this respect.



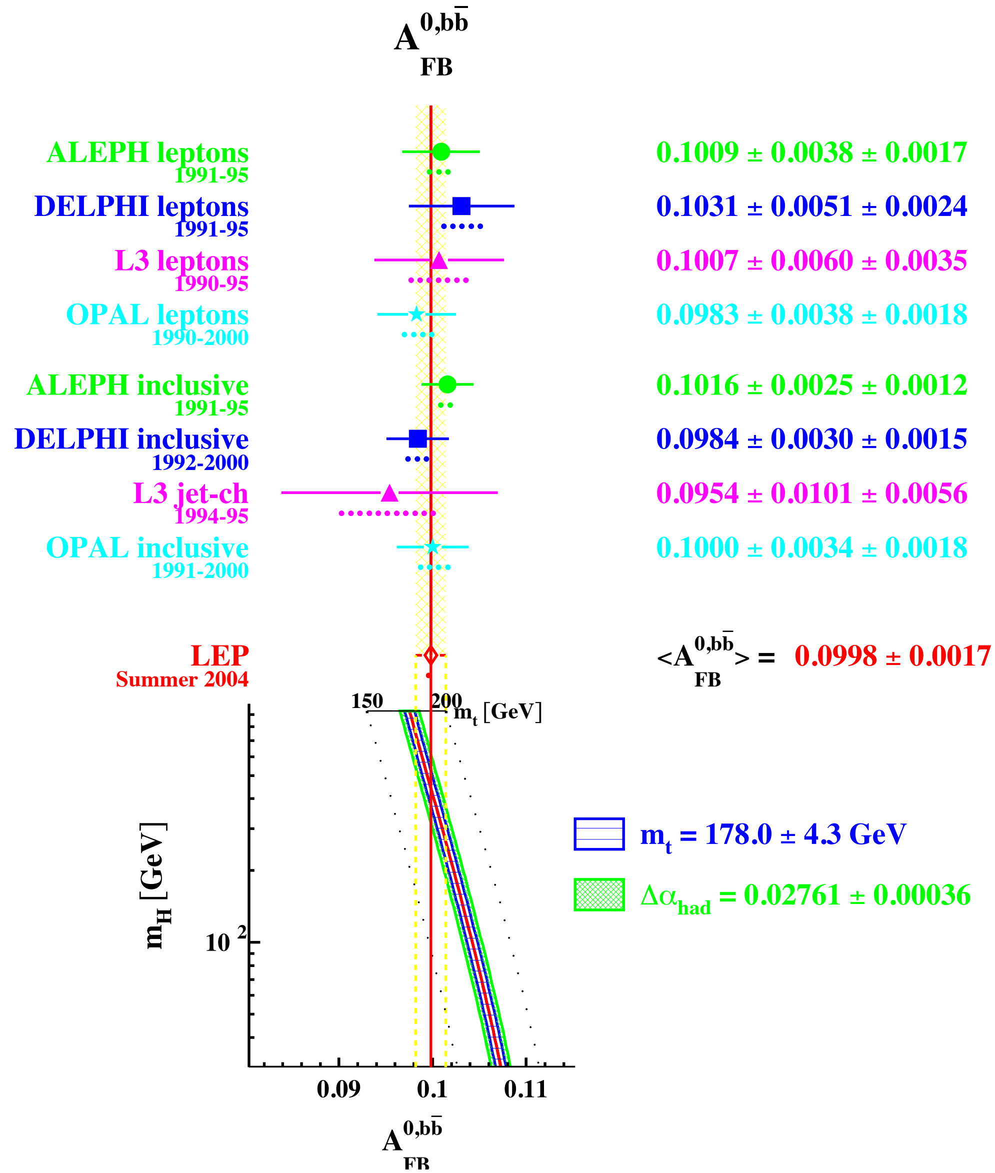
Indications that the SM Higgs mass is something between 100 and 200 GeV are definitely quite strong.

Very difficult to build extensions of the standard model in which the effects of a heavier Higgs are compensated by some kind of new physics.

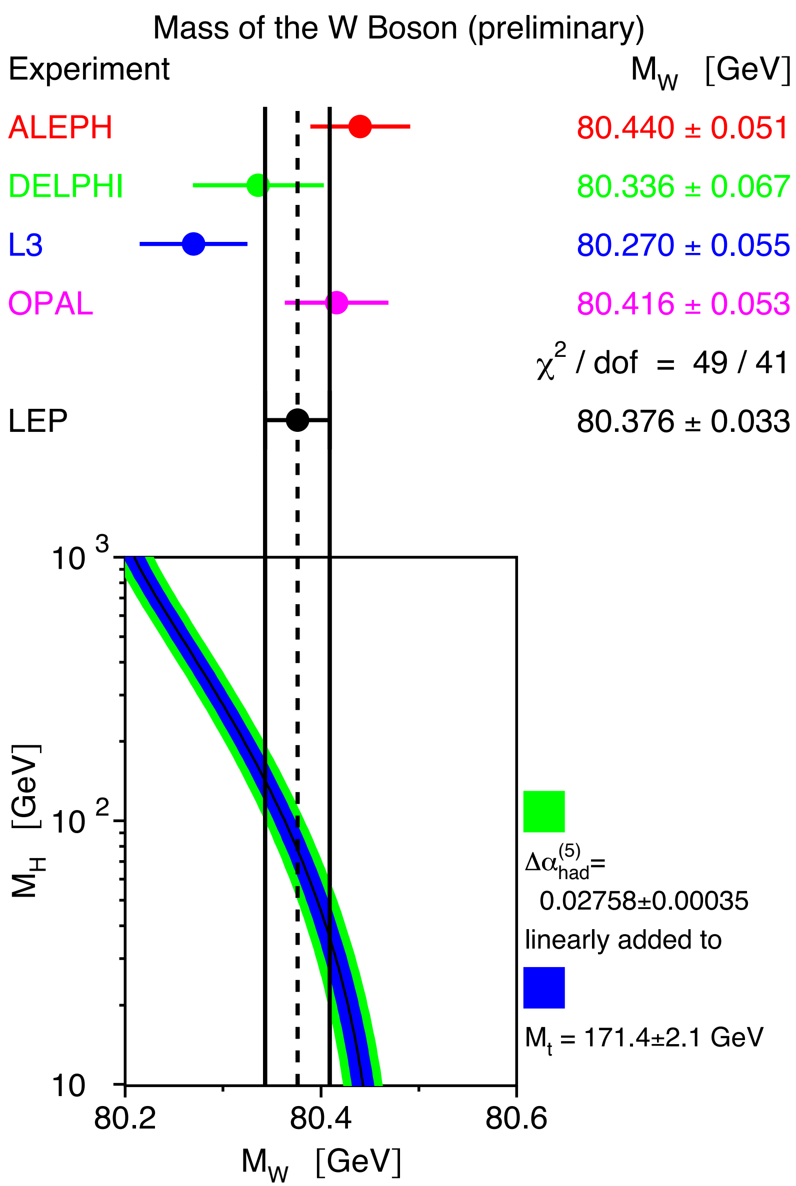
Warning: **the global SM fit is not particularly good** $[P(\chi^2) \sim 25\%]$**. Lepton asymmetries and** m_W **favour values of** m_H **below the exclusion limit** ~ 114 **GeV; hadron (especially** b**) asymmetries prefer much higher values of** m_H**.**



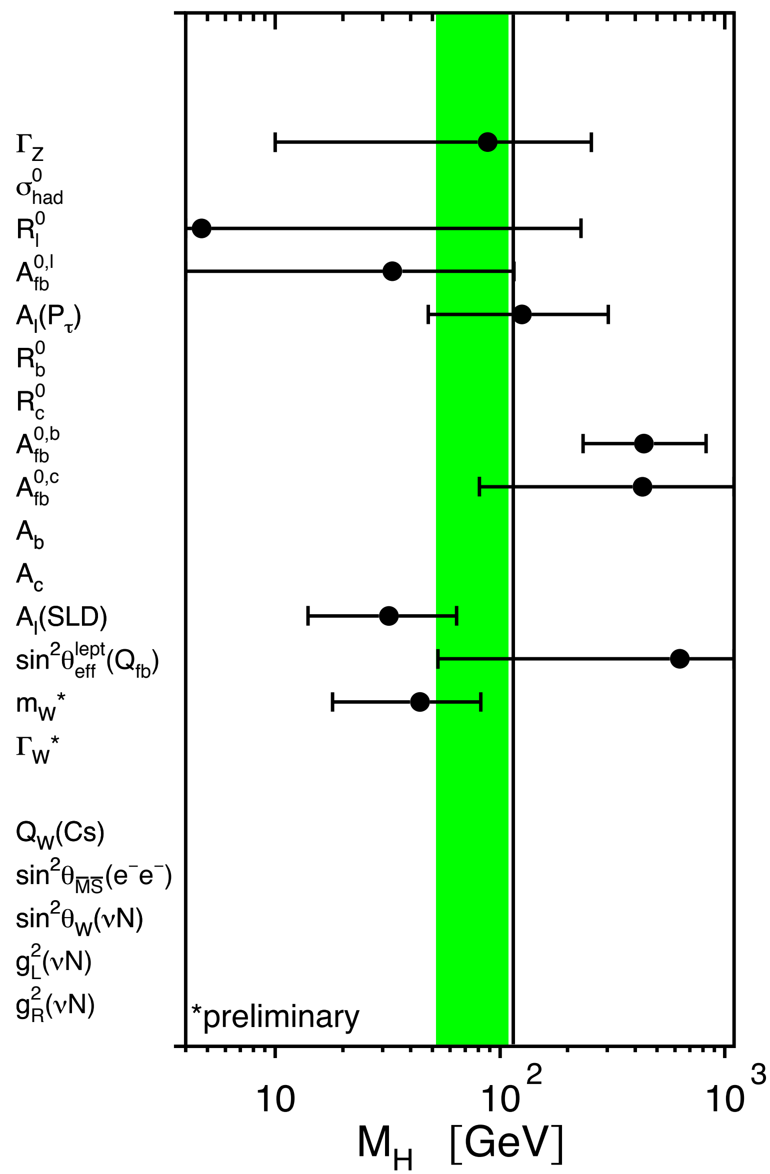
Forward-Backward Pole Asymmetry
Experiment
$A^{0,l}_{Fl}$
ALEPH 0.0173 ± 0.0016
DELPHI 0.0187 ± 0.0019
L3 0.0192 ± 0.0024
OPAL 0.0145 ± 0.0017
χ^2 / dof = 3.9 / 3
LEP 0.0171 ± 0.0010
common error 0.0003
$\Delta\alpha^{(5)}_{had}$=
0.02761±0.00036
linearly added to
M_t = 178.0±4.3 GeV
M_H [GeV]
$A^{0,l}_{FB}$



$A^{0,b\bar{b}}_{FB}$
ALEPH leptons 1991-95 0.1009 ± 0.0038 ± 0.0017
DELPHI leptons 1991-95 0.1031 ± 0.0051 ± 0.0024
L3 leptons 1990-95 0.1007 ± 0.0060 ± 0.0035
OPAL leptons 1990-2000 0.0983 ± 0.0038 ± 0.0018
ALEPH inclusive 1991-95 0.1016 ± 0.0025 ± 0.0012
DELPHI inclusive 1992-2000 0.0984 ± 0.0030 ± 0.0015
L3 jet-ch 1994-95 0.0954 ± 0.0101 ± 0.0056
OPAL inclusive 1991-2000 0.1000 ± 0.0034 ± 0.0018
LEP Summer 2004
$\langle A^{0,b\bar{b}}_{FB} \rangle$ = 0.0998 ± 0.0017
m_t [GeV]
m_H [GeV]
m_t = 178.0 ± 4.3 GeV
$\Delta\alpha_{had}$ = 0.02761 ± 0.00036



Mass of the W Boson (preliminary)
Experiment
M_W [GeV]
ALEPH
80.440 ± 0.051
DELPHI
80.336 ± 0.067
L3
80.270 ± 0.055
OPAL
80.416 ± 0.053
χ² / dof = 49 / 41
LEP
80.376 ± 0.033
10³
10²
10
M_H [GeV]
80.2
80.4
80.6
M_W [GeV]
Δα_had^(5)=
0.02758±0.00035
linearly added to
M_t = 171.4±2.1 GeV



Γ_Z
σ⁰_had
R⁰_l
A_fb^0,l
A_l(P_τ)
R⁰_b
R⁰_c
A_fb^0,b
A_fb^0,c
A_b
A_c
A_l(SLD)
sin²θ_eff^lept(Q_fb)
m_W*
Γ_W*
Q_W(Cs)
sin²θ_MS(e⁻e⁻)
sin²θ_W(νN)
g²_L(νN)
g²_R(νN)
*preliminary
10
10²
10³
M_H [GeV]

The possibility of a strongly interacting Higgs

Since

$$\lambda = \frac{1}{2}\frac{m_H^2}{v^2}$$

a large value of m_H corresponds to a large value of λ, eventually outside the perturbative domain. What should we expect if this is the case?

A first consequence:

$$\Gamma_{H\to VV} = \frac{3}{32\pi^2}\frac{m_H^3}{v^2} \sim m_H$$

for $m_H \sim 1.4$ TeV. No longer a narrow resonance.

Also, unitarity is violated at tree level in $V_L V_L$ scattering for m_H above 1 TeV.

A question arises:

Does a large value of λ generate large radiative corrections on observables?

The answer is no: the typical example is the ratio

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W}$$

which is 1 at tree level, and receives a one-loop correction

$$\Delta\rho = -\frac{11 g^2}{48\pi^3} \tan^2\theta_W \log\frac{m_H^2}{m_W^2}$$

which grows only logarithmically with m_H^2.

This arises from a symmetry property of the scalar potential, called the custodial symmetry.

A different way to state the problem: after the inclusion of radiative corrections,

$$\mathcal{L}_{\rm mass} = \frac{1}{2} m_W^2 (W^{1\mu} W^1_\mu + W^{2\mu} W^2_\mu) + \frac{1}{2} (W_3^\mu \ \ B^\mu) \begin{bmatrix} M^2 & M'^2 \\ M'^2 & M''^2 \end{bmatrix} \begin{pmatrix} W_{3\mu} \\ B_\mu \end{pmatrix}.$$

with $M'^2 = MM''$**,** $M^2 + M''^2 = m_Z^2$**. Hence**

$$\tan\theta_W = \frac{\sqrt{m_Z^2 - M^2}}{M}.$$

and

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = \frac{m_W^2}{M^2},$$

that is, $\rho = 1$ **only if** $M^2 = m_W^2$**.**

The reason is that the scalar potential posesses an $O(4)$ invariance, larger than the gauge symmetry. Indeed, the scalar field

$$\tilde{\phi} = \left(\begin{array}{c} \phi_0^* \\ -\phi^- \end{array} \right) = \epsilon \phi^*$$

can be shown to be an $SU(2)_L$ doublet with $Y_{\tilde{\phi}} = -1$. Hence the matrix

$$\mathcal{H} = \left[\tilde{\phi} \;\; \phi \right] = \left[\begin{array}{cc} \phi^{0\,*} & \phi^+ \\ -\phi^- & \phi^0 \end{array} \right]$$

transforms as

$$\mathcal{H}(x) \to \exp \left[\frac{ig}{2} \tau^a \alpha^a(x) \right] \mathcal{H}(x)$$

$$\mathcal{H}(x) \to \mathcal{H}(x) \exp \left[-\frac{ig'}{2} \beta(x) \tau_3 \right] .$$

under gauge transformations.

The scalar potential can be written

$$V(\phi) = \frac{1}{2} m^2 \,\mathrm{Tr}\left(\mathcal{H}^\dagger \mathcal{H}\right) + \frac{1}{4}\lambda \left[\mathrm{Tr}\left(\mathcal{H}^\dagger \mathcal{H}\right)\right]^2$$

which is invariant under the $SU(2)_L \times SU_R(2)$ transformations

$$\mathcal{H} \to U \mathcal{H} V^\dagger$$

where $V \in SU(2)$.

We also have

$$(D_\mu \phi)^\dagger D^\mu \phi = \frac{1}{2} \mathrm{Tr}\left[(D_\mu \mathcal{H})^\dagger D^\mu \mathcal{H}\right],$$

with

$$D_\mu \mathcal{H} = \partial_\mu \mathcal{H} - \frac{ig}{2} W_\mu^a \tau^a \mathcal{H} + \frac{ig'}{2} B_\mu \mathcal{H} \tau_3$$

Clearly

$$\begin{aligned} D_\mu \mathcal{H} &\rightarrow (U D_\mu U^\dagger)(U \mathcal{H} V^\dagger) \\ &= U(\partial_\mu \mathcal{H} - \frac{ig}{2} W_\mu^a \tau^a \mathcal{H}) V^\dagger + \frac{ig'}{2} B_\mu U \mathcal{H} V^\dagger \tau_3 \\ &\neq U(D_\mu \mathcal{H}) V^\dagger \end{aligned}$$

because of the last term.

The full lagrangian has a custodial symmetry only for $g' = 0$.

Due to spontaneous breaking of $SU(2)_L$, the vacuum expectation value

$$\langle 0|\mathcal{H}|0\rangle = \frac{v}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is not $O(4)$ invariant. However, there is a residual $O(3) \sim SU(2)$ symmetry

$$\mathcal{H} \to U\mathcal{H}U^{\dagger}$$

that leave the vacuum expectation value unchanged. The only mass term for the W_μ^i fields allowed by this residual symmetry is proportional to $W_\mu^i W_i^\mu$, which in turn implies $M^2 = m_w^2$ and $\rho = 1$.

If the Higgs boson is too heavy to be produced directly, the study of longitudinal gauge bosons can be useful.

Scattering amplitudes can be computed in the standard way, but it proves easier and more instructive to use the so called equivalence theorem:

Scattering amplitudes for longitudinal vector bosons are equal to the corresponding amplitudes for would-be goldstone bosons, up to corrections of order m_V/E:

$$\mathcal{M}(W_L^+, W_L^-, Z_L, \ldots) = \mathcal{M}(G^+, G^-, G_3, \ldots) + \mathcal{O}\left(\frac{m_W}{E}\right)$$

We can build an effective theory, which describes correctly the dynamics of longitudinal vector bosons (i.e. Goldstone bosons G_i) in the range of intermediate energies $m_W \ll E \ll m_H$.

[A simple proof of the equivalence theorem is based on the use of the 't Hooft gauge-fixing term.]

In this regime, the dynamics of Goldstone bosons is governed by the scalar potential. By analogy with chiral effective theories in the strong interactions, we may find a nonlinear realization of the custodial symmetry, such that the Goldstone fields are removed from the scalar potential, and take up only derivative couplings among themselves.

This can be done as follows. Since

$$\mathcal{H}^\dagger \mathcal{H} = \begin{pmatrix} \phi^\dagger \phi & 0 \\ 0 & \phi^\dagger \phi \end{pmatrix}$$

we may parametrize $\mathcal{H}$ as $|\phi|$ times a unitary matrix Σ:

$$\mathcal{H} = \frac{v + H(x)}{\sqrt{2}} \Sigma(x); \qquad \Sigma = \exp\left[\frac{iG_a(x)\tau_a}{v}\right]$$

so that

$$V(\phi) = \frac{1}{2} m^2 \,\mathrm{Tr}\left(\mathcal{H}^\dagger \mathcal{H}\right) + \frac{1}{4}\lambda \left[\mathrm{Tr}\left(\mathcal{H}^\dagger \mathcal{H}\right)\right]^2$$

does not contain the fields $G_a(x)$ any more.

The effective lagrangian for Goldstone bosons is therefore

$$\mathcal{L} = \frac{v^2}{4} \mathrm{Tr} \left[(D_\mu \Sigma)^\dagger D^\mu \Sigma \right]$$

up to terms involving the Higgs field $H(x)$, which we neglect by the assumption that m_H is so large that it does not enter the low-energy dynamics of Goldstone bosons.

Goldstone bosons couplings are now purely derivative: amplitudes are computed as expansions in powers of momenta.

3. Breaking of accidental symmetries

Consider $n_q = n_\ell = 1$ for simplicity. Then

$$\mathcal{L}_{SM} = \mathcal{L}_{\text{Yang-Mills}} + \sum_{r=1}^{5} \bar{\psi}_r \, i\not{D}_r \, \psi_r + \mathcal{L}_{\text{Higgs}}$$

has a $[U(1)]^5$ global invariance:

$$\psi_r \to e^{i\alpha_r} \, \psi_r$$

The corresponding conserved currents are

$$\begin{array}{lcl}
J_1^\mu = \bar{u}_L \gamma^\mu u_L + \bar{d}_L \gamma^\mu d_L & & J_Y^\mu = \sum_{r=1}^5 \frac{Y_r}{2} J_r^\mu \\
J_2^\mu = \bar{u}_R \gamma^\mu u_R & & J_\ell^\mu = J_4^\mu + J_5^\mu \equiv \bar{\nu}\gamma^\mu \nu + \bar{e}\gamma^\mu e \\
J_3^\mu = \bar{d}_R \gamma^\mu d_R & \to & J_{\ell 5}^\mu = J_5^\mu - J_4^\mu \equiv \bar{\nu}\gamma^\mu \gamma_5 \nu + \bar{e}\gamma^\mu \gamma_5 e \\
J_4^\mu = \bar{\nu}_L \gamma^\mu \nu_L + \bar{e}_L \gamma^\mu e_L & & J_b^\mu = \frac{1}{3}(J_1^\mu + J_2^\mu + J_3^\mu) \equiv \frac{1}{3}(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d) \\
J_5^\mu = \bar{e}_R \gamma^\mu e_R & & J_{b5}^\mu = J_2^\mu + J_3^\mu - J_1^\mu \equiv \bar{u}\gamma^\mu \gamma_5 u + \bar{d}\gamma^\mu \gamma_5 d.
\end{array}$$

Conserved charges:

Y	**local symmetry**
$N_L - N_{\bar{L}}$	**OK**
$N_B - N_{\bar{B}}$	**OK**
$N_L + N_{\bar{L}}$	**not observed**
$N_B + N_{\bar{B}}$	**not observed**

With N families, the symmetry is much larger: generation mixings also allowed,

$$\psi_r^i \to U_r^{ij}\,\psi_r^j; \quad U_r^\dagger U_r = I$$

A global $[U(N)]^5$ symmetry which is not present in observed phenomena. This is called an accidental symmetry.

Accidental symmetries are an accidental consequence of gauge invariance and renormalizability. For example, fermion mass terms would break accidental symmetries explicitly:

$$-m\bar{\psi}\psi = -m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$$

but they are forbidden by gauge invariance.

Accidental symmetries can be broken (while preserving gauge invariance) by fermion couplings to ϕ:

$$\mathcal{L}_{SM} = \mathcal{L}_{\text{Yang-Mills}} + \sum_{i=1}^{N} \sum_{r=1}^{5} \bar{\psi}_r^i \, i \not{D}_r \, \psi_r^i + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}$$

$$\psi_1^i \equiv Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix}; \quad \psi_2^i \equiv u_R^i; \quad \psi_3^i \equiv d_R^i; \quad \psi_4^i \equiv L_L^i = \begin{pmatrix} \nu_L^i \\ \ell_L^i \end{pmatrix}; \quad \psi_5^i \equiv \ell_R^i$$

$$\mathcal{L}_{\text{Yukawa}} = -\bar{Q}_L^i \, h_u^{ij} \, u_R^j \, \tilde{\phi} - \bar{Q}_L^i \, h_d^{ij} \, d_R^j \, \phi - \bar{L}_L^i \, h_\ell^{ij} \, \ell_R^j \, \phi + \text{h.c.}$$

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \qquad \tilde{\phi} = \begin{pmatrix} \phi_0^* \\ -\phi^- \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

- $\mathcal{L}_{\text{Yukawa}}$ **is allowed by Lorentz invariance, gauge symmetry and renormalizability.**
- **Each term in** $\mathcal{L}_{\text{Yukawa}}$ **breaks part of the accidental symmetry explicitly; for example, the first term is not invariant under independent** $U(N)$ **rotation of right-handed up quarks and left-handed quark doublets**

$$Q_L \to U\, Q_L; \qquad u_R \to V\, u_R$$

(although still invariant under the subgroup $U = V$**).**

- **Important remark: For the same reason, removing one or more term from** $\mathcal{L}_{\text{Yukawa}}$ **increases the symmetry of the theory. Yukawa couplings are protected from receiving large radiative corrections.**

In matrix notation

$$\mathcal{L}_{\text{Yukawa}} = -\bar{Q}_L\, h_u\, u_R\, \tilde{\phi} - \bar{Q}_L\, h_d\, d_R\, \phi - \bar{L}_L\, h_\ell\, \ell_R\, \phi + \text{h.c.}$$

h_u, h_d, h_ℓ **are generic complex** $N \times N$ **matrices.**

A theorem in linear algebra: Any generic complex squared matrix h **can be diagonalized by a bi-unitary transformation**

$$\hat{h} = U^\dagger\, h\, V$$

where U, V **are unitary matrices, and** $\hat{h}$ **is diagonal with real positive entries.**

Thus, for example, we may redefine the lepton fields by

$$L_L \to U\, L_L; \qquad \ell_R \to V\, \ell_R$$

with U, V **such that**

$$\hat{h}_\ell = U^\dagger\, h_\ell\, V$$

is diagonal with real and positive entries.

The theory is otherwise unaffected, because this operation leaves the rest of $\mathcal{L}_{SM}$ unchanged. Hence, in the leptonic sector,

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}}^{\text{lept}} &= -\bar{L}_L\,\hat{h}_\ell\,\ell_R\,\phi + \text{h.c.} \to -\frac{1}{\sqrt{2}}\bar{\ell}_L\,\hat{h}_\ell\,\ell_R\,(v+H) + \text{h.c.} \\ &= -m_e\bar{e}e - m_\mu\bar{\mu}\mu - m_\tau\bar{\tau}\tau - \frac{H}{\sqrt{2}}(\hat{h}_e\bar{e}e + \hat{h}_\mu\bar{\mu}\mu + \hat{h}_\tau\bar{\tau}\tau)\end{aligned}$$

- **Lepton masses $m_\ell^i = \frac{\hat{h}_\ell^i\,v}{\sqrt{2}}$ are generated**
- **The original global symmetry is broken, but a residual $[U(1)]^3$ invariance**

  $$\ell^i \to e^{i\alpha_i}\,\ell^i$$

  is still present. This symmetry corresponds to the conservation of individual (e,μ,τ) leptonic numbers.

The same argument does not apply to the hadron sector:

$$\mathcal{L}_{\text{Yukawa}}^{\text{hadr}} = -\bar{Q}_L \, h_u \, u_R \, \tilde{\phi} - \bar{Q}_L \, h_d \, d_R \, \phi + \text{h.c.}$$

We may transform the quark fields

$$u_L \to U_L \, u_L; \quad d_L \to V_L \, d_L; \quad u_R \to U_R \, u_R; \quad u_R \to V_R \, u_R$$

with $U_{L,R}, V_{L,R}$ **chosen so that**

$$\hat{h}_u = U_L^\dagger \, h_u \, U_R; \qquad \hat{h}_d = V_L^\dagger \, h_d \, V_R$$

are diagonal, but this is *not* **a symmetry for the rest of the Lagrangian.**

Only one term is affected by such a rotation: the charged-current interaction term in the hadron sector

$$\begin{aligned}\mathcal{L}_{cc} &= \frac{g}{\sqrt{2}}\left[\bar{u}_L\,\gamma^\mu\,d_L\,W^+_\mu + \bar{d}_L\,\gamma^\mu\,u_L\,W^-_\mu\right] \\ &\to \frac{g}{\sqrt{2}}\left[\bar{u}_L\,\gamma^\mu\,(U_L^\dagger V_L) d_L\,W^+_\mu + \bar{d}_L\,\gamma^\mu\,(V_L^\dagger U_L) u_L\,W^-_\mu\right]\end{aligned}$$

The matrix

$$V = U_L^\dagger V_L$$

is a unitary $N \times N$ matrix, usually called the Cabibbo-Kobayashi-Maskawa matrix.

With the Yukawa couplings in diagonal form we have

$$\begin{aligned}\mathcal{L}^{\text{hadr}}_{\text{Yukawa}} &= -\bar{Q}_L\,\hat{h}_u\,u_R\,\tilde{\phi} - \bar{Q}_L\,\hat{h}_d\,d_R\,\phi + \text{h.c.} \\ &\to -\frac{1}{\sqrt{2}}(v+H)\left[\bar{u}_L\,\hat{h}_u\,u_R + \bar{d}_L\,\hat{h}_d\,d_R\right] + \text{h.c.}\end{aligned}$$

- **Quark mass terms appear:**

$$m_u^i = \frac{\hat{h}_u^i v}{\sqrt{2}} \qquad m_d^i = \frac{\hat{h}_d^i v}{\sqrt{2}}$$

- **The original global symmetry is lost; the residual symmetry is now a $U(1)$ symmetry**

$$u_L^i \to e^{i\alpha}\,u_L^i; \qquad d_L^i \to e^{i\alpha}\,d_L^i; \qquad u_R^i \to e^{i\alpha}\,u_R^i; \qquad d_R^i \to e^{i\alpha}\,d_R^i$$

with a common phase α for all flavours, because of the CKM mixing matrix. Baryon number conservation.

The entries of the CKM matrix are fundamental parameters of the theory: they must be extracted from experiments.

How many independent numbers does V contain? A generic $N \times N$ unitary matrix depends on N^2 independent real parameters. Some (N_A) of them can be thought of as rotation angles in the N-dimensional space of generations, and they are as many as the coordinate planes in N dimensions:

$$N_A = \begin{pmatrix} N \\ 2 \end{pmatrix} = \frac{1}{2}\, N\, (N-1).$$

The remaining

$$\hat{N}_P = N^2 - N_A = \frac{1}{2}\, N\,(N+1)$$

parameters are complex phases. Some can be removed by a redefinition of left-handed quarks:

$$u_L^f \to e^{i\alpha_f}\, u_L^f; \qquad d_L^g \to e^{i\beta_g}\, d_L^g$$

which leaves all terms in$\mathcal{L}_{\rm SM}$ **unchanged except** $\mathcal{L}_c^{\rm hadr}$**, and therefore amount to a redefinition of the CKM matrix:**

$$V_{fg} \to e^{i(\beta_g - \alpha_f)}\, V_{fg}$$

The $2N$ **constants** α_f, β_g **can be chosen so that** $2N-1$ **phases are eliminated from the matrix** V**, since there are** $2n-1$ **independent differences** $\beta_g - \alpha_f$**. The number of really independent complex phases in** V **is therefore**

$$N_P = \hat{N}_P - (2N-1) = \frac{1}{2}(N-1)(N-2)$$

To summarize, the total number of independent parameters in the CKM matrix is

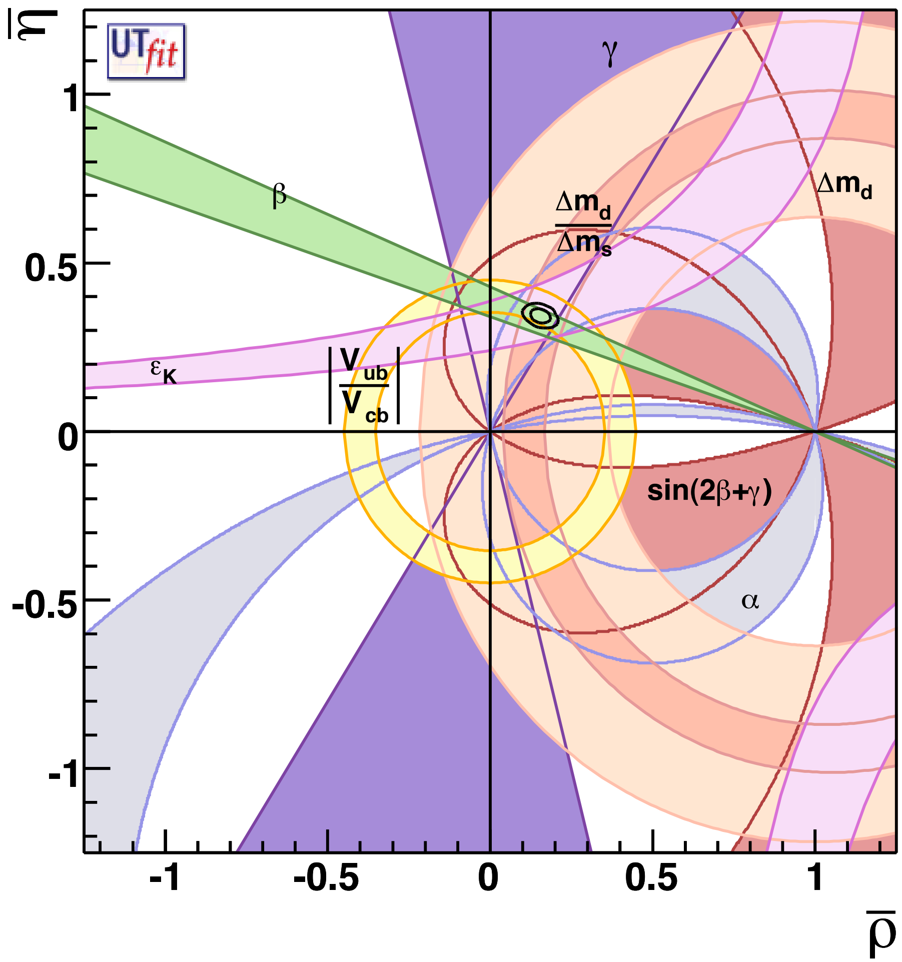
$$N_A + N_P = (N-1)^2; \qquad N_P = \frac{1}{2}(N-1)(N-2)$$

Comments:

- **with $N = 1$ or $N = 2$ the CKM matrix can be made real. In particular, for $N = 2$ it is fixed by one rotation angle, the Cabibbo angle.**
- **The first case with non-trivial phases is $N = 3$, which corresponds to $N_P = 1$.**
- **The presence of complex coupling constants implies violation of the CP symmetry.**

Much effort devoted to investigations in the flavour sector. A subject of special interest: CP violation in B systems and the unitarity relation

$$V_{ud}V_{ub}^{*} + V_{cd}V_{cb}^{*} + V_{td}V_{tb}^{*} = 0$$



Unitary triangle fit in the SM in the $\bar{\rho}$ - $\bar{\eta}$ plane

UTfit Collaboration

Very nice:

- **Fermion masses generated**
- **All global symmetries broken except global baryon number and individual lepton numbers (no right-handed neutrinos)**
- **FCNC effects suppressed (flavour mixing confined in the charged-current sector)**
- **CP violated for $N \geq 3$**

Not easy to achieve the same result in different contexts.

(Slightly) beyond the Standard Model: neutrino masses

In the standard model, neutrinos are massless. With good reasons: the experimental upper bounds on neutrino masses are

$$m_{\nu_e} \leq 3 \text{ eV}; \qquad m_{\nu_\mu} \leq 0.19 \text{ MeV}; \qquad m_{\nu_\tau} \leq 18.2 \text{ MeV},$$

so $m_\nu \ll m_f$, and $m_\nu = 0$ is an excellent approximation.

However: non-zero neutrino masses are now a solid experimental evidence, thus we must ask how to modify the standard model in order to keep this evidence into account.

Standard neutrinos are massless because right-handed neutrinos do not exist (more precisely, they transform trivially under the gauge group, and therefore undergo no interaction: they are sterile objects).

Let us now assume right-handed neutrinos do exist (only one generation, for simplicity) with a covariant derivative term

$$\bar{\nu}_R \, i \not{D} \nu_R \equiv \bar{\nu}_R \, i \not{\partial} \, \nu_R .$$

A Dirac mass term can be generated as in the case of up-type quarks:

$$\mathcal{L}_{\text{Yukawa}} \to \mathcal{L}_{\text{Yukawa}} - h_\nu \left[\bar{\ell}_L \, \tilde{\phi} \, \nu_R + \bar{\nu}_R \, \tilde{\phi}^\dagger \, \ell_L \right]$$

contains a term

$$-m \, \left(\bar{\nu}_L \, \nu_R + \bar{\nu}_R \, \nu_L \right) ; \qquad m = \frac{h_\nu v}{\sqrt{2}}$$

The see-saw mechanism

Why are neutrino masses so much smaller than other fermions' masses? Indeed, the experimental bound implies

$$\frac{h_\nu}{h_e} = \frac{m}{m_e} \lesssim 10^{-6}$$

difficult to understand.

However, right-handed neutrinos also admit a Majorana mass term:

$$-\frac{1}{2}\, M \,\left(\bar{\nu}_R^c\, \nu_R + \bar{\nu}_R\, \nu_R^c\right)$$

where $\nu_R^c = \gamma_0 \gamma_2 \bar{\nu}_R^T$ is the charge-conjugated spinor (not true for other fermions, e.g. ν_L, because of gauge invariance).

Majorana mass terms induce violation of lepton number conservation, typically suppressed by inverse powers of M. It is natural to assume that M is of the order of the energy scale characteristic of the unknown phenomena (e.g. the effects of grand unification) experienced by right-handed neutrinos.

The most general neutrino mass term:

$$\mathcal{L}_{\nu\,\mathrm{mass}} = -\frac{1}{2}\left(\bar{\nu}_L^c\ \bar{\nu}_R\right)\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}\begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \mathrm{h.c.}$$

diagonalized by a linear transformation

$$\begin{pmatrix} 0 & m \\ m & M \end{pmatrix} = \mathcal{U}^T \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \mathcal{U}$$

with $\mathcal{U}$ unitary, and m_1, m_2 real and positive:

$$\mathcal{U} = \begin{pmatrix} i\cos\theta & -i\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \qquad \tan 2\theta = \frac{2m}{M}$$

and

$$m_1 = \frac{1}{2}\left(\sqrt{M^2+4m^2} + M\right); \qquad m_2 = \frac{1}{2}\left(\sqrt{M^2+4m^2} - M\right).$$

For $m \ll M$, $\theta \simeq m/M$, and

$$m_1 \simeq M; \qquad m_2 \simeq \frac{m^2}{M}$$

One eigenstate not observed at low energy; the other is lighter than ordinary fermions by a factor m/M.

This is the *see-saw* mechanism.

The mass term takes the form

$$\mathcal{L}_{\nu\,\mathrm{mass}} = -\frac{1}{2}\,m_1\,\left(\bar{\nu}_1^c\,\nu_1 + \bar{\nu}_1\,\nu_1^c\right) - \frac{1}{2}\,m_2\,\left(\bar{\nu}_2^c\,\nu_2 + \bar{\nu}_2\,\nu_2^c\right),$$

where

$$\nu_1 = \nu_L\,\sin\theta + \nu_R^c\,\cos\theta$$
$$\nu_2 = -i\nu_L\,\cos\theta + i\nu_R^c\,\sin\theta$$

General case: N species of left-handed neutrinos, ($N = 3$ as far as we know), plus K right-handed neutrinos (not necessarily $N = K$).

m is a $K \times N$ matrix, and M a $K \times K$ matrix.

Choosing $K = N$ we have

$$\begin{aligned}\mathcal{L}_Y^{\rm lept} \quad &= \quad - \left[\overline{\ell}_L \, \phi \, h_{\rm E} \, e_R + \overline{e}_R \, \phi^\dagger \, h_{\rm E}^\dagger \, \ell_L\right] \\ & \quad\quad - \left[\bar{\ell}_L \, \tilde{\phi} \, h_{\rm N} \, \nu_R + \bar{\nu}_R \, \tilde{\phi}^\dagger \, h_{\rm N}^\dagger \, \ell_L\right]\end{aligned}$$

The Majorana mass terms for right-handed neutrinos are

$$-\frac{1}{2} \, \left(\bar{\nu}_R^{\prime c} \, M \, \nu_R + \bar{\nu}_R \, M^\dagger \, \nu_R^{\prime c}\right)$$

Lepton flavour eigenstates are linear combinations of mass eigenstates. Neutrinos produced with a definite flavour (e.g. nuclear β decays in the Sun produce electron neutrinos)

A neutrino beam of definite flavour, is a linear combination of mass eigenstates:

$$|\nu_\alpha\rangle = \sum_{i=1}^{n} U^*_{\alpha i} |\nu_i\rangle$$

with U a unitary matrix. Time evolution in the rest frame is given by

$$|\nu_i(\tau)\rangle = e^{-im_i\tau} |\nu_i(0)\rangle$$

or, in the laboratory frame,

$$|\nu_i(t)\rangle = e^{-i(E_i t - p_i L)} |\nu_i(0)\rangle$$

where L is the distance travelled in the time interval t.

Since neutrinos are almost massless,

$$L \simeq t; \qquad E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2E}$$

where $E \simeq p_i \simeq p_j$**. Hence,**

$$|\nu_\alpha(L)\rangle \simeq \sum_{i=1}^{n} U_{\alpha i}^* \exp\left(-i\frac{m_i^2}{2E}L\right)|\nu_i(0)\rangle$$

The probability amplitude of observing the flavour β **at distance** L **is given by**

$$\begin{aligned}\langle\nu_\beta|\nu_\alpha(L)\rangle &= \sum_{i=1}^{n} U_{\alpha i}^* \exp\left(-i\frac{m_i^2}{2E}L\right)\sum_{j=1}^{n} U_{\beta j}\langle\nu_j|\nu_i\rangle \\ &= \sum_{i=1}^{n} \xi_i^{\alpha\beta} \exp\left(-i\epsilon_i L\right),\end{aligned}$$

where we have used the unitarity of U**, and we have defined**

$$\xi_i^{\alpha\beta} = U_{\alpha i}^* U_{\beta i}; \qquad \epsilon_i = \frac{m_i^2}{2E}.$$

The corresponding probability is given by

$$P_{\alpha\beta}(L) = |\langle\nu_\beta|\nu_\alpha(L)\rangle|^2 = \delta\alpha\beta - 4\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathrm{Re}\left(\xi_i^{\alpha\beta}\,\xi_j^{*\alpha\beta}\right)\sin^2\frac{1}{2}(\epsilon_j-\epsilon_i)L$$
$$-2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathrm{Im}\left(\xi_i^{\alpha\beta}\,\xi_j^{*\alpha\beta}\right)\sin(\epsilon_j-\epsilon_i)L.$$

Not very rigorous: Quantum states with definite momentum have an infinite uncertainty in position, and therefore it makes no sense to talk about observation at distance L. We should introduce wave packets, and check that a sizable overlap among packets survives at distance L from the source.

If this is the case, the oscillation probability is correctly given by the above formula.

A simple case: CP invariance + mixing between two flavours. In this case

$$\xi_1^{12} = \xi_1^{21} = -\cos\theta_{12}\sin\theta_{12}$$
$$\xi_2^{12} = \xi_2^{21} = +\cos\theta_{12}\sin\theta_{12}.$$

and therefore

$$P_{12}(L) = P_{21}(L) = \sin^2 2\theta_{12}\,\sin^2\frac{L\Delta m_{12}^2}{4E}.$$

The results of neutrino oscillation experiments are usually displayed in the form of allowed regions in the $(\Delta m_{12}^2, \theta_{12})$ **plane. The following units are often adopted:**

$$\frac{L\Delta m^2}{4E} \simeq 1.27\,\frac{\Delta m^2(\mathrm{eV}^2)\,L(\mathrm{km})}{E(\mathrm{GeV})}.$$

To summarize:

- **Massive neutrinos bring us out of the Standard Model**
- **Heavy sterile neutrinos + see-saw mechanism: a satisfactory scenario**
- **New parameters needed, but no radical modification**

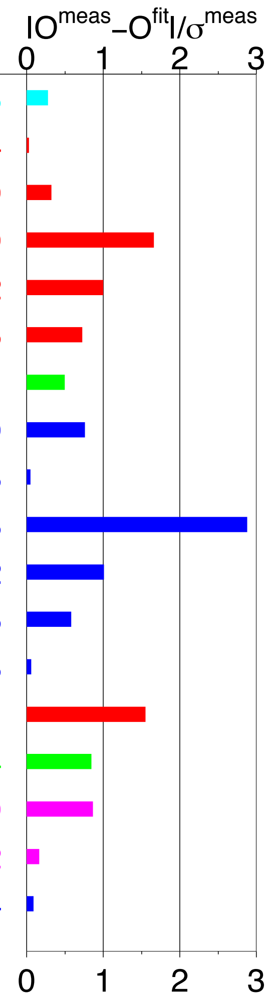
First experimental confirmations

- **1974 charm quark and weak neutral currents observed**
- **1977 bottom quark**
- **1983 W and Z bosons observed**
- **1994 top quark observed**

Precision data in excellent agreement with SM predictions. The standard model tested at the level of one-loop corrections.

$$N_\nu = 2.9841 \pm .0083$$

	Measurement	Fit
$\Delta\alpha_{had}^{(5)}(m_Z)$	0.02758 ± 0.00035	0.02768
m_Z [GeV]	91.1875 ± 0.0021	91.1874
Γ_Z [GeV]	2.4952 ± 0.0023	2.4959
σ_{had}^0 [nb]	41.540 ± 0.037	41.479
R_l	20.767 ± 0.025	20.742
$A_{fb}^{0,l}$	0.01714 ± 0.00095	0.01645
$A_l(P_\tau)$	0.1465 ± 0.0032	0.1481
R_b	0.21629 ± 0.00066	0.21579
R_c	0.1721 ± 0.0030	0.1723
$A_{fb}^{0,b}$	0.0992 ± 0.0016	0.1038
$A_{fb}^{0,c}$	0.0707 ± 0.0035	0.0742
A_b	0.923 ± 0.020	0.935
A_c	0.670 ± 0.027	0.668
A_l(SLD)	0.1513 ± 0.0021	0.1481
$\sin^2\theta_{eff}^{lept}(Q_{fb})$	0.2324 ± 0.0012	0.2314
m_W [GeV]	80.399 ± 0.023	80.379
Γ_W [GeV]	2.085 ± 0.042	2.092
m_t [GeV]	173.3 ± 1.1	173.4



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4. Open questions

1. **Why so many (18) free parameters?**
   - **Is there a grand unification?**
   - **What is the origin of flavour mixing?**
2. **Why is the Higgs boson so light?**
   - **Hierarchy and Naturalness**
3. **Cosmology-related questions:**
   - **the value of the cosmological constant**
   - **baryogenesis**
   - **dark matter**
4. **Gravity**