Electroweak LHC: High-energy lepton colliders

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Theory and Phenomenology of Fundamental Interactions

UNIVERSITY AND INFN · BOLOGNA

The Ohio State University

Our goal and the dream machine

A lot of particle physics is missing in the Standard Model

- Why Electroweak Symmetry Breaking occurs?
  What is the history of the Electroweak Phase Transition?
- The reason for the Hierarchy in Fermion Masses and their Flavor Structure
- The Nature of Dark Matter
- The origin of the Matter-Antimatter Asymmetry
- The generation of Neutrino Masses
- The cause of the Universe's accelerated expansion - Dark Energy
- What are the quantum properties of Gravity?
- What caused Cosmic Inflation after the Big Bang?

The SM is silent about all above, BSM physics is at the core of it all

The colliders

Our goal is to "**Address the Big Questions**" and to "**Explore the unknown**"

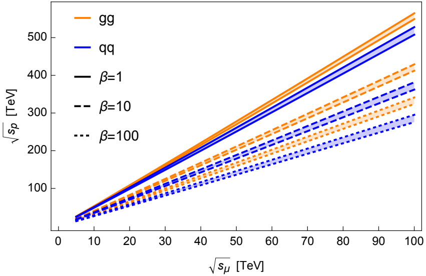
- Study known phenomena at high energies looking for indirect evidence of BSM physics
  Higgs Factories $\Rightarrow$ Probe TeV scale via precision measurements
- Search for direct evidence of BSM physics at the energy frontier
  Directly reach the multi-TeV scale

Current Colliders

- Hadron colliders collide **composite particles** $\Rightarrow$ To reach high energies
  Generate large QCD backgrounds and you use a fraction of the energy of beam for physics
- Lepton colliders collide **fundamental particles** $\Rightarrow$ To reach high precisions
  Exploit the full energy and avoid large QCD backgrounds

Dream machine: A multi-TeV level lepton collider

Get use of the full machine energy



Discovery reach: $M \sim \frac{\sqrt{s}}{2}$

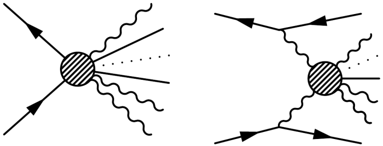
[2103.14043 "Muon Smasher's Guide"]

muC@10 TeV $\sim$ pp@70 TeV

10 TeV is not the limit

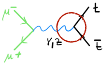
More than lepton collisions:

Two mechanisms: Annihilation VS Fusion



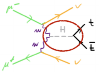
⇒ VBF collider:

$\sqrt{s} \lesssim$ 1-5 TeV



$\sigma_s \sim \frac{1}{s}$ ⟷ $\sigma_s \sim \frac{1}{M^2} \log^n \frac{s}{M}$

$\sqrt{s} \gtrsim$ 1-5 TeV



Need to resume the large Logs

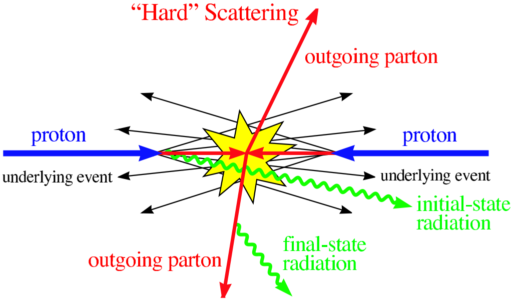
⇒ **The partonic picture is needed**

The partonic picture

[T. Han, YM, K. Xie, 2007.14300, 2103.09844]

Hadron colliders and the Parton Distribution Function (PDF)

- **Recall the hadron colliders: the $\mathrm{Sp\bar{p}S}$, the Tevatron, or the LHC**



- **Hadrons are composite**
  a, b are the "partons" from the beam particles A and B.
- **PDFs**
  $f_{a/A}, f_{b/B}$ are the probabilities to find a parton a (b) from the beam particle A (B) with a momentum fraction x_a (x_b).

- **Factorization formalism : PDFs $\otimes$ partonic cross sections**

$$\sigma(AB \to X) = \sum_{a,b} \int \mathrm{d}x_a \mathrm{d}x_b f_{a/A}(x_a, Q) f_{b/B}(x_b, Q) \hat{\sigma}(ab \to X)$$

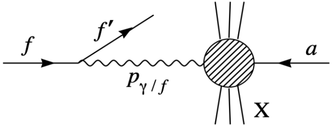
"Parton" of a lepton

Leptons are elementary particles $\Rightarrow$ **"Equivalent photon approximation (EPA)"**

- Treat photon as a parton constituent in the electron

$$\sigma(\ell^- + a \to \ell^- + X) = \int \mathrm{d}x\, f_{\gamma/\ell}\hat{\sigma}(\gamma a \to X)$$

$$f_{\gamma/\ell,\mathrm{EPA}}(x_\gamma, Q^2) = \frac{\alpha}{2\pi}\frac{1+(1-x_\gamma)^2}{x_\gamma}\ln\frac{Q^2}{m_\ell^2}$$



[C. F. von Weizsacker, Z. Phys. 88, 612 (1934)]

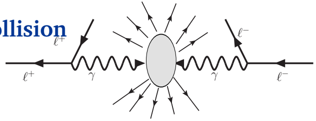
[E. J. Williams, Phys. Rev. 45, 729 (1934)]

- At lepton colliders

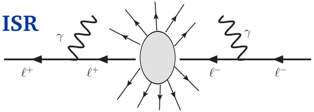
$$\sigma(\ell^+\ell^- \to F + X) = \int_{\tau_0}^{1} d\tau \sum_{ij} \frac{d\mathcal{L}_{ij}}{d\tau}\,\hat{\sigma}(ij \to F),\ \tau = \hat{s}/s$$

$$\frac{d\mathcal{L}_{ij}}{d\tau} = \frac{1}{1+\delta_{ij}}\int_\tau^1 \frac{d\xi}{\xi}\left[f_i(\xi,Q^2)f_j\left(\frac{\tau}{\xi},Q^2\right) + (i \leftrightarrow j)\right]$$

$\gamma\gamma$ **collision**



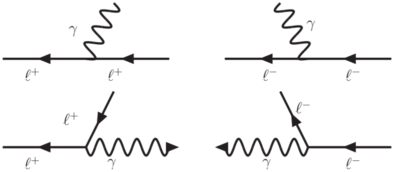
ISR



People have been doing:

- $\ell^+\ell^-$ annihilation

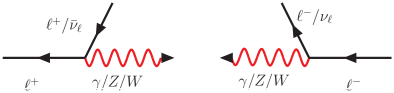
- EPA and ISR



- "Effective W Approx." (EWA)

[G. Kane, W. Repko, and W. Rolnick, PLB 148 (1984) 367]

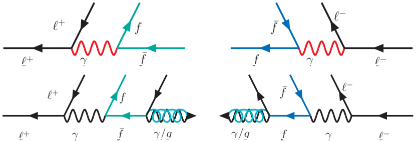
[S. Dawson, NPB 249 (1985) 42]



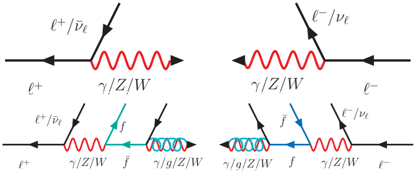
We will add [T. Han, Y. Ma, K.Xie 2007.14300, 2103.09844]

[F. Garosi, D. Marzocca, S. Trifinopoulos 2303.16964]

- Above μ_{QCD}: QED⊗QCD
  q/g emerge



- Above $\mu_{\mathrm{EW}} = M_Z$: EW⊗QCD
  EW partons / corrections to the above



In the end, everything is parton, i.e. need the full SM PDFs.

The PDF evolution: DGLAP

- The DGLAP equations

$$\frac{\mathrm{d} f_i}{\mathrm{d}\log Q^2} = \sum_I \frac{\alpha_I}{2\pi} \sum_j P^I_{ij} \otimes f_j$$

- The initial conditions

$$f_{\ell/\ell}(x, m_\ell^2) = \delta(1-x)$$

- Three regions and two matchings
  - $m_\ell < Q < \mu_{\mathrm{QCD}}$: QED
  - $Q = \mu_{\mathrm{QCD}} \lesssim 1$ GeV: $f_q \propto P_{q\gamma} \otimes f_\gamma, f_g = 0$
  - $\mu_{\mathrm{QCD}} < Q < \mu_{\mathrm{EW}}$: QED⊗QCD
  - $Q = \mu_{\mathrm{EW}} = M_Z$: $f_\nu = f_t = f_W = f_Z = f_{\gamma Z} = 0$
  - $\mu_{\mathrm{EW}} < Q$: EW⊗QCD.

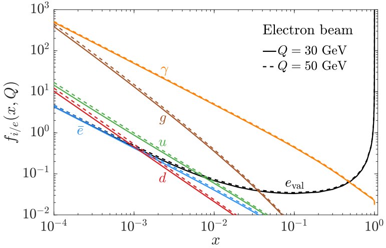
$$\begin{pmatrix} f_B \\ f_{W^3} \\ f_{BW^3} \end{pmatrix} = \begin{pmatrix} c_W^2 & s_W^2 & -2c_W s_W \\ s_W^2 & c_W^2 & 2c_W s_W \\ c_W s_W & -c_W s_W & c_W^2 - s_W^2 \end{pmatrix} \begin{pmatrix} f_\gamma \\ f_Z \\ f_{\gamma Z} \end{pmatrix}$$

- We work in the (B, W) basis. The technical details can be referred to the backup slides.

The QED⊗QCD PDFs for lepton colliders

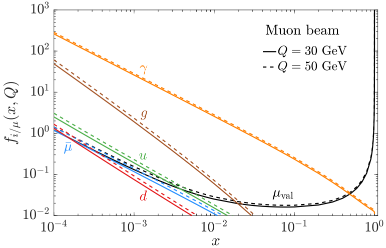
- **Electron PDFs**: $f_{e_{\rm val}},\ f_\gamma,\ f_{\ell_{\rm sea}},\ f_q,\ f_g$
- Scale uncertainty: 10% for $f_{g/e}$
- The averaged momentum fractions $\langle x_i \rangle = \int x f_i(x) \mathrm{d}x$

$Q(e^\pm)$	$e_{\rm val}$	γ	ℓsea	q	g
30 GeV	96.6	3.20	0.069	0.080	0.023
50 GeV	96.5	3.34	0.077	0.087	0.026
M_Z	96.3	3.51	0.085	0.097	0.028



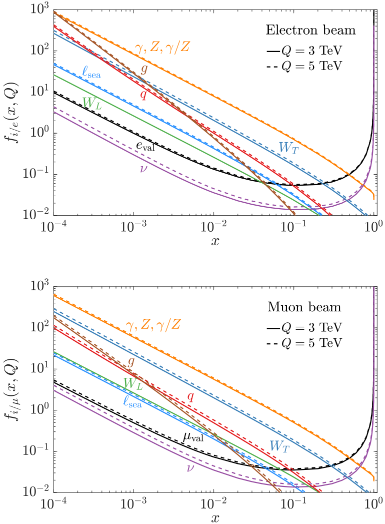
- **Muon PDFs**: $f_{\mu_{\rm val}},\ f_\gamma,\ f_{\ell_{\rm sea}},\ f_q,\ f_g$
- Scale uncertainty: 20% for $f_{g/\mu}$
- The averaged momentum fractions $\langle x_i \rangle = \int x f_i(x) \mathrm{d}x$

$Q(\mu^\pm)$	$\mu_{\rm val}$	γ	ℓsea	q	g
30 GeV	98.2	1.72	0.019	0.024	0.0043
50 GeV	98.0	1.87	0.023	0.029	0.0051
M_Z	97.9	2.06	0.028	0.035	0.0062



The PDFs of a lepton beyond the EW scale

- **All SM particles are partons**
  [T. Han, Y. Ma, K.Xie 2007.14300, 2103.09844]
  - The sea leptonic and quark PDFs show up
    $\nu = \sum_i (\nu_i + \bar{\nu}_i)$,
    $\ell\text{sea} = \bar{\mu} + \sum_{i\neq\mu} (\ell_i + \bar{\ell}_i)$,
    $q = \sum_{i=d}^{t} (q_i + \bar{q}_i)$
    There is even neutrino due to the EW sector
- W_L does not evolve at the leading order.
- The EW correction is not small: $\sim 50\%$ (100%) for $f_{d/e}$ $(f_{d/\mu})$ due to the relatively **large SU(2) gauge coupling**. [T. Han, Y. Ma, K.Xie 2103.09844]
- Scale uncertainty: $\sim 15\%$ (20%) between $Q = 3$ TeV and $Q = 5$ TeV



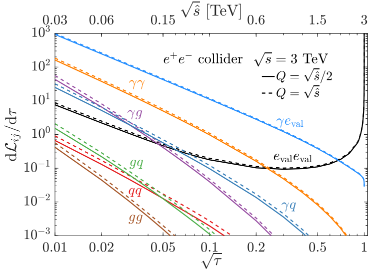
Parton luminosities at high-energy lepton colliders

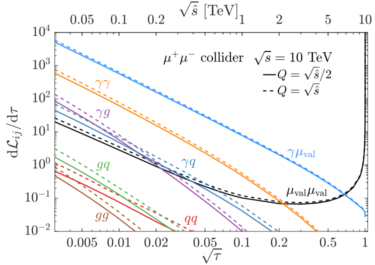
A 3 TeV e^+e^- machine and a 10 TeV $\mu^+\mu^-$ machine

- Partonic luminosities for

$$\ell^+\ell^-,\ \gamma\ell,\ \gamma\gamma,\ qq,\ \gamma q,\ \gamma g,\ gq,\ \text{and}\ gg$$

- $\gamma\gamma$ gives the largest partonic luminosity
- The luminosity of $\gamma g + \gamma q$ is $\sim 50\%$ (20%) of $\gamma\gamma$
- The luminosities of qq, gq, and gg are $\sim 2\%$ (0.5%) of $\gamma\gamma$
- Given the stronger QCD coupling, **sizable QCD cross sections are expected**.
- Scale uncertainty is $\sim 20\%$ (50%) for photon (gluon) initiated processes.





The SM expectation

[T. Han, YM, K. Xie, 2007.14300, 2103.09844]

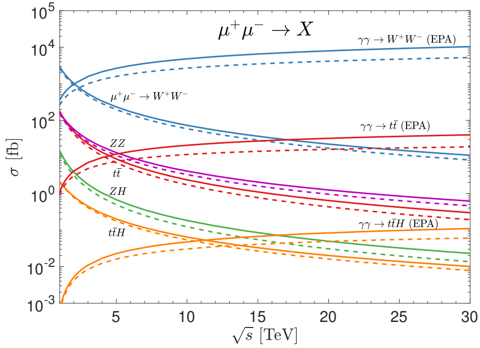
What do we get if the machine is turned on?

- What is the SM physics picture?
- What is the largest background signal?
- Where can we see the possible BSM physics?

Apply EPA at high-energy lepton colliders

A high-energy muon collider at first glance

What do people expect from a high-energy lepton (muon) collider?



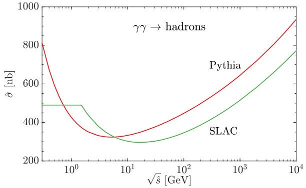
[T. Han, YM, K.Xie 2007.14300]

Some "commonsense":

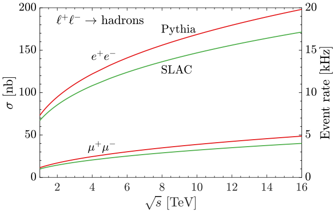
- The annihilations decrease as $1/s$.
- ISR needs to be considered, which can give over 10% enhancement.
- The fusions increase as $\ln^p(s)$, which take over at high energies.
- The large collinear logarithm $\ln(s/m_\ell^2)$ needs to be resummed, set $Q = \sqrt{\hat{s}}/2$,
- $\gamma\gamma \to W^+W^-$ production has the largest cross section.

Photon induced hadronic production at high-energy lepton colliders

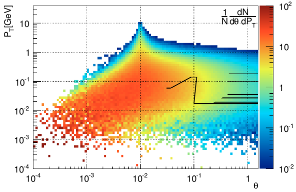
- **Model-dependent $\hat{\sigma}_{\gamma\gamma}$**



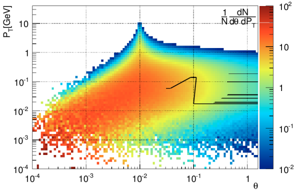
- $\sigma_{\ell\ell}$ may reach nano-barns



- The events populate at low p_T regime



(a) Pythia sample

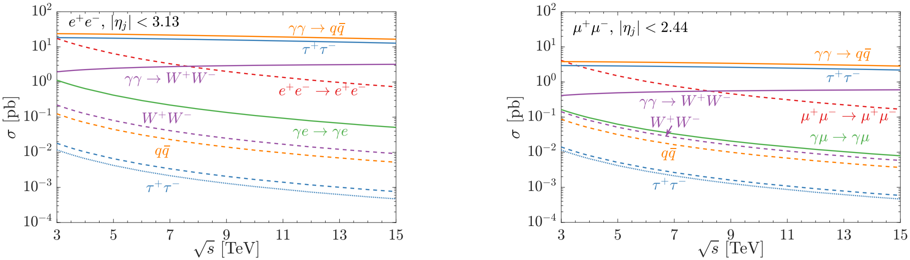


(b) SLAC sample

[T. Barklow, D. Dannheim, M. O. Sahin, and D. Schulte, LCD-2011-020]

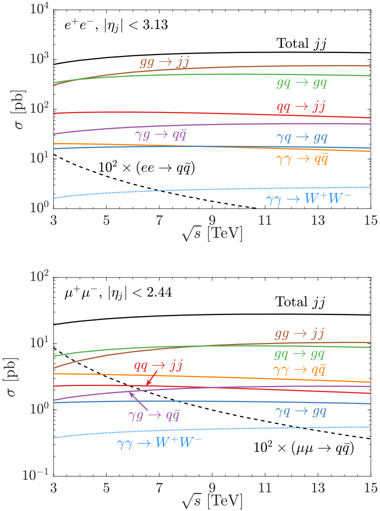
What are the dominant processes in the high p_T range?

- Detector angle & Threshold: $\theta_{\text{cut}} = 5^\circ\ (10^\circ) \iff |\eta| < 3.13(2.44)$, $m_{ij} > 20$ GeV
- To separate from the nonperturbative hadronic production: $p_T > \left(4 + \frac{\sqrt{s}}{3\,\text{TeV}}\right)$ GeV
  - Leading-order: $\ell^+\ell^- \to \ell^+\ell^-,\ \tau^+\tau^-,\ q\bar{q},\ W^+W^-$, and $\gamma\ell \to \gamma\ell$
  - $\gamma\gamma$ scatterings: $\gamma\gamma \to \tau^+\tau^-,\ q\bar{q},\ W^+W^-$



The full background: Di-Jet production at high-energy lepton colliders

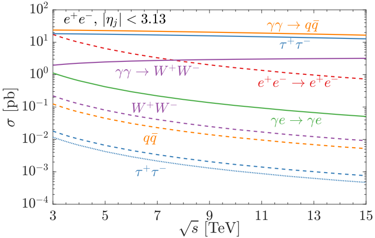
- Consider all the "partons"
  $\Rightarrow$ **perturbatively computable processes**
  $$\gamma\gamma \to q\bar{q},\ \gamma g \to q\bar{q},\ \gamma q \to gq,$$
  $$qq \to qq\,(gg),\ gq \to gq \text{ and } gg \to gg\,(q\bar{q}).$$
- Large $\alpha_s \ln(Q^2)$ brings a $6\% \sim 15\%$ ($30\% \sim 40\%$) enhancement if $Q = 2Q$
- The QCD contributions result in total cross section.
- gg initiated cross sections are large for the **multiplicity**
- gq initiated cross sections are large for the **luminosity**.
- $\gamma\gamma$ gives smaller cross sections than the EPA does.

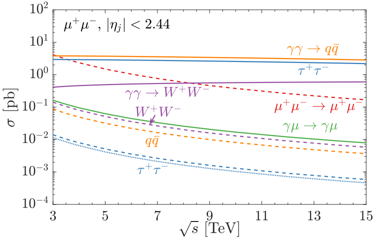


Compare the new with the old

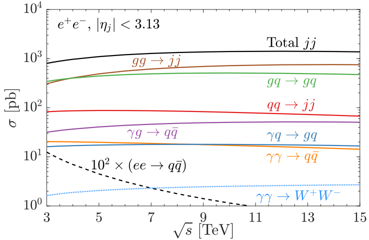
Quark/gluon initiated jet production dominates

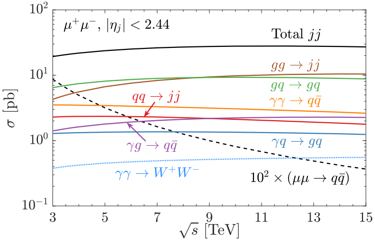
Before:





After:

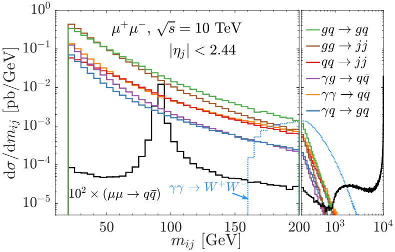


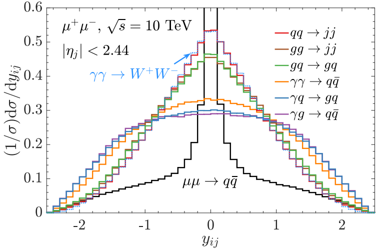


Di-jet distributions at a 10 TeV muon collider

Rather a conservative set up: $\theta = 10^\circ$

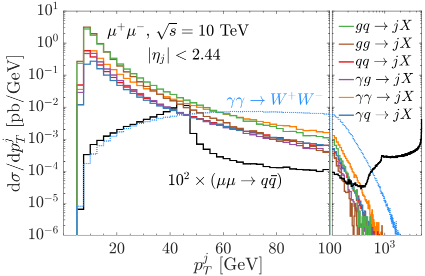
- Some physics:
  Two different mechanisms: $\mu^+\mu^-$ **annihilation** VS **Fusion processes**
  - Annihilation is more than 2 orders of magnitude smaller than fusion process.
  - Annihilation peaks at $m_{ij} \sim \sqrt{s}$;
  - Fusion processes peak near m_{ij} threshold.
  - Annihilation is very central, spread out due to ISR;
  - Fusion processes spread out, especially for γq and γg initiated ones.

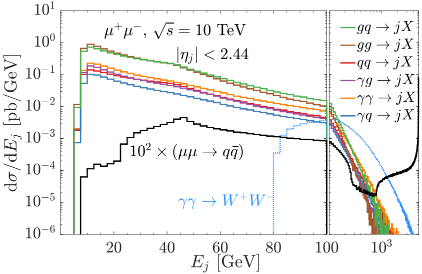




Inclusive jet distributions at a 10 TeV muon collider

Important guidelines for future analysis





We expect

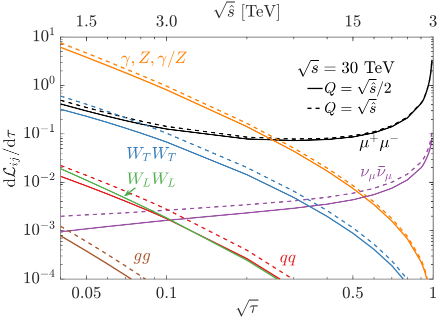
- Jet production dominates over WW production until $p_T > 60$ GeV;
- WW production takes over around energy ~ 200 GeV.

The SM EW sector, as well as any possible BSM, can only be seen in the high p_T (E_j) range.

The full picture a multi-TeV lepton collider: An EW version of LHC

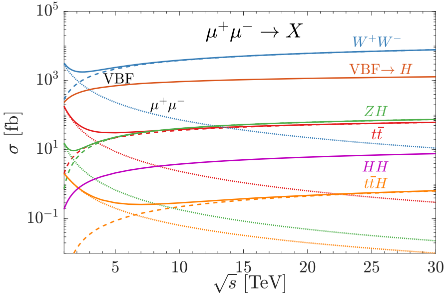
- All SM particles are partons
- We are allowed to determine the partons with their different polarizations

The EW parton luminosities of a 30 TeV muon collider



Just like in hadronic collisions:

$\mu^+\mu^- \to$ exclusive particles + remnants



Compare the "EW LHC" with LHC

pp VS $\mu\mu$

$$\mathcal{L}_{W^+_{\lambda_1} W^-_{\lambda_2}} = \int_\tau^1 \frac{d\xi}{\xi} f_{W_{\lambda_1}}(\xi, \mu_f) f_{W_{\lambda_1}}(\frac{\tau}{\xi}, \mu_f)$$

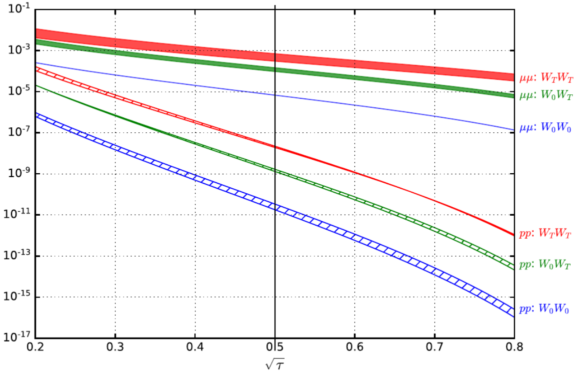
Consider the two colliders in the same ring

$$\sqrt{s}_{\mu\mu} = \sqrt{s}_{pp}$$

For $2 \to 1$ processes, take a benchmark

$$\sqrt{\tau} = \frac{M}{\sqrt{s}} = \frac{1}{2}$$

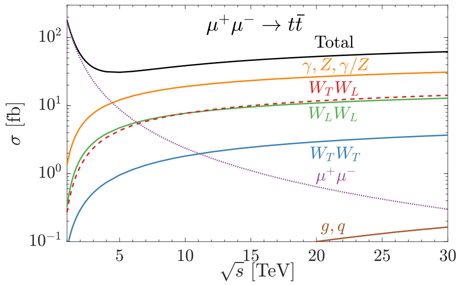
The ratio $\mu\mu/pp$ is larger than 10^4!

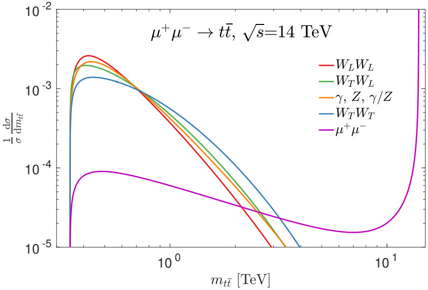


[2005.10289]

One example: $t\bar{t}$ production at a future muon collider

- Two different mechanisms: **Annihilation** and **Fusion**
- The VBF processes exceed the $\mu^+\mu^-$ annihilation at high energies
- The EW PDF formalism allows to determine different partons/polarizations
- The resummation effects lie in the tails.





Summary and prospects

Muon collider is a fantastic platform - full of physics opportunities

- It combines the advantages of proton and of e^+e^- colliders
- It is an amazing precision tool but also can be discovery machine

A multi-TeV level muon collider is an Electroweak versions of the LHC

- Two classes of processes: $\mu^+\mu^-$ annihilation VS fusions
- The scale is far above the EW scale, all the SM particles are “partons”
- Quark and gluon appear as partons of the muon via the DGLAP evolution
- The EW PDF formalism allows to determine the polarization of the partons

  The PDF will be included in the main stream event generators soon!

The main background is the jet production

- Low p_T range: non-perturbative $\gamma\gamma$ initiated hadronic production dominates

  [Chen, Barklow, and Peskin, hep-ph/9305247; Drees and Godbole, PRL 67, 1189,T. Barklow, etal, LCD-2011-020]
- High p_T range, q and g initiated jet production dominates [T. Han, Y. Ma, K.Xie 2103.09844]

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One example in precision physics: The Muon-Higgs Coupling

[T. Han, W. Kilian, N. Kreher, YM, T. Striegl, J. Reuter, and K. Xie, 2108.05362]

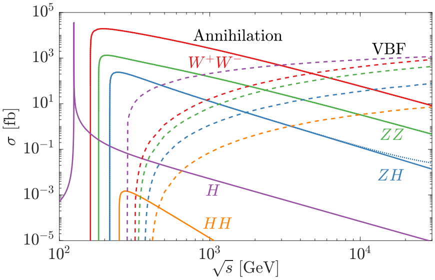
[E. Celada, T. Han, W. Kilian, N. Kreher, YM, F. Maltoni, D. Pagani, T. Striegl, J. Reuter, and K. Xie, in progress]

- Physics: We actually do not know whether the SM mass-generation mechanism applies just to the heavy particles, or also to the 1st/2nd generations.
- Logical possibility: Muon mass not (only) generated by SM Higgs.
  $\Rightarrow$ **Why not have an arbitrary Yukawa coupling?**

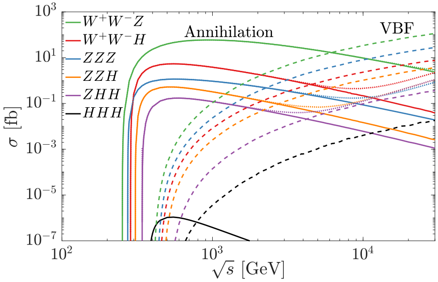
Multi-boson final states and the Muon-Higgs coupling

- **SM**: $\lambda(\mathrm{Muon}-\mathrm{Higgs}) \sim y_\mu^{\mathrm{SM}} = \sqrt{2} m_\mu^{\mathrm{SM}}/v$
- **Possible BSM physics**: $m_\mu = m_\mu^{\mathrm{SM}}$, $\lambda(\mathrm{Muon}-\mathrm{Higgs}) \sim \kappa_\mu y_\mu^{\mathrm{SM}}$, e.g. $\kappa_\mu = 0$

Two-boson final states

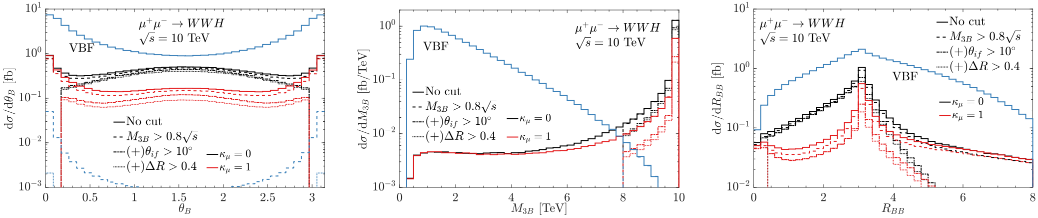


Three-boson final states



New physics signal shows up in the high energy region

WWH at a 10 TeV muon collider: Kinematics



- Background (VBF) is much larger than signal (annihilation)
- VBF events accumulate around threshold, and mostly forward
- Annihilation in the rest frame (central, and $M \sim \sqrt{s}$ spread by ISR)
- Annihilation also has forward dominance, due to the gauge splitting $W \to WH$

WWH at a 10 TeV muon collider: Cuts

Cut flow	$\kappa_\mu = 1$	w/o ISR	$\kappa_\mu = 0$ (2)	CVBF	NVBF
σ [fb]	WWH				
No cut	0.24	0.21	0.47	2.3	7.2
$M_{3B} > 0.8\sqrt{s}$	0.20	0.21	0.42	$5.5 \cdot 10^{-3}$	$3.7 \cdot 10^{-2}$
$10° < \theta_B < 170°$	0.092	0.096	0.30	$2.5 \cdot 10^{-4}$	$2.7 \cdot 10^{-4}$
$\Delta R_{BB} > 0.4$	0.074	0.077	0.28	$2.1 \cdot 10^{-4}$	$2.4 \cdot 10^{-4}$
# of events	740	770	2800	2.1	2.4
S/B	2.8				

- Integrated luminosity $\mathcal{L} = (\sqrt{s}/10\ \text{TeV})^2 \cdot 10\,\text{ab}^{-1}$ [1901.06150]
- $S = N_{\kappa_\mu} - N_{\kappa_\mu=1}$, $B = N_{\kappa_\mu=1} + N_{\text{VBF}}$.
- VBF and ISR are mostly excluded by invariant mass cut.
- Angular cut also weaken VBF further.

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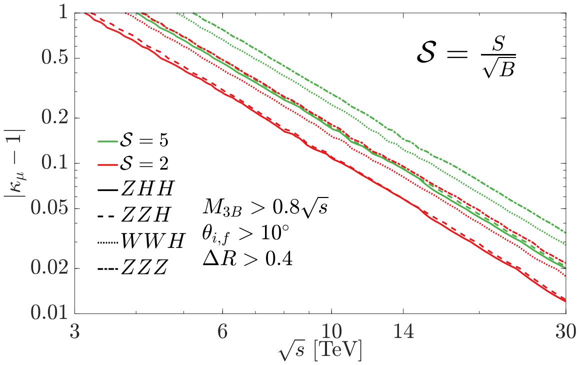
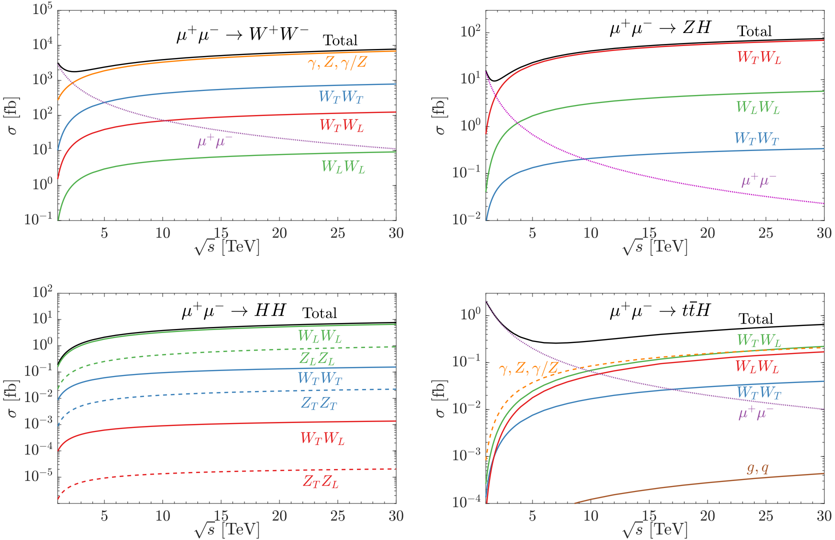
Test the muon Yukawa: statistical sensitivity

- The most sensitive channels are ZHH and ZZH, similar probes due to GBET.
- Taking $S = 2$ criterion, we can test the muon-Higgs coupling up to 10% (1%) precision at a 10 (30) TeV muon collider, corresponding to new physics scale $\Lambda_{\rm NP} \sim 30 - 100$ TeV.



Other processes: $W^+W^-, ZH, HH, t\bar{t}H$



The DGLAP

[T. Han, YM, K. Xie, 2007.14300, 2103.09844]

Solving the DGLAP: Singlet and Non-singlet PDFs

The singlets

$$f_L = \sum_{i=e,\mu,\tau} (f_{\ell_i} + f_{\bar{\ell}_i}),\ f_U = \sum_{i=u,c} (f_{u_i} + f_{\bar{u}_i}),\ f_D = \sum_{i=d,s,b} (f_{d_i} + f_{\bar{d}_i})$$

The non-singlets

- The only non-trivial singlet $f_{e,NS} = f_e - f_{\bar{e}}$
- the leptons $f_{\ell_i,NS} = f_{\ell_i} - f_{\bar{\ell}_i}(i=2,3),\ f_{\ell,12} = f_{\bar{e}} - f_{\bar{\mu}},\ f_{\ell,13} = f_{\bar{e}} - f_{\bar{\tau}};$
- the up-type quarks $f_{u_i,NS} = f_{u_i} - f_{\bar{u}_i},\ f_{u,12} = f_u - f_c;$
- and the down-type quarks $f_{d_i,NS} = f_{d_i} - f_{\bar{d}_i},\ f_{d,12} = f_d - f_s,\ f_{d,13} = f_d - f_b.$

Reconstruction:

$$f_e = \frac{f_L + (2N_\ell - 1) f_{e,NS}}{2N_\ell},\ f_{\bar{e}} = f_\mu = f_{\bar{\mu}} = f_\tau = f_{\bar{\tau}} = \frac{f_L - f_{e,NS}}{2N_\ell}.$$

$$f_u = f_{\bar{u}} = f_c = f_{\bar{c}} = \frac{f_U}{2N_u},\ f_d = f_{\bar{d}} = f_s = f_{\bar{s}} = f_b = f_{\bar{b}} = \frac{f_D}{2N_d}.$$

The QED⊗QCD case

- The singlets and gauge bosons

$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2}\begin{pmatrix} f_L \\ f_U \\ f_D \\ f_\gamma \\ f_g \end{pmatrix} = \begin{pmatrix} P_{\ell\ell} & 0 & 0 & 2N_\ell P_{\ell\gamma} & 0 \\ 0 & P_{uu} & 0 & 2N_u P_{u\gamma} & 2N_u P_{ug} \\ 0 & 0 & P_{dd} & 2N_d P_{d\gamma} & 2N_d P_{dg} \\ P_{\gamma\ell} & P_{\gamma u} & P_{\gamma d} & P_{\gamma\gamma} & 0 \\ 0 & P_{gu} & P_{gd} & 0 & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} f_L \\ f_U \\ f_D \\ f_\gamma \\ f_g \end{pmatrix}$$

- The non-singlets

$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2} f_{NS} = P_{ff} \otimes f_{NS}.$$

- The averaged momentum fractions of the PDFs: $f_{\ell_{\mathrm{val}}},\ f_\gamma,\ f_{\ell_{\mathrm{sea}}},\ f_q,\ f_g$

$$\langle x_i \rangle = \int x f_i(x)\mathrm{d}x,\ \sum_i \langle x_i \rangle = 1$$

$$\frac{\langle x_q \rangle}{\langle x_{\ell\mathrm{sea}} \rangle} \lesssim \frac{N_c\left[\sum_i (e^2_{u_i} + e^2_{\bar{u}_i}) + \sum_i (e^2_{d_i} + e^2_{\bar{d}_i})\right]}{e^2_{\ell_{\mathrm{val}}} + \sum_{i\neq \ell\mathrm{val}} (e^2_{\ell_i} + e^2_{\bar{\ell}_i})} = \frac{22/3}{5}$$

The DGLAP for the full SM

$$\frac{\mathrm{d}}{\mathrm{d}L}\begin{pmatrix} f_L^{0\pm} \\ f_Q^{0\pm} \\ f_E^{0\pm} \\ f_U^{0\pm} \\ f_D^{0\pm} \\ f_B^{0\pm} \\ f_W^{0\pm} \\ f_g^{0\pm} \end{pmatrix} = \begin{pmatrix} P_{LL}^{0\pm} & 0 & 0 & 0 & 0 & P_{LB}^{0\pm} & P_{LW}^{0\pm} & 0 \\ 0 & P_{QQ}^{0\pm} & 0 & 0 & 0 & P_{QB}^{0\pm} & P_{QW}^{0\pm} & P_{Qg}^{0\pm} \\ 0 & 0 & P_{EE}^{0\pm} & 0 & 0 & P_{EB}^{0\pm} & 0 & 0 \\ 0 & 0 & 0 & P_{UU}^{0\pm} & 0 & P_{UB}^{0\pm} & 0 & P_{Ug}^{0\pm} \\ 0 & 0 & 0 & 0 & P_{DD}^{0\pm} & P_{DB}^{0\pm} & 0 & P_{Dg}^{0\pm} \\ P_{BL}^{0\pm} & P_{BQ}^{0\pm} & P_{BE}^{0\pm} & P_{BU}^{0\pm} & P_{BD}^{0\pm} & P_{BB}^{0\pm} & 0 & 0 \\ P_{WL}^{0\pm} & P_{WQ}^{0\pm} & 0 & 0 & 0 & 0 & P_{WW}^{0\pm} & 0 \\ 0 & P_{gQ}^{0\pm} & 0 & P_{gU}^{0\pm} & P_{gD}^{0\pm} & 0 & 0 & P_{gg}^{0\pm} \end{pmatrix} \otimes \begin{pmatrix} f_L^{0\pm} \\ f_Q^{0\pm} \\ f_E^{0\pm} \\ f_U^{0\pm} \\ f_D^{0\pm} \\ f_B^{0\pm} \\ f_W^{0\pm} \\ f_g^{0\pm} \end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}L}\begin{pmatrix} f_L^{1\pm} \\ f_Q^{1\pm} \\ f_W^{1\pm} \\ f_{BW}^{1\pm} \end{pmatrix} = \begin{pmatrix} P_{LL}^{1\pm} & 0 & P_{LW}^{1\pm} & P_{LM}^{1\pm} \\ 0 & P_{QQ}^{1\pm} & P_{QW}^{1\pm} & P_{QM}^{1\pm} \\ P_{WL}^{1\pm} & P_{WQ}^{1\pm} & P_{WW}^{1\pm} & 0 \\ P_{ML}^{1\pm} & P_{MQ}^{1\pm} & 0 & P_{MM}^{1\pm} \end{pmatrix} \otimes \begin{pmatrix} f_L^{1\pm} \\ f_Q^{1\pm} \\ f_W^{1\pm} \\ f_{BW}^{1\pm} \end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}L} f_W^{2\pm} = P_{WW}^{2\pm} \otimes f_{WW}^{2\pm}$$

The splitting functions can be found in [Chen et al. 1611.00788, Bauer et al. 1703.08562,1808.08831]