Karlsruher Institut für Technologie

Local Analytic Sector Subtraction: status and perspectives

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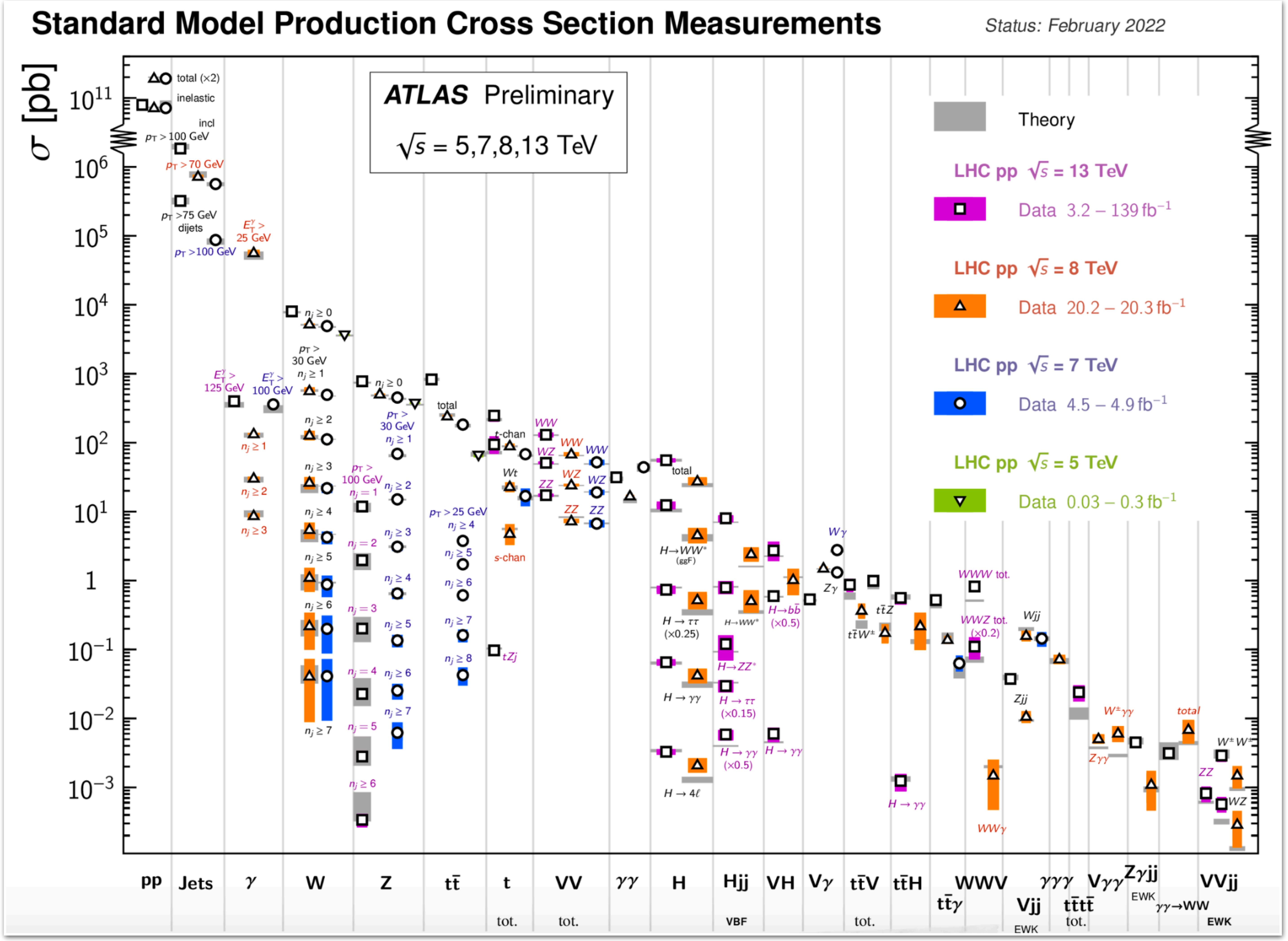
In collaboration with: Bertolotti, Magnea, Pelliccioli, Ratti, Torrielli, Uccirati
Based on: *JHEP 12(2018)107, JHEP 02(2021)037, arXiv 2212.11190*

Take-home message

Local Analytic Sector Subtraction provides a fully local infrared subtraction scheme at NNLO for generic coloured massless final states.

Landscape

LHC continues to confirm the Standard Model



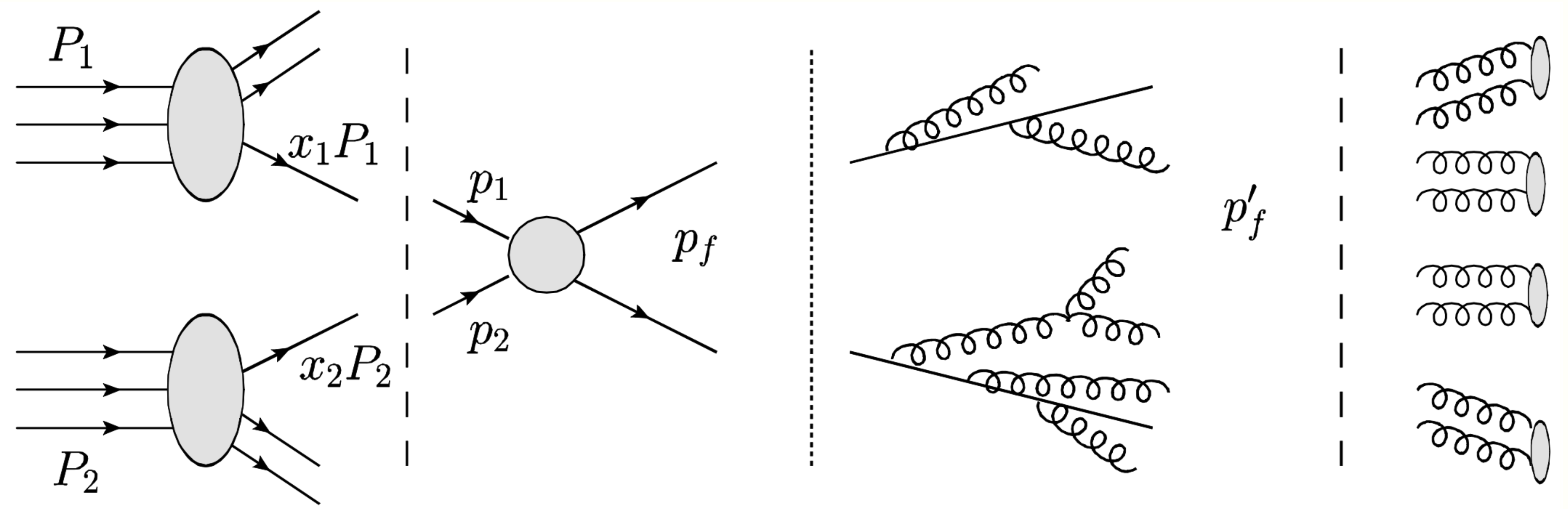
- Direct search for BSM: many proposal, no obvious candidate
- Indirect search for BSM: small corrections to SM

High precision theoretical predictions

Pillars of precision calculations

The success of a percent level phenomenology program relies on our ability to interpret and predict the outcome LHC measurement.

[Snowmass'2021 whitepaper]



[Phys. Proc. 51(2014)25-30]

Hard collisions at the LHC are described in terms of quark and gluon cross sections

→ Collinear factorisation theorem *[Collins, Soper, Sterman 0409313]*

Typical precision at NNLO with 5-15% uncertainties

$$d\sigma = \sum_{ij} \int dx_1\, dx_2\, f_{i/p}(x_1) f_{j/p}(x_2)\, d\hat{\sigma}_{ij}(x_1 x_2 s) \left(1 + \mathcal{O}\left(\frac{\Lambda^n_{\rm QCD}}{Q^n}\right)\right), \quad n \geq 1$$

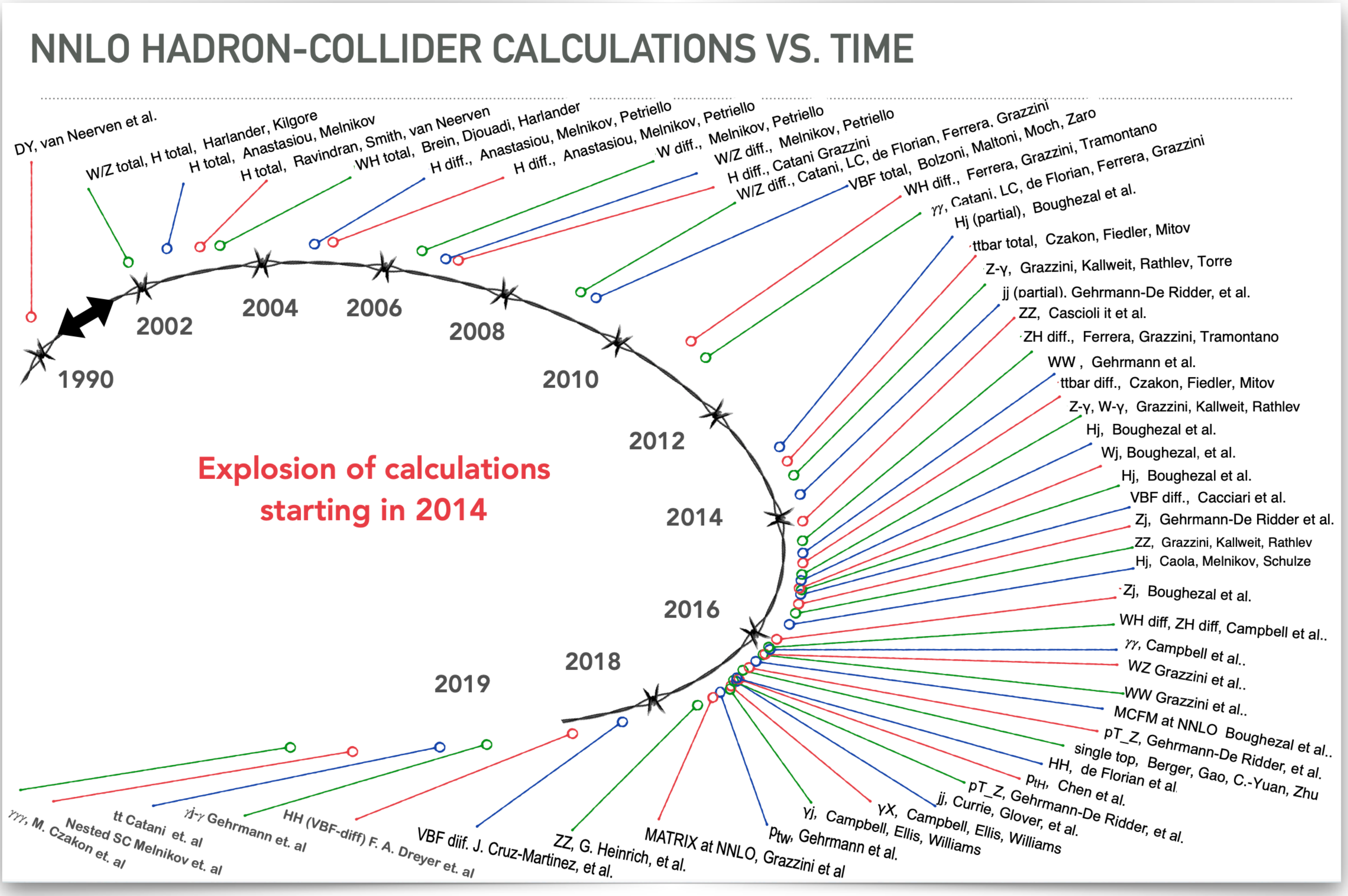
Parton distribution functions
$\pm(3-5)\,\%$

Hard scattering
(perturbative quantum field theory)
aim for few % level!

Non perturbative effects
(fragmentation, hadronisation)
$\sim \%\,(?)$

Motivations

[Cieri, WG1 meeting '19]



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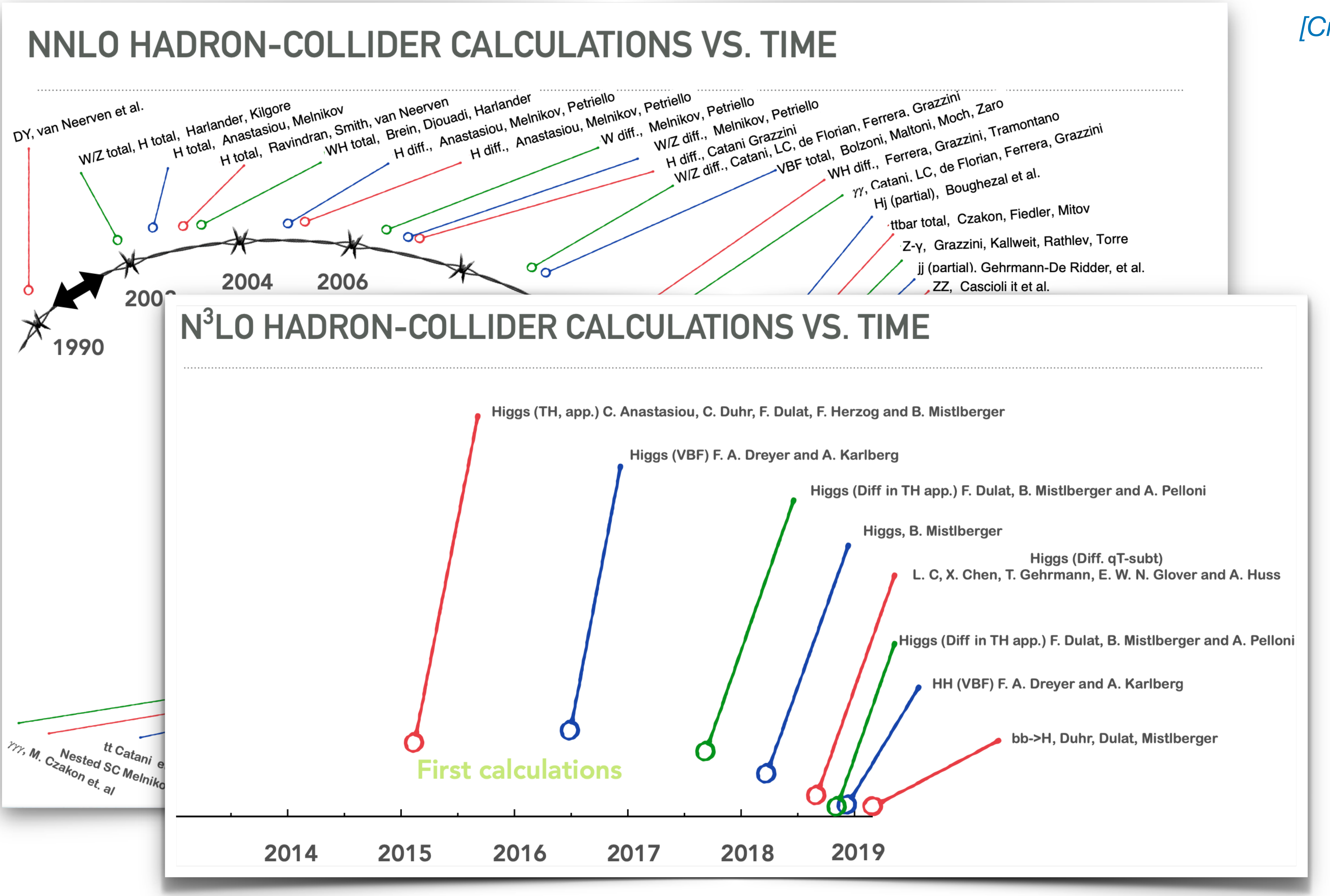
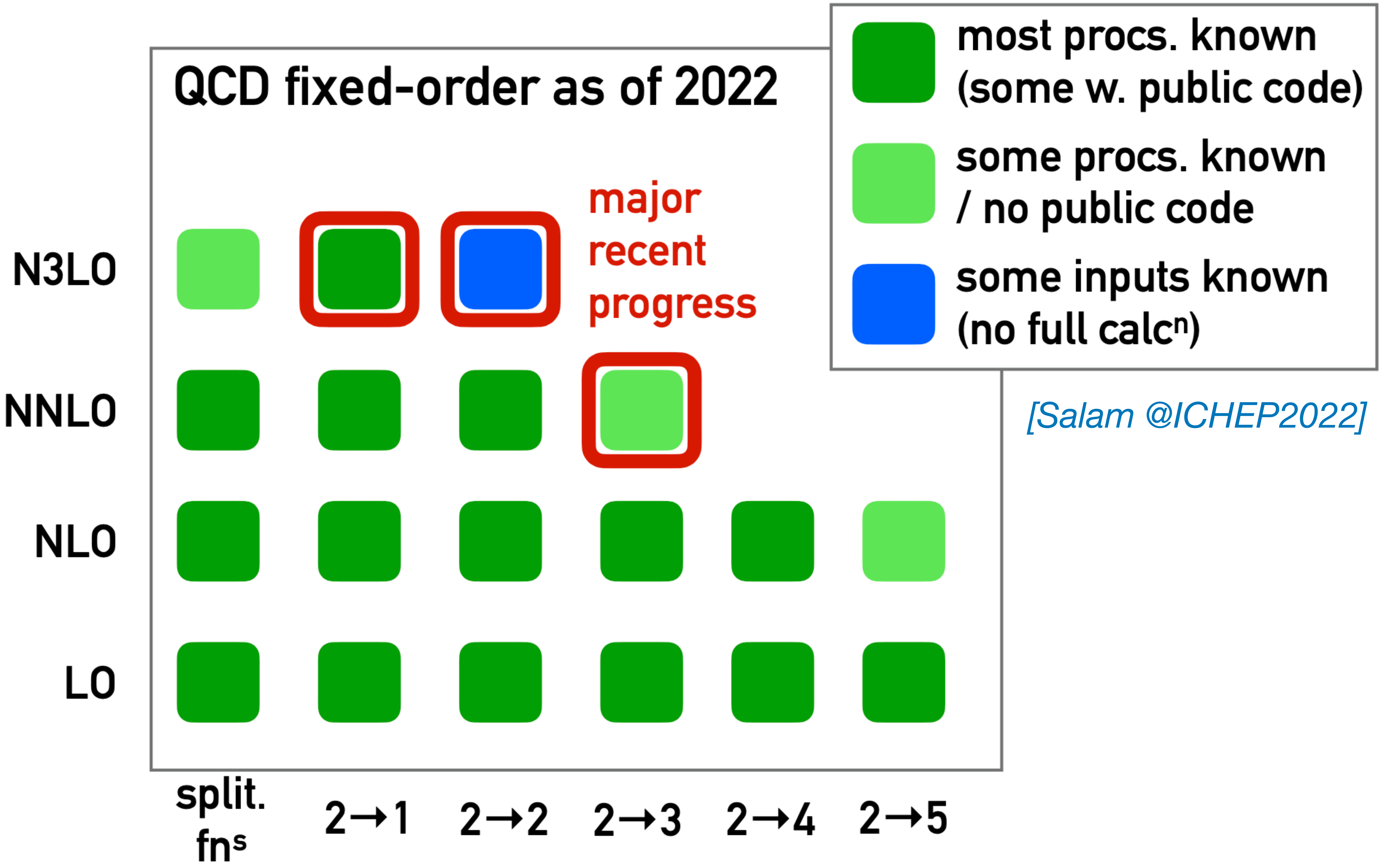


Motivations



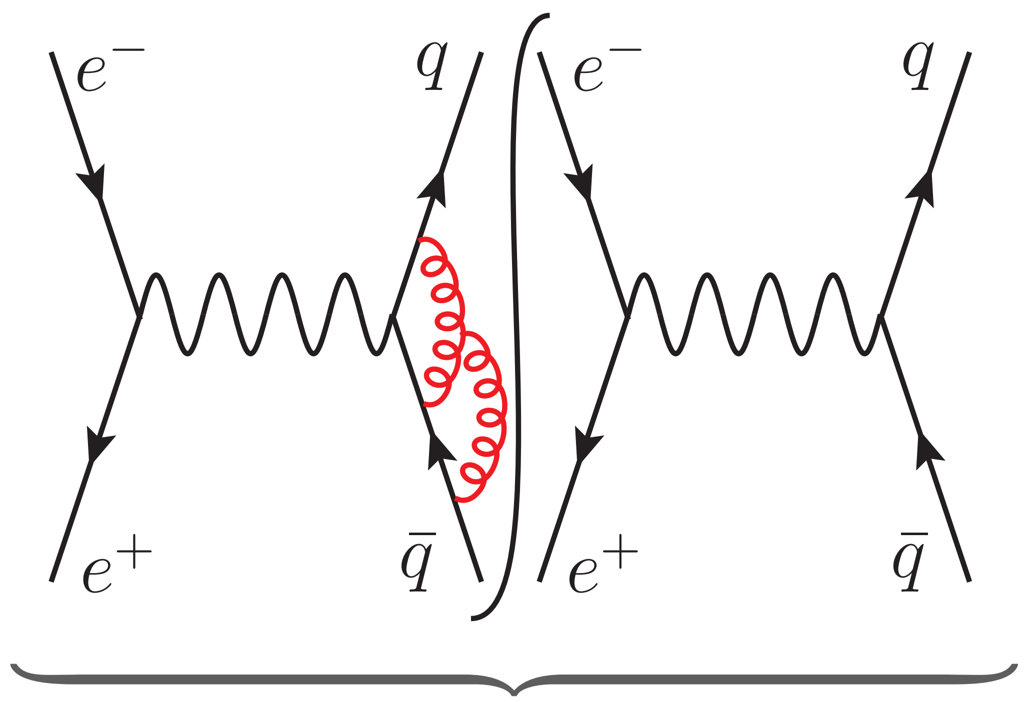
[Salam @ICHEP2022]

NNLO generalities

Ingredients for NNLO correction to $pp \to X$

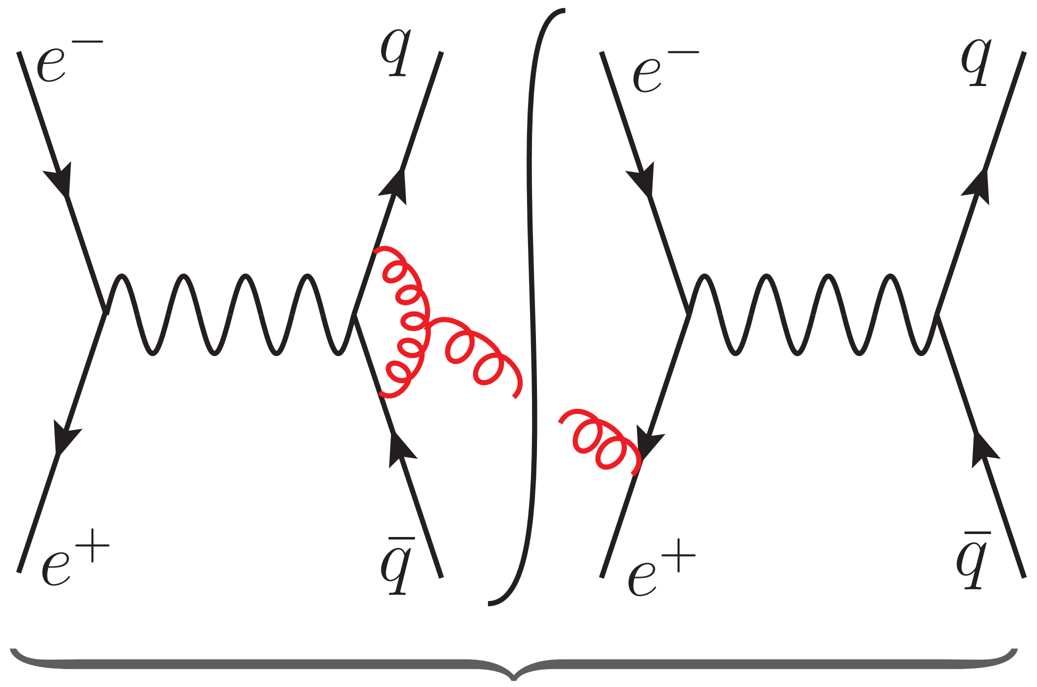
- **two-loop** matrix element for $ff \to X$
- **one-loop** matrix element for $ff \to X + f'$
- **tree-level** matrix element for $ff \to X + f'f'$

$$\frac{d\sigma_{\text{NNLO}}}{dX} = \int d\Phi_{n+2}\, RR\, \delta_{n+2}(X) + \int d\Phi_{n+1}\, RV\, \delta_{n+1}(X) + \int d\Phi_n\, VV\, \delta_n(X)$$



Explicit poles

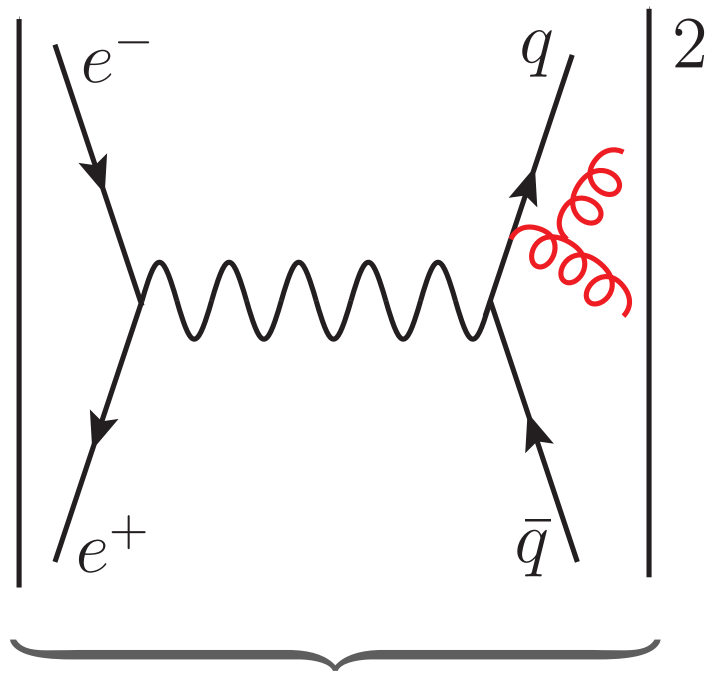
- Significant progress in calculations of **two-loop amplitudes** (both analytic and numerical methods)
- Almost all relevant amplitudes for $2 \to 2$ massless processes
- First results for $2 \to 3$ amplitudes



Explicit poles from virtual corrections

Phase space singularities

- **One-loop amplitudes in degenerate kinematics**
- OpenLoops, Recola



Well defined in the non-degenerate kinematics

- **Real emission corrections finite in the bulk of the allowed PS**
- IR singularities arise upon integration over energies and angles of emitted partons

The problem

1. **Extract** infrared $1/\epsilon$ poles in d-dimension **without integrating over the resolved phase space**

   → **fully differential** predictions for IR-safe observables

2. **Cancel** the $1/\epsilon$ poles stemming from the phase space integration against the poles of the **virtual contributions**

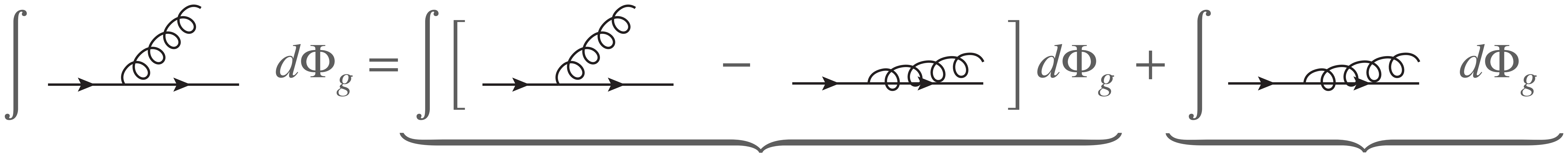
Fully general solution?

- **Phase space singularities** of the real radiation
- Explicit **poles from virtual contributions**

} Known **independently** of the hard subprocess

→ A general procedure seems to be practicable, although non-trivial to implement

- **Slicing**
- **Subtraction** ← This talk



Finite in d=4, integrable numerically

exposes the same $1/\epsilon$ poles as the virtual correction

Well established schemes at NLO

- **Catani-Seymour (CS)** *[9602277]*
- **Frixione-Kunst-Signer (FKS)** *[9512328]*
- Nagy-Soper *[1012.4948]*

Currently implemented in full generality in fast and efficient NLO generators *[Gleisberg, Krauss '07, Frederix, Gehrmann, Greiner '08, Hasegawa, Moch, Uwer '09, Frederix, Frixione, Maltoni, Stelzer '09, Alioli, Nason, Oleari, Re '10, Reuter et al. '16]*

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NLO SOLVED

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NLO SOLVED

What about NNLO?

Extraction of real-emission singularities was the main bottleneck for NNLO predictions.

Example: di-jet two-loop amplitudes ~ 20 years ago *[Anastasiou, Glover, Oleari, Tejeda-Yeomans '01]*,

di-jet production at NNLO ~ 5 ago *[Currie, De Ridder, Gehrmann, Glover, Huss, Pires '17]*

Two-loop QCD corrections to massless identical quark scattering*

2001

C. Anastasiou[a], E. W. N. Glover[a], C. Oleari[b] and M. E. Tejeda-Yeomans[a]

We therefore expect that the problem of the analytic cancellation of the infrared divergences will soon be addressed thereby enabling the construction of numerical programs to provide next-to-next-to-leading order QCD estimates of jet production in hadron collisions.

2017

Precise predictions for dijet production at the LHC

J. Currie[a], A. Gehrmann-De Ridder[b,c], T. Gehrmann[c], E.W.N. Glover[a], A. Huss[b], J. Pires[d]
[a] *Institute for Particle Physics Phenomenology, University of Durham, Durham DH1 3LE, UK*
[b] *Institute for Theoretical Physics, ETH, CH-8093 Zürich, Switzerland*
[c] *Department of Physics, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland*
[d] *Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 Munich, Germany*

Why is NNLO so difficult?

At NLO two main strategies have been implemented

Catani Seymour:

- Counterterm contribution: reproduces the **IR singularities** related to a dipole in **all of the phase space** [complicated structure]
- Full counterterm: sum of **contributions**, each **parametrised differently**
- **Analytic integration** of each term [non trivial, complicated structure of the counterterm]

FKS:

- **Partition** of the radiative phase space with sector functions
- **Different parametrisation** for each sector
- **Analytic integration**, after getting rid of sector functions [non trivial, non optimised parametrisation]

Detail informations of NNLO kernels also available ~ 20 years ago

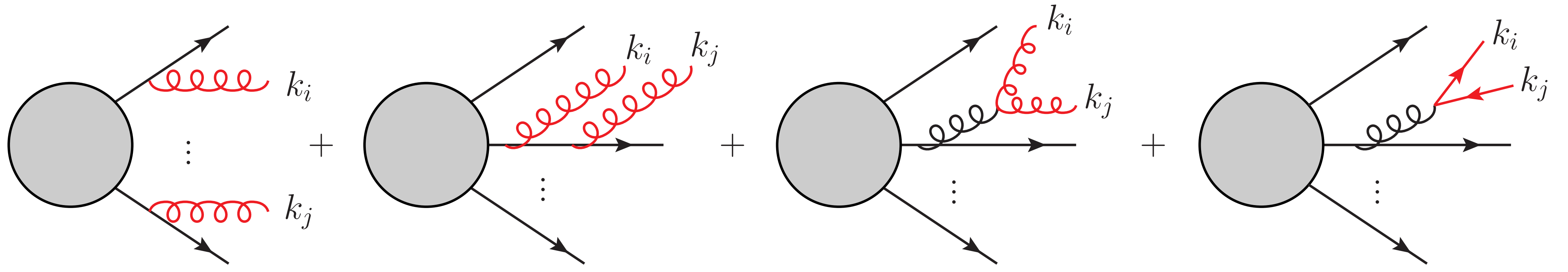
(N3LO kernels partially available *[Catani, Colferai, Torrini 1908.01616, Del Duca, Duhr, Haindl, Lazopoulos Michel 1912.06425, Dixon, Herrmann, Kai Yan, Hua Xing Zhu 1912.09370, Yu Jiao Zhu 2009.08919 …]*)

Why is NNLO so difficult?

Under IR singular limits, the radiative matrix element squared factorises into (universal kernel) x (lower multiplicity matrix elements)

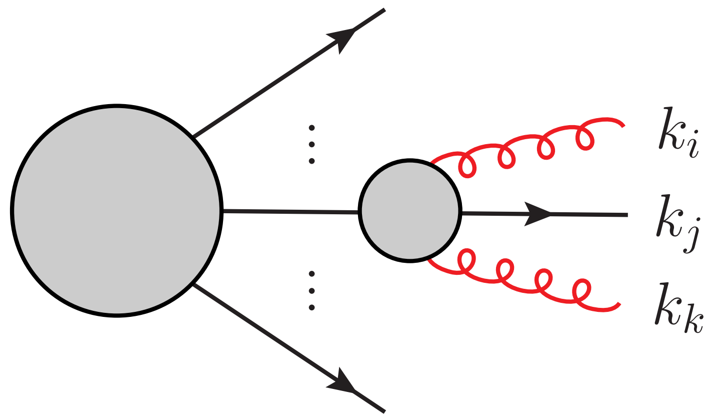
Double soft limit *[Catani, Grazzini 9903516,9810389]*

$$\lim_{k_i,\, k_j \to 0} RR_{n+2}\big(\{k\}_n\,, k_i, k_j\big) \sim \mathrm{Eik}\big(\{k\}_n\,, k_i, k_j\big) \otimes B_n\big(\{k\}_n\big)$$



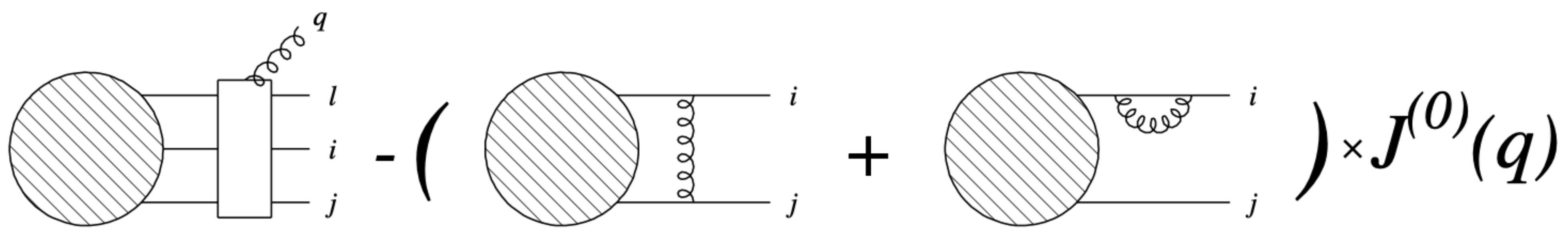
Triple collinear limit *[Catani, Grazzini 9903516,9810389]*

$$\lim_{k_i \| k_j \| k_k} RR_{n+2}\big(\{k\}_{n-1}, k_i, k_j, k_k\big) \sim \frac{1}{s_{ijk}^2} P\big(k_i, k_j, k_k\big) \otimes B_n\big(\{k\}_{n-1}, k_{ijk}\big)$$



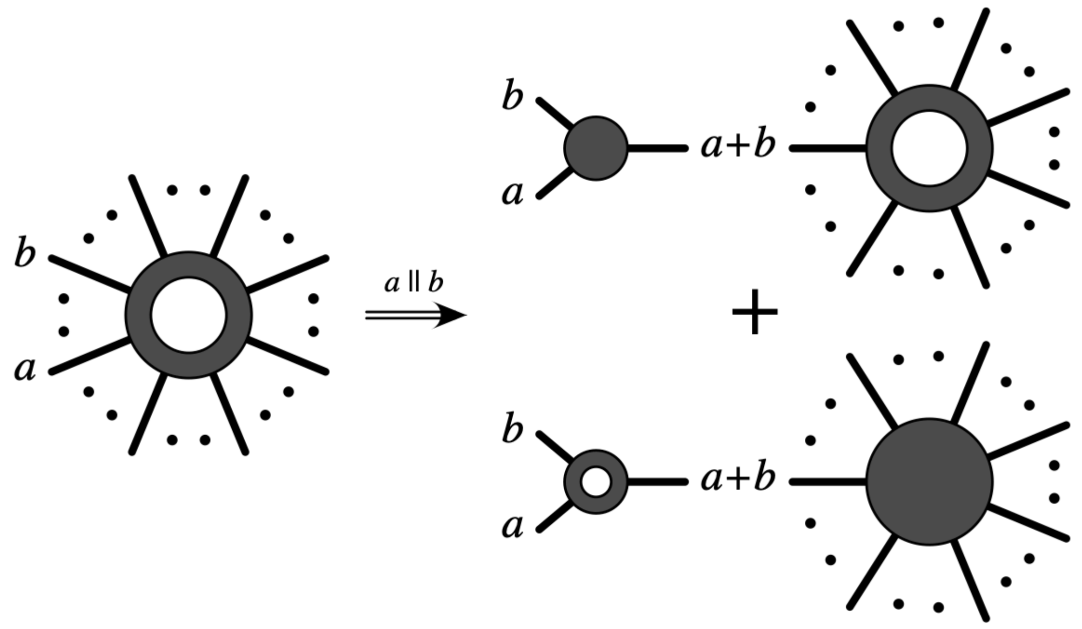
One loop single soft limit *[Catani, Grazzini 0007142]*

$$\lim_{k_i \to 0} RV_{n+1}\big(\{k\}_n\,, k_i\big) \sim \mathrm{Eik}\big(\{k\}_n\,, k_i\big) \otimes V_n\big(\{k\}_n\big) + \widetilde{\mathrm{Eik}}\big(\{k\}_n\,, k_i\big) \otimes B_n\big(\{k\}_n\big)$$



One loop single collinear limit *[Kosower 9901201, Bern, Del Duca, Kilgore, Schmidt 9903516]*

$$\lim_{k_i \| k_{\to} 0} RV_{n+1}\big(\{k\}_n\,, k_i\big) \sim \frac{1}{s_{ij}} \Big[P\big(k_i, k_j\big) \otimes V_n\big(\{k\}_n\big) + \widetilde{P}\big(k_i, k_j\big) \otimes B_n\big(\{k\}_n\big) \Big]$$



Recipe for a subtraction scheme

The construction of a subtraction scheme involves several well-defined steps:

- clear understanding of which **singular configurations** do actually contribute: find regions of the phase space which lead to non-integrable singularities of the matrix element,
- define simplified versions of the matrix element squared to be used in the subtraction terms,
- understanding how to deal with multiple radiators and overlapping singularities (first time at NNLO),
- find a way to **integrate the subtraction terms** in d-dimensions.

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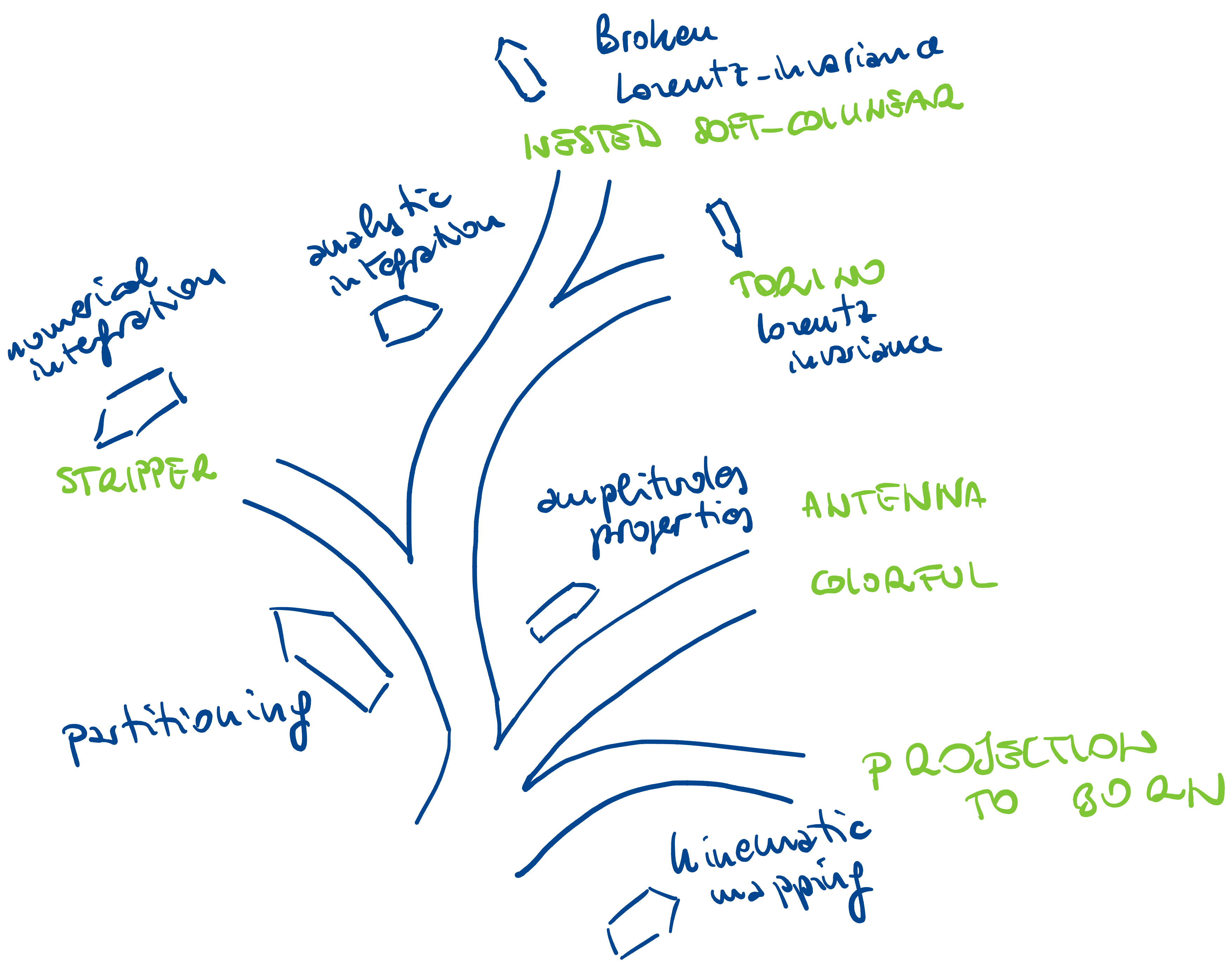

Many schemes are available:

Antenna *[Gehermann-De Ridder et al. 0505111]*

ColorfullNNLO *[Del Duca et al. 1603.08927]*

Nested-soft-collinear *[Caola et al. 1702.01352]*

STRIPPER *[Czakon 1005.0274]*

Analytic Analytic Sector *[Magnea et al. 1806.09570]*

Geometric IR subtraction *[Herzog 1804.07949]*

Unsubtraction *[Sborlini et al. 1608.01584]*

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Universal Factorisation *[Sterman et al. 2008.12293]*

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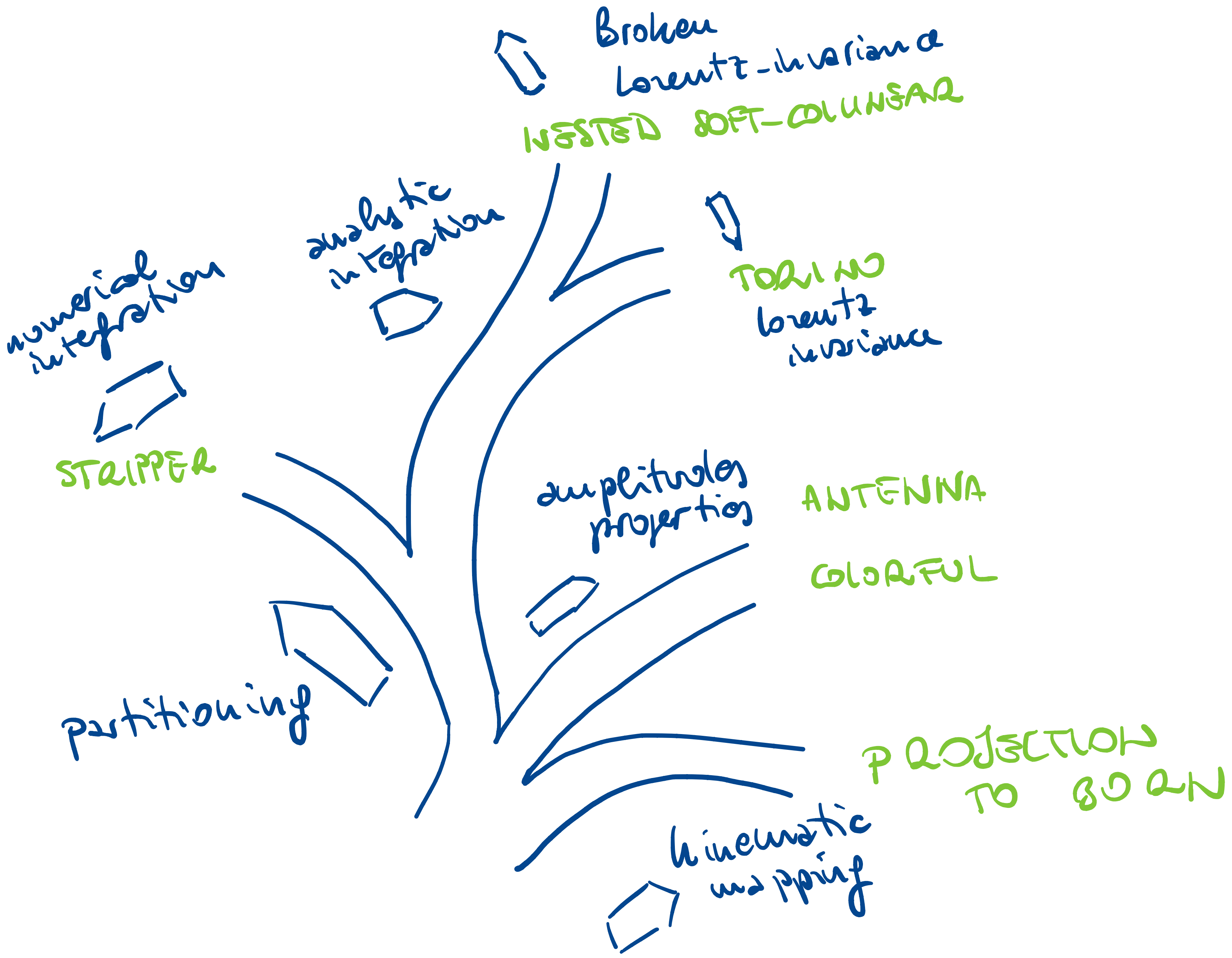

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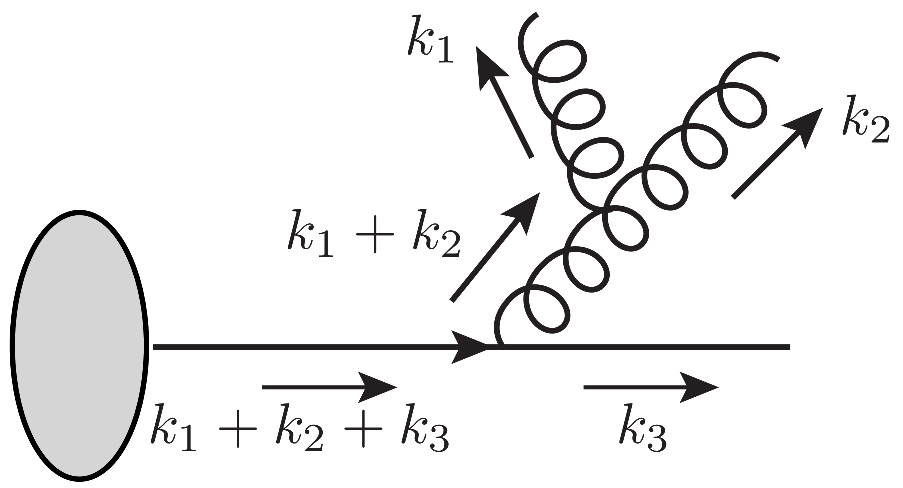
None of the existing subtraction schemes satisfies **all the '5 criteria'**

1) **Physical transparency**
2) **Generality**
3) **Locality**
4) **Analyticity**
5) **Efficiency**

Common problems

1. Clear understanding of which singular configurations do actually contribute



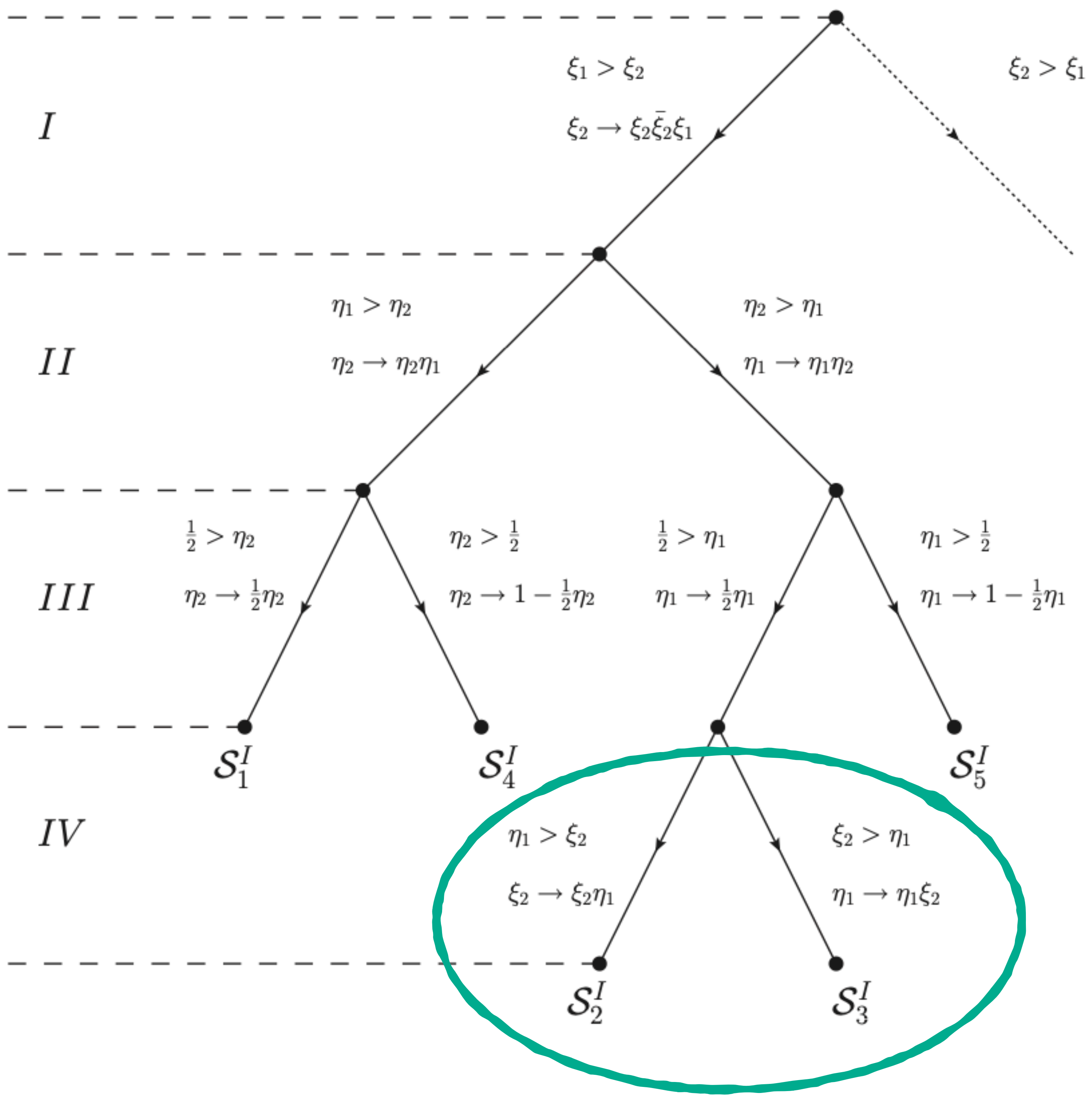
$$\sim \frac{1}{(k_1+k_2)^2} \frac{1}{(k_1+k_2+k_3)^2} = \frac{1}{2k_1 \cdot k_2} \frac{1}{2k_1 \cdot k_2 + 2k_1 \cdot k_3 + 2k_2 \cdot k_3} \iff k_1 \to 0 \text{ and } k_2 \parallel k_3$$

Entangled soft-collinear limits of diagrams can not be treated in a process-independent way.
Do non-commutative limits actually contribute?

STRIPPER *[Czakon 1005.0274]* was implemented taking into account all the possible choices of soft and collinear limits order -> redundant configurations were included.

Gauge invariant amplitudes are free of entangled singularities thanks to color coherence: soft parton does not resolve angles of the collinear partons *[Caola et al. 1702.01352]*.

Soft-collinear limits can be described by taking the known soft and collinear limits sequentially.

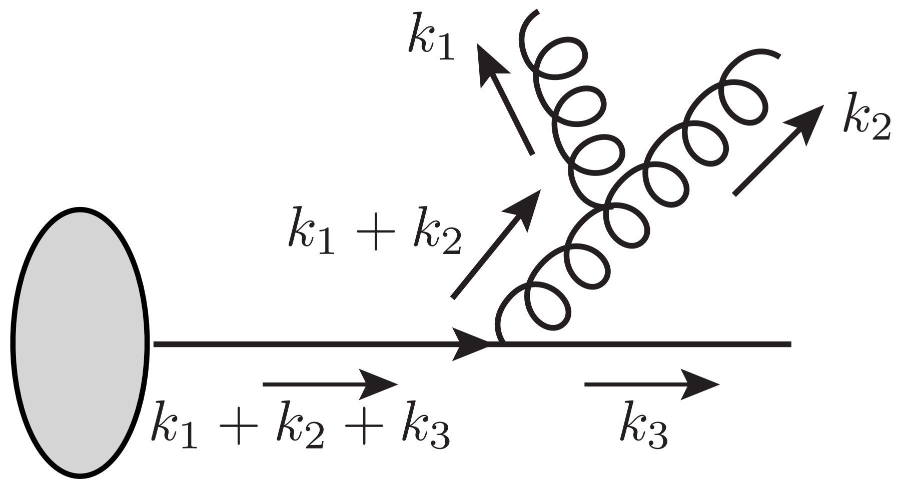


Common problems

2. Get to the point where the problem is well defined

a) Identify the overlapping singularities

b) Regulate them



$$\sim \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2)} \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2) + E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3) + E_2 E_3 (1 - \vec{n}_2 \cdot \vec{n}_3)}$$

Soft origin

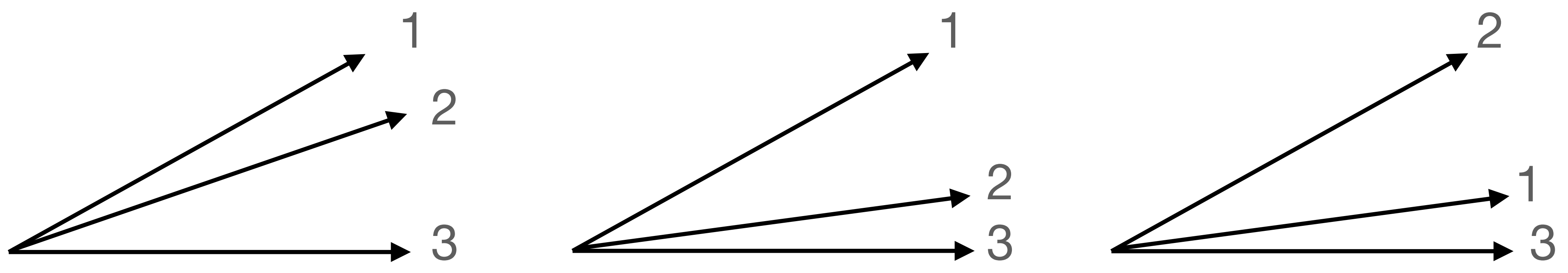
$$E_1 \to 0 \quad E_2 \to 0 \quad E_1, E_2 \to 0$$

Collinear origin

$$\vec{n}_1 \parallel \vec{n}_2 \quad \vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3$$

$$E_1 \ll E_2 \,, \quad E_2 \ll E_1$$

Includes **strongly ordered** configurations



$\vec{n}_1 \cdot \vec{n}_2 < \vec{n}_1 \cdot \vec{n}_3$ $\quad$ $\vec{n}_2 \cdot \vec{n}_3 < \vec{n}_1 \cdot \vec{n}_3$ $\quad$ $\vec{n}_1 \cdot \vec{n}_3 < \vec{n}_2 \cdot \vec{n}_3$

Soft and collinear modes do not intertwine: soft subtraction can be done globally. Collinear singularities have still to be regulated. Strongly ordered configurations have to be properly taken into account.

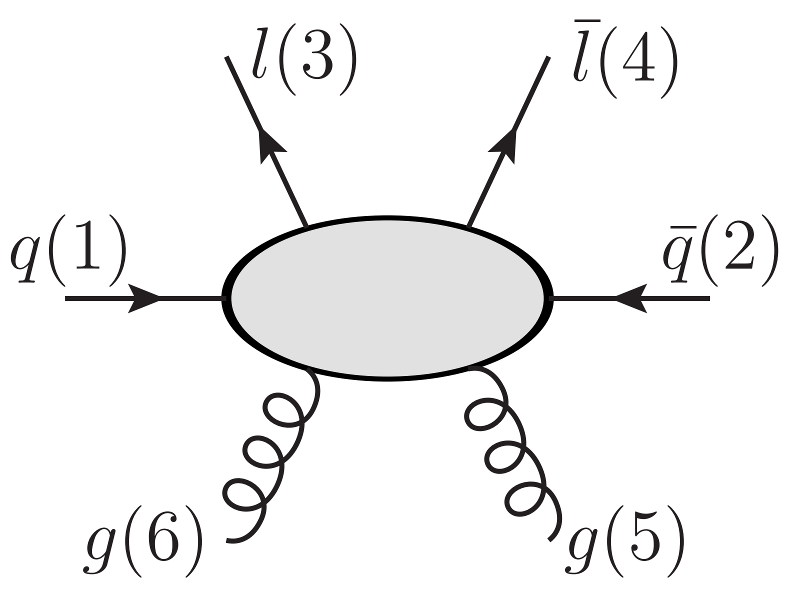
Phase space partitions

Efficient way to simplify the problem: introduce **partition functions** (following FKS philosophy):

- **Unitary partition**
- Select a **minimum number of singularities** in each sector
- Do **not affect** the **analytic integration** of the counterterms

Definition of partition functions benefits from remarkable degree of **freedom**: different approaches can be implemented

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ \, g \, g$ [Caola, Melnikov, Röntsch 1702.01352]

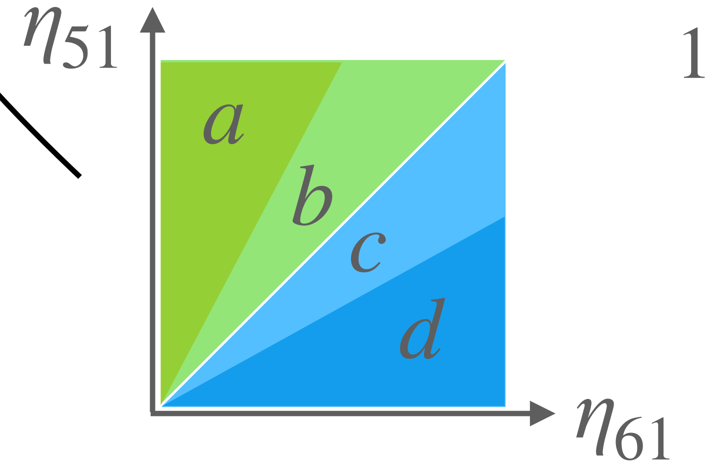


$$1 = \omega^{51,61} + \omega^{52,62} + \omega^{51,62} + \omega^{52,61}$$

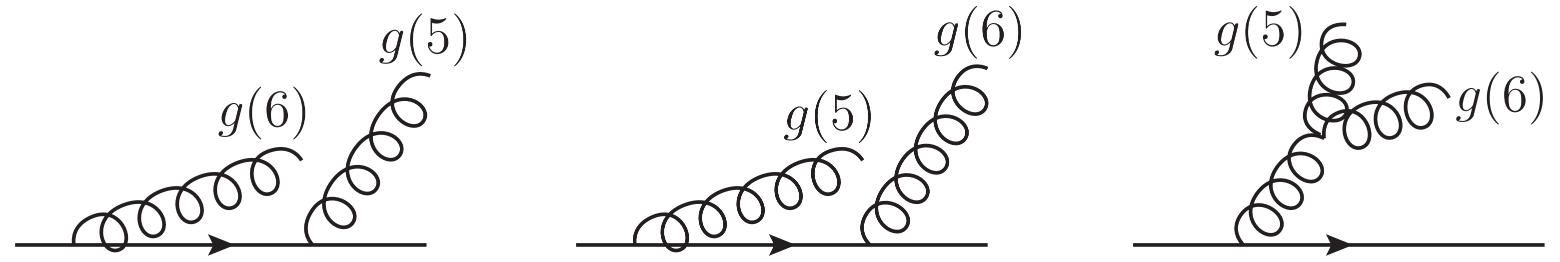
$$\rho_{ab} = 1 - \cos\vartheta_{ab} \, , \, \eta_{ab} = \rho_{ab}/2$$

$$\omega^{51,62} = \frac{\rho_{25}\,\rho_{16}\,\rho_{56}}{d_5\,d_6\,d_{5612}} \qquad \omega^{51,61} = \frac{\rho_{25}\,\rho_{26}}{d_5\,d_6}\left(1 + \frac{\rho_{15}}{d_{5621}} + \frac{\rho_{16}}{d_{5612}}\right)$$

$$\omega^{52,62} = \frac{\rho_{15}\,\rho_{16}}{d_5\,d_6}\left(1 + \frac{\rho_{25}}{d_{5621}} + \frac{\rho_{26}}{d_{5612}}\right) \qquad \omega^{52,61} = \frac{\rho_{15}\,\rho_{26}\,\rho_{56}}{d_5\,d_6\,d_{5621}}$$



$$1 = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right)$$



Advantages:

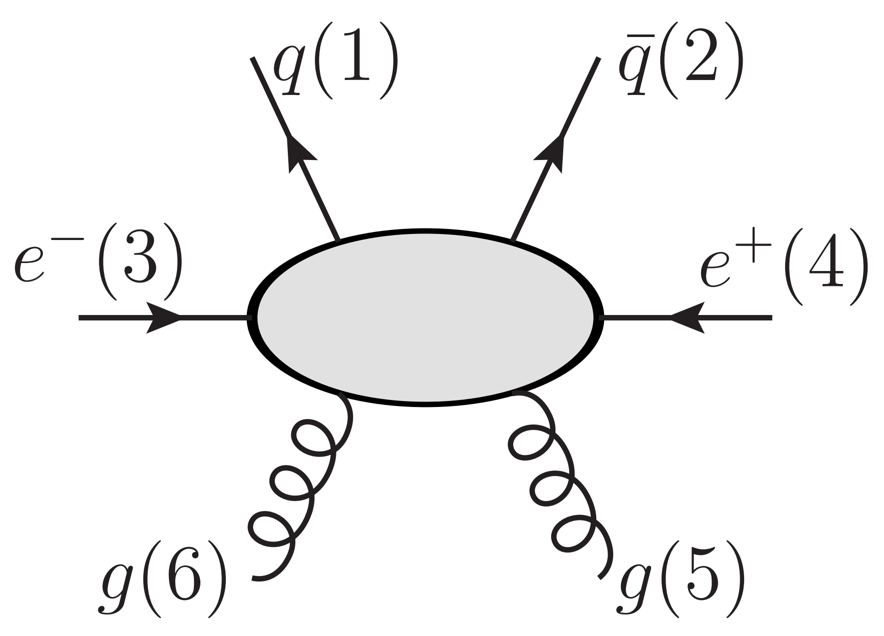
1. Simple definition
2. Structure of collinear singularities fully defined
3. **Minimum number of sector**

Disadvantages:

1. Partition based on **angular ordering** → **Lorentz invariance not preserved**
2. Theta function

Phase space partitions

Examples: Local Analytic Sector Subtraction $e^+e^- \to \gamma^* \to q\bar{q} + g\,g$ *[Magnea, C.S-S. et al. 1806.09570]*



$$1 = \mathcal{W}_{1225} + \mathcal{W}_{1226} + \mathcal{W}_{1252} + \mathcal{W}_{1256} + \ldots + \mathcal{W}_{6152}$$

$$\mathcal{W}_{abcd} = \frac{\sigma_{abcd}}{\sum_{m,n,p,q} \sigma_{mnpq}} \qquad \sigma_{abcd} = \frac{1}{(e_a\, w_{ab})^\alpha} \frac{1}{(e_c + \delta_{bc}\, e_a)\, w_{cd}}, \quad \alpha > 1$$

$$e_i \propto s_{qi}\,, \qquad w_{ij} \propto \frac{s_{ij}}{s_{qi}\, s_{qj}} \qquad q^\mu = (\sqrt{s}, \vec{0}), \qquad s_{ab} = 2k_a \cdot k_b$$

Advantages:

1. Compact definition
2. Triple-collinear sectors do not require further partition
3. Structure of collinear singularities fully defined
4. Valid for arbitrary number of FS partons
5. **Defined in terms of Lorentz invariants**

Disadvantages:

1. Numerous sectors → consequence of being fully general
   → non minimal structure
2. Non-trivial recombination before integration

Common problems

3. Solve the PS integrals

The problem is now well defined:

A. **Singular kernels** and their nested limits have to be **subtracted from the double real correction** to get integrable object

$$\int d\Phi_{n+2} RR_{n+2} = \int d\Phi_{n+2} \left[RR_{n+2} - K_{n+2}\right] + \int d\Phi_{n+2} K_{n+2} \qquad K_{n+2} \supset C_{ij}, C_{kl}, S_i, S_{ij}, C_{ijk}$$

B. **Counterterms** have to be **integrated over the unresolved phase space**

$$I = \int \mathrm{PS}_{\mathrm{unres.}} \otimes \mathrm{Limit} \otimes \mathrm{Constraints}$$

The ‘Limit’ component is universal and known. The phase space is well defined. Constraints may vary depending on the scheme.

Several kinematic structures have to be integrated **analytically** over a 6-dim PS.

Different approximations and techniques can be applied: the result assume different forms according on the integration strategy.

Two main structure are the most complicated ones and affect most of the physical processes:

- **Double soft**
- **Triple collinear**

Details of the calculation: NLO as a playground

Local Analytic Sector Subtraction

Go **back to NLO** to implement a new scheme featuring **key properties** that can be exported at **exported at NNLO**.
(This talk: massless partons, FSR only, arbitrary number of FS particles)

$$\frac{d\sigma_{\rm NLO}}{dX} = \lim_{d\to 4} \left\{ \int d\Phi_n\, V\,\delta_n(X) + \int d\Phi_{n+1}\, R\,\delta_{n+1}(X) \right\}$$

X IR safe observable

$$\frac{d\sigma^{\rm NLO}_{ct}}{dX} = \int d\Phi_{n+1}\, K$$

Counterterm

$$I = \int d\Phi_{\rm rad}\, K$$

Integrated Counterterm

$$\frac{d\sigma^{\rm NLO}}{dX} = \int d\Phi_n \left(V + I \right) \delta_n(X) + \int d\Phi_{n+1} \left(R\,\delta_{n+1}(X) - K\,\delta_n(X) \right)$$

Properties of the scheme:

Analytically calculable (possibly with standard techniques)

Minimal structure and **simple integration**

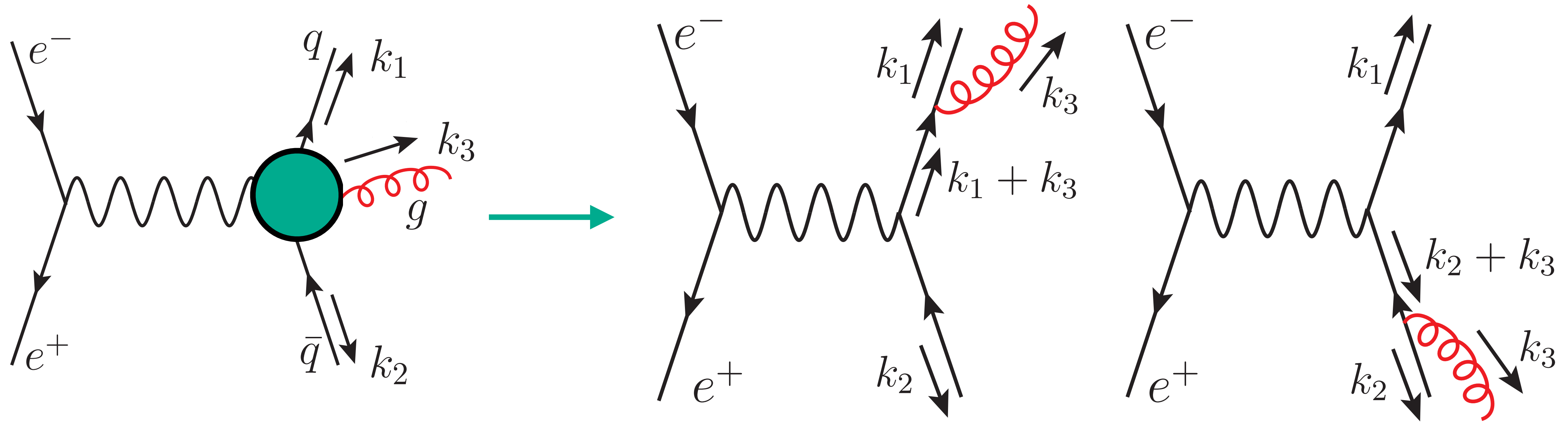
Requirements:

Choose an **optimise parametrisation** of the phase space

Organise all the overlapping **singularities** and choose an **appropriate kinematics**

Ingredients of the subtraction

- *Projection operators: extract from the real-radiation matrix element its leading soft and collinear limits*



$$R \sim \frac{1}{(k_1+k_3)^2} + \frac{1}{(k_2+k_3)^2} \sim \frac{1}{E_1\,E_3\,(1-\vec{n}_1\cdot\vec{n}_3)} + \frac{1}{E_2\,E_3\,(1-\vec{n}_2\cdot\vec{n}_3)}$$

$$R \to \infty \quad \begin{cases} E_3 \to 0 & \to \mathbf{S}_3 \\ \vec{n}_1 \parallel \vec{n}_3 & \to \mathbf{C}_{13} = \mathbf{C}_{31} \\ \vec{n}_2 \parallel \vec{n}_3 & \to \mathbf{C}_{23} = \mathbf{C}_{32} \end{cases}$$

soft

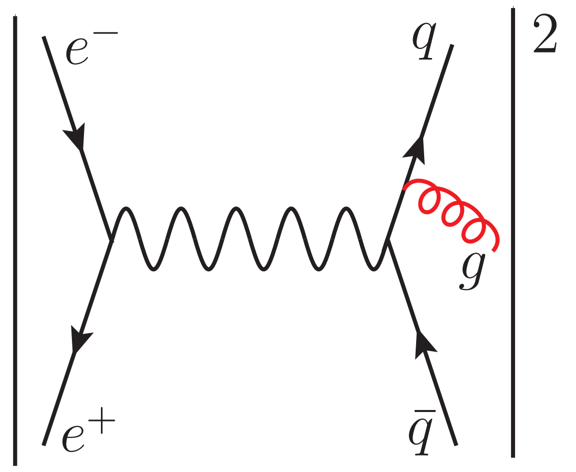
collinear

Singular limits have universal form, independent of the resolved subprocess *[Altarelli, Parisi '77]*

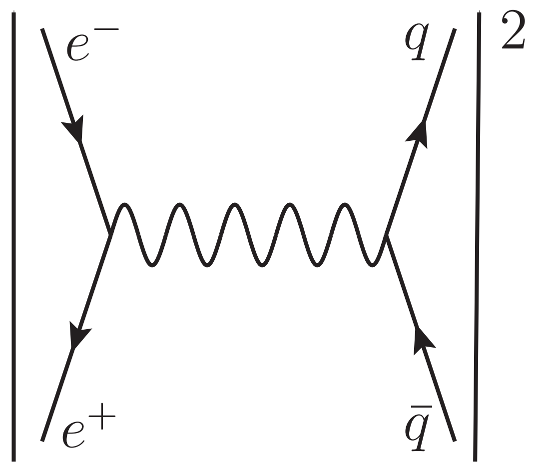
$$S_i\,R(\{k\}) \propto \sum_{a,c\neq i} \frac{s_{cd}}{s_{ci}\,s_{di}}\,B\left(\{k\}_{\not{i}}\right)$$

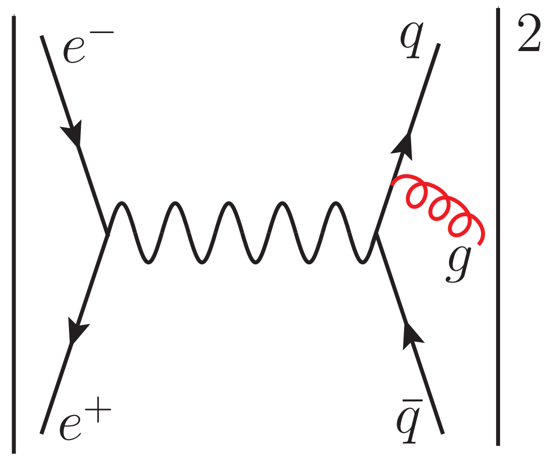
$$C_{ij}\,R(\{k\}) \propto \frac{1}{s_{ij}}\,P^{\mu\nu}_{ij}(s_{ir},s_{jr})\;B^{\mu\nu}\left(\{k\}_{\not{i}\not{j}},k_{ij}\right)$$

$$S_i\,C_{ij}\,R(\{k\}) \propto \frac{s_{jr}}{s_{ij}\,s_{ir}}\,B\left(\{k\}_{\not{i}}\right)$$

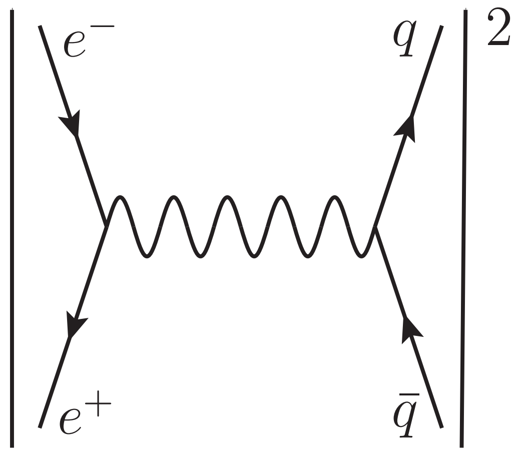


$$\underset{E_3\to 0}{\sim}\; 2C_F\,g_s^2\,\frac{k_1\cdot k_2}{(k_1\cdot k_3)\,(k_2\cdot k_3)}$$





$$\underset{k_1\parallel k_3}{\sim}\; C_F\,g_s^2\,\frac{1}{k_1\cdot k_3}\,P_{qg}$$



Ingredients of the subtraction

- *Phase space partitioning (FKS): multiple singular configuration that overlap*
  - **Unitary partition**
  - Select a **minimum number of singularities** in each sector: set of kinematic weights smoothly damping all radiative singularities but those due to particle i becoming soft, or collinear to j
  - Do **not affect** the **analytic integration** of the counterterms

Sector functions $\mathcal{W}_{ij}$:

$$R = \sum_{i,j} R\,\mathcal{W}_{ij} = R\,\mathcal{W}_{31} + R\,\mathcal{W}_{32} + \dots$$

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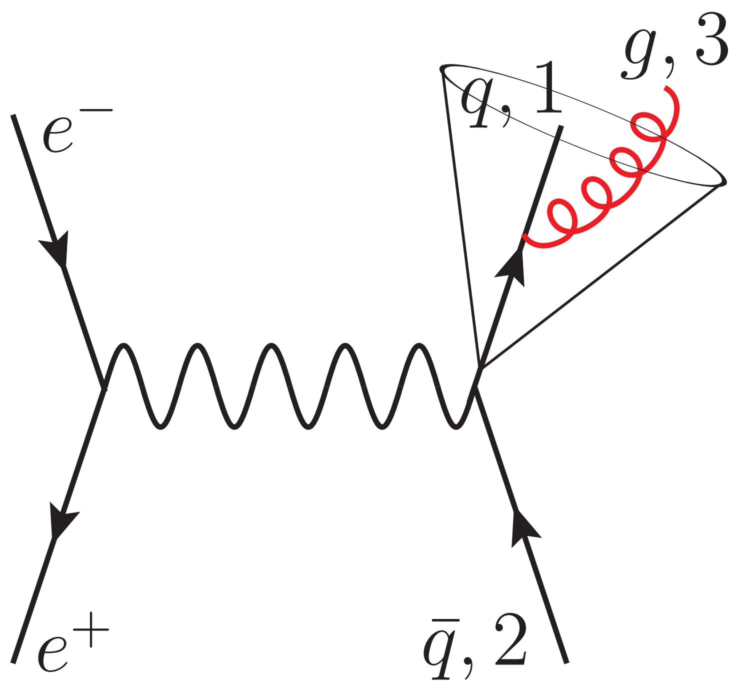

Damp: $\vec{n}_2 \parallel \vec{n}_3$

Enhance: $\vec{n}_1 \parallel \vec{n}_3$

$\mathcal{W}_{31} \sim \dfrac{1}{s_{31}}$

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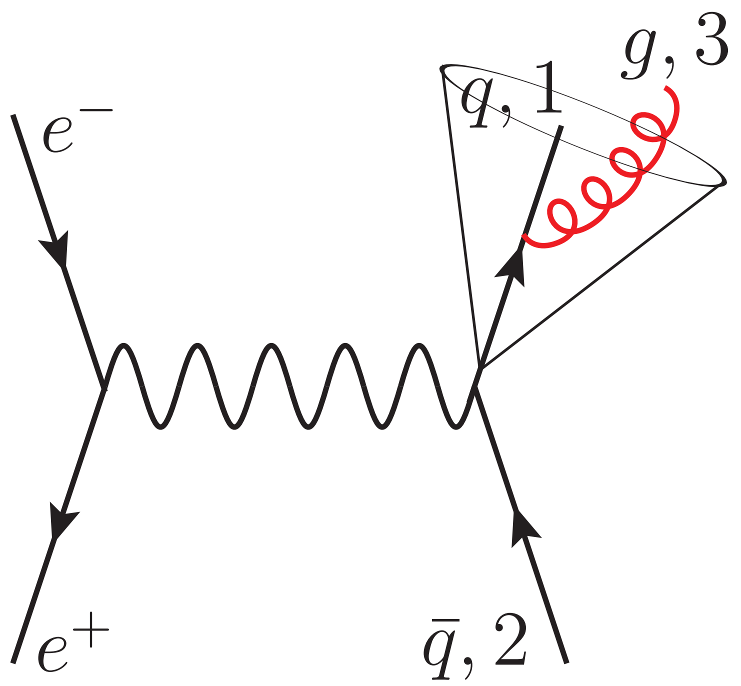

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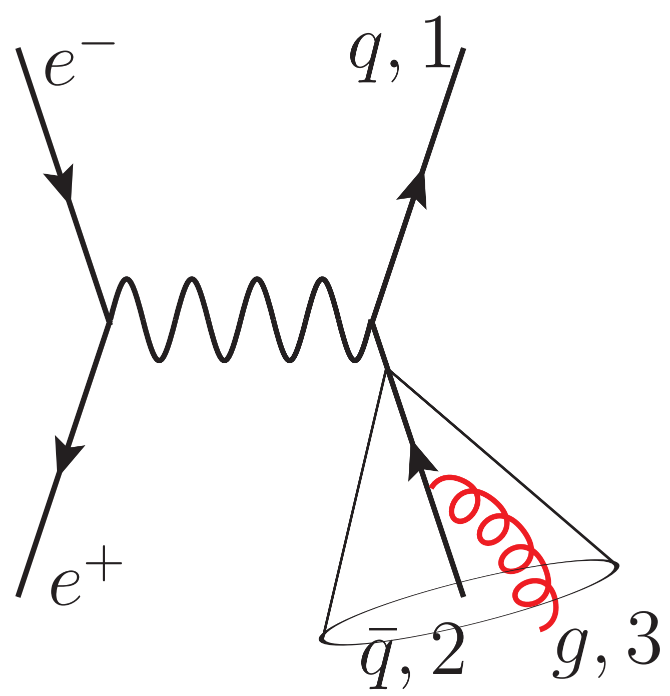

Sector functions at NLO in the analytic sector subtraction

Sector functions $\mathcal{W}_{ij}$:

1) Select the minimum number of singularities

$$\mathbf{S}_i \mathcal{W}_{ab} = 0 \,, \qquad \forall i \neq a \qquad\qquad \mathbf{C}_{ij} \mathcal{W}_{ab} = 0 \,, \qquad \forall a, b \notin \{i, j\} \,.$$

2) Sum properties

$$\sum_{i, j \neq i} \mathcal{W}_{ij} = 1 \qquad \mathbf{S}_i \sum_{j \neq i} \mathcal{W}_{ij} = 1 \,, \qquad \mathbf{C}_{ij} \sum_{a, b \in \{ij\}} \mathcal{W}_{ab} = 1 \,.$$

3) Explicit form

$$CM : q^\mu = (\sqrt{s}, \vec{0}) \,, \qquad e_i = \frac{s_{qi}}{s} \,, \qquad \omega_{ij} = \frac{s \, s_{ij}}{s_{qi} \, s_{qj}} \,, \qquad \mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{k, l \neq k} \sigma_{kl}} \,, \qquad \sigma_{ij} = \frac{1}{e_i \, \omega_{ij}}$$

$$\mathbf{S}_i \mathcal{W}_{ab} = \delta_{ia} \frac{1/\omega_{ab}}{\sum_{c \neq a} 1/\omega_{ac}} \,, \qquad \mathbf{C}_{ij} \mathcal{W}_{ab} = \left(\delta_{ia} \delta_{jb} + \delta_{ib} \delta_{ja} \right) \frac{e_b}{e_a + e_b}$$

The idea of mappings

$$\int d\Phi_{n+1}\left(R_{n+1} - K_{n+1}\right) \xrightarrow{\{k\}_{n+1} \to \{\bar{k}_n\}^{(abc)}} \int d\Phi_{n+1}\left(R_{n+1} - \overline{K}_{n+1}\right)$$

$$S_i\, R_{n+1}(\{k\}) \propto \sum_{a,c\neq i} \frac{s_{cd}}{s_{ci}\, s_{di}} B_n(\{k\}_{\not i}) \qquad \bar{S}_i\, R_{n+1}(\{k\}) \propto \sum_{a,c\neq i} \frac{s_{cd}}{s_{ci}\, s_{di}} B_n(\{\bar{k}\}^{(icd)})$$

$$C_{ij}\, R_{n+1}(\{k\}) \propto \frac{1}{s_{ij}} P^{\mu\nu}_{ij}(s_{ir}, s_{jr})\, B^{\mu\nu}_n(\{k\}_{\not i \not j}, k_{ij}) \qquad \bar{C}_{ij}\, R_{n+1}(\{k\}) \propto \frac{1}{s_{ij}} P^{\mu\nu}_{ij}(s_{ir}, s_{jr})\, B^{\mu\nu}_n(\{\bar{k}\}^{(ijr)})$$

$$S_i\, C_{ij}\, R_{n+1}(\{k\}) \propto \frac{s_{jr}}{s_{ij}\, s_{ir}} B_n(\{k\}_{\not i}) \qquad \bar{S}_i\, \bar{C}_{ij}\, R_{n+1}(\{k\}) \propto \frac{s_{jr}}{s_{ij}\, s_{ir}} B_n(\{\bar{k}\}^{(ijr)})$$

Why a mapping?

1. $\{k\}_{\not i}$ is a set of n momenta that do not satisfy n-body momentum conservation away from the exact S_i limit
2. $\{k\}_{\not i \not j}, k_{ij}$ is a set of n momenta where $k_{ij} = k_i + k_j$ is off-shell away from the exact C_{ij} limit
3. Factorise the $n+1$-body PS intro a n-body and radiation phase space is necessary to integrate K only in the latter

Collinear limit: single mapping > *dipole = (ijr)*
Soft limit: different mapping for each contribution > *dipole = (icd)*

The idea of mappings

Factorise the phase space $d\Phi_{n+1} = d\overline{\Phi}_n \, d\overline{\Phi}_{\text{rad}}$

On-shell particle **conserving momentum** in the entire PS

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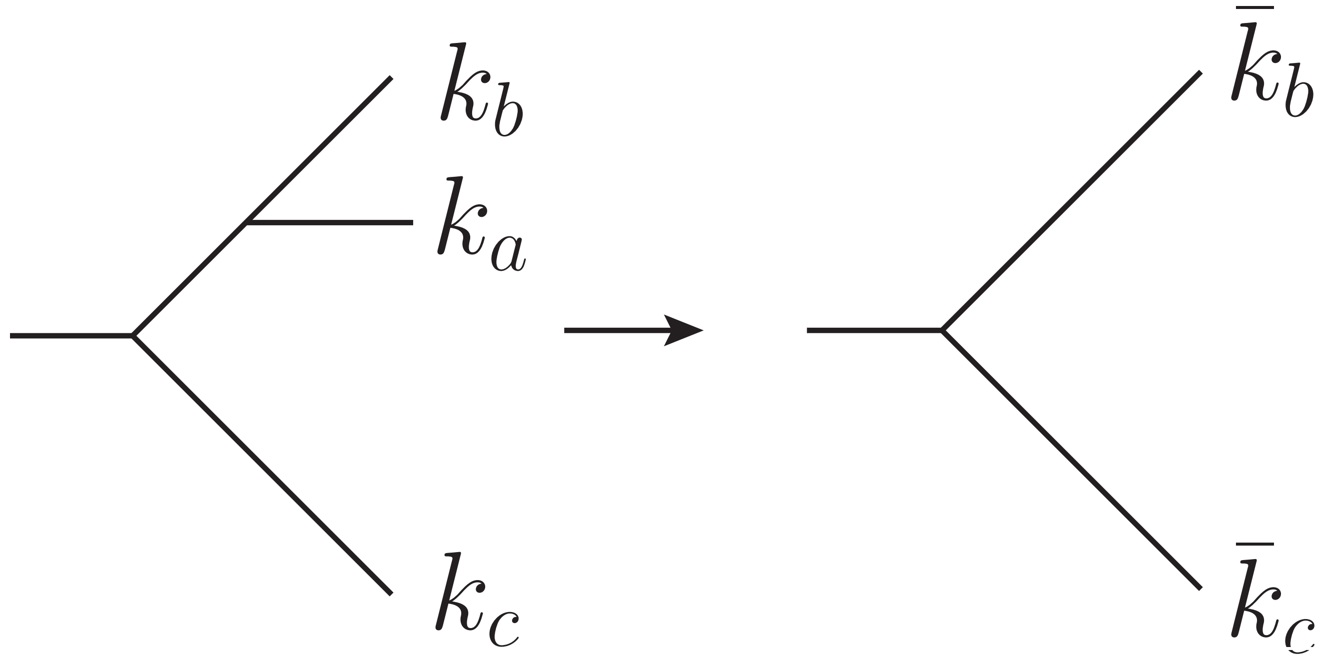
On-shell particle **conserving momentum** in the entire PS

↓

Mapped kinematics $\{\bar{k}\}^{(abc)} = \left\{ \{k\}_{\not a \not b \not c}, \bar{k}_b^{(abc)}, \bar{k}_c^{(abc)} \right\}$

$\bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c$



Different ways to combine momenta, depending on the **choice** of the dipole (abc)

→ Freedom to choose the momenta to **simplify the integration**

The idea of mappings

Factorise the phase space $d\Phi_{n+1} = d\overline{\Phi}_n \, d\overline{\Phi}_{\text{rad}}$

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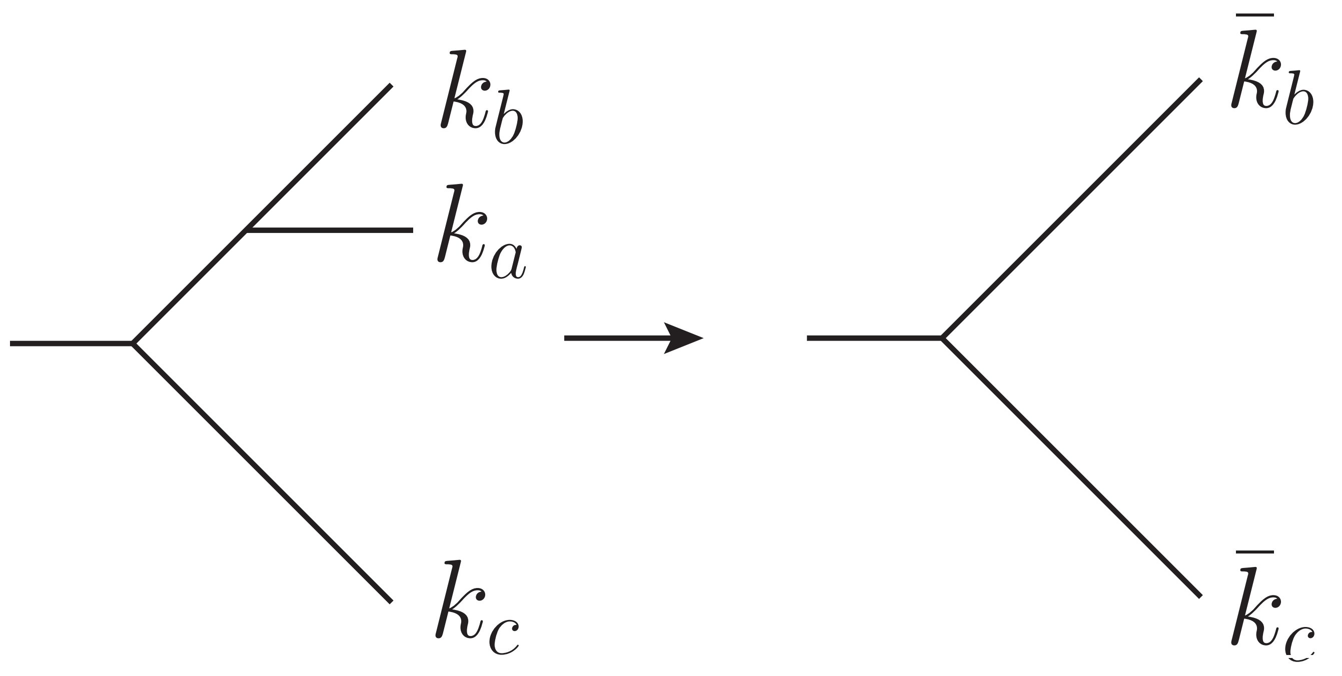

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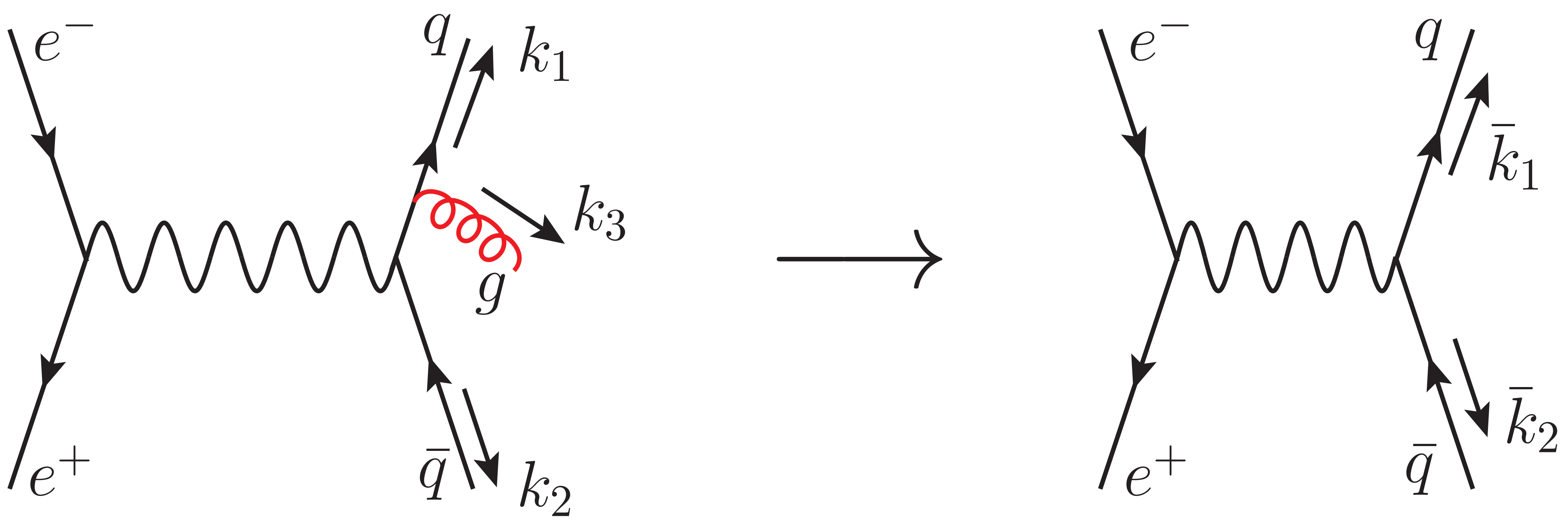
Different ways to combine momenta, depending on the **choice** of the dipole (abc)

→ Freedom to choose the momenta to **simplify the integration**

$$k_1, k_2, k_3, \; k_i^2 = 0$$

$$\bar{k}_2^{(312)} = \frac{s_{312}}{s_{32} + s_{12}} k_2$$

$$\bar{k}_1^{(312)} = k_3 + k_1 - \frac{s_{31}}{s_{32} + s_{12}} k_2$$



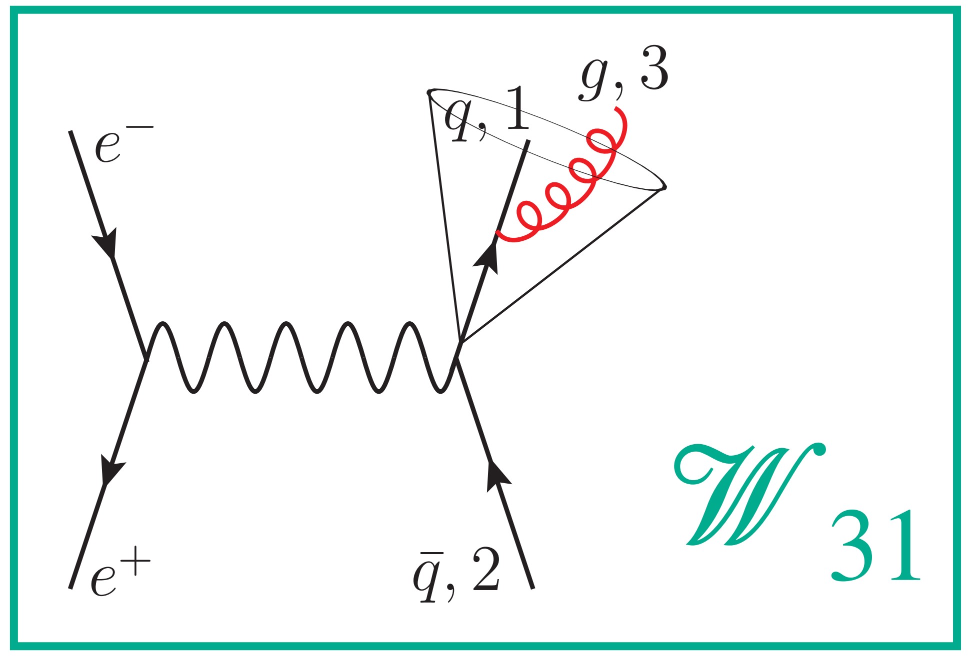
Ingredients of the subtraction

- *Candidate counterterm:*

Defined **sector by sector** as the collection of all the contributing limits (correct multiplicity!)

iterative definition $\left(1-\overline{\mathbf{S}}_3\right)\left(1-\overline{\mathbf{C}}_{13}\right) R\,\mathcal{W}_{31} = \text{finite}$

$$K_{31} = \left[\overline{\mathbf{S}}_3 + \overline{\mathbf{C}}_{13}\left(1-\overline{\mathbf{S}}_3\right)\right] R\,\mathcal{W}_{31} \;\rightarrow\; R\,\mathcal{W}_{31} - K_{31} = \text{finite}$$



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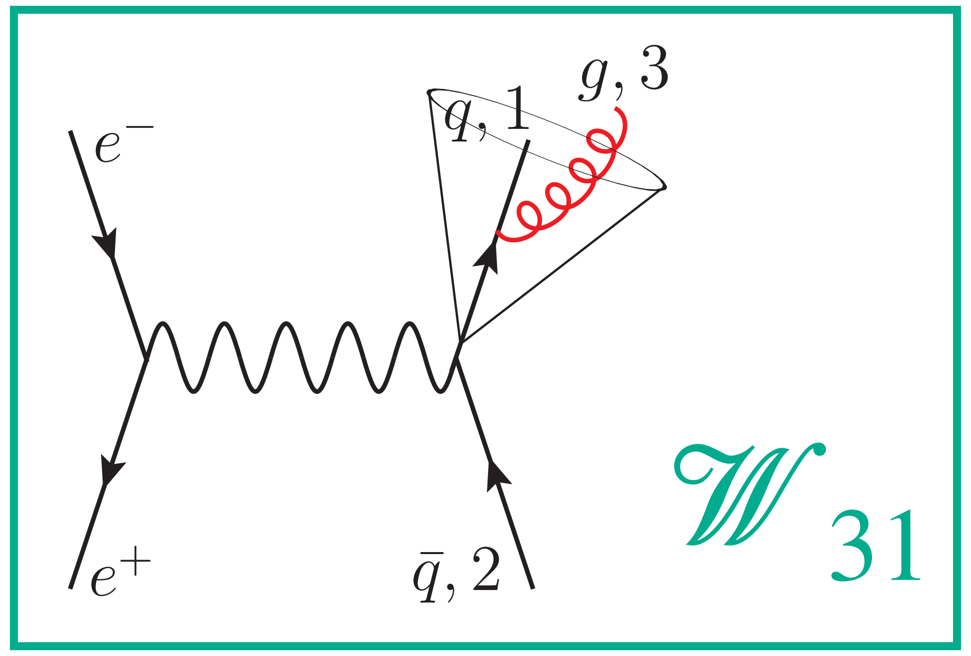

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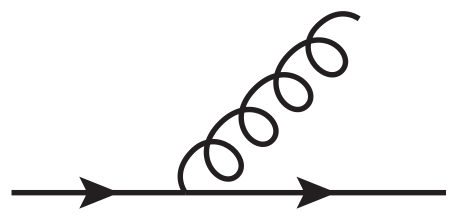
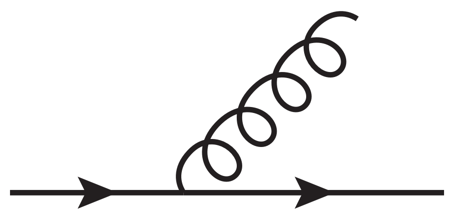
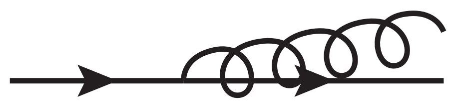
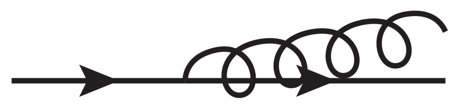
$\int$  $\mathcal{W}_{ij}\, d\Phi_{n+1} = \int \Big[$  $\mathcal{W}_{ij}\; -$  $\Big]\, d\Phi_{n+1} + \int$  $d\Phi_{n+1}$

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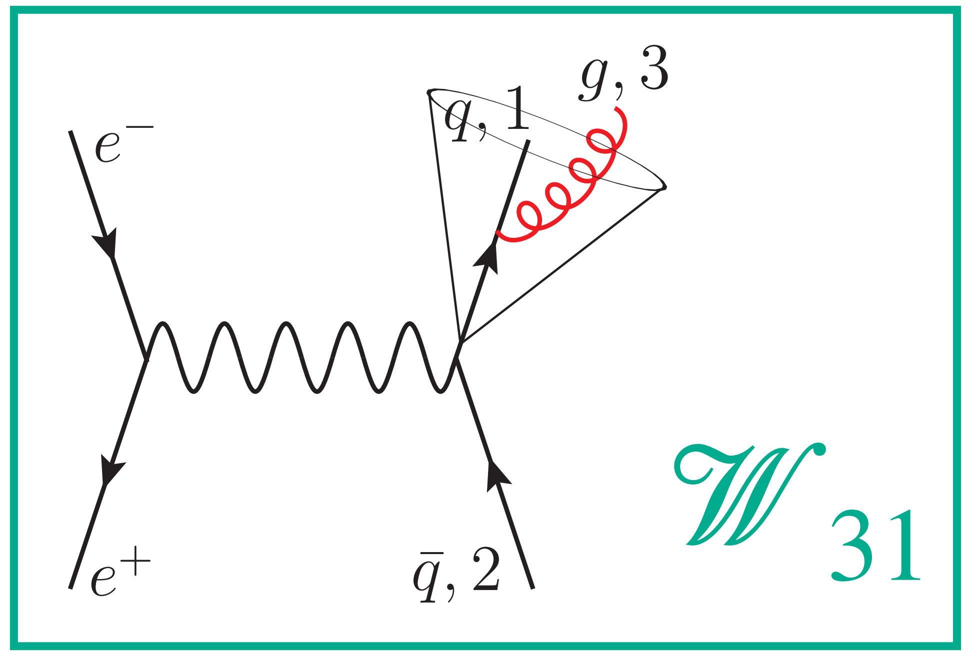

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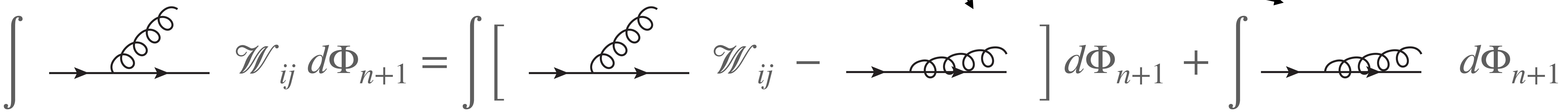


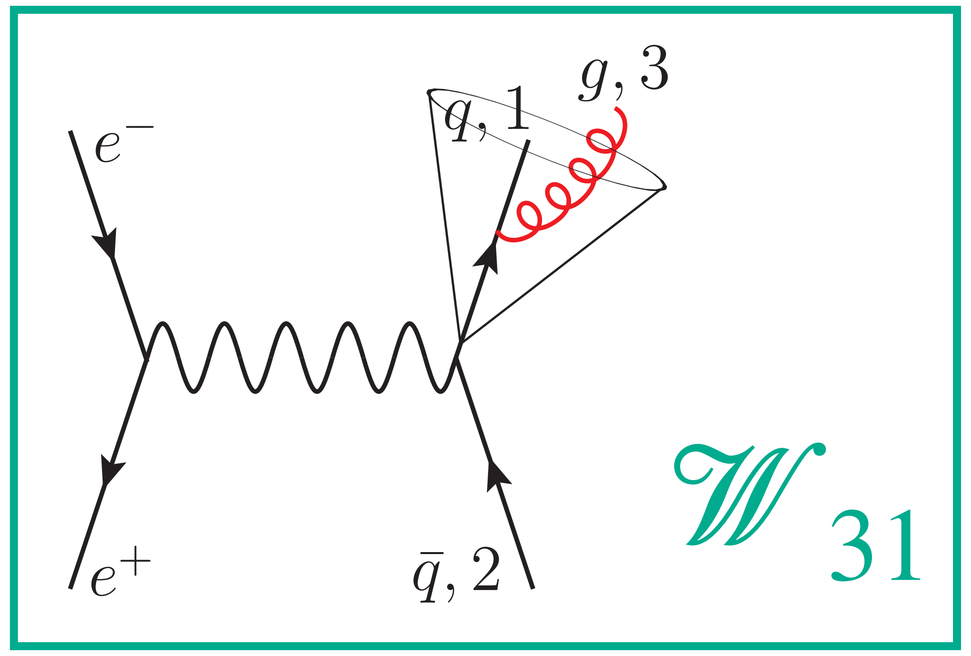
Featuring **optimised remapping** for integration $\{k_{n+1}\} \rightarrow \{k_n\}^{(abc)}$

(abc) **according to the invariants** appearing in the kernel

$$\overline{\mathbf{S}}_i R(\{k\}) \propto \sum_{c,d\neq i} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{k\}^{(icd)}\right) \longrightarrow$$ Different mapping for each contribution

$$\overline{\mathbf{C}}_{ij} R(\{k\}) \propto \frac{1}{s_{ij}} P_{ij}^{\mu\nu} B_{\mu\nu}\left(\{k\}^{(ijr)}\right) \longrightarrow$$ Single mapping

Ingredients of the subtraction



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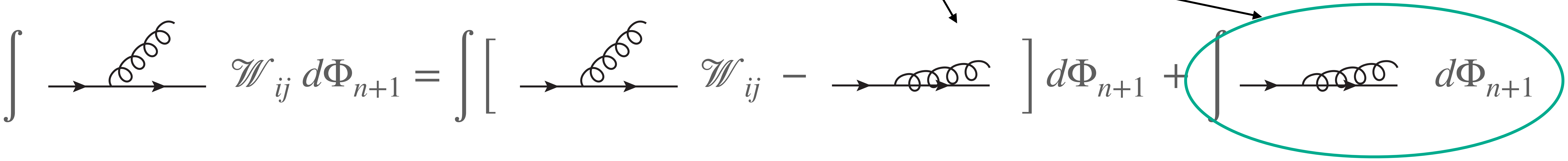

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Ingredients of the subtraction

- *Analytic integration:*

Parametrisation of the phase space

$$d\Phi_{n+1} = d\Phi_n^{(abc)} \times d\Phi_{\rm rad}\left(s_{bc}^{(abc)}; y, z, \phi\right)$$

$$s_{ab} = y\, s_{bc}^{(abc)}$$
$$s_{ac} = z(1-y)\, s_{bc}^{(abc)}$$
$$s_{bc} = (1-z)(1-y)\, s_{bc}^{(abc)}$$

Radiative phase space:

$$d\Phi_{\rm rad}^{(abc)} \propto \left(s_{bc}^{(abc)}\right)^{1-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, (1-y)\left[(1-y)^2\, y\, (1-z)\, z\right]^{-\epsilon}$$

Kernel to integrate:

$$\overline{\mathbf{S}}_i\, R\left(\{k\}\right) \propto \sum_{c,d\neq i} \frac{s_{cd}}{s_{ic}\, s_{id}}\, B_{cd}\left(\{k\}^{(icd)}\right)$$

Freedom to **adapt** the **parametrisation to the kernel** → Exact analytic integration

$$I^s \propto \sum_{c,d\neq i} \int d\Phi_{\rm rad}^{(icd)} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{k\}^{(icd)}\right) = \sum_{c,d\neq i} \left(s_{bc}^{(abc)}\right)^{-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, (1-y)\left[(1-y)^2\, y\, (1-z)\, z\right]^{-\epsilon} \frac{1-z}{z} B_{cd}\left(\{k\}^{(icd)}\right)$$

$$= \sum_{c,d\neq i} \left(s_{bc}^{(abc)}\right)^{-\epsilon} \frac{(4\pi)^{\epsilon-2}\,\Gamma(1-\epsilon)\,\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} B_{cd}\left(\{k\}^{(icd)}\right)$$

General remarks:

1. Different parametrisation for the soft and for the hard-collinear counterterm
2. Each contribution to the soft is parametrised differently to simplify the integration

Lesson from NLO

- **Unitary partition** of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$
- Collection of relevant IRC limits for a given sector
- **Catani-Seymour** final-state **dipole mapping**
- Promotion to counterterms: **improved limits**
- **Locality of the cancellation** ensured by **consistency relations**

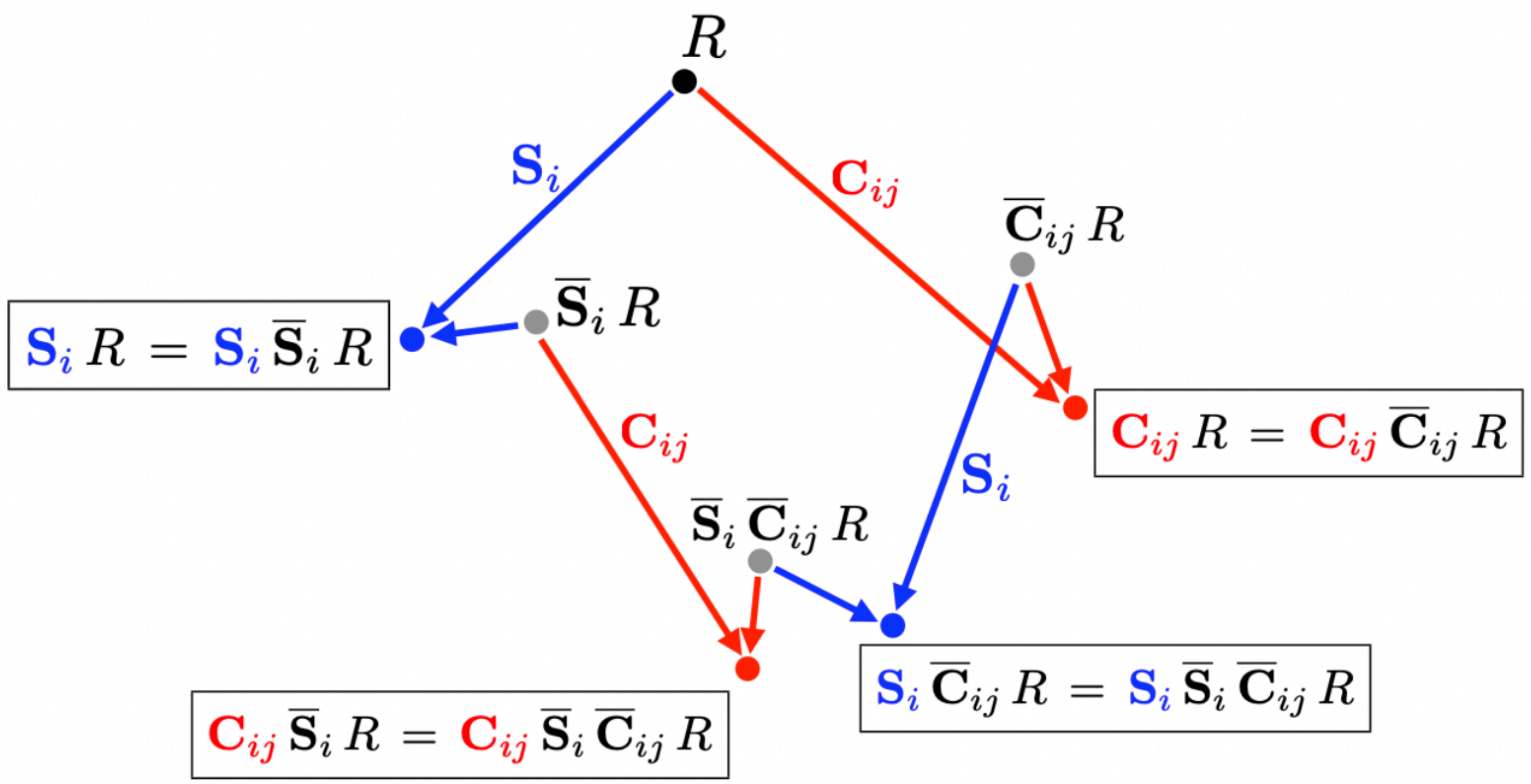
$$\mathbf{S_i}\,R = \mathbf{S}_i\left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right)R$$

$$\mathbf{C_{ij}}\,R = \mathbf{C}_{ij}\left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right)R$$

As well as

$$\mathbf{S_i}\,\mathcal{W}_{ij} = \mathbf{S}_i\overline{\mathbf{S}}_i\,\mathcal{W}_{ij}$$

$$\mathbf{C_{ij}}\,\mathcal{W}_{ij} = \mathbf{C}_{ij}\overline{\mathbf{C}}_{ij}\,\mathcal{W}_{ij}$$



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- Promotion to counterterms: **improved limits**
- **Locality of the cancellation** ensured by **consistency relations**
- $\mathcal{W}_{ij}$ **sum rules+ mapping adaptation = simple analytic counterterm integration**

$$K = \sum_{i,j} K_{ij} \propto \bar{\mathbf{S}}_i R \Big[\underbrace{\sum_j \bar{\mathbf{S}}_i \mathcal{W}_{ij}}_{=1} \Big] + \bar{\mathbf{C}}_{ij} R \Big[\underbrace{\bar{\mathbf{C}}_{ij} \big(\mathcal{W}_{ij} + \mathcal{W}_{ji} \big)}_{=1} \Big] - \bar{\mathbf{S}}_i \bar{\mathbf{C}}_{ij} R \Big[\underbrace{\bar{\mathbf{S}}_i \bar{\mathbf{C}}_{ij} \mathcal{W}_{ij}}_{=1} \Big]$$

$$\Longrightarrow K = \sum_i \bar{\mathbf{S}}_i R + \sum_{i,j \neq i} \bar{\mathbf{C}}_{ij} \big(1 - \bar{\mathbf{S}}_i \big) R$$

Remarks:

1. The integrated counterterm has to **match the poles of V**, which is **not split** into sectors
2. The sector functions would have made the **integration** much **more involved**

Details of the calculation: NNLO

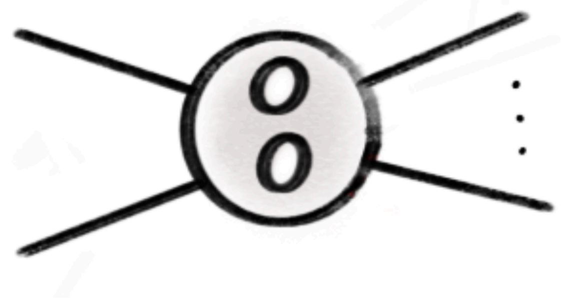
Exploring the framework

$$\frac{d\sigma}{dX} = \frac{d\sigma_{\mathrm{LO}}}{dX} + \frac{d\sigma_{\mathrm{NLO}}}{dX} + \boxed{\frac{d\sigma_{\mathrm{NNLO}}}{dX}} + \dots$$

Arbitrary number of massless QCD final-state emissions

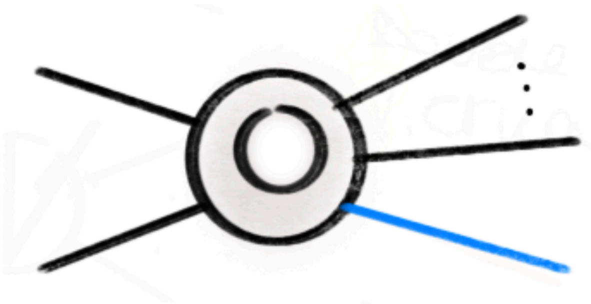
X_i = IRC-safe observable computed with i-body kinematics, $\delta_{X_i} = \delta(X - X_i)$

$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \int d\Phi_n \, VV \, \delta_{X_n}$$



Up to $1/\epsilon^4$ explicit poles

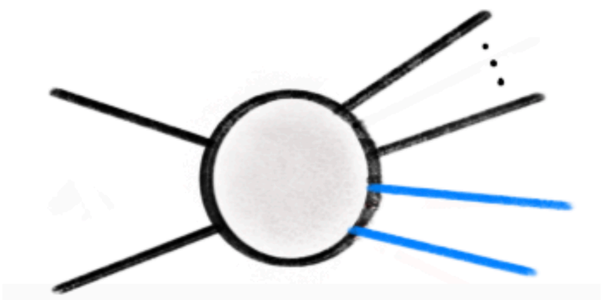
$$+ \int d\Phi_{n+1} \, RV \, \delta_{X_{n+1}}$$



Up to $1/\epsilon^2$ explicit poles

Singular in PS

$$+ \int d\Phi_{n+2} \, RR \, \delta_{X_{n+2}}$$

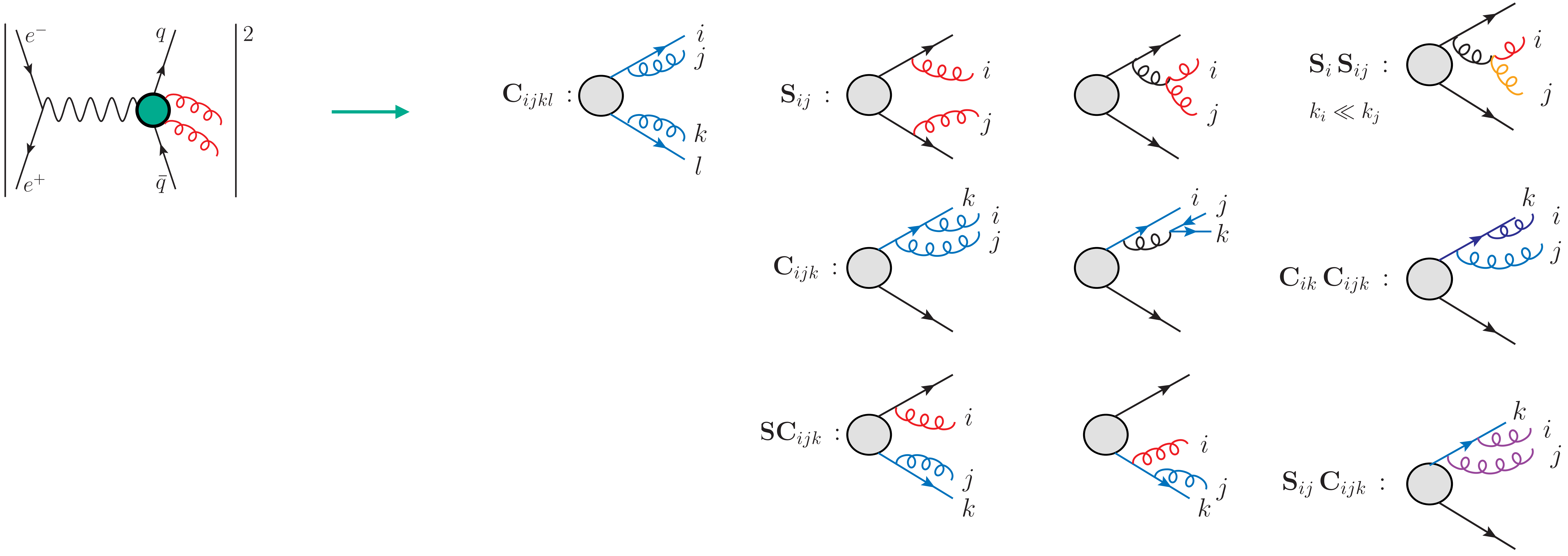


Singular in PS

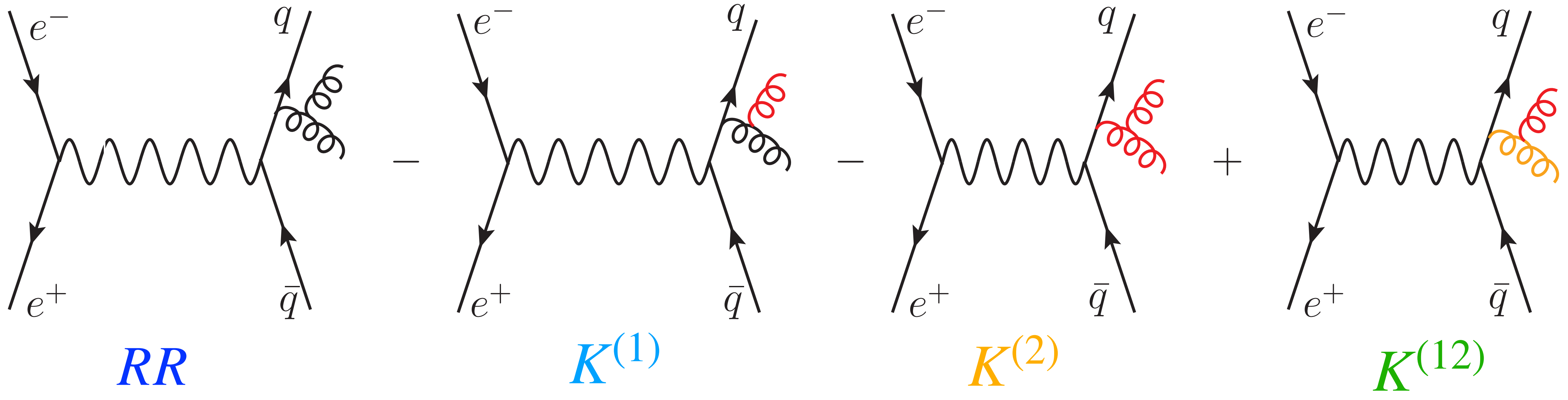
Each ingredient requires **specific treatment** and encodes **difficulties** to overcome

Subtracting RR singularities

<u>First step:</u> **divide the singular configurations into single-unresolved, double unresolved, and strongly ordered**



- *Many different* ***singular configuration****s arise and* ***overlap: 3 distinct counterterms are necessary***



$$\int d\Phi_{n+2}\Big[\, RR\ \delta_{n+2} - K^{(1)}\ \delta_{n+1} - \left(K^{(2)} - K^{(12)} \right)\delta_n \Big]$$

Subtracting RR singularities

$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \int d\Phi_n \, VV \, \delta_{X_n}$$

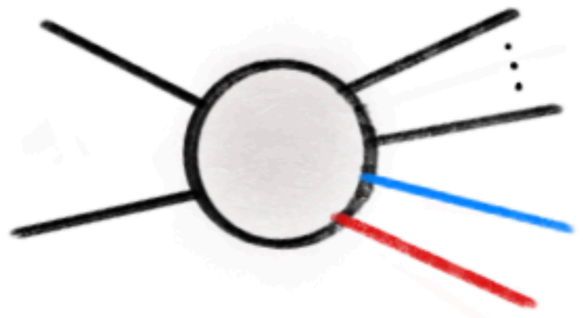
$$+ \int d\Phi_{n+1} \, RV \, \delta_{X_{n+1}}$$

$$+ \int d\Phi_{n+2} \left[RR \, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right.$$

- *Different counterterms account for different configurations and degree of divergence*

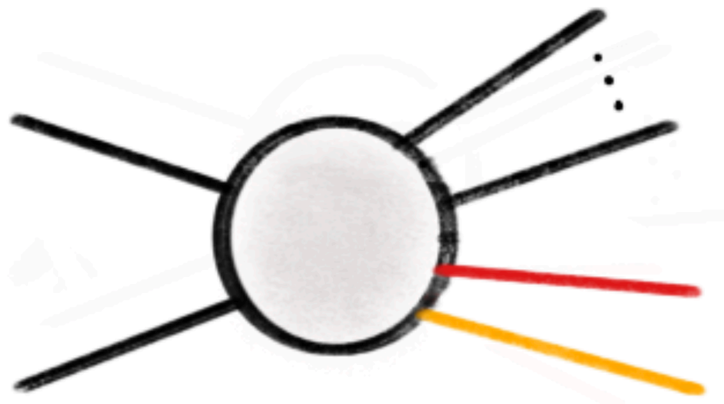
Single unresolved

$$K^{(1)}$$



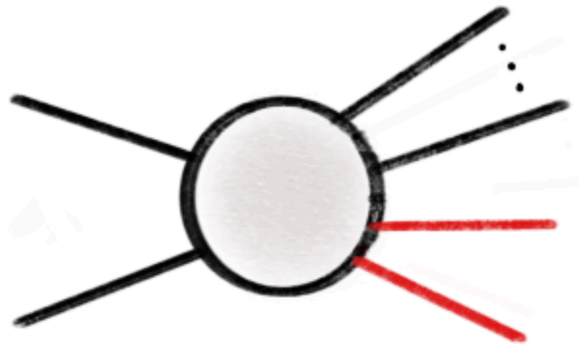
Strongly-ordered double unresolved

$$K^{(12)}$$



Double unresolved

$$K^{(2)}$$

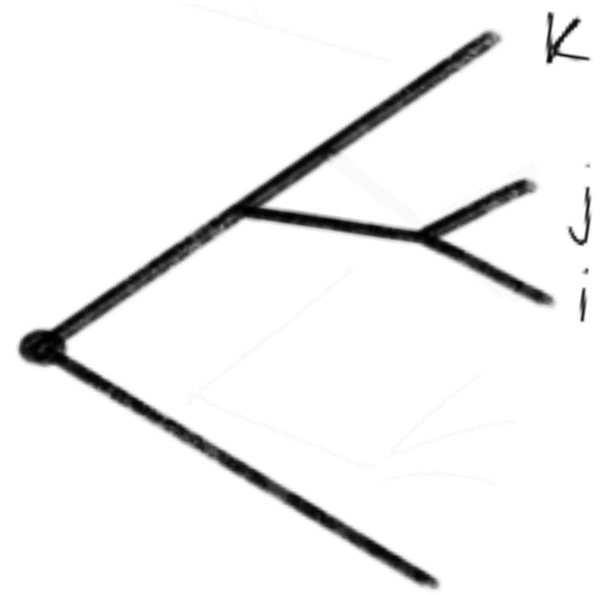


Sector functions at NNLO

Second step: unitary partition of double-unresolved phase space Φ_{n+2} into sectors $\mathcal{W}_{ijkl}$

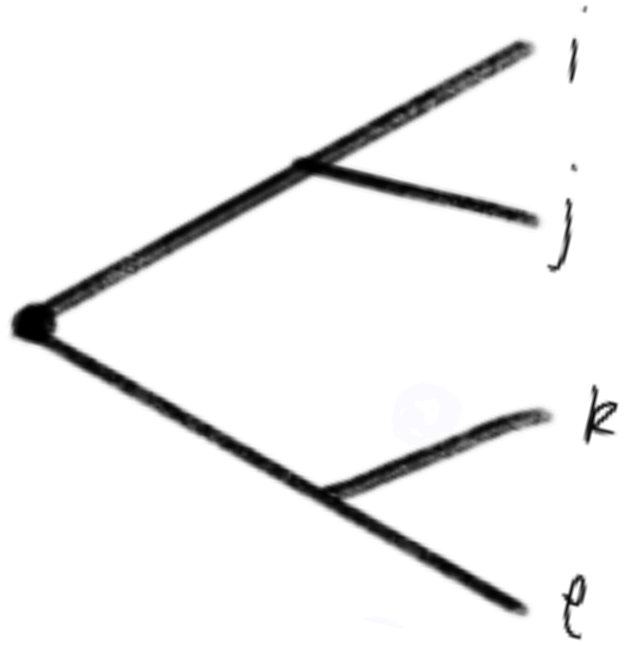
$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l \neq i,k}} RR\,\mathcal{W}_{ijkl}\,, \qquad \text{with} \qquad \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l \neq i,k}} \mathcal{W}_{ijkl} = 1$$

- ***3 topologies*** collecting all types of singularities



$\mathcal{W}_{ijjk}\,, \qquad i \neq j \neq k$

$\mathcal{W}_{ijkj}\,, \qquad i \neq j \neq k$



$\mathcal{W}_{ijkl}\,, \qquad i \neq j \neq k \neq l$

Singularities selected: $\mathcal{W}_{abcd} \quad \begin{cases} a, c & \rightarrow soft \\ ab, cd & \rightarrow collinear \end{cases}$

Possible realisation of the desired properties:

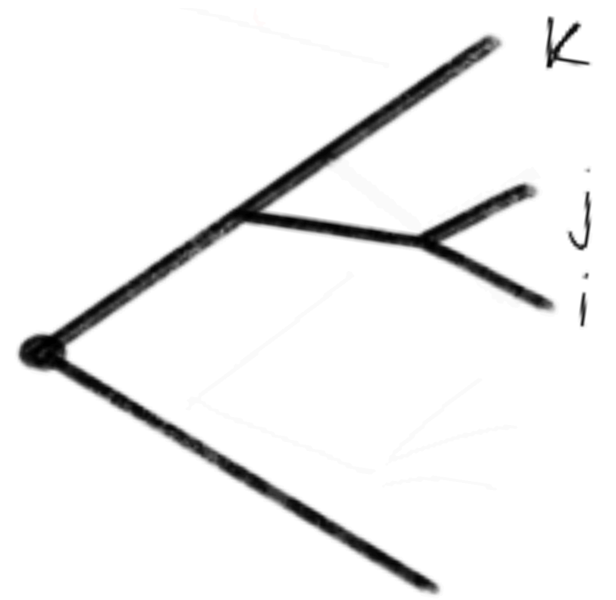
$$\mathcal{W}_{abcd} = \frac{\sigma_{abcd}}{\sigma}\,, \quad \sigma = \sum_{a,b\neq a} \sum_{\substack{c\neq a \\ d\neq a,c}} \sigma_{abcd} \quad \Longrightarrow \quad \sum_{a,b\neq a} \sum_{\substack{c\neq a \\ d\neq a,c}} \mathcal{W}_{abcd} = 1\,, \qquad \sigma_{abcd} = \frac{1}{(e_a\, w_{ab})^{\alpha}} \frac{1}{(e_c + \delta_{bc}\, e_a)\, w_{cd}}\,, \quad \alpha > 1$$

Sector functions at NNLO

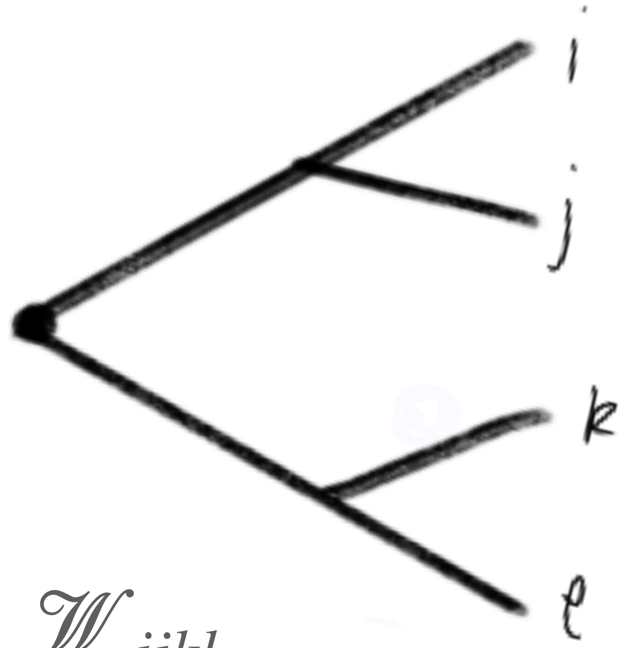
<u>Second step:</u> **unitary partition of double-unresolved phase space Φ_{n+2} into sectors $\mathcal{W}_{ijkl}$**

$$RR = \sum_{i,j\neq i} \sum_{\substack{k \neq i \\ l \neq i,k}} RR\,\mathcal{W}_{ijkl}\,, \qquad \text{with} \qquad \sum_{i,j\neq i} \sum_{\substack{k \neq i \\ l \neq i,k}} \mathcal{W}_{ijkl} = 1$$

- ***List of contributing limits in each topology.***



$\mathcal{W}_{ijkj}$ $\mathcal{W}_{ijjk}$



$\mathcal{W}_{ijkl}$

		Single unresolved		Double unresolved			
$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ij}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	
$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	$\mathbf{SC}_{kij}$
$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijkl}$	$\mathbf{SC}_{ikl}$	$\mathbf{SC}_{kij}$

- ***Sum rules:*** *limits of sector functions still form a unitary partition.*

$$\mathbf{S}_{ik}\Big(\sum_{b\neq i}\sum_{d\neq i,k}\mathcal{W}_{ibkd} + \sum_{b\neq k}\sum_{d\neq k,i}\mathcal{W}_{kbid}\Big) = 1$$

$$\dots$$

$$\mathbf{C}_{ijk} \sum_{abc\in\pi(ijk)} \Big(\mathcal{W}_{abbc} + \mathcal{W}_{abcb}\Big) = 1$$

$\mathbf{S}_{ij}$	**double-soft** partons i and j
$\mathbf{C}_{ijk}$	**triple-collinear** partons (i, j, k)
$\mathbf{C}_{ijkl}$	**double-collinear** partons (i, j) and (k, l)
$\mathbf{SC}_{ijk}$	**soft** partons i and **collinear** partons (j, k)

- ***NLO-factorisation:*** *$\mathcal{W}_{abcd}$ factorise into products of NLO-type sector function under single-unresolved limits.*

Limits collection

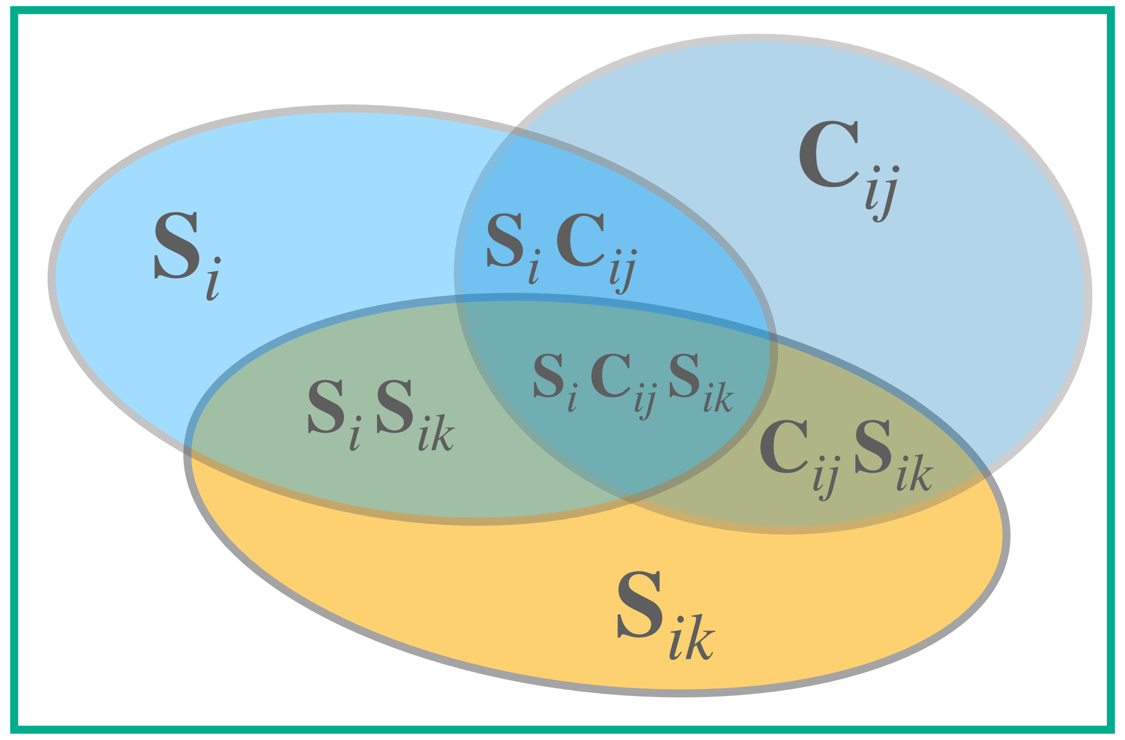
Third step: collect the limited relevant IRC limits for each topology

$$RR\mathcal{W}_\tau - \left[\mathbf{L}^{(1)}_{ij} + \mathbf{L}^{(2)}_{\tau} - \mathbf{L}^{(1)}_{ij}\mathbf{L}^{(2)}_{\tau}\right] RR\mathcal{W}_\tau \quad \rightarrow \quad \text{integrable}$$

Single unresolved

Double unresolved

Overlapping



$$\mathbf{L}^{(1)}_{ij} = \mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)$$

$$(\tau = ijjk, ijkj, ijkl)$$

$$\mathbf{L}^{(2)}_{ijjk} = \mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})$$

$$\mathbf{L}^{(2)}_{ijkj} = \mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ijk}+\mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})$$

$$\mathbf{L}^{(2)}_{ijkj} = \mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ikl}+\mathbf{SC}_{kij})(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijkl})$$

- ***Limits action:*** *singular limits act on both sector functions $\mathcal{W}_{abcd}$ and matrix elements*

$$\mathbf{L}\,RR\mathcal{W}_{ijkl} = (\mathbf{L}\,RR)\,(\mathbf{L}\mathcal{W}_{ijkl})$$

Universal, and independent on the number of coloured partons

Dependence on the choice of partition functions

Singular structure of the RR

- ***Limits on matrix elements:*** *under IRC limits RR factorises into (universal kernel)×(lower multiplicity matrix elements)* *[Catani, Grazzini 9810389, 9908523]*

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not i \not j}\big) \right]$$

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s_{ijk}^2}\, P^{\mu\nu}_{ijk}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{ijk}\big)$$

$$\mathbf{C}_{ijkl}\, RR(\{k\}) \propto \frac{1}{s_{ij}\, s_{kl}}\, P^{\mu\nu}_{ij}\big(s_{ir}, s_{jr}\big)\, P^{\rho\sigma}_{kl}\big(s_{kr'}, s_{lr'}\big)\, B_{\mu\nu\rho\sigma}\big(\{k\}_{\not i \not j \not k \not l}, k_{ij}, k_{kl}\big)$$

$$\mathbf{SC}_{ijk}\, RR(\{k\}) = \mathbf{CS}_{jki} RR(\{k\}) \propto \frac{1}{s_{jk}} \sum_{c,d\neq i} P^{\mu\nu}_{jk}\, I^{(i)}_{cd}\, B^{cd}_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{jk}\big)$$

$I^{(i)}_{cd}$ = single eikonal
$I^{(ij)}_{cd}$ = double eikonal
$P^{\mu\nu}_{ij}$ = single splitting
$P^{\mu\nu}_{ijk}$ = triple splitting

} Functions of **Lorentz invariants**

Singular structure of the RR

- ***Limits on matrix elements:*** *under IRC limits RR factorises into (universal kernel)×(lower multiplicity matrix elements)* *[Catani, Grazzini 9810389, 9908523]*

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I_{cd}^{(i)}\, I_{ef}^{(j)}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + I_{cd}^{(ij)}\, B_{cd}\big(\{k\}_{\not i \not j}\big) \right]$$

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s_{ijk}^2}\, P_{ijk}^{\mu\nu}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{ijk}\big)$$

$$\mathbf{C}_{ijkl}\, RR(\{k\}) \propto \frac{1}{s_{ij}\, s_{kl}}\, P_{ij}^{\mu\nu}\big(s_{ir}, s_{jr}\big)\, P_{kl}^{\rho\sigma}\big(s_{kr'}, s_{lr'}\big)\, B_{\mu\nu\rho\sigma}\big(\{k\}_{\not i \not j \not k \not l}, k_{ij}, k_{kl}\big)$$

$$\mathbf{SC}_{ijk}\, RR(\{k\}) = \mathbf{CS}_{jki} RR(\{k\}) \propto \frac{1}{s_{jk}} \sum_{c,d\neq i} P_{jk}^{\mu\nu}\, I_{cd}^{(i)}\, B_{\mu\nu}^{cd}\big(\{k\}_{\not i \not j \not k}, k_{jk}\big)$$

Born-level kinematics does **not satisfy the mass-shell condition and momentum conservation**

Momentum mapping needed!

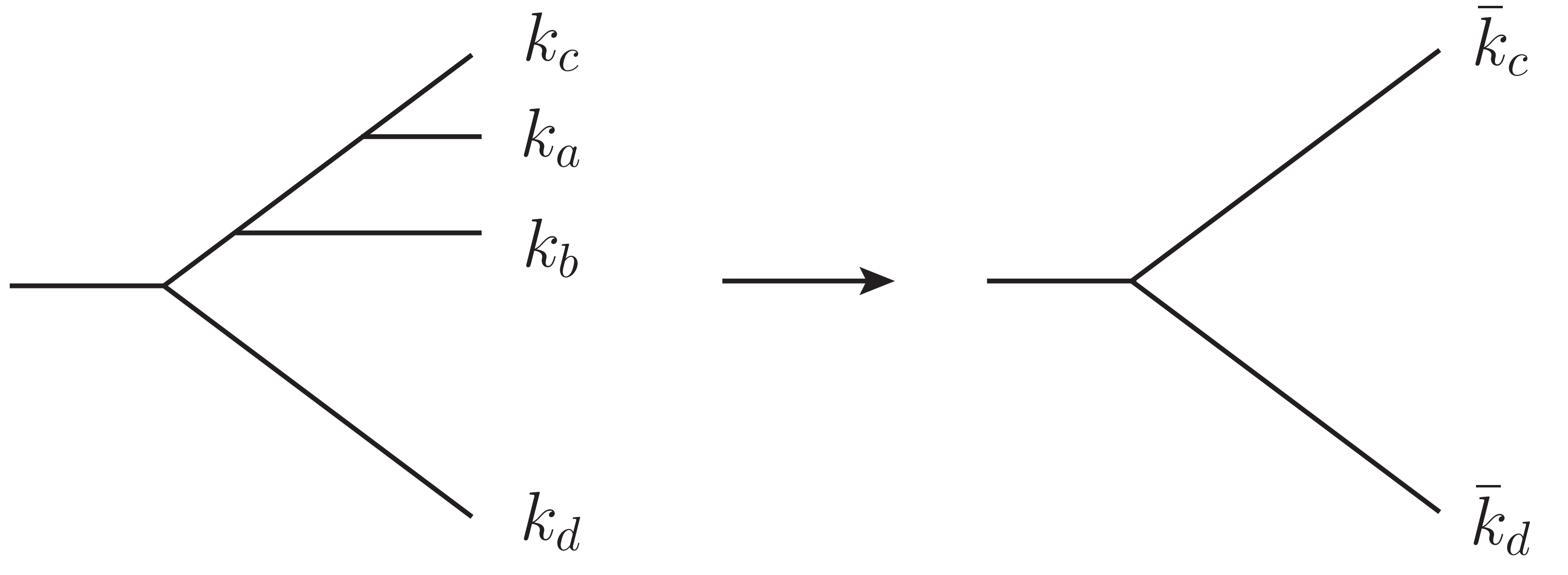
$I_{cd}^{(i)}$ = single eikonal
$I_{cd}^{(ij)}$ = double eikonal
$P_{ij}^{\mu\nu}$ = single splitting
$P_{ijk}^{\mu\nu}$ = triple splitting

} Functions of **Lorentz invariants**

NNLO momentum mapping

- ***Momentum mappings:*** *minimal set of involved momenta and complete factorisation of the phase space*

1. One-step mapping



$$\left\{\bar{k}_n^{(abcd)}\right\} = \left\{k_{\not a \not b \not c \not d}, \bar{k}_c^{(abcd)}, \bar{k}_d^{(abcd)}\right\}$$

$$d\Phi_{n+2} = d\Phi_n^{(abcd)} \cdot d\Phi_{\mathrm{rad},2}^{(abcd)} = d\Phi_n^{(abcd)} \cdot d\Phi_{\mathrm{rad},2}\left(\bar{s}_{cd}^{(abcd)}; y, z, \phi, y', z', x'\right)$$

$$\int d\Phi_{\mathrm{rad},2} \propto \left(\bar{s}_{cd}^{(abcd)}\right)^{2-2\epsilon} \int_0^1 dw' \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \left[w'(1-w')\right]^{-1/2-\epsilon} \left[y'(1-y')^2\, z'(1-z')\, y^2(1-y)^2\, z(1-z)\right]^{-\epsilon} (1-y')\, y(1-y)$$

2. Two-step mapping

$$\left\{\bar{k}_n^{(acd,bef)}\right\} = \left\{\bar{k}_{\not a \not b \not e \not f}^{(acd)}, \bar{k}_e^{(acd,bef)}, \bar{k}_f^{(acd,bef)}\right\}$$

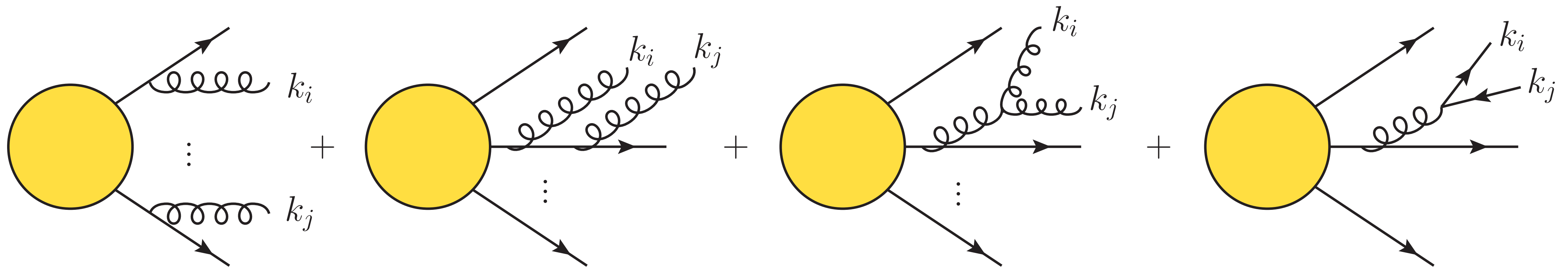
$$d\Phi_{n+2} = d\Phi_n^{(abcd)} \cdot d\Phi_{\mathrm{rad}}^{(acd)} \cdot d\Phi_{\mathrm{rad}}^{(bef)} = d\Phi_n^{(acd,bef)} \cdot d\Phi_{\mathrm{rad}}\left(\bar{s}_{ef}^{(acd,bef)}; y, z, \phi\right) \cdot d\Phi_{\mathrm{rad}}\left(\bar{s}_{cd}^{(acd)}; y', z', \phi'\right)$$

$$d\Phi_{\mathrm{rad},2}^{(acd,bef)} \propto \left(\bar{s}_{cd}^{(acd,bef)} \bar{s}_{ef}^{(acd,bef)}\right)^{1-\epsilon} \int_0^\pi d\phi'\,(\sin\phi')^{-2\epsilon} \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \left[y'(1-y')^2\, z'(1-z')\, y(1-y)^2\, z(1-z)\right]^{-\epsilon} (1-y')(1-y)$$

Adaptive mapping

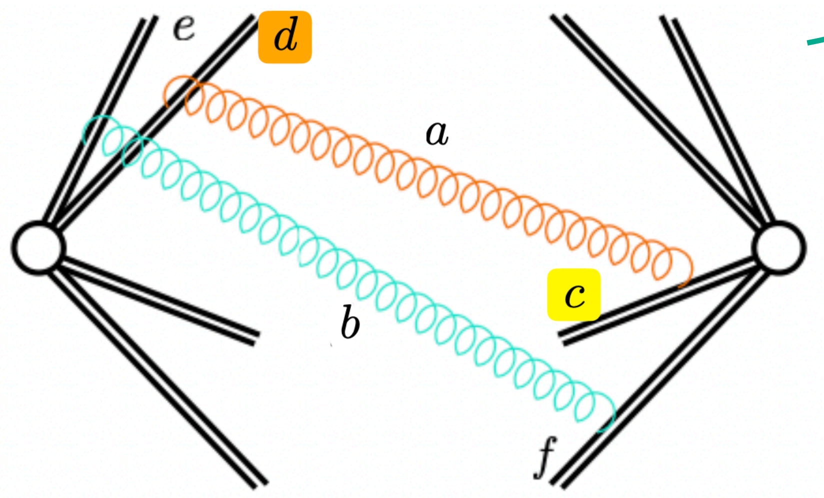
- ***Freedom in choosing the mapping:*** *adaptive parametrisation tuned to the specific kernel*

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not i \not j}\big) \right]$$



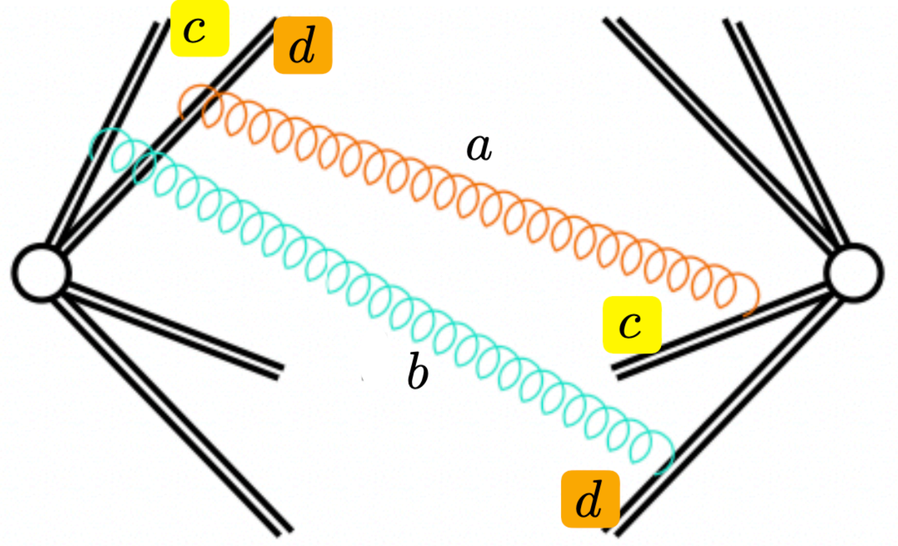
Freedom to map each term of the sum separately, adapting the choice to the invariants appearing in the kernel itself

$$\bar{\mathbf{S}}_{ij}\, RR(\{k\}) \propto \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \Bigg[\sum_{\substack{e\neq i,j,c,d \\ f\neq i,j,c,d}} I^{(i)}_{cd}\, \bar{I}^{(j)(icd)}_{ef} B_{cdef}\Big(\big\{\bar{k}^{(icd,jef)}\big\}\Big) + 4 \sum_{e\neq i,j,c,d} I^{(i)}_{cd}\, \bar{I}^{(j)(icd)}_{ed} B_{cded}\Big(\big\{\bar{k}^{(icd,jed)}\big\}\Big)$$
$$+ 2\, I^{(i)}_{cd}\, I^{(j)}_{cd} B_{cdcd}\Big(\big\{\bar{k}^{(ijcd)}\big\}\Big) + \Big(I^{(ij)}_{cd} - \frac{1}{2} I^{(ij)}_{cc} - \frac{1}{2} I^{(ij)}_{dd}\Big) B_{cd}\Big(\big\{\bar{k}^{(ijcd)}\big\}\Big) \Bigg]$$



$$\{k\} \to \{\bar{k}\}^{(acd,bef)}$$

$$d\Phi_{n+2} = d\Phi^{(abcd)}_{n} \cdot d\Phi^{(acd)}_{\rm rad} \cdot d\Phi^{(bef)}_{\rm rad}$$



$$\{k\} \to \{\bar{k}\}^{(abcd)}$$

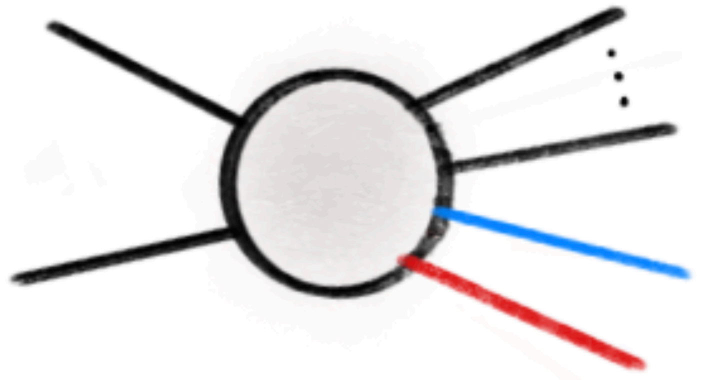
$$d\Phi_{n+2} = d\Phi^{(abcd)}_{n} \cdot d\Phi^{(abcd)}_{\rm rad,2}$$

Counterterm definition

Forth step: **Promotion of the collected limits to counterterms. Improved limits adapting momenta mapping to each kernel**, while tuning action on sector functions when necessary.

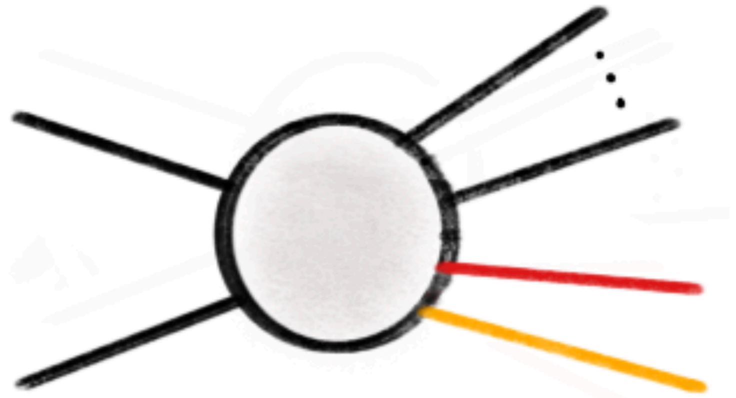
Single unresolved

$$K^{(1)} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(1)}_{ij} \, RR \, \mathcal{W}_{ijkl}$$



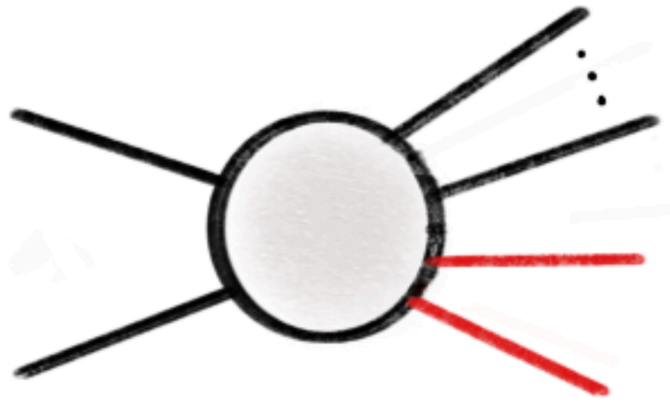
Strongly-ordered double unresolved

$$K^{(12)} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(1)}_{ij} \, \overline{\mathbf{L}}^{(2)}_{ijkl} \, RR \, \mathcal{W}_{ijkl}$$



Double unresolved (uniform)

$$K^{(2)} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(2)}_{ijkl} \, RR \, \mathcal{W}_{ijkl}$$



$$K^{(2)} = \Bigg\{ \sum_{i,k>i} \overline{\mathbf{S}}_{ij} + \sum_{i,j>i} \sum_{k>j} \overline{\mathbf{C}}_{ijk}\left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right) + \sum_{i,j>i} \sum_{\substack{k>i \\ k\neq j}} \sum_{\substack{l>k \\ l\neq j}} \overline{\mathbf{C}}_{ijkl}\Big[1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{jl} - \overline{\mathbf{SC}}_{ikl}\left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{il}\right)$$

$$- \overline{\mathbf{SC}}_{jkl}\left(1 - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{jl}\right) - \overline{\mathbf{SC}}_{kij}\left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right) - \overline{\mathbf{SC}}_{lij}\left(1 - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jl}\right)\Big] + \sum_{i,j>i} \sum_{\substack{k>j \\ k\neq i}} \overline{\mathbf{SC}}_{ijk}\left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik}\right)\left(1 - \overline{\mathbf{C}}_{ijk}\right) \Bigg\} RR$$

Counterterm definition

- ***Locality of the cancellation*** *ensured by* ***consistency relations***

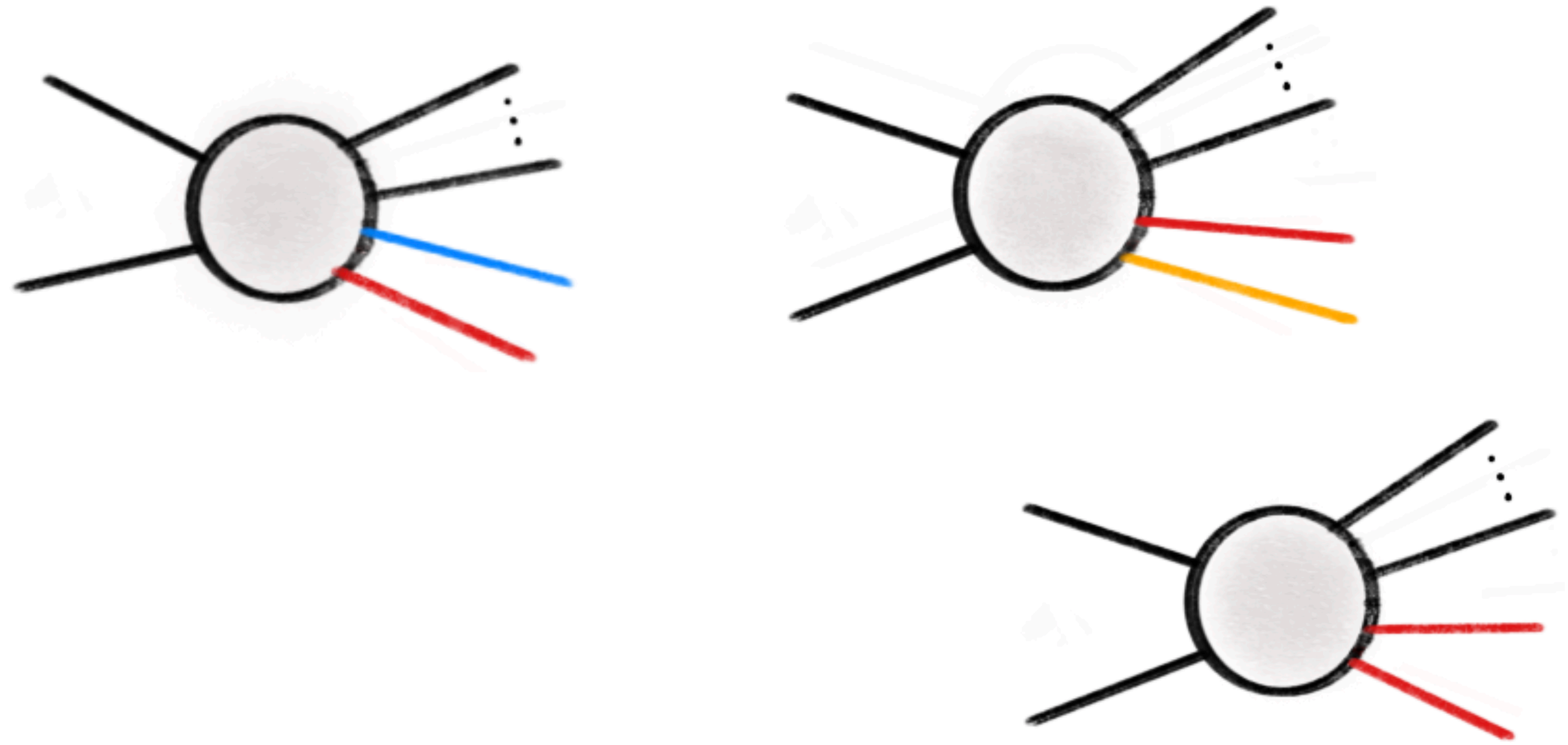
    - **Tower of nested limits** that have “horizontal” and “vertical” consistency relations.

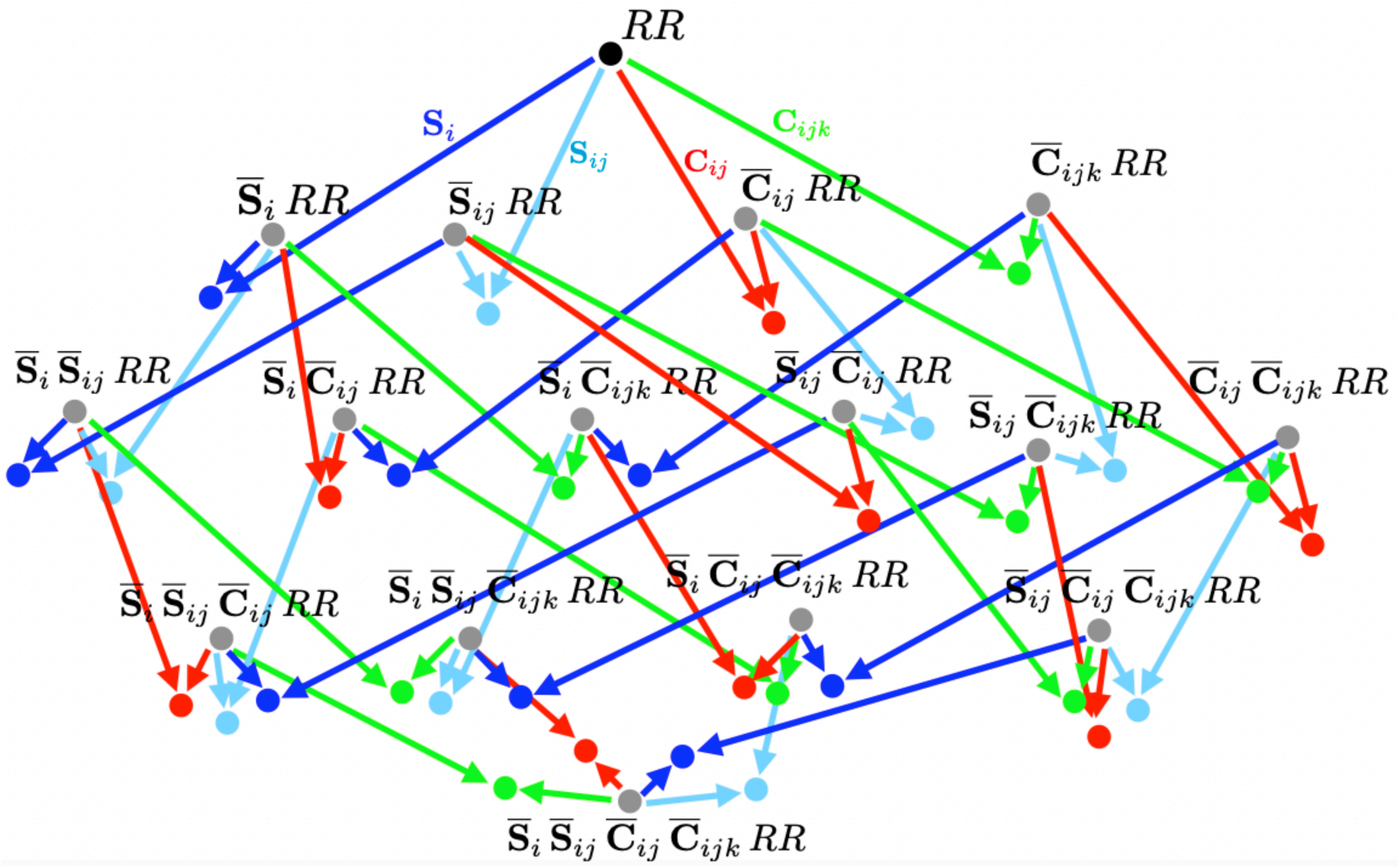
    - **Consistency** relations have to **hold simultaneously** for **all the mapped limits**.

    - The **number of consistency relations grows rapidly** as the number of unresolved limits increases.

    - **Inconsistencies at the bottom** of the tower usually require a **redefinition** of the mapped limits **at the top** (and, as a consequence, of the entire cascade).

    - The definition of consistent mapped limits has to be set <u>**once for all**</u>, and is almost process-independent.





Selection of displayed limits

$\mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ij} \quad \mathbf{C}_{ijk}$

Integration of the double-real counterterms

<u>Fifth step:</u> conterterms **integration.** Great advantage from choosing the **appropriate mapping**, and **phase-space parametrisation**

$$\frac{d\sigma_{\text{NNLO}}}{dX} = \int d\Phi_n \, VV \, \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \, RV \, \delta_{X_{n+1}}$$

$$+ \int d\Phi_{n+2} \left[RR \, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right.$$

→ **Finite by construction and integrable in $d = 4$**

- ***3 different integrated counterterms:*** *different phase-space and complexity*

$$I^{(1)} = \int d\Phi_{\text{rad},1} \, K^{(1)} \,, \qquad I^{(2)} = \int d\Phi_{\text{rad},2} \, K^{(2)} \,, \qquad I^{(12)} = \int d\Phi_{\text{rad}} \, K^{(12)} \,,$$

Integration of the double-real counterterms

Fifth step: conterterms **integration.** Great advantage from choosing the **appropriate mapping**, and **phase-space parametrisation**

$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \int d\Phi_n \Big(VV + I^{(2)} \qquad \Big)\, \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \Big[\Big(RV + I^{(1)} \Big) \delta_{X_{n+1}} - \Big(\qquad + I^{(12)} \Big)\, \delta_{X_n}$$

$$+ \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)} \Big)\, \delta_{X_n}$$

→ **Finite by construction and integrable in $d = 4$**

- ***3 different integrated counterterms:*** *different phase-space and complexity*

$$I^{(1)} = \int d\Phi_{\mathrm{rad},1}\, K^{(1)}\,, \qquad I^{(2)} = \int d\Phi_{\mathrm{rad},2}\, K^{(2)}\,, \qquad I^{(12)} = \int d\Phi_{\mathrm{rad}}\, K^{(12)}\,,$$

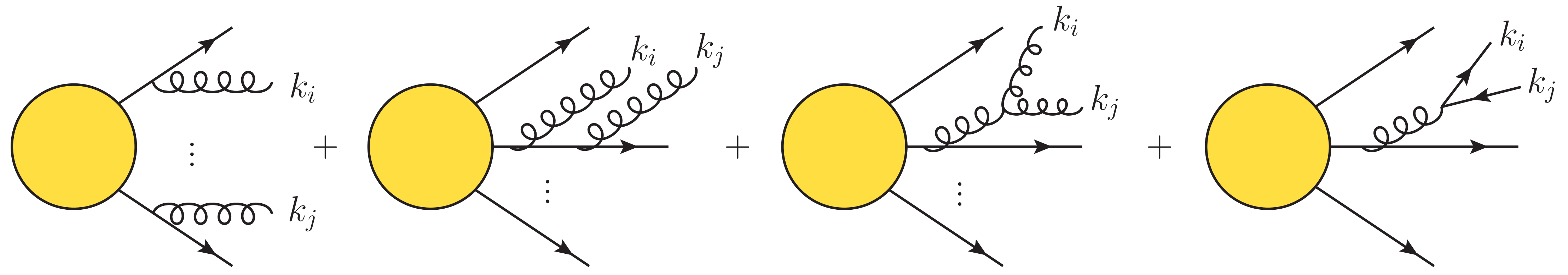
NNLO complexity: highly non trivial!

- ***Analytic integration*** *via* ***standard techniques*** *→ sectors sum rules + mapping adaptation [Magnea, C-SS et al. 2010.14493]*
- ***No approximations*** *→* ***simple and compact results*** *(at most simple* ***logarithmic dependence*** *on Mandelstam invariants)*

Integration of the double-real counterterms: example

- ***Freedom in choosing the mapping:*** *adaptive parametrisation tuned to the specific kernel [Magnea, C-SS et al. 2010.14493]*

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \Big[\sum_{e,f\neq i,j} I_{cd}^{(i)} I_{ef}^{(j)} B_{cdef}\big(\{k\}_{\not i \not j}\big) + I_{cd}^{(ij)} B_{cd}\big(\{k\}_{\not i \not j}\big) \Big]$$



We are free to map each term of the sum separately, adapting the choice to the invariants appearing in the kernel itself

$$\overline{\mathbf{S}}_{ij}\, RR(\{k\}) \propto \sum_{\substack{c \neq i,j \\ d \neq i,j,c}} \Bigg[\sum_{\substack{e \neq i,j,c,d \\ f \neq i,j,c,d}} I_{cd}^{(i)}\, \bar{I}_{ef}^{(j)(icd)} B_{cdef}\Big(\big\{\bar{k}^{(icd,jef)}\big\}\Big) + 4 \sum_{e \neq i,j,c,d} I_{cd}^{(i)}\, \bar{I}_{ed}^{(j)(icd)} B_{cded}\Big(\big\{\bar{k}^{(icd,jed)}\big\}\Big)$$
$$+ 2\, I_{cd}^{(i)}\, I_{cd}^{(j)} B_{cdcd}\Big(\big\{\bar{k}^{(ijcd)}\big\}\Big) + \Big(I_{cd}^{(ij)} - \frac{1}{2} I_{cc}^{(ij)} - \frac{1}{2} I_{dd}^{(ij)}\Big) B_{cd}\Big(\big\{\bar{k}^{(ijcd)}\big\}\Big)\Bigg]$$

The PS parametrisation follows the mapping structure

$$I^{(2)}_{\mathrm{SS},cdef} = \int d\Phi_{\mathrm{rad},2}\, I_{cd}^{(i)}\, \bar{I}_{ef}^{(j),(icd)} = \int d\overline{\Phi}^{(icd,jef)}_{\mathrm{rad}}\, \bar{I}_{ef}^{(j),(icd)} \int d\Phi^{(icd)}_{\mathrm{rad}}\, I_{cd}^{(i)} = \frac{(4\pi)^{\epsilon-2}}{\big(\bar{s}_{cd}^{(icd,jef)}\big)^{\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\, \Gamma(2-3\epsilon)} \frac{(4\pi)^{\epsilon-2}}{\big(\bar{s}_{ef}^{(icd,jef)}\big)^{\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\, \Gamma(2-3\epsilon)}$$

Some of the double-soft kernel structures feature a NLOxNLO complexity → integration exact in ϵ

The most difficult part arises from the pure NNLO current.

Integration of the double-real counterterms: example

$$\int d\Phi_{n+2}\,\bar{\mathbf{S}}_{ij}\,RR(\{k\}) \propto \int d\Phi^{(ijcd)}_{n+2}\, I^{(ij)}_{cd}\, B_{cd}\Big(\{\bar{k}^{(ijcd)}\}\Big)$$

$$I^{(gg)(ij)}_{cd} = \frac{(1-\epsilon)(s_{ic}\,s_{jd} + s_{id}\,s_{jc}) - 2s_{ij}\,s_{cd}}{s_{ij}^2\,(s_{ic}+s_{jc})(s_{id}+s_{jd})} + s_{cd}\,\frac{s_{ic}\,s_{jd} + s_{id}\,s_{jc} - s_{ij}\,s_{cd}}{s_{ij}\,s_{ic}\,s_{id}\,s_{jd}\,s_{jc}}\left[1 - \frac{1}{2}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic}+s_{jc})(s_{id}+s_{jd})}\right]$$

Mapping: $\{\bar{k}\}^{(ijcd)}$.

Catani-Seymour parameters y', z', y, z:

$$s_{ij} = y'\,y\,\bar{s}^{(ijcd)}_{cd}\,, \qquad s_{ic} = z'(1-y')\,y\,\bar{s}^{(ijcd)}_{cd}\,,$$

$$s_{cd} = (1-y')(1-y)(1-z)\,\bar{s}^{(ijcd)}_{cd} \qquad s_{jc} = (1-y')(1-z')\,y\,\bar{s}^{(ijcd)}_{cd}\,,$$

$$s_{id} = (1-y)\Big[y'(1-z')(1-z)+z'z-2(1-2x')\sqrt{y'z'(1-z')z(1-z)}\Big]\,\bar{s}^{(ijcd)}_{cd}\,,$$

$$s_{jd} = (1-y)\Big[y'z'(1-z)+(1-z')z+2(1-2x')\sqrt{y'z'(1-z')z(1-z)}\Big]\,\bar{s}^{(ijcd)}_{cd}\,.$$

Use partial fractioning to isolate complicated denominators $\quad \dfrac{1}{s_{id}\,s_{jd}} = \dfrac{1}{s_{id}+s_{jd}}\left(\dfrac{1}{s_{id}} + \dfrac{1}{s_{jd}}\right)$

Use symmetries of the 4-partons of the phase space *[De Ridder, Gehrmann, Heinrich 0311276]* $\quad \dfrac{1}{s_{id}\,s_{jd}} = \dfrac{1}{s_{id}+s_{jd}}\left(\dfrac{1}{s_{id}} + \dfrac{1}{s_{jd}}\right) \xrightarrow{k_i\leftrightarrow k_j} \dfrac{1}{s_{id}\,s_{jd}} = \dfrac{1}{s_{id}+s_{jd}}\,\dfrac{2}{s_{jd}}$

Parametrise the PS using Catani-Seymour parameters

$$\int d\Phi^{(ijcd)}_{\mathrm{rad},2} = 2^{-4\epsilon}\,N^2(\epsilon)\left(\bar{s}^{(ijcd)}_{cd}\right)^{2-2\epsilon}\int_0^1 dx'\int_0^1 dy'\int_0^1 dz'\int_0^1 dx\,[x(1-x)]^{-1/2-\epsilon}\int_0^1 dy\int_0^1 dz\,[x'(1-x')]^{-1/2-\epsilon}\left[y'(1-y')^2\,z'(1-z')\,y^2(1-y)^2\,z(1-z)\right]^{-\epsilon}(1-y')\,y(1-y)$$

Integration of the double-real counterterms: example

- *How the result looks like:*

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_{ij}\,RR = \frac{1}{2}\,\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg\{\sum_{e\neq i,j,c,d}\Bigg[\sum_{f\neq i,j,c,d,e}\int d\Phi_n^{(icd,jef)}\,J_{\mathrm{s}\otimes\mathrm{s}}^{ijcdef}\,\bar{B}_{cdef}^{(icd,jef)} + 4\int d\Phi_n^{(icd,jed)}\,J_{\mathrm{s}\otimes\mathrm{s}}^{ijcde}\,\bar{B}_{cded}^{(icd,jed)}\Bigg] + \int d\Phi_n^{(ijcd)}\Big[2\,J_{\mathrm{s}\otimes\mathrm{s}}^{ijcd}\,\bar{B}_{cdcd}^{(ijcd)} + J_{\mathrm{ss}}^{ijcd}\,\bar{B}_{cd}^{(ijcd)}\Big]\Bigg\},$$

$$J_{\mathrm{s}\otimes\mathrm{s}}^{ijcdef} \equiv \mathcal{N}_1^2\int d\Phi_{\mathrm{rad},2}^{(icd,jef)}\,\mathcal{E}_{cd}^{(i)}\,\mathcal{E}_{ef}^{(j)} \equiv J_{\mathrm{s}\otimes\mathrm{s}}^{(4)}\left(\bar{s}_{cd}^{(icd,jef)},\bar{s}_{ef}^{(icd,jef)}\right)f_{ij}^{gg},$$

$$J_{\mathrm{s}\otimes\mathrm{s}}^{ijcde} \equiv \mathcal{N}_1^2\int d\Phi_{\mathrm{rad},2}^{(icd,jed)}\,\mathcal{E}_{cd}^{(i)}\,\mathcal{E}_{ed}^{(j)} \equiv J_{\mathrm{s}\otimes\mathrm{s}}^{(3)}\left(\bar{s}_{cd}^{(icd,jed)},\bar{s}_{ed}^{(icd,jed)}\right)f_{ij}^{gg},$$

$$J_{\mathrm{s}\otimes\mathrm{s}}^{ijcd} \equiv \mathcal{N}_1^2\int d\Phi_{\mathrm{rad},2}^{(ijcd)}\,\mathcal{E}_{cd}^{(i)}\,\mathcal{E}_{cd}^{(j)} \equiv J_{\mathrm{s}\otimes\mathrm{s}}^{(2)}\left(\bar{s}_{cd}^{(ijcd)}\right)f_{ij}^{gg},$$

$$J_{\mathrm{ss}}^{ijcd} \equiv \mathcal{N}_1^2\int d\Phi_{\mathrm{rad},2}^{(ijcd)}\,\mathcal{E}_{cd}^{(ij)} \equiv 2\,T_R\,J_{\mathrm{ss}}^{(\mathrm{q\bar{q}})}\left(\bar{s}_{cd}^{(ijcd)}\right)f_{ij}^{q\bar{q}} - 2\,C_A\,J_{\mathrm{ss}}^{(\mathrm{gg})}\left(\bar{s}_{cd}^{(ijcd)}\right)f_{ij}^{gg},$$

Integration of the double-real counterterms: example

- ***How the result looks like:***

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_{ij}\,RR = \frac{1}{2}\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg\{\sum_{e\neq i,j,c,d}\Bigg[\sum_{f\neq i,j,c,d,e}\int d\Phi_n^{(icd,jef)}\,J_{\mathrm{s}\otimes\mathrm{s}}^{ijcdef}\,\bar{B}_{cdef}^{(icd,jef)} + 4\int d\Phi_n^{(icd,jed)}\,J_{\mathrm{s}\otimes\mathrm{s}}^{ijcde}\,\bar{B}_{cded}^{(icd,jed)}\Bigg] + \int d\Phi_n^{(ijcd)}\Big[2\,J_{\mathrm{s}\otimes\mathrm{s}}^{ijcd}\,\bar{B}_{cdcd}^{(ijcd)} + J_{\mathrm{ss}}^{ijcd}\,\bar{B}_{cd}^{(ijcd)}\Big]\Bigg\},$$

$$J_{\mathrm{s}\otimes\mathrm{s}}^{(4)}(s,s') = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon}\Bigg[\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(16-\frac{7}{6}\pi^2\right)\frac{1}{\epsilon^2}+\left(60-\frac{14}{3}\pi^2-\frac{50}{3}\zeta_3\right)\frac{1}{\epsilon} + 216-\frac{56}{3}\pi^2-\frac{200}{3}\zeta_3+\frac{29}{120}\pi^4+\mathcal{O}(\epsilon)\Bigg],$$

$$J_{\mathrm{s}\otimes\mathrm{s}}^{(3)}(s,s') = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon}\Bigg[\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(17-\frac{4}{3}\pi^2\right)\frac{1}{\epsilon^2}+\left(70-\frac{16}{3}\pi^2-\frac{68}{3}\zeta_3\right)\frac{1}{\epsilon} + 284-\frac{68}{3}\pi^2-\frac{272}{3}\zeta_3+\frac{13}{90}\pi^4+\mathcal{O}(\epsilon)\Bigg],$$

$$J_{\mathrm{s}\otimes\mathrm{s}}^{(2)}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg[\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(18-\frac{3}{2}\pi^2\right)\frac{1}{\epsilon^2}+\left(76-6\pi^2-\frac{74}{3}\zeta_3\right)\frac{1}{\epsilon} + 312-27\pi^2-\frac{308}{3}\zeta_3+\frac{49}{120}\pi^4+\mathcal{O}(\epsilon)\Bigg],$$

$$J_{\mathrm{ss}}^{(q\bar{q})}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg[\frac{1}{6}\frac{1}{\epsilon^3}+\frac{17}{18}\frac{1}{\epsilon^2}+\left(\frac{116}{27}-\frac{7}{36}\pi^2\right)\frac{1}{\epsilon}+\frac{1474}{81}-\frac{131}{108}\pi^2-\frac{19}{9}\zeta_3+\mathcal{O}(\epsilon)\Bigg]$$

$$J_{\mathrm{ss}}^{(gg)}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg[\frac{1}{2}\frac{1}{\epsilon^4}+\frac{35}{12}\frac{1}{\epsilon^3}+\left(\frac{487}{36}-\frac{2}{3}\pi^2\right)\frac{1}{\epsilon^2}+\left(\frac{1562}{27}-\frac{269}{72}\pi^2-\frac{77}{6}\zeta_3\right)\frac{1}{\epsilon} + \frac{19351}{81}-\frac{3829}{216}\pi^2-\frac{1025}{18}\zeta_3-\frac{23}{240}\pi^4+\mathcal{O}(\epsilon)\Bigg].$$

Subtracting RV singularities

Sixth step: regularisation of the second line → delicate interplay between different counterterms *[Magnea, C-SS et al. 2212.11190]*

$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \int d\Phi_n \Big(VV + I^{(2)} \qquad \Big)\, \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \Big[\Big(RV + I^{(1)} \Big) \delta_{X_{n+1}} - \Big(\qquad + I^{(12)} \Big)\, \delta_{X_n}$$

$$+ \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)} \Big)\, \delta_{X_n}$$

- ***Intricate cancellation pattern*** *involving both* ***poles and phase-space singularities***

$RV + I^{(1)} \rightarrow$ finite in ϵ → ***Still singular in PS***

$I^{(1)} - I^{(12)} \rightarrow$ integrable → ***Contains poles in*** ϵ

→ *Need for a* ***couterterm*** *to* ***compensate:***

the PS singularities of $RV + I^{(1)}$

AND

the explicit poles of $I^{(1)} - I^{(12)}$

Subtracting RV singularities

Sixth step: regularisation of the second line → delicate interplay between different counterterms

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\rm RV)} + I^{(12)} \right) \delta_{X_n} \right.$$

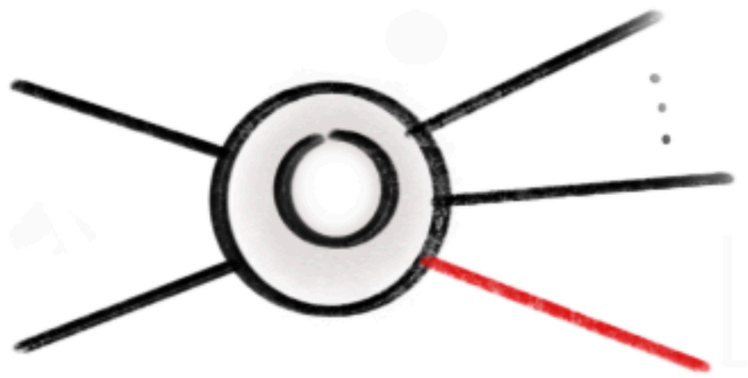
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right.$$

$RV + I^{(1)}$ → finite in ϵ

$I^{(1)} - I^{(12)}$ → integrable

- ***Intricate cancellation pattern*** *involving both* ***poles and phase-space singularities***

1loop single unresolved



$K^{(\rm RV)}$

integrable in Φ_{n+1}

integrable in Φ_{n+1}

$$\int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\rm RV)} + I^{(12)} \right) \delta_{X_n} \right.$$

finite in ϵ — finite in ϵ

- ***Analytic check of the second line finiteness and integrability***

Subtracting RV singularities

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$

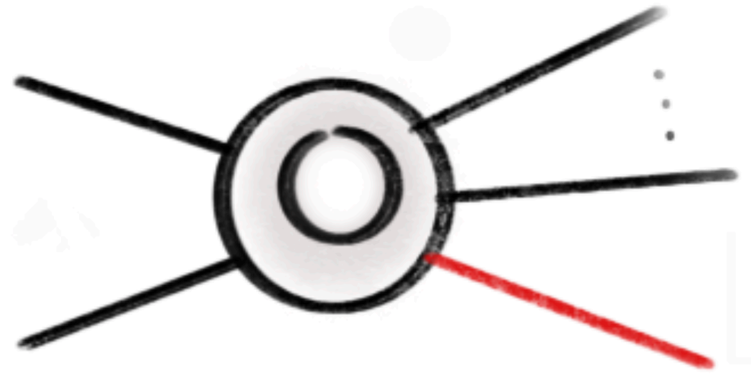
$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \qquad \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \right.$$

$$\left. + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right. \right.$$

- ***Intricate cancellation pattern*** *involving both* ***poles and phase-space singularities***

1loop single unresolved



$$K_{ij}^{(\mathbf{RV})} \equiv K_{ij,\,\mathrm{expected}}^{(\mathbf{RV})} + \Delta_{ij} = \left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i \right) \right] RV\, \mathcal{W}_{ij} + \Delta_{ij}$$

Subtracting RV singularities

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \right) \delta_{X_n}$$

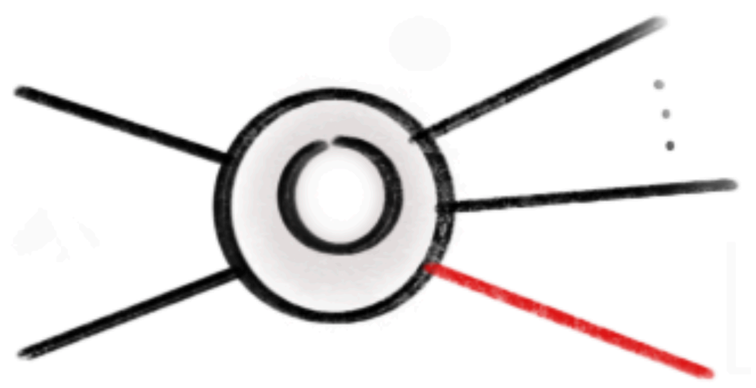
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\rm RV)} + I^{(12)} \right) \delta_{X} \right.$$

$$+ \int d\Phi_{n+2} \left[RR \, \delta_{X_{n+}} \right.$$

$$\Delta_{\mathrm{S},i} = -\frac{\alpha_s}{2\pi} \mathcal{N}_1 \sum_{\substack{c\neq i \\ d\neq i,c}} \mathcal{E}^{(i)}_{cd} \Bigg\{ \frac{1}{2\epsilon^2} \sum_{\substack{e\neq i,c \\ f\neq i,c,e}} \left[\left(\frac{s_{ef}}{\bar{s}^{(icd)}_{ef}} \right)^{-\epsilon} - 1 \right] \bar{B}^{(icd)}_{efcd} + \frac{1}{\epsilon^2} \sum_{e\neq i,d} \left[\left(\frac{s_{ed}}{\bar{s}^{(icd)}_{ed}} \right)^{-\epsilon} - 1 \right] \bar{B}^{(icd)}_{edcd}$$
$$+ \left[\left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon} \right) 2\, C_{f_c} + \frac{\gamma^{\rm hc}_c}{\epsilon} \right] \left(\bar{B}^{(icd)}_{cd} - \bar{B}^{(idc)}_{cd} \right) \Bigg\}$$
$$- \frac{\alpha_s}{2\pi} \mathcal{N}_1 \sum_{\substack{k\neq i \\ c\neq i,k,r}} \mathcal{E}^{(i)}_{cr} \frac{\gamma^{\rm hc}_k}{\epsilon} \left(\bar{B}^{(irc)}_{cr} - \bar{B}^{(icr)}_{cr} \right), \qquad r = r_{ik}\,.$$

- ***Intricate cancellation pattern*** *involving bo*

1loop single unresolved



$$K^{(\mathbf{RV})}_{ij} \equiv K^{(\mathbf{RV})}_{ij,\,\rm expected} + \Delta_{ij} = \left[\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij} \left(1 - \bar{\mathbf{S}}_i \right) \right] RV\, \mathcal{W}_{ij} + \Delta_{ij}$$

Combination with double virtual

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\rm RV)} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\rm RV)} + I^{(12)} \right) \delta_{X_n} \right.$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right.$$

$$I^{(\rm RV)} = \int d\Phi_{\rm rad}\, K^{(\rm RV)}$$

- ***Most** of the contributions to $I^{(\rm RV)}$ can be computed using **NLO-like strategy***
- ***Non-trivial integrals arise from triple-color-correlated component*** $B_{lmp} = \sum_{a,b,c} f_{abc}\, \mathcal{A}_n^{(0)*}\, \mathbf{T}_l^a\, \mathbf{T}_m^b\, \mathbf{T}_p^c\, \mathcal{A}_n^{(0)}$

$$\mathbf{S}_i\, RV = -\mathcal{N}_1 \sum_{\substack{l\neq i\\ m\neq i}} \left[\mathcal{I}_{lm}^{(i)}\, V_{lm}(\{k\}_{\not i}) - \frac{\alpha_s}{2\pi} \left(\widetilde{\mathcal{I}}_{lm}^{(i)} + \mathcal{I}_{lm}^{(i)} \frac{\beta_0}{2\epsilon} \right) B_{lm}(\{k\}_{\not i}) + \alpha_s \sum_{p\neq i,l,m} \widetilde{\mathcal{I}}_{lmp}^{(i)} B_{lmp}(\{k\}_{\not i}) \right]$$

$$\widetilde{\mathcal{I}}_{lmp}^{(i)} = \delta_{f_i g} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\epsilon\,\Gamma(1-2\epsilon)} \frac{s_{lm}}{s_{il}s_{im}} \left(\frac{e^{\gamma_E} \mu^2 s_{mp}}{s_{im}s_{ip}} \right)^{\epsilon}$$

→ ***Technique used for NNLO double-unresolved kernels***

Combination with double virtual

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$

$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1}$$

$$+ \int d\Phi_n$$

$$\tilde{J}_{\mathrm{s}}^{\,\mathrm{tripole}}(s,\xi) = \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\left[\frac{3}{8}\frac{1}{\epsilon^3} + \left(\frac{3}{2} - \frac{1}{4}\ln\xi\right)\frac{1}{\epsilon^2} + \left(7 - \frac{19}{48}\pi^2 - \ln\xi + \frac{1}{4}\ln^2\xi\right)\frac{1}{\epsilon} + 32 - \frac{19}{12}\pi^2 - 10\zeta_3 - \left(4 - \frac{\pi^2}{24}\right)\ln\xi + \ln^2\xi - \frac{1}{6}\ln^3\xi - \mathrm{Li}_3(-\xi) + \mathcal{O}(\epsilon)\right].$$

- ***Most*** *of the contributions to* $I^{(\mathrm{RV})}$ *can be computed using* ***NLO-like strategy***
- ***Non-trivial integrals arise from triple-color-correlated component*** $B_{lmp} = \sum_{a,b,c} f_{abc}\, \mathcal{A}_n^{(0)*}\, \mathbf{T}_l^a\, \mathbf{T}_m^b\, \mathbf{T}_p^c\, \mathcal{A}_n^{(0)}$

$$\mathbf{S}_i\, RV = -\mathcal{N}_1 \sum_{\substack{l\neq i\\ m\neq i}} \left[\mathcal{I}_{lm}^{(i)}\, V_{lm}(\{k\}_{\not i}) - \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\widetilde{\mathcal{I}}_{lm}^{(i)} + \mathcal{I}_{lm}^{(i)}\frac{\beta_0}{2\epsilon}\right) B_{lm}(\{k\}_{\not i}) + \alpha_{\mathrm{S}} \sum_{p\neq i,l,m} \widetilde{\mathcal{I}}_{lmp}^{(i)} B_{lmp}(\{k\}_{\not i}) \right]$$

$$\widetilde{\mathcal{I}}_{lmp}^{(i)} = \delta_{f_i g}\, \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\epsilon\,\Gamma(1-2\epsilon)}\, \frac{s_{lm}}{s_{il}s_{im}} \left(\frac{e^{\gamma_E}\mu^2 s_{mp}}{s_{im}s_{ip}}\right)^{\epsilon}$$

→ ***Technique used for NNLO double-unresolved kernels***

Combination with double virtual

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\rm RV)} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\rm RV)} + I^{(12)} \right) \delta_{X_n} \right.$$

$$\left. + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right. \right.$$

- ***Explicit poles of*** VV *extracted by looking at the* ***factorisation*** *properties of* ***virtual amplitudes.***
- ***Poles cancellation verified analytically for an arbitrary number of final state partons.***
- ***Finite result*** *is* ***compact*** *and features* ***simple dependence on kinematic*** *invariants.*
- *At most* Li_3 *contribute.*

Combination with double virtual

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$ *[Magnea, C-SS et al. 2212.11190]*

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\rm RV)} \right) \delta_{X_n}$$

$$\begin{aligned}
VV + I^{(2)} + I^{(\rm RV)} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ & \Bigg[I^{(0)} + \sum_j I^{(1)}_j \, \mathbf{L}_{jr} + \sum_j I^{(2)}_j \, \mathbf{L}^2_{jr} + \frac{1}{2} \sum_{j,l\neq j} \gamma_j^{\rm hc} \, \gamma_l^{\rm hc} \, \mathbf{L}_{jr'} \mathbf{L}_{lr'} \Bigg] \mathbf{B} \\
& + \sum_j \Big[I^{(0)}_{jr} + I^{(1)}_{jr} \, \mathbf{L}_{jr} \Big] \mathbf{B}_{jr} - 2\,(1-\zeta_2) \sum_{j,c\neq j,r} \gamma_j^{\rm hc} \, (2 - \mathbf{L}_{cr}) \, \mathbf{B}_{cr} \\
& + \sum_{c,d\neq c} \mathbf{L}_{cd} \Bigg[I^{(0)}_{cd} + I^{(1)}_{cd} \, \mathbf{L}_{cd} + \frac{\beta_0}{12} \mathbf{L}^2_{cd} + (4 - \mathbf{L}_{cd}) \sum_j \gamma_j^{\rm hc} \, \mathbf{L}_{jr} \Bigg] \mathbf{B}_{cd} \\
& + \sum_{c,d\neq c} \Bigg[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\,\zeta_4 + 2(1-\zeta_3)\,\mathbf{L}_{cd} \Bigg] \mathbf{B}_{cdcd} \\
& + (1-\zeta_2) \sum_{\substack{c,d\neq c \\ e\neq d}} \mathbf{L}_{cd} \, \mathbf{L}_{ed} \, \mathbf{B}_{cded} + \sum_{\substack{c,d\neq c \\ e,f\neq e}} \mathbf{L}_{cd} \, \mathbf{L}_{ef} \Bigg[1 - \frac{1}{2} \mathbf{L}_{cd} \left(1 - \frac{1}{8} \mathbf{L}_{ef} \right) \Bigg] \mathbf{B}_{cdef} \\
& + \pi \sum_{\substack{c,d\neq c \\ e\neq c,d}} \Bigg[\ln\frac{s_{ce}}{s_{de}} \, \mathbf{L}^2_{cd} + \frac{1}{3} \ln^3 \frac{s_{ce}}{s_{de}} + 2\,{\rm Li}_3 \left(-\frac{s_{ce}}{s_{de}} \right) \Bigg] \mathbf{B}_{cde} \Bigg\} \\
+ \left(\frac{\alpha_s}{2\pi}\right) \Bigg\{ & \Bigg[\Sigma_\phi - \sum_j \gamma_j^{\rm hc} \, \mathbf{L}_{jr} \Bigg] \mathbf{V}^{\rm fin} + \sum_{c,d\neq c} \mathbf{L}_{cd} \left(2 - \frac{1}{2} \mathbf{L}_{cd} \right) \mathbf{V}^{\rm fin}_{cd} \Bigg\} + \mathbf{VV}^{\rm fin}
\end{aligned}$$

- ***Explicit pol***
- ***Poles canc***
- ***Finite resul***
- *At most* Li_3

Combination with double virtual

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$ *[Magnea, C-SS et al. 2212.11190]*

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\rm RV)} \right) \delta_{X_n}$$

$$VV + I^{(2)} + I^{(\rm RV)} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ \Bigg[I^{(0)} + \sum_j I_j^{(1)} \mathbf{L}_{jr} + \sum_j I_j^{(2)} \mathbf{L}_{jr}^2$$

$$+ \sum_j \left[I_{jr}^{(0)} + I_{jr}^{(1)} \mathbf{L}_{jr} \right] \mathbf{B}_{jr} - 2\,(1-$$

$$+ \sum_{c,d\neq c} \mathbf{L}_{cd} \left[I_{cd}^{(0)} + I_{cd}^{(1)} \mathbf{L}_{cd} + \frac{\beta_0}{12} \mathbf{L}_{cd}^2 \right.$$

$$+ \sum_{c,d\neq c} \Big[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\,\zeta_4 + 2($$

$$+ (1-\zeta_2) \sum_{\substack{c,d\neq c\\ e\neq d}} \mathbf{L}_{cd}\,\mathbf{L}_{ed}\,\mathbf{B}_{cded} + \sum_{\substack{c,d\neq c\\ e,f\neq e}}$$

$$+ \pi \sum_{\substack{c,d\neq c\\ e\neq c,d}} \Big[\ln\frac{s_{ce}}{s_{de}} \mathbf{L}_{cd}^2 + \frac{1}{3} \ln^3 \frac{s_{ce}}{s_{de}} + 2$$

$$+ \left(\frac{\alpha_s}{2\pi}\right) \Bigg\{ \Big[\Sigma_\phi - \sum_j \gamma_j^{\rm hc} \mathbf{L}_{jr} \Big] \mathbf{V}^{\rm fin} + \sum_{c,d\neq c} \mathbf{L}_{cd}$$

- ***Explicit pol***
- ***Poles canc***
- ***Finite resul***
- *At most* Li_3

$$I^{(0)} = N_q^2 C_F^2 \left[\frac{101}{8} - \frac{141}{8}\zeta_2 + \frac{245}{16}\zeta_4\right] + N_g N_q C_F \left[C_A\left(\frac{13}{3} - \frac{125}{6}\zeta_2 + \frac{245}{8}\zeta_4\right) + \beta_0\left(\frac{77}{12} - \frac{53}{12}\zeta_2\right)\right]$$
$$+ N_g^2 \left[C_A^2\left(\frac{20}{9} - \frac{13}{3}\zeta_2 + \frac{245}{16}\zeta_4\right) + \beta_0^2\left(\frac{73}{72} - \frac{1}{8}\zeta_2\right) + C_A\beta_0\left(-\frac{1}{9} - \frac{11}{3}\zeta_2\right)\right]$$
$$+ N_q C_F \left[C_F\left(\frac{53}{32} - \frac{57}{8}\zeta_2 + \frac{1}{2}\zeta_3 + \frac{21}{4}\zeta_4\right) + C_A\left(\frac{677}{432} + \frac{5}{3}\zeta_2 - \frac{25}{2}\zeta_3 + \frac{47}{8}\zeta_4\right)\right.$$
$$\left. + \beta_0\left(\frac{5669}{864} - \frac{85}{24}\zeta_2 - \frac{11}{12}\zeta_3\right)\right]$$
$$+ N_g \left[C_F C_A\left(-\frac{737}{48} + 11\zeta_3\right) + C_F\beta_0\left(\frac{67}{16} - 3\zeta_3\right) + \beta_0^2\left(\frac{73}{72} - \frac{3}{8}\zeta_2\right)\right.$$
$$\left. + C_A^2\left(-\frac{4289}{216} + \frac{15}{2}\zeta_2 - 14\zeta_3 + \frac{89}{8}\zeta_4\right) + C_A\beta_0\left(\frac{647}{54} - \frac{53}{8}\zeta_2 - \frac{11}{12}\zeta_3\right)\right]$$

$$I_j^{(1)} = \delta_{f_a\{q,\bar q\}} C_F \left[N_q C_F\left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A\left(\frac{1}{3} - \frac{7}{4}\zeta_2\right) + \frac{2}{3} N_g \beta_0\right.$$
$$\left. + C_F\left(-\frac{3}{8} - 4\zeta_2 + 2\zeta_3\right) + C_A\left(\frac{25}{12} - 3\zeta_2 + 3\zeta_3\right) + \beta_0\left(\frac{1}{24} + \zeta_2\right)\right]$$
$$+ \delta_{f_a g}\left[N_q C_F C_A (10 - 7\zeta_2) - N_q C_F \beta_0\left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A^2\left(\frac{4}{3} - 7\zeta_2\right) + N_g C_A \beta_0\left(\frac{7}{3} + \frac{7}{4}\zeta_2\right)\right.$$
$$\left. - \frac{2}{3}(N_g + 1)\beta_0^2 + \frac{11}{4} C_F C_A - \frac{3}{4} C_F \beta_0 + C_A^2\left(\frac{28}{3} - \frac{23}{2}\zeta_2 + 5\zeta_3\right) - C_A\beta_0\left(\frac{2}{3} - \frac{5}{2}\zeta_2\right)\right]$$

$$I_j^{(2)} = \frac{1}{8}(15\,C_A - 7\,\beta_0 - 15) C_{f_j} - \frac{1}{4}(5\,C_A - 2\,\beta_0)\gamma_j + 2\,\zeta_2\, C_{f_j}^2$$

$$I_{jr}^{(0)} = (-1 + 3\zeta_2 - 2\zeta_3) C_A - \frac{1}{2}(13 + 10\zeta_2 + 2\zeta_3) C_{f_j} + (5 + 2\zeta_3)\gamma_j$$

$$I_{jr}^{(1)} = (1 - \zeta_2) C_A + \frac{1}{2}(4 + 7\zeta_2) C_{f_j} - (2 + \zeta_2)\gamma_j$$

$$I_{cd}^{(0)} = \left(\frac{20}{9} - 2\zeta_2 - \frac{7}{2}\zeta_3\right) C_A + \frac{31}{9}\,\beta_0 + 2\,\Sigma_\phi + 8\,(1 - \zeta_2)\, C_{f_d}$$

$$I_{cd}^{(1)} = -\left(\frac{1}{3} - \frac{1}{2}\zeta_2\right) C_A - \frac{11}{12}\,\beta_0 - \frac{1}{2}\,\Sigma_\phi$$

Take home message

1. **Phenomenology** requires **higher order corrections.**

2. To obtain fully **differential results** a **subtraction scheme** is needed.

3. **Local Analytic Sector Subtraction** is designed to address the fundamental requirements for an **optimal subtraction scheme**.

4. The main **building blocks** of the schemes are now **available** for an **arbitrary number of final state partons** (partition, integrated counterterm, mappings, …)

5. **Poles cancellation** has been proved **analytically in full generality**, and the **finite remainder** appears to be fairly **compact and simple**.

What’s next?

1. **Implementation and test of NNLO formula in a numerica framework** (massless FSR and ISR at NLO already implemented *[Bertolotti, Torrielli, Uccirati, Zaro 2209.09123]*)

2. Generalisation to **initial-state coloured particles at NNLO** for LHC applications.

3. Extension to **massive partons:** less singular limits, but more involved integrals.

What's next?

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3. Extension to **massive partons:** less singular limits, but more involved integrals.

Thank you for your attention!

Backup

Is percent precision a reality?

- Frontiers of experimental precision
  - Determination of the interaction luminosity

[CMS-LUMI-17-003]

Source	2015 [%]	2016 [%]
Total normalization uncertainty	1.3	1.0
Total integration uncertainty	1.0	0.7
Total uncertainty	1.6	1.2

[ATLAS-CONF-2019-021]

Data sample	2015+16	2017	2018	Comb.
Integrated luminosity (fb^{-1})	36.2	44.3	58.5	139.0
Total uncertainty (fb^{-1})	0.8	1.0	1.2	2.4 → 1.7%

  - Resolution on observed energy of particles and hadronic jets

[JINST 12 P02014]

The final uncertainties on the jet energy scale are below 3% across the phase space considered by most analyses ($p_T > 30$ GeV and $|\eta| < 5.0$). In the barrel region we reach an uncertainty below 1% for $p_T > 30$ GeV, when excluding the jet-flavor uncertainties, provided separately for different jet-flavor mixtures. At its lowest, the core uncertainty (excluding optional time-dependent and flavor systematics) is 0.32% for jets with p_T between 165 and 330 GeV, and $|\eta| < 0.8$. These results set a new benchmark for jet energy scale determination at hadron colliders.

[Phys. Rev. D 96, 072002]

The uncertainty in the jet energy scale is consistent with previous results in 2011 using 7 TeV data, and is at a level of 4.5% at 20 GeV, 1% at 200 GeV, and 2% at 2 TeV for an inclusive dijet sample. The uncertainties are fairly constant with respect to η, and a dedicated uncertainty is introduced for $2.0 < |\eta| < 2.6$ to account for details in the calorimeter energy reconstruction. A new method for combining

  - Statistical limitations are expected to be overcome by HL-LHC

[CERN-2019-007]

(HL-LHC). The HL-LHC will collide protons against protons at 14 TeV centre-of-mass energy with an instantaneous luminosity a factor of five greater than the LHC and will accumulate ten times more data, resulting in an integrated luminosity of 3 ab^{-1}.

Integration of the double-real counterterms: example

$$\int d\Phi_{n+2}\, \bar{\mathbf{S}}_{ij}\, RR(\{k\}) \propto \int d\Phi_{n+2}^{(ijcd)}\, I_{cd}^{(ij)}\, B_{cd}\Big(\{\bar{k}^{(ijcd)}\}\Big)$$

$$I_{cd}^{(gg)(ij)} = \frac{(1-\epsilon)(s_{ic}\,s_{jd} + s_{id}\,s_{jc}) - 2 s_{ij}\,s_{cd}}{s_{ij}^2\,(s_{ic}+s_{jc})(s_{id}+s_{jd})} + s_{cd}\,\frac{s_{ic}\,s_{jd} + s_{id}\,s_{jc} - s_{ij}\,s_{cd}}{s_{ij}\,s_{ic}\,s_{id}\,s_{jd}\,s_{jc}}\left[1 - \frac{1}{2}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic}+s_{jc})(s_{id}+s_{jd})}\right]$$

$$\int d\Phi_{n+2}^{(ijcd)}\, \frac{s_{ij}\,s_{cd}^2}{s_{ij}\,s_{ic}\,s_{id}\,s_{jd}\,s_{jc}} \propto \int_0^1 \frac{dx'\,dy'\,dz'\,dx\,dy\,dz\,(z-1)^2(1-y)^{1-2\epsilon}\,y^{-2\epsilon-1}(1-y')^{1-2\epsilon}\,y'^{-\epsilon}[(1-z)z]^{-\epsilon}[(1-z')z']^{-\epsilon-1}}{[x(1-x)x'(1-x')]^{\epsilon+1/2}(y'(z-1)-z)\Big(y'z'(1-z)+(1-z')z+2(2x'-1)\sqrt{y'(z-1)z(z'-1)z'}\Big)}$$

- Integrate over x $\to$ simple Beta functions
- Integrate over y $\to$ simple Beta function
- Integrate over x' $\to$ Master Integral $I_{x'}$ $\to$ Hypergeometric and Theta functions
- Integrate over z' $\to$ partial fractioning $\frac{I_{x'}}{[z'(1-z')]^{1+\epsilon}} = \frac{I_{x'}}{[z'(1-z')]^{\epsilon}}\left[\frac{1}{z} + \frac{1}{1-z}\right]$
  $\to$ Master Integral $I_{x'z'} + J_{x'z'}$ $\to$ Hypergeometric functions
- Integrate over z $\to$ Integral representation of Hyp. $\to$ auxiliary t variable
- Integrate over y' $\to$ poles extraction

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not i \not j}\big) \right]$$

$$I^{(i)}_{cd} = \frac{s_{cd}}{s_{ic}\, s_{id}} \qquad I^{(ij)}_{cd} = 2\, T_R\, I^{(q\bar{q})(ij)}_{cd} - 2\, C_A\, I^{(gg)(ij)}_{cd} \qquad s_{ab} = 2 p_a \cdot p_b$$

$$I^{(q\bar{q})(ij)}_{cd} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}^2(s_{ic} + s_{jc})(s_{id} + s_{jd})} \qquad I^{(gg)(ij)}_{cd} = \frac{(1-\epsilon)(s_{ic}s_{jd} + s_{id}s_{jc}) - 2 s_{ij}s_{cd}}{s_{ij}^2(s_{ic} + s_{jc})(s_{id} + s_{jd})} + s_{cd}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}} \left[1 - \frac{1}{2}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic} + s_{jc})(s_{id} + s_{jd})} \right]$$

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s_{ijk}^2}\, P^{\mu\nu}_{ijk}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{ijk}\big) \qquad P^{\mu\nu}_{ijk} B_{\mu\nu} = P_{ijk}\, B + Q^{\mu\nu}_{ijk}\, B_{\mu\nu}$$

$$P^{(3g)}_{ijk} = C_A^2 \Bigg\{ \frac{(1-\epsilon) s_{ijk}^2}{4 s_{ij}^2} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 + \frac{s_{ijk}}{s_{ij}} \left[4\, \frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1 - z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2} z_k + \frac{3}{2} \right]$$

$$+ \frac{s_{ijk}^2}{2 s_{ij} s_{ik}} \left[\frac{2 z_i z_j z_{ik}(1 - 2 z_k)}{z_k z_{ij}} + \frac{1 + 2 z_i (1 + z_i)}{z_{ik} z_{ij}} + \frac{1 - 2 z_i z_{jk}}{z_j z_k} + 2 z_j z_k + z_i (1 + 2 z_i) - 4 \right] + \frac{3(1-\epsilon)}{4} \Bigg\} + perm\,.$$

$$z_a = \frac{s_{ar}}{s_{ir} + s_{jr} + s_{kr}}\,, \quad z_{ab} = z_a + z_b$$

$$Q^{(3g)\mu\nu}_{ijk} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \Bigg\{ \left[\frac{2 z_j}{z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} \right) \frac{1}{s_{ik}} \right] \tilde{k}_i^2\, q_i^{\mu\nu} + \left[\frac{2 z_i}{z_k} \frac{1}{s_{ij}} - \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_j^2\, q_j^{\mu\nu} - \left[\frac{2 z_i z_j}{z_{ij} z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_k^2\, q_k^{\mu\nu} \Bigg\} + perm\,.$$

Key problem: several **different invariants** combined into **non-trivia**l and various **structures**, to be integrated over a **6-dim PS.**

Double real singular kernels:

Universal NNLO splitting *[Catani, Grazzini 9903516,9810389] [Campbell, Glover 9710255]*

$$\mathbf{S}_{ij}\, RR\big(\{k\}\big) \propto \sum_{c,d\neq i,j} \Bigg[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not i \not j}\big)\Bigg]$$

$$I^{(ij)}_{cd} = 2\, T_R\, I^{(q\bar q)(ij)}_{cd} - 2\, C_A\, I^{(gg)(ij)}_{cd}$$

$$I^{(q\bar q)(ij)}_{cd} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}^2 (s_{ic}+s_{jc})(s_{id}+s_{jd})}$$

$$I^{(gg)(ij)}_{cd} = \frac{(1-\epsilon)(s_{ic}s_{jd} + s_{id}s_{jc}) - 2 s_{ij}s_{cd}}{s_{ij}^2 (s_{ic}+s_{jc})(s_{id}+s_{jd})} + s_{cd}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}} \left[1 - \frac{1}{2}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic}+s_{jc})(s_{id}+s_{jd})}\right]$$

$$\mathbf{C}_{ijk}\, RR\big(\{k\}\big) \propto \frac{1}{s_{ijk}^2}\, P^{\mu\nu}_{ijk}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{ijk}\big)$$

$$P^{\mu\nu}_{ijk} B_{\mu\nu} = P_{ijk}\, B + Q^{\mu\nu}_{ijk}\, B_{\mu\nu}$$

$$P^{(3g)}_{ijk} = C_A^2 \left\{ \frac{(1-\epsilon)s_{ijk}^2}{4 s_{ij}^2} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 + \frac{s_{ijk}}{s_{ij}} \left[4\, \frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1 - z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2} z_k + \frac{3}{2} \right] + \frac{s_{ijk}^2}{2 s_{ij} s_{ik}} \left[\frac{2 z_i z_j z_{ik}(1 - 2 z_k)}{z_k z_{ij}} + \frac{1 + 2 z_i(1 + z_i)}{z_{ik} z_{ij}} + \frac{1 - 2 z_i z_{jk}}{z_j z_k} + 2 z_j z_k + z_i(1 + 2 z_i) - 4 \right] + \frac{3(1-\epsilon)}{4} \right\} + perm\,.$$

$$Q^{(3g)\mu\nu}_{ijk} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \left\{ \left[\frac{2 z_j}{z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} \right) \frac{1}{s_{ik}} \right] \tilde k_i^2\, q_i^{\mu\nu} + \left[\frac{2 z_i}{z_k} \frac{1}{s_{ij}} - \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}} \right) \frac{1}{s_{ik}} \right] \tilde k_j^2\, q_j^{\mu\nu} - \left[\frac{2 z_i z_j}{z_{ij} z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}} \right) \frac{1}{s_{ik}} \right] \tilde k_k^2\, q_k^{\mu\nu} \right\} + perm\,.$$

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$$\mathbf{S}_{ij}\, RR\big(\{k\}\big) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I_{cd}^{(i)} I_{ef}^{(j)} B_{cdef}\big(\{k\}_{\not i \not j}\big) + I_{cd}^{(ij)} B_{cd}\big(\{k\}_{\not i \not j}\big) \right] \qquad I_{cd}^{(ij)} = 2\, T_R\, I_{cd}^{(q\bar{q})(ij)} - 2\, C_A\, I_{cd}^{(gg)(ij)}$$

$$I_{cd}^{(q\bar{q})(ij)} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}^2(s_{ic}+s_{jc})(s_{id}+s_{jd})} \qquad I_{cd}^{(gg)(ij)} = \frac{(1-\epsilon)(s_{ic}s_{jd}+s_{id}s_{jc}) - 2s_{ij}s_{cd}}{s_{ij}^2(s_{ic}+s_{jc})(s_{id}+s_{jd})} + s_{cd}\,\frac{s_{ic}s_{jd}+s_{id}s_{jc}-s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}}\left[1-\frac{1}{2}\,\frac{s_{ic}s_{jd}+s_{id}s_{jc}}{(s_{ic}+s_{jc})(s_{id}+s_{jd})}\right]$$

$$\mathbf{C}_{ijk}\, RR\big(\{k\}\big) \propto \frac{1}{s_{ijk}^2}\, P_{ijk}^{\mu\nu}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{ijk}\big) \qquad P_{ijk}^{\mu\nu} B_{\mu\nu} = P_{ijk}\, B + Q_{ijk}^{\mu\nu}\, B_{\mu\nu}$$

$$P_{ijk}^{(3g)} = C_A^2 \left\{ \frac{(1-\epsilon)s_{ijk}^2}{4s_{ij}^2}\left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}}\right)^2 + \frac{s_{ijk}}{s_{ij}}\left[4\,\frac{z_iz_j - 1}{z_{ij}} + \frac{z_iz_j - 2}{z_k} + \frac{(1-z_kz_{ij})^2}{z_iz_kz_{jk}} + \frac{5}{2}\,z_k + \frac{3}{2}\right] + \frac{s_{ijk}^2}{2s_{ij}s_{ik}}\left[\frac{2z_iz_jz_{ik}(1-2z_k)}{z_kz_{ij}} + \frac{1+2z_i(1+z_i)}{z_{ik}z_{ij}} + \frac{1-2z_iz_{jk}}{z_jz_k} + 2z_jz_k + z_i(1+2z_i) - 4\right] + \frac{3(1-\epsilon)}{4} \right\} + perm\,.$$

$$Q_{ijk}^{(3g)\mu\nu} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \left\{ \left[\frac{2z_j}{z_k}\frac{1}{s_{ij}} + \left(\frac{z_jz_{ik}}{z_kz_{ij}} - \frac{3}{2}\right)\frac{1}{s_{ik}}\right] \tilde{k}_i^2\, q_i^{\mu\nu} + \left[\frac{2z_i}{z_k}\frac{1}{s_{ij}} - \left(\frac{z_jz_{ik}}{z_kz_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}}\right)\frac{1}{s_{ik}}\right] \tilde{k}_j^2\, q_j^{\mu\nu} - \left[\frac{2z_iz_j}{z_{ij}z_k}\frac{1}{s_{ij}} + \left(\frac{z_jz_{ik}}{z_kz_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}}\right)\frac{1}{s_{ik}}\right] \tilde{k}_k^2\, q_k^{\mu\nu} \right\} + perm\,.$$

Key problem: several **different invariants** combined into **non-trivial** and various **structures**, to be integrated over a **6-dim PS.**

↘

Key solution: split the **different structures** according to the contributing Lorentz invariants and **tune the mapping !**

Double real singular kernels:

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$$\mathbf{S}_{ij}\, RR\big(\{k\}\big) \propto \sum_{c,d\neq i,j} \Bigg[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not{i}\not{j}}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not{i}\not{j}}\big) \Bigg] \qquad I^{(ij)}_{cd} = 2\,T_R\, I^{(q\bar{q})(ij)}_{cd} - 2\,C_A\, I^{(gg)(ij)}_{cd}$$

$$I^{(q\bar{q})(ij)}_{cd} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s^2_{ij}(s_{ic} + s_{jc})(s_{id} + s_{jd})} \qquad I^{(gg)(ij)}_{cd} = \frac{(1-\epsilon)(s_{ic}s_{jd} + s_{id}s_{jc}) - 2s_{ij}s_{cd}}{s^2_{ij}(s_{ic} + s_{jc})(s_{id} + s_{jd})} + s_{cd}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}}\left[1 - \frac{1}{2}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic} + s_{jc})(s_{id} + s_{jd})}\right]$$

$$\mathbf{C}_{ijk}\, RR\big(\{k\}\big) \propto \frac{1}{s^2_{ijk}}\, P^{\mu\nu}_{ijk}\left(s_{ir}\,, s_{jr}\,, s_{kr}\right) B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{ijk}\big) \qquad P^{\mu\nu}_{ijk}B_{\mu\nu} = P_{ijk}\, B + Q^{\mu\nu}_{ijk}\, B_{\mu\nu}$$

$$P^{(3g)}_{ijk} = C_A^2 \left\{ \frac{(1-\epsilon)s^2_{ijk}}{4s^2_{ij}} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 + \frac{s_{ijk}}{s_{ij}} \left[4\,\frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1 - z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2}\, z_k + \frac{3}{2} \right] + \frac{s^2_{ijk}}{2s_{ij}s_{ik}} \left[\frac{2z_i z_j z_{ik}(1-2z_k)}{z_k z_{ij}} + \frac{1 + 2z_i(1+z_i)}{z_{ik}z_{ij}} + \frac{1 - 2z_i z_{jk}}{z_j z_k} + 2z_j z_k + z_i(1+2z_i) - 4 \right] + \frac{3(1-\epsilon)}{4} \right\} + perm\,.$$

$$Q^{(3g)\mu\nu}_{ijk} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \left\{ \left[\frac{2z_j}{z_k}\frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} \right) \frac{1}{s_{ik}} \right] \tilde{k}_i^2\, q_i^{\mu\nu} + \left[\frac{2z_i}{z_k}\frac{1}{s_{ij}} - \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_j^2\, q_j^{\mu\nu} - \left[\frac{2z_i z_j}{z_{ij}z_k}\frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_k^2\, q_k^{\mu\nu} \right\} + perm\,.$$

How the results look like:

$$\int d\Phi_{n+2}\, \overline{\mathbf{C}}_{ijk}\, RR \;=\; \int d\Phi_n\big(\bar{k}^{(ijrk)}\big)\, J_{\rm cc}\big(\bar{s}^{ijkr}_{kr}\big)\, B\big(\bar{k}^{(ijrk)}\big)$$

$$J^{(3g)}_{\rm cc}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_A^2 \left[\frac{15}{\epsilon^4} + \frac{63}{\epsilon^3} + \left(\frac{853}{3} - 22\,\pi^2 \right) \frac{1}{\epsilon^2} + \left(\frac{10900}{9} - \frac{275}{3}\,\pi^2 - 376\,\zeta_3 \right) \frac{1}{\epsilon} + \frac{180739}{36} - \frac{3736}{9}\,\pi^2 - 1555\,\zeta_3 + \frac{41}{10}\,\pi^4 + \mathcal{O}(\epsilon) \right]$$