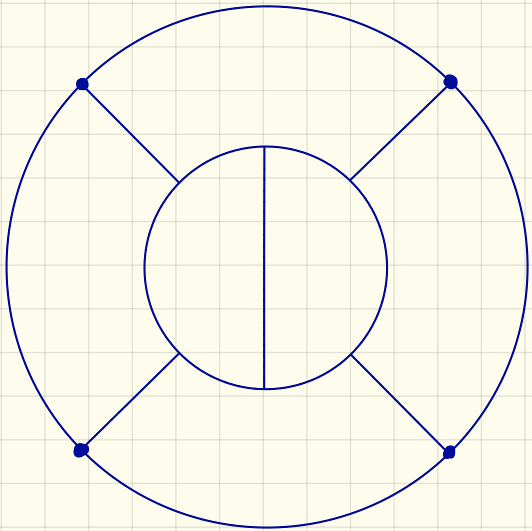
String / Supergravity Loops in AdS from CFT



Based on work with

F. Aprile, R. Glew, P. Heslop, D. Nandan, H. Paul, K. Rigatos,
F. Sanfilippo, M. Santagata, A. Stewart

Recent huge progress in calculating amplitudes in flat space.

gauge theories, gravity, ...

string theory

Can we make similar progress on curved backgrounds?

Natural place to start: AdS_{d+1}

Large isometry group $SO(2,d) \simeq$ conformal transformations on boundary

AdS / CFT

Large radius $\longrightarrow$ Flat space limit

$$\text{AdS} \longleftrightarrow \text{CFT}$$

particles	Local operators
graviton $g_{\mu\nu}$	$T_{\mu\nu}$ energy-momentum tensor
Amplitudes $A(g_1 \dots g_n)$	$\langle T(x_1) \dots T(x_n) \rangle$ correlators
loop expansion	$\frac{1}{c}$ expansion

$$\text{IIB on } AdS_5 \times S^5 \longleftrightarrow \mathcal{N}=4 \quad SU(N) \quad SYM$$

supergravity multiplet — Energy-momentum multiplet

$\Phi, \ldots, g_{\mu\nu}, \ldots \phi, \chi$ — $\frac{1}{2}$ BPS ops (short) $\left\{ \underbrace{tr(\phi^{\{I}\phi^{J\}})}_{\mathcal{O}_2}, \ldots, T_{\mu\nu}, \ldots, \mathcal{L} \right.$

Kaluza-Klein modes — $\mathcal{O}_3 = tr(\phi^{\{I}\phi^J\phi^{K\}}), \mathcal{O}_4, \ldots$

Multi-particle SG states — short OR long $\left\{ \mathcal{O}_2 \mathcal{O}_2, \ldots \right.$

Excited string states — long $\left\{ K = tr(\phi^I\phi^I), \ldots \right.$

Amplitudes $A(\Phi, \ldots, \Phi)$ — $\langle \mathcal{O}_2(x_1) \ldots \mathcal{O}_2(x_n) \rangle$ correlators

string corrections — $1/\sqrt{\lambda}$ expansion, $\lambda = g^2 N$

loop expansion — $\frac{1}{c} \sim \frac{1}{N^2-1}$ expansion

SYM moduli space

excited string states decouple

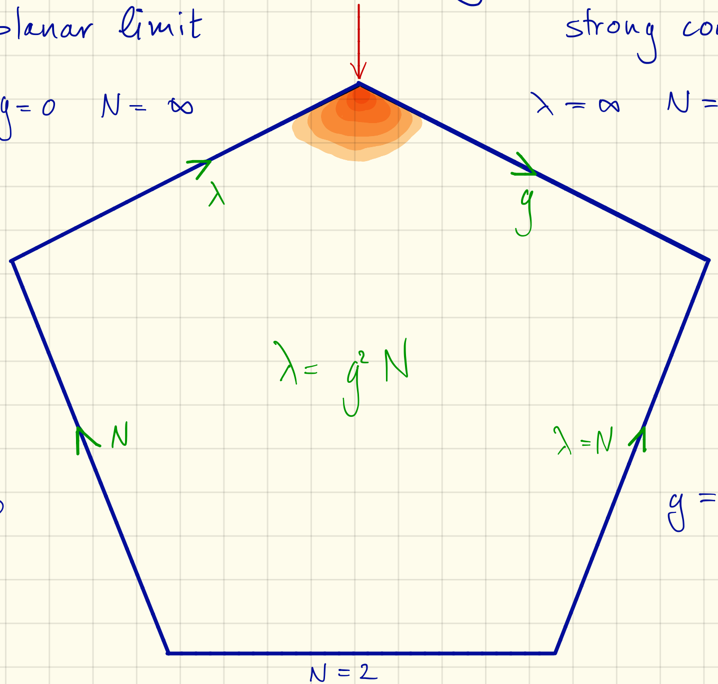
Supergravity

large N 't Hooft planar limit

$g = 0 \quad N = \infty$

strong coupling, large N

$\lambda = \infty \quad N = \infty$



(free theory) $g = \lambda = 0$

$g = 1$ (self-dual)

SU(2) theory

Focus on $\frac{1}{2}$ - BPS single-particle operators

$$\mathcal{O}_p(x,y) = \mathrm{tr}\left(\phi^{I_1} \cdots \phi^{I_p}\right)(x)\, y^{I_1} \cdots y^{I_p} \qquad y^2 = 0$$

$$+ \ \cdots$$

multi-trace terms ($\frac{1}{N}$ suppressed)

s.t. $\langle \mathcal{O}_p \ [\mathcal{O}_{q_1} \cdots \mathcal{O}_{q_\ell}]\rangle = 0$

$$\mathcal{O}_2 = \mathrm{tr}\,\phi^2$$

$$\mathcal{O}_3 = \mathrm{tr}\,\phi^3$$

$$\mathcal{O}_4 = \mathrm{tr}\,\phi^4 - \frac{2N^2-3}{N(N^2+1)}[\mathcal{O}_2\,\mathcal{O}_2]$$

$$\vdots$$

Two-point functions:

$$\langle O_p \, O_q \rangle = \delta_{pq} \, R_p \, \frac{y_{12}^2}{x_{12}^2} \qquad R_p = p^2 (p-1) \left[\frac{1}{(N+1-p)_{p-1}} - \frac{1}{(N+1)_{p-1}} \right]^{-1}$$

$$= p N^p + O(N^{p-2})$$

Three-point functions:

$$\langle O_p \, O_{q_1} \, O_{q_2} \rangle = 0 \qquad q_1 + q_2 = p \quad \text{extremal}$$

$$= q_1 q_2 R_p \qquad q_1 + q_2 = p+2$$

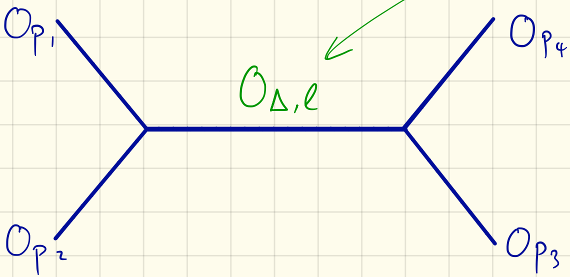
$$\vdots$$

$$= \qquad q_1 + q_2 = p+4$$

No dependence on g — protected ops.

Four-point functions $\sim$ four-point SG amplitudes

Operator Product Expansion:



exchanged operators can be short (protected) OR long (unprotected)

all short exchanges

depends on g

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle = \langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\text{free}} + \langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\text{int}}$$

$$\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle_{\text{int}} = \underbrace{\mathcal{P}}_{\text{dimensions}} \times \underbrace{\mathcal{I}}_{\text{Supersymmetry}} \times \underbrace{\mathcal{H}(u,v;\tilde{u},\tilde{v})}_{\text{dynamics}}$$

$$\mathcal{I} = (x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y})$$

$$u = x\bar{x} = \frac{X_{12}^2 X_{34}^2}{X_{13}^2 X_{24}^2} \qquad v = (1-x)(1-\bar{x}) = \frac{X_{14}^2 X_{23}^2}{X_{13}^2 X_{24}^2}$$

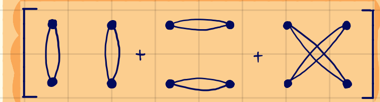
$$\tilde{u} = y\bar{y} = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2} \qquad \tilde{v} = (1-y)(1-\bar{y}) = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}$$

e.g. $$\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle = \frac{y_{12}^4 y_{34}^4}{X_{12}^4 X_{34}^4} \times \mathcal{I} \times \mathcal{H}(u,v)$$

just a fuction of u,v

Expansion around supergravity limit

$$\langle O_2\, O_2\, O_2\, O_2 \rangle =$$



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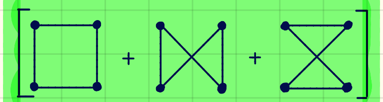


$\frac{1}{c}$

$$+\, \mathcal{P}\mathcal{I}\left[\, \mathcal{H}^{(1)}(u,v) + \mathcal{H}^{(2)}(u,v) + \dots \right]$$

$\frac{1}{c^2}$

disconnected

tree-level string

(AdS Virasoro-Shapiro)

one loop

$$\mathcal{H}^{(1)}(u,v) = \underbrace{\mathcal{H}^{(1,0)}(u,v)}_{\text{Tree-level supergravity}} + \lambda^{-3/2}\, \mathcal{H}^{(1,3)}(u,v) + \lambda^{-5/2}\, \mathcal{H}^{(1,5)}(u,v) + \dots$$

string corrections

all charges $\langle O_{p_1} O_{p_2} O_{p_3} O_{p_4} \rangle$

(Rastelli, Zhou 2016)

Tree level supergravity:

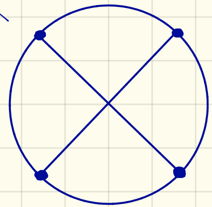
(Mellin rep.)

$$\mathcal{H}(u,v,\tilde{u},\tilde{v}) \sim \int ds\, dt \sum_{\check{s},\check{t}} u^{s} v^{t} \tilde{u}^{\check{s}} \tilde{v}^{\check{t}}\; \Gamma \;\frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)}$$

contain double poles $\longrightarrow$ Mack gamma functions

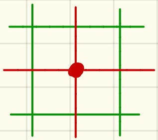
$s+\check{s}$ $\qquad\qquad$ $\mathbf{s}+\mathbf{t}+\mathbf{u}=-4$

$$\sim \sum \bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v)$$



(Position space rep.)

$$\mathcal{D}\;\bar{\Phi}^{(1)}(u,v)$$



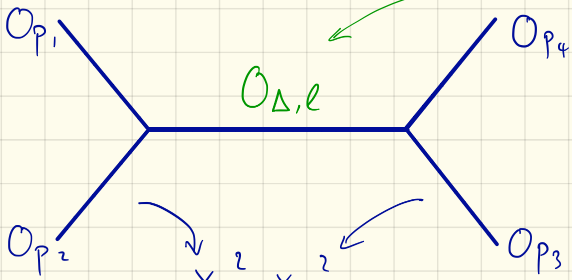
$\log u$ discontinuity

$$\Phi^{(1)}(u,v) = \frac{2\left(\mathrm{Li}_2(x) - \mathrm{Li}_2(\bar{x})\right) + \log u \left(\log(1-x) - \log(1-\bar{x})\right)}{x-\bar{x}}$$

coeffs $= \dfrac{\text{poly}}{(x-\bar{x})^7}$

one-loop 4d four-mass box fn.

Recall OPE:



Large N:

Double-trace ops

Two-particle bound states

e.g. $[O_2 O_2]$ in singlet

$$u = \frac{x_{12}^2 \, x_{34}^2}{x_{13}^2 \, x_{24}^2}$$

$$\mathcal{H} \sim \sum C_{O_{p_1} O_{p_2} O_{\Delta,\ell}} \, C_{O_{\Delta_\ell} O_{p_3} O_{p_4}} \, G_{\Delta,\ell,[aba]}(u,v,\check{u},\check{v})$$

contains $u^{\Delta} = u^{\Delta_0 + \frac{1}{N^2}\gamma + \dots} = u^{\Delta_0}\left(1 + \frac{1}{N^2}\gamma \log u + \dots\right)$

From $\log u$ disc can read off leading anom. dims.

Many double trace ops with same quantum numbers: Mixing

Double-trace spectrum

(Aprile, JMD, Heslop, Paul 2018)

$$\left[\mathcal{O}_p \, \Box^n \, \partial^\ell \, \mathcal{O}_q \right]_{[a\,b\,a]}$$

$$M_t = (t-1)(t+a)(t+a+b+1)(t+2a+b+2)$$

$$t = \frac{\Delta_0 - \ell - b}{2} - a$$

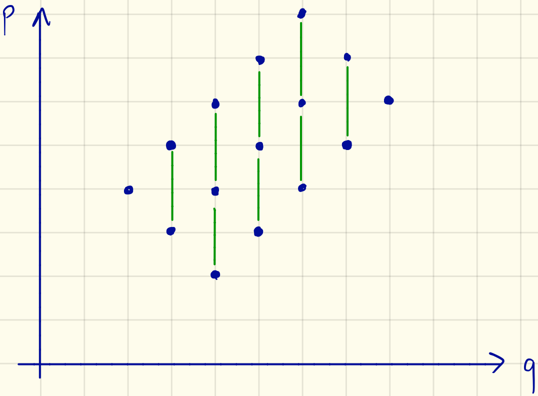
$$\Delta = p + q + 2n + \ell - \frac{4}{N^2} \frac{M_t \, M_{t+\ell+1}}{\left(\ell + 2p - 2 - a - \frac{1+(-1)^{a+\ell}}{2}\right)_6} + \ldots$$

Rational eigenvalues!

Degeneracy only partially lifted!

Ten-dimensional conformal symmetry

(Caron-Huot, Trinh 2018)



Recall we made use of

$$u^{\Delta} = u^{\Delta_0 + \frac{1}{N^2}\gamma + \dots} = u^{\Delta_0}\left(1 + \frac{1}{N^2}\gamma \log u + \frac{1}{2N^4}\gamma^2 \log^2 u + \dots\right)$$

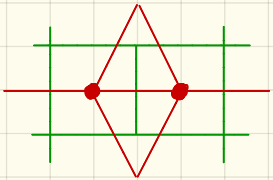
predict double $\log u$ at $\frac{1}{N^4}$

Find crossing symmetric functions which match double log in all OPE channels?

(Aprile, JMD, Heslop, Paul 2017, 2019) (Alday, Zhou 2019) Mellin rep.

Yes! Involves $\Phi^{(2)}(u,v) = \dfrac{\mathrm{Li}_4(x) - \mathrm{Li}_4(\bar{x}) + \dots}{x - \bar{x}}$ & derivatives

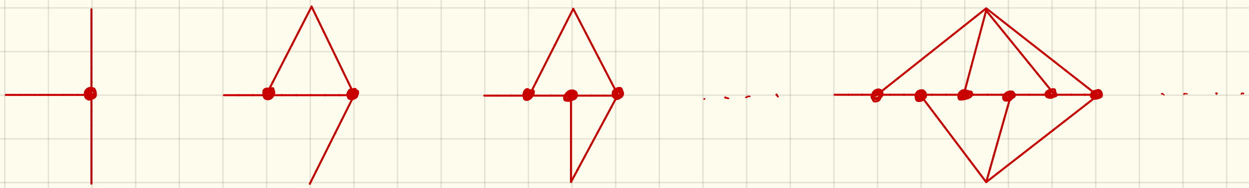
(Two-loop ladder integral)



for (2222) coeffs are $\dfrac{\text{poly}}{(x-\bar{x})^{15}}$

Same reasoning: can predict maximal power of $\log u$ at any order.

The natural class of functions which capture leading log: <u>zigzags</u>



(JMD 2012)

also: (Schnetz 2012

Brown, Schnetz 2012)

All given in terms of single-valued harmonic polylogs.

In fact, simple anomalous dimensions imply simpler form of leading log:

$$\gamma \sim \frac{M_t \, M_{t+\ell+1}}{\left(\ell + 2p - 2 - a - \frac{1+(-1)^{a+\ell}}{2}\right)_6} \longleftrightarrow \Delta^{(8)}$$

$$\mathcal{H}(u,v)\Big|_{\frac{1}{N^{2n}}\log^n u} \sim \sum \gamma^n \, G_{\Delta_0,\ell}(u,v)$$

$$\|$$

$$\left(\Delta^{(8)}\right)^{n-1} \text{(simpler function)}$$

Suggests e.g. (2222) $\mathcal{H}^{(n)}(u,v) = \left(\Delta^{(8)}\right)^{n-1} \mathcal{P}^{(n)}(u,v) + \dots$

simpler 'preamplitude'

<u>n=2</u>: $\mathcal{H}^{(2)}(u,v) = \Delta^{(8)} \mathcal{P}^{(2)}(u,v) + \mathcal{H}^{(1)}(u,v)$

(Aprile, JMD, Heslop, Paul 2019)

Still $\Phi^{(2)}(u,v)$ & derivatives but

coefficients simpler $\frac{\text{poly}}{(x-\bar{x})^7}$ vs $\frac{\text{poly}}{(x-\bar{x})^{15}}$

Using this idea can even go to two loops (Huang, Yuan 2022)
(JMD, Paul 2022)

$$\mathcal{H}^{(3)}(u,v) = \left(\Delta^{(8)}\right)^2 \mathcal{P}^{(3)}(u,v) + \alpha\, \mathcal{H}^{(2)}(u,v) + \beta\, \mathcal{H}^{(1)}(u,v)$$

Crossing properties $\quad \Delta^{(8)}\Big|_{\substack{x \to \frac{x}{x-1} \\ u \to \frac{u}{v}}} = \Delta^{(8)}$

Two-loop bonus: $\quad \left(\Delta^{(8)}\right)^2\Big|_{\substack{x \to 1-x \\ u \to v}} = \frac{u^4}{v^4}\left(\Delta^{(8)}\right)^2$

$\hookrightarrow$ Can impose crossing symmetry directly on $\mathcal{P}^{(3)}(u,v)$.

- Match leading $\log^3 u$ disc (zigzags)
- No twist < 4 ops, match subleading disc @ twist 4
- Match flat space limit (Bissi, Georgoudis 2021)
- No poles at $x = \bar{x}$

Spectrum corrections

At twist 4, only one operator at a given spin: $O_2 \partial^\ell O_2 \big|_{[000]}$

No mixing problem to solve $\longrightarrow$ can read off anomalous dimension

$$\Delta = 4 + \ell + 2\left(a\gamma^{(1)} + a^2 \gamma^{(2)} + a^3 \gamma^{(3)} + \dots \right) \qquad a \equiv \frac{1}{N^2-1}$$

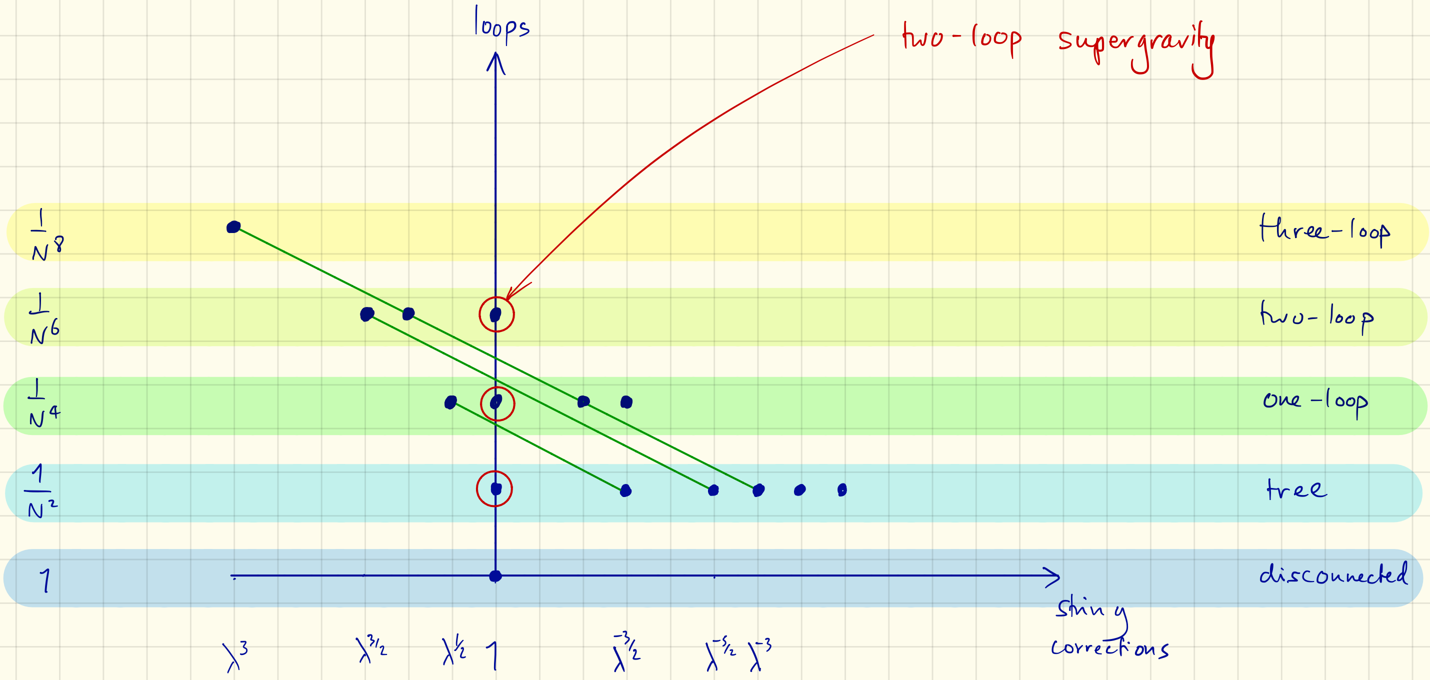
$$\gamma^{(1)} = -\frac{48}{(\ell+1)(\ell+6)}$$ (Dolan, Osborn 2004)

(Aprile, JMD, Heslop, Paul
Alday, Bissi
Alday, Caron-Huot 2017)

$$\gamma^{(2)} = \frac{1344(\ell-7)(\ell+14)}{(\ell-1)(\ell+1)^2(\ell+6)^2(\ell+8)} - \frac{2304(2\ell+7)}{(\ell+1)^3(\ell+6)^3} - \frac{1080}{7}\delta_{\ell,0}$$

$$\gamma^{(3)} = C_3\left(S_{-3} - S_3 - 2S_{1,-2} + 3\zeta_3\right) + C_2 S_{-2} + C_1 S_1 + C_0$$

(JMD, Paul 2022)



Many related papers:

Alday, Bissi, Perlmutter 2018

Abl, Heslop, Lipstein 2020

Alday, Hansen, Silva 2022, 2023

Binder, Chester, Pufu, Wang 2019

Aprile, JMD, Paul, Santagata 2020

+ many more ...

Ten-dimensional conformal symmetry

$$\breve{O}_p = \frac{O_p}{\sqrt{p R_p}}$$

$$\frac{H^{(1,0)}_{2222}(u_{10}, v_{10})}{(x_{12}^2 - y_{12}^2)^4 (x_{34}^2 - y_{34}^2)^4} = \frac{1}{I\,(x_{13}^2 x_{24}^2 y_{13}^2 y_{24}^2)^2} \sum_{\vec{p}} \langle \breve{O}_{p_1} \breve{O}_{p_2} \breve{O}_{p_3} \breve{O}_{p_4} \rangle$$

$$u_{10} = \frac{(x_{12}^2 - y_{12}^2)(x_{34}^2 - y_{34}^2)}{(x_{13}^2 - y_{13}^2)(x_{24}^2 - y_{24}^2)} \qquad v_{10} = \frac{(x_{14}^2 - y_{14}^2)(x_{23}^2 - y_{23}^2)}{(x_{13}^2 - y_{13}^2)(x_{24}^2 - y_{24}^2)}$$

i.e. the (2222) correlator generates all charges for for tree-level supergravity.

equiv:

$$\mathcal{H}^{(1,0)}_{p_1 p_2 p_3 p_4}(u, v, \breve{u}, \breve{v}) = \mathcal{D}_{p_1 p_2 p_3 p_4} \mathcal{H}^{(1,0)}_{2222}(u, v)$$

Consequence :