Luminosity Spectra

Thorsten Ohl

Institute for Theoretical Physics and Astrophysics
University of Würzburg

http://physik.uni-wuerzburg.de/ohl

ECFA Higgs Factories: 2nd Topical Meeting on Generators

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∴ wanted: smooth parametrization of CAIN and Guinea-Pig output that allows efficient generation of an arbitrary set of random numbers with (as far as possible) the same distribution

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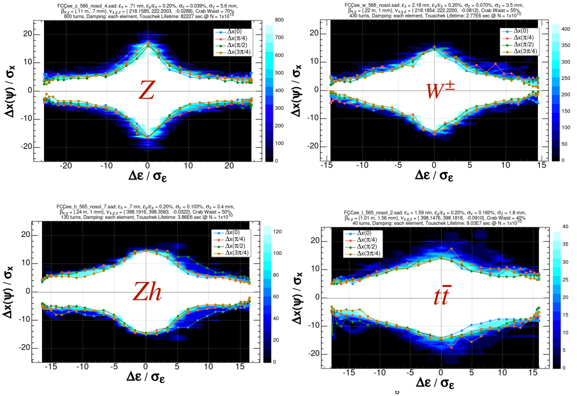
- need “blessed” (x_1, x_2) samples for different designs

Daniel Schulte's 11th commandment: Thou shallst not use results of beam-beam simulations without quality control by accelerator physicists

- Interaction of beam transport and beamstrahlung

Dynamic aperture (*z*-*x*)



FCC WEEK 2023 05 – 09 June

- At the CDR, the dynamic aperture (DA) and beam-beam were estimated separately. Then the estimation of the beam lifetimes was not good enough, esp. including the beamstrahlung.
- Thus it had not been noticed until recent that at some betatron tunes, the beam lifetime suffered a lot by beam-beam & beamstrahlung.
- Also the blowup of the vertical emittance, or the required lattice emittance, were not properly estimated at the CDR.
- All such issues are addressed this time, but the resulting luminosity is reduced by more than 15%, on top of the reductions due to the shorter circumference (-7%) and less damping between IPs (-7%).

June 6, 2023, K. Oide

[Katsunobu Oide, FCC Week, June 2023]

- Improvements wrt. the CDR

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[Katsunobu Oide, FCC Week, June 2023]

- parameters (for reference)

Parameters

FCC-ee collider parameters as of June 3, 2023.

Beam energy	[GeV]	45.6	80	120	182.5
Layout		PA31-3.0			
# of IPs		4			
Circumference	[km]	90.658816			
Bend. radius of arc dipole	[km]	9.936			
Energy loss / turn	[GeV]	0.0394	0.374	1.89	10.42
SR power / beam	[MW]	50			
Beam current	[mA]	1270	137	26.7	4.9
Colliding bunches / beam		15880	1780	440	60
Colliding bunch population	$[10^{11}]$	1.51	1.45	1.15	1.55
Hor. emittance at collision ε_x	[nm]	0.71	2.17	0.71	1.59
Ver. emittance at collision ε_y	[pm]	1.4	2.2	1.4	1.6
Lattice ver. emittance $\varepsilon_{y,\text{lattice}}$	[pm]	0.75	1.25	0.85	0.9
Arc cell		Long 90/90		90/90	
Momentum compaction α_p	$[10^{-6}]$	28.6		7.4	
Arc sext families		75		146	
$\beta^*_{x/y}$	[mm]	110 / 0.7	220 / 1	240 / 1	1000 / 1.6
Transverse tunes $Q_{x/y}$		218.158 / 222.200	218.186 / 222.220	398.192 / 398.358	398.148 / 398.182
Chromaticities $Q'_{x/y}$		0 / +5	0 / +2	0 / 0	0 / 0
Energy spread (SR/BS) σ_δ	[%]	0.039 / 0.089	0.070 / 0.109	0.104 / 0.143	0.160 / 0.192
Bunch length (SR/BS) σ_z	[mm]	5.60 / 12.7	3.47 / 5.41	3.40 / 4.70	1.81 / 2.17
RF voltage 400/800 MHz	[GV]	0.079 / 0	1.00 / 0	2.08 / 0	2.1 / 9.38
Harm. number for 400 MHz		121200			
RF freqeuncy (400 MHz)	MHz	400.786684			
Synchrotron tune Q_s		0.0288	0.081	0.032	0.091
Long. damping time	[turns]	1158	219	64	18.3
RF acceptance	[%]	1.05	1.15	1.8	2.9
Energy acceptance (DA)	[%]	±1.0	±1.0	±1.6	-2.8/+2.5
Beam crossing angle at IP $\pm\theta_x$	[mrad]	±15			
Piwinski angle $(\theta_x \sigma_{z,\text{BS}})/\sigma^*_x$		21.7	3.7	5.4	0.82
Crab waist ratio	[%]	70	55	50	40
Beam-beam ξ_x/ξ_y [a]		0.0023 / 0.096	0.013 / 0.128	0.010 / 0.088	0.073 / 0.134
Lifetime (q + BS + lattice)	[sec]	15000	4000	6000	6000
Lifetime (lum) [b]	[sec]	1340	970	840	730
Luminosity / IP	$[10^{34}/\text{cm}^2\text{s}]$	140	20	5.0	1.25
Luminosity / IP (CDR, 2 IP)	$[10^{34}/\text{cm}^2\text{s}]$	230	28	8.5	1.8

[a] incl. hourglass.
[b] only the energy acceptance is taken into account for the cross section

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[Katsunobu Oide, FCC Week, June 2023]

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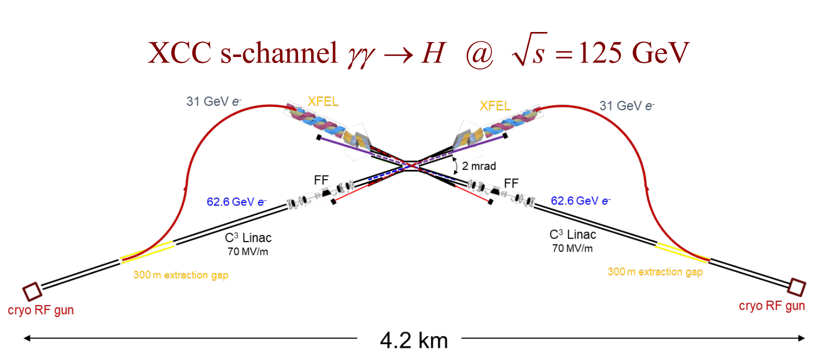
- ☺ however, Oide-san has graciously agreed to provide us with the results of their beam optics simulations that we can feed into Guinea-Pig to simulate the pieces missing for beamstrahlung and get comprehensive luminosity spectra
- stay tuned!

- cf. Lindsey Gray's talk later today

- It's back as XCC (XFEL based Compton Collider) (cf. [2203.08484])



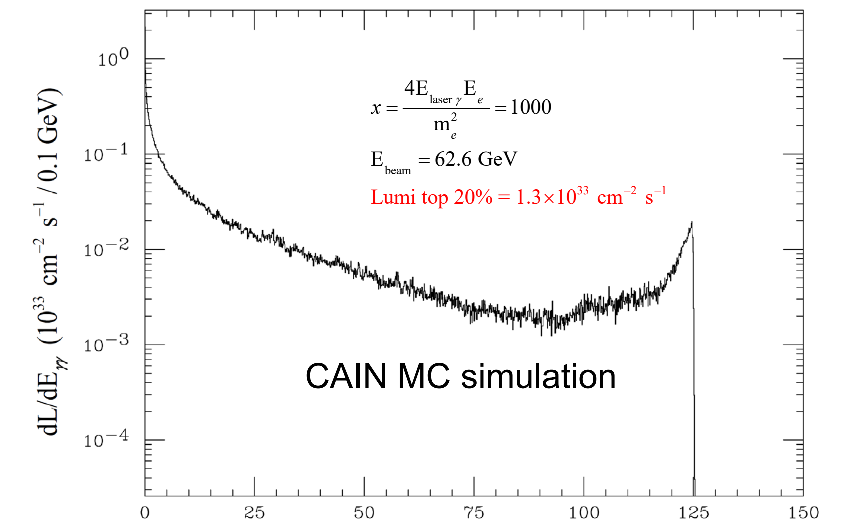
[Tim Barklow, LCWS23, May 2023]

- parameters for reference

Final Focus parameters	Approx. value	XFEL parameters	Approx. value
Electron energy	62.8 GeV	Electron energy	31 GeV
Electron beam power	0.57 MW	Electron beam power	0.28 MW
β_x/β_y	0.03/0.03 mm	normalized emittance	120 nm
$\gamma\epsilon_x/\gamma\epsilon_y$	120/120 nm	RMS energy spread $\langle\Delta\gamma/\gamma\rangle$	0.05%
σ_x/σ_y at e^-e^- IP	5.4/5.4 nm	bunch charge	1 nC
σ_z	20 μm	Linac-to-XFEL curvature radius	133 km
bunch charge	1 nC	Undulator B field	$\gtrsim$ 1 T
Rep. Rate at IP	240 × 38 Hz	Undulator period λ_u	9 cm
σ_x/σ_y at IPC	12.1/12.12 nm	Average β function	12 m
$\mathcal{L}_{\text{geometric}}$	9.7×10^{34} cm^2 s^{-1}	x-ray λ (energy)	1.2 nm (1 keV)
δ_E/E	0.05%	x-ray pulse energy	0.7 J
L^* (QD0 exit to e^- IP)	1.5m	pulse length	40 μm
d_{cp} (IPC to IP)	60 μm	$a_{\gamma x}/a_{\gamma y}$ (x/y waist)	21.2/21.2 nm
QD0 aperture	9 cm diameter	non-linear QED ξ^2	0.10
Site parameters	Approx. value		
crossing angle	2 mrad		
total site power	85 MW		
total length	3.0 km		

[Tim Barklow, LCWS23, May 2023]

- $\gamma\gamma$ energy distribution ($E_{\gamma\gamma}$/GeV)



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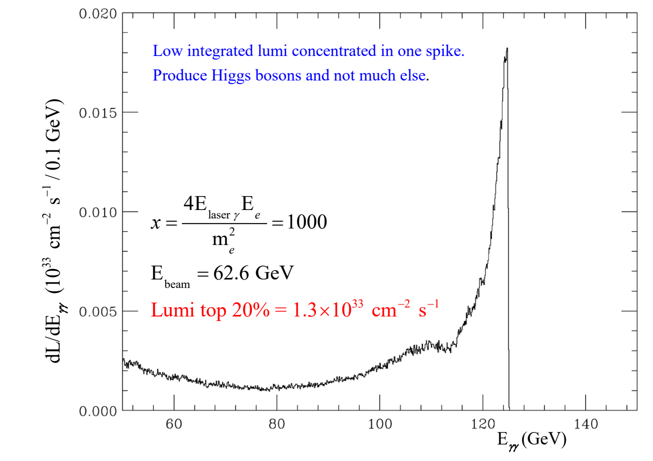
- as with the TESLA $\gamma\gamma$ collider, simple parametrization will not suffice.



[Tim Barklow, LCWS23, May 2023]

- Can it be shrunk and made more powerful?

Replace 62.5 GeV C^3 e- beam w/ 7500 GeV PWFA e- beam and simulate $\gamma\gamma$ Collisions using CAIN MC

Technology	PWFA	$\gamma\gamma$ PWFA
Aspect Ratio	Round	Round
CM Energy	15	15
Single beam energy (TeV)	7.5	7.5
Gamma	1.47E+07	1.4E+07
Emittance X (mm mrad)	0.1	0.12
Emittance Y (mm mrad)	0.1	0.12
Beta* X (m)	1.50E-04	0.30E-04
Beta* Y (m)	1.50E-04	0.30E-04
Sigma* X (nm)	1.01	0.48
Sigma* Y (nm)	1.01	0.48
N_bunch (num)	5.00E+09	6.2E+09 then later switch to 5.00E+09
Freq (Hz)	7725	7725
Sigma Z (um)	5	5
Geometric Lumi (cm-2 s-1)	1.50E+36	6.58E+36

[Tim Barklow, LCWS23, May 2023]

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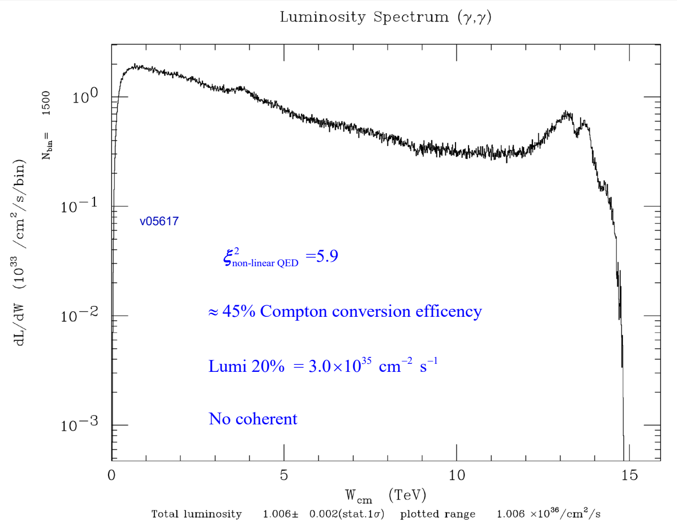

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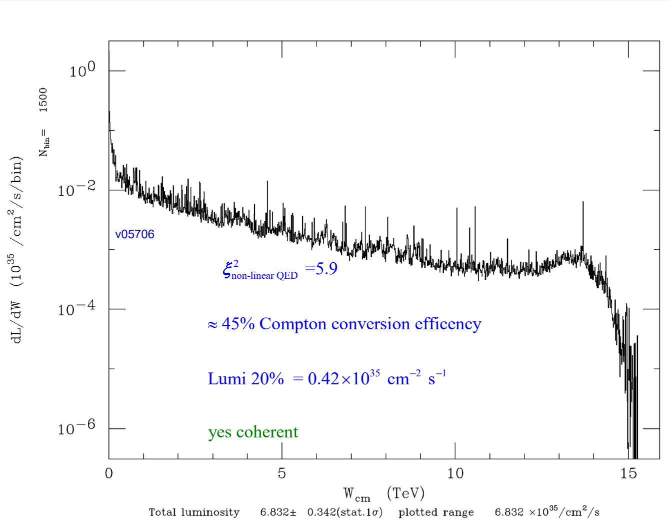

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  - `circe2_tool.opt` for generating smooth distributions from your own Guinea-Pig and CAIN output

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- the work on a promised next generation of CIRCE using tricks from machine learning to describe the luminosity spectra has not yet started, but
  - CIRCE2 works well for practical purposes
  - CIRCE2 can be used without access to GPUs!

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- 😊 realistic luminosity spectra for FCC are finally on the horizon!
- even if outside of ECFA's scope: work on C^3 is starting!
- even if further outside of ECFA's scope: keep an open mind about $\gamma\gamma$-colliders and keep the software flexible enough to handle their luminosity spectra!

— BACKUP SLIDES —

- read TDR.circe and generate 1000000 (x_1, x_2) pairs for unpolarized electron-positron pairs

```
program girce2
  type(circe2_state) :: c2s
  type(rng_t) :: rng
  integer :: i, ierror
  real(kind=default), dimension(2) :: x
  call circe2_load (c2s, "TDR.circe", "ILC", 500.0_default, ierror)
  do i = 1, 1000000
     call circe2_generate (c2s, rng, x, [11, -11], [0, 0])
     print *, x, 1.0_default
  end do
end program girce2
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- even simpler: use it from inside WHIZARD as

```
sqrts = 500
beams = "e-", "e+" => circe2
$circe2_file = "TDR.circe"
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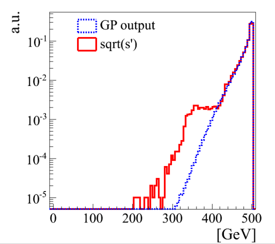

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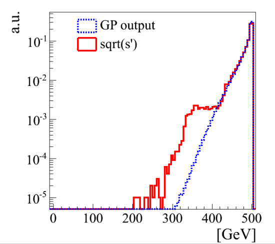

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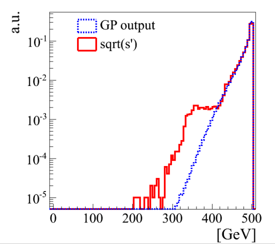

[Moritz Habermehl]

- `iterations = 2` appears to be safe
- histograms must be smoothed
  - ∵ limited statistics from CAIN or Guinea-Pig
  - ∴ monitor smoothing to avoid oversmoothing

- basic example of CIRCE2 input

```
{ file = "TDR.circe"     # name of the output file
  { design = "ILC"       # there can be more than one design per file
     roots = 500         #                                  energy
     scale = 250         # map [0,250] → [0,1]
     bins = 100          # use 100 bins in each direction
    { pid/1 = electron   # first and second particle
      pid/2 = positron
      pol = 0            # both particles unpolarized
      events = "guinea_pig/out/ILC_500_unpolarized.data"
      columns = 2        # read only the first two columns
      lumi = 8.008e33
      min = 0
      max = 1.05         # allow 5% energy spread at the upper end
} } }
```

will generate a fixed width histogram with weights according to Guinea-Pig output:

```
  $ head guinea_pig/out/ILC_500_unpolarized.data
249.435 250.16 405.499 -0.67215 32.2081 193 2.31349e-05 ...
249.791 250.109 -406.506 5.4995 61.3885 267 7.91127e-06 ...
...
```

- more sophisticated CIRCE2 input

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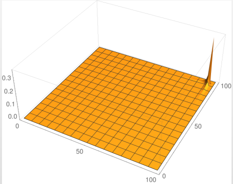
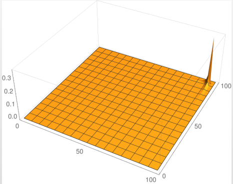
      columns = 2
      lumi = 8.008e33
      min = 0
      max = 1.05
      iterations = 10
} } }
```

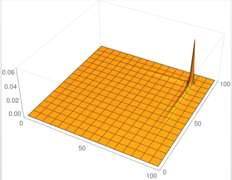
will generate a variable width histogram with weights according to Guinea-Pig output performing 10 iterations of adapting the bin widths to minimize the variance of the weights

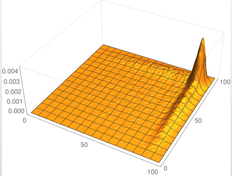
- `iterations` = 0, 1, 2, 3, 4, 5, 6, 7, 8:



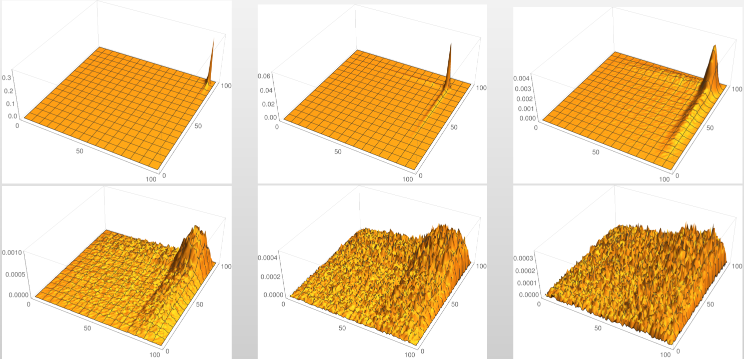
- `iterations` $= 0, 1, 2, 3, 4, 5, 6, 7, 8$:



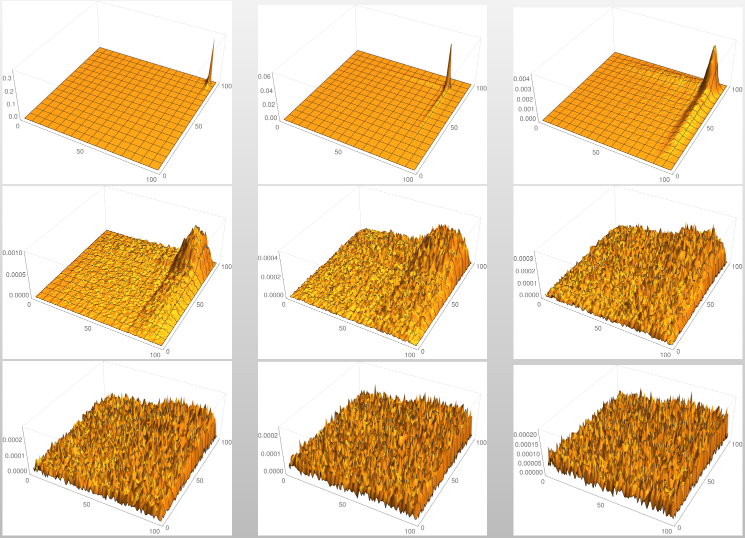




- `iterations` $= 0, 1, 2, 3, 4, 5, 6, 7, 8$:



- `iterations` = 0, 1, 2, 3, 4, 5, 6, 7, 8:



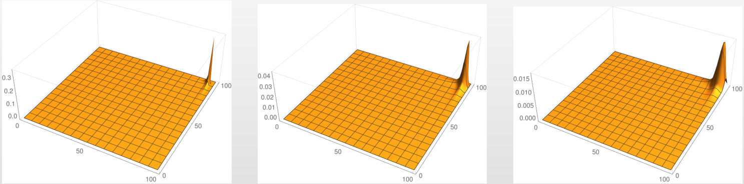
(171.306 Guinea-Pig events in 10.000 bins)

- more sophisticated CIRCE2 input

```
{ file = "TDR.circe"
  { design = "ILC"
     roots = 500
     scale = 250
     bins = 100
    { pid/1 = electron
      pid/2 = positron
      pol = 0
      events = "guinea_pig/out/ILC_500_unpolarized.data"
      columns = 2
      lumi = 8.008e33
      min = 0
      max = 1.05
      iterations = 4
      smooth = 5 [0.00,1.05] [0.00,1.05]
} } }
```

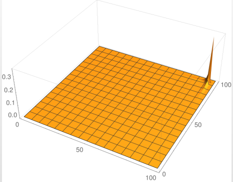
applies a Gaussian smearing

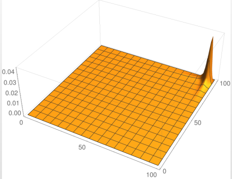
- iterations $= 0$ and smooth $= 0, 3, 5$:

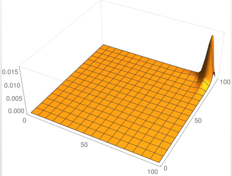


CIRCE2: Smoothing Grids

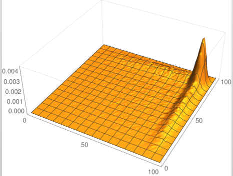
- `iterations` $= 0$ and `smooth` $= 0, 3, 5$:

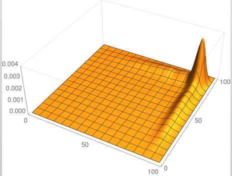


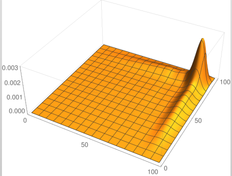




- `iterations` $= 2$ and `smooth` $= 0, 3, 5$:

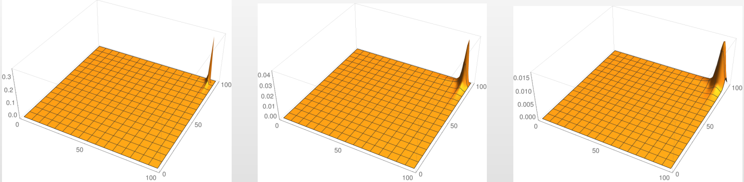




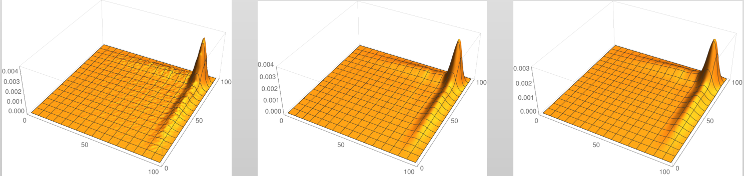


CIRCE2: Smoothing Grids

- $\texttt{iterations} = 0$ and $\texttt{smooth} = 0, 3, 5$:



- $\texttt{iterations} = 2$ and $\texttt{smooth} = 0, 3, 5$:



- $\texttt{iterations} = 4$ and $\texttt{smooth} = 0, 3, 5$: