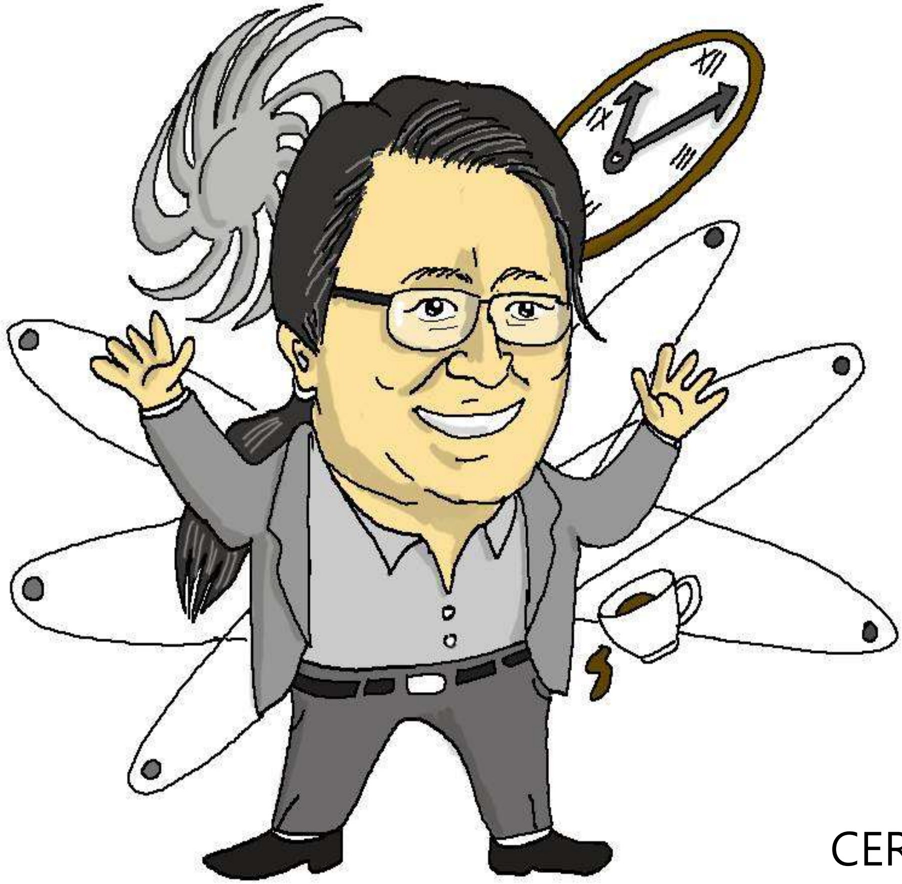

# Introduction to Cosmology

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CERN-Korea Summer Student Program, 2023.06.27

# Plan of Lectures

- Contents
  1. General Relativity
  2. Observed Universe
  3. Expanding Universe
  4. Thermal History
  5. Inhomogeneity
  6. Inflation

- References
  - Scott Dodelson, <Modern Cosmology>, Academic Press (2003)
  - Steven Weinberg, <Cosmology>, Oxford (2008)
  - 김항배, <우주, 시공간과 물질>, 컬처룩 (2017)

# Units and Conversion Factors

- Basic units
  - **Natural units / Planck units** $\hbar = c = k_{\mathrm{B}} = 1$ / $\hbar = c = k_{\mathrm{B}} = G = 1$
  - Reduced Planck mass $M_{\mathrm{P}} = (8\pi G)^{-1/2} = 2.4 \times 10^{18}\,\mathrm{GeV}$
  - Mass unit – Solar mass $M_{\odot} = 2 \times 10^{30}\,\mathrm{kg}$
  - Length unit – parsec (pc) $1\,\mathrm{pc} = 3.26\,\text{light-year} = 3.1 \times 10^{16}\,\mathrm{m}$

- Conversion factors in Natural units
  - Energy-Mass $1\,\mathrm{eV}^{-1}/c^2 = 1.78 \times 10^{-36}\,\mathrm{kg}$
  - Energy-Time $1\,\mathrm{eV}^{-1}\hbar = 6.58 \times 10^{-16}\,\mathrm{s}$
  - Energy-Length $1\,\mathrm{eV}^{-1}\hbar c = 1.97 \times 10^{-7}\,\mathrm{m}$
  - Energy-Temperature $1\,\mathrm{eV}/\mathrm{k_B} = 1.16 \times 10^{4}\,\mathrm{K}$

# General Relativity

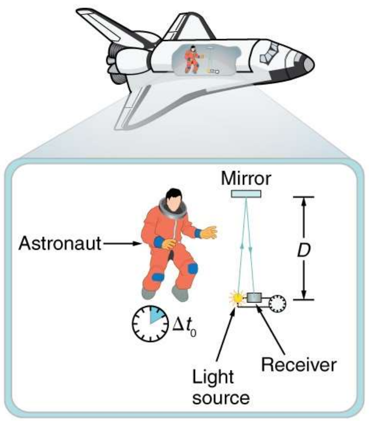
# Relativity

- Relativity – Symmetry of Spacetime ↔ Freedom of choice of the coordinate systems
  - Newton introduced the **inertial frame (absolute spacetime)** to describe the motion of particles.
  - **All inertial frames are equivalent.** – Galilean Relativity and Special Relativity (SR)
    - Gal.R and SR are distinguished by how the coordinates are transformed between two inertial frames.
    - Galilean transformation and Lorentz transformation
  - **All freely falling frames are equivalent.** – General Relativity (GR)
    - Inertial frames (SR) are local.
    - Price to pay for choosing general (non-freely-falling) frames – (Apparent) **Gravity appears**.

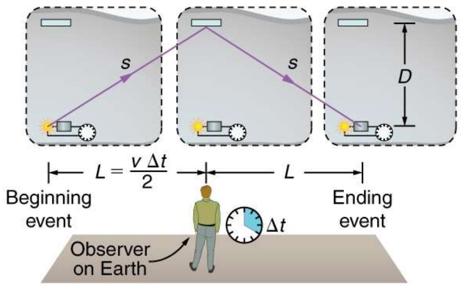
- Symmetry of Spacetime – Time and Distance between two events
  - Galilean Relativity (Absolute Spacetime) – same to all observer, independent of motion
  - SR – dependent on the motion of observers relative to the two events
  - GR – dependent on the position and the time, as well as on the relative motion
  - Spacetime intervals are same to all observers. – General covariance !

# Special Relativity

- Special Relativity
  - Conflict between Newtonian Mechanics and Maxwellian Electromagnetism
    – The **speed of light** is same to all observer. → Invariance of spacetime interval $ds^2 = -c^2dt^2 + d\vec{r}^2$
  - Time and distance between two events depend on observers' relative motion.
    → **Time dilation** and **Length contraction** → **Lorentz transformation**

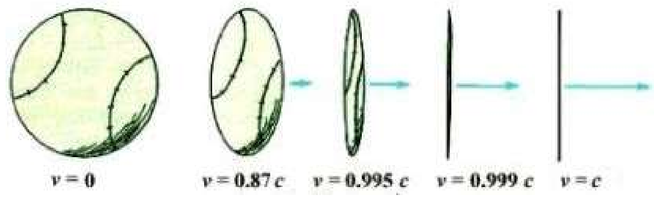



$$T = \frac{T_0}{\sqrt{1-(v/c)^2}}$$




- Proper time – measured at clock's rest frame
- Proper length – measured at object's rest frame

$$L = L_0\sqrt{1-(v/c)^2}$$




- Conversion between Matter (Mass) and Energy $E = \dfrac{mc^2}{\sqrt{1-(v/c)^2}}$

# Coordinate change – Vector and Tensor

- Classification of physical quantities by the properties under coordinate transformations
  - Symmetry of spacetime → Group theory – **Group** and its **Representations**
  - SR → Lorentz Group SO(1,3) → Representations : Scalar, Vector, Tensor, Spinor
- Spatial Rotation, SO(3) : $x_B^i = O^i_j\, x_A^j \quad \text{where} \quad \delta_{ij} O^i_k O^j_l = \delta_{kl}$
  - Vector and Tensor : $V_B^i = O^i_j\, V_A^j, \quad T_B^{ij\cdots} = O^i_k O^j_l \cdots T_A^{kl\cdots}$
  - Scalar : time, mass, …, scalar product $\vec{V}\cdot\vec{U} = \delta_{ij} V^i U^j$

$$x^i,\ v^i \equiv \frac{dx^i}{dt},\ a^i \equiv \frac{dv^i}{dt}$$

$$\partial_i \equiv \frac{\partial}{\partial x^i},\quad \partial_{Bi} = (O^{-1})^j_i\, \partial_{Aj}$$

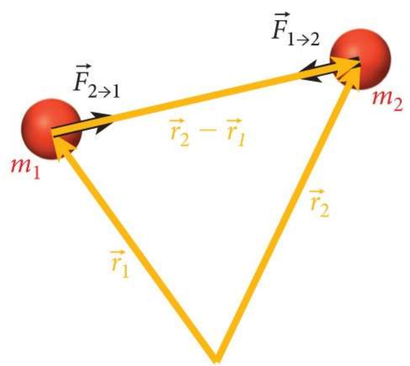
- Special Relativity – Lorentz transformation, SO(1,3) : $x_B^\mu = \Lambda^\mu_\nu\, x_A^\nu \quad \text{where} \quad \eta_{\mu\nu}\Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma}$
  - Contravariant vector and Covariant vector $V_B^\mu = \Lambda^\mu_\nu\, V_A^\nu, \quad V_{B\mu} = (\Lambda^{-1})^\nu_\mu\, V_{A\nu}$ $\qquad x^\mu,\ \partial_\mu$
- General coordinate transformation $x_B^\mu = x_B^\mu(x_A^\nu) \qquad dx_B^\mu = \frac{\partial x_B^\mu}{\partial x_A^\nu}\, dx_A^\nu, \quad \partial_{B\mu} = \frac{\partial x_A^\nu}{\partial x_B^\mu}\, \partial_{A\nu}$
- Covariance of Physical Laws → manifestly expressed by Tensor Equations

# Newton's Gravity and Special Relativity

- Newtonian Gravity (NG)
  - Force acting between two masses
  - Gravitational field and potential

$$\vec{F}(\vec{r}) = -G\frac{m_1 m_2}{r^2}\,\hat{r}$$

$$\nabla^2\phi(\vec{r}) = 4\pi G\,\rho(\vec{r})$$




$$\vec{F}(\vec{r}) = m_1\,\vec{g}(\vec{r}), \quad \vec{g}(\vec{r}) = -G\,\frac{m_2}{r^2}\,\hat{r}$$

$$\vec{g}(\vec{r}) = -\vec{\nabla}\phi(\vec{r}), \quad \phi(\vec{r}) = -G\,\frac{m_2}{r}$$

- Newtonian Gravity and Special Relativity (SR)
  - NG does not fit into SR. – Poisson equation is not covariant under Lorentz transformations (LT).
    - ✓ LHS – Laplacian, it does not involve time derivative. NG acts with infinite propagation speed.
    - ✓ LHS – Gravitational potential, is it scalar, vector, or tensor under LT?
    - ✓ RHS – Mass density, probably 00 component of 2nd rank tensor (energy-momentum tensor)
  - What is the gravity theory consistent with SR? – Complete tensor equation !

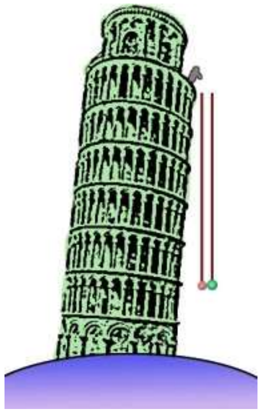
# Equivalence Principle

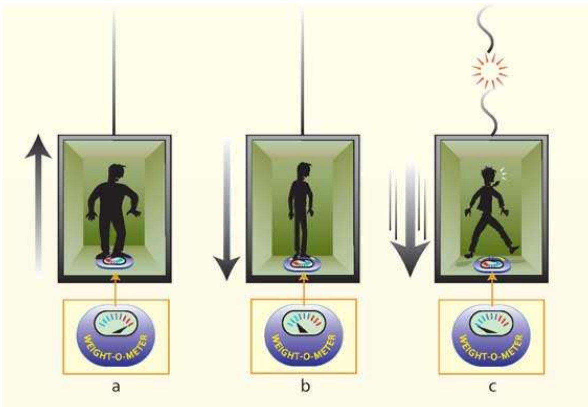
- Properties of the gravitational force
- Weak Equivalence Principle: **Inertial Mass = Gravitational Mass**
  $\Rightarrow$ Motion due to gravity is independent of mass.
  - **Gravitational Force = Inertial Force ?**
    - ✓ Gravity is absent in freely falling frames, which can be regarded as inertial frames, in which SR and known laws of physics holds.
    - ✓ **Local inertial frame** – Inertial frames hold only **locally** !!

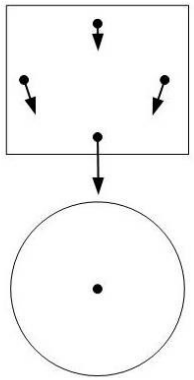
$$m \frac{d^2\vec{r}}{dt^2} = m\,\vec{g}(\vec{r})$$

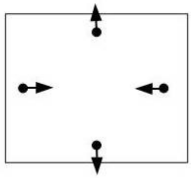
| Inertial Mass | Gravitational Mass |
|---|---|
| Law of Motion | Law of Gravitation |

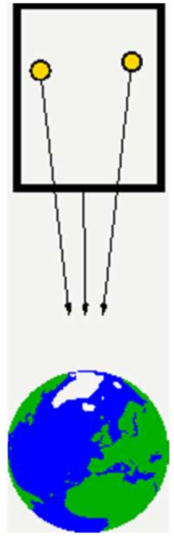
- **Einstein's Equivalence Principle – All local inertial frames are equivalent.**





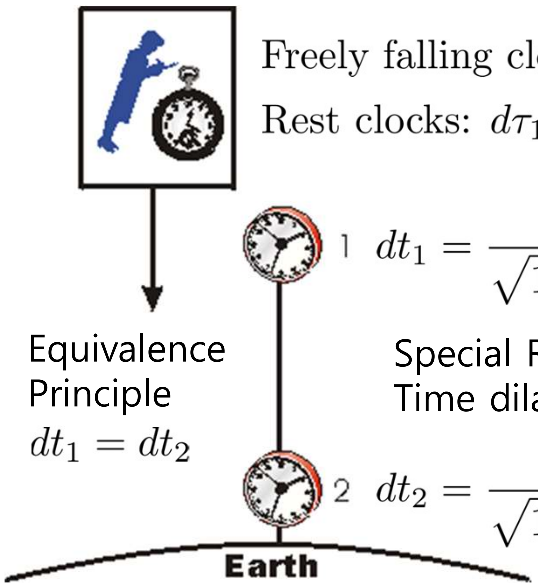







# Gravity as Spacetime geometry

- Consequences of Einstein's equivalence principle
- Gravity changes time and length !
  - Demonstration – Comparison of elapsed times of a clock in the freely falling elevator and two clocks at rest in the gravitational field




Freely falling clock: $dt_1$, $dt_2$

Rest clocks: $d\tau_1$, $d\tau_2$

1 $$dt_1 = \frac{d\tau_1}{\sqrt{1 - v_1^2/c^2}}$$

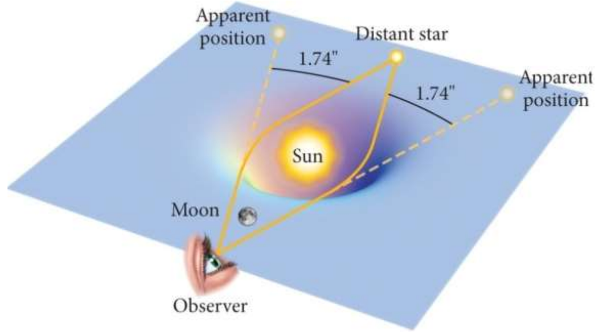
Equivalence Principle
$dt_1 = dt_2$

Special Relativity
Time dilation

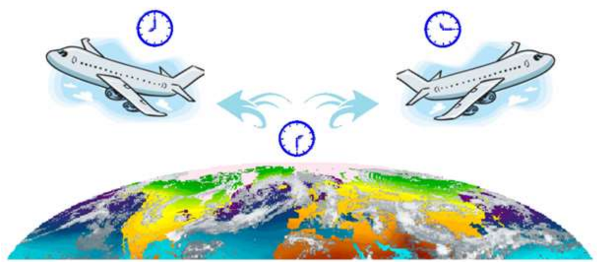
2 $$dt_2 = \frac{d\tau_2}{\sqrt{1 - v_2^2/c^2}}$$

$$\frac{d\tau_1}{d\tau_2} = \sqrt{\frac{1 - v_1^2/c^2}{1 - v_2^2/c^2}} \approx 1 + \frac{v_2^2 - v_1^2}{2c^2} = 1 + \frac{\phi_1 - \phi_2}{c^2}$$

Clock runs faster at higher gravitational potential.




Hafele and Keating Experiment



# Metric, Connection, and Curvature

- Coordinates ( $x^\mu$ ) – label the position in space.
- Metric ( $g_{\mu\nu}$ ) – defines the distance.

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) = (d\theta, d\phi)\begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}\begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = g_{\mu\nu}\, dx^\mu dx^\nu$$

Metric is a second-rank tensor.

- Connection ( $\Gamma^\mu{}_{\nu\lambda}$ ) – defines the parallel transport of vectors.
  - It determines how to differentiate the vector field in the curved space.
  - Once the metric is specified, the connection is completely determined.

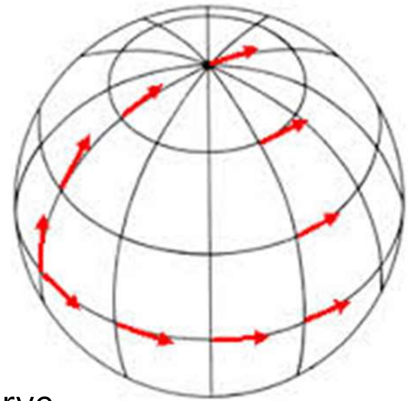
$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda \qquad \Gamma^\mu{}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}\left[\partial_\nu g_{\lambda\rho} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}\right]$$

Levi-Civita Connection

- Curvature ( $R^\rho{}_{\sigma\mu\nu}$ ) – measures how much the space is curved.
  - It is defined by the change of vectors when parallel-transported along a closed curve.

$$[\nabla_\mu, \nabla_\nu]\, V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma \qquad R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

- There are infinite number of different ways of setting the coordinate system on a space.
- Components of Metric, Connection, and Curvature depends on the coordinate system. Metric and Curvature are tensor, while Connection is not.
- The geometry of a space is independent of the choice of a coordinate system.



In curved space, the vector parallel-transported along a closed curve differs from the original vector.

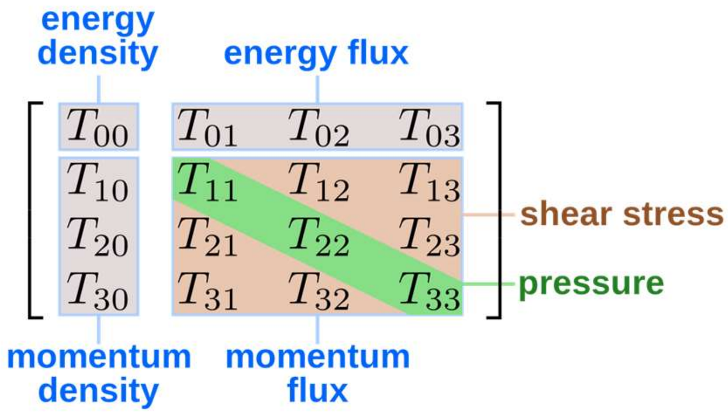
# Einstein equation

- Einstein tensor (spacetime curvature) $\boldsymbol{G_{\mu\nu} = 8\pi G\, T_{\mu\nu}}$ Stress-energy tensor (matter distribution)

Ricci tensor and scalar $\quad R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}\,,\ \ R = R^{\mu}{}_{\mu}$

Einstein Tensor $\quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$

conserved symmetric 2nd rank tensor $\quad \nabla^{\mu} G_{\mu\nu} = 0 \ \Leftrightarrow\ \nabla^{\mu} T_{\mu\nu} = 0$




- Newton's Gravity and Einstein's Gravity (General Relativity)

| Newton's Gravity | Einstein's Gravity |
|---|---|
| Mass → Gravitational Force | Matter → Spacetime Geometry |
| $\vec{F} = \frac{G m_1 m_2}{r^2}\,\hat{r}$ | $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ |

$$\nabla^2 \phi(\vec{r}) = 4\pi G\, \rho(\vec{r})$$

- Metric ↔ Gravitational potential
- Connection ↔ Gravitational field
- Curvature ↔ Tidal force

# General Relativity and Cosmology

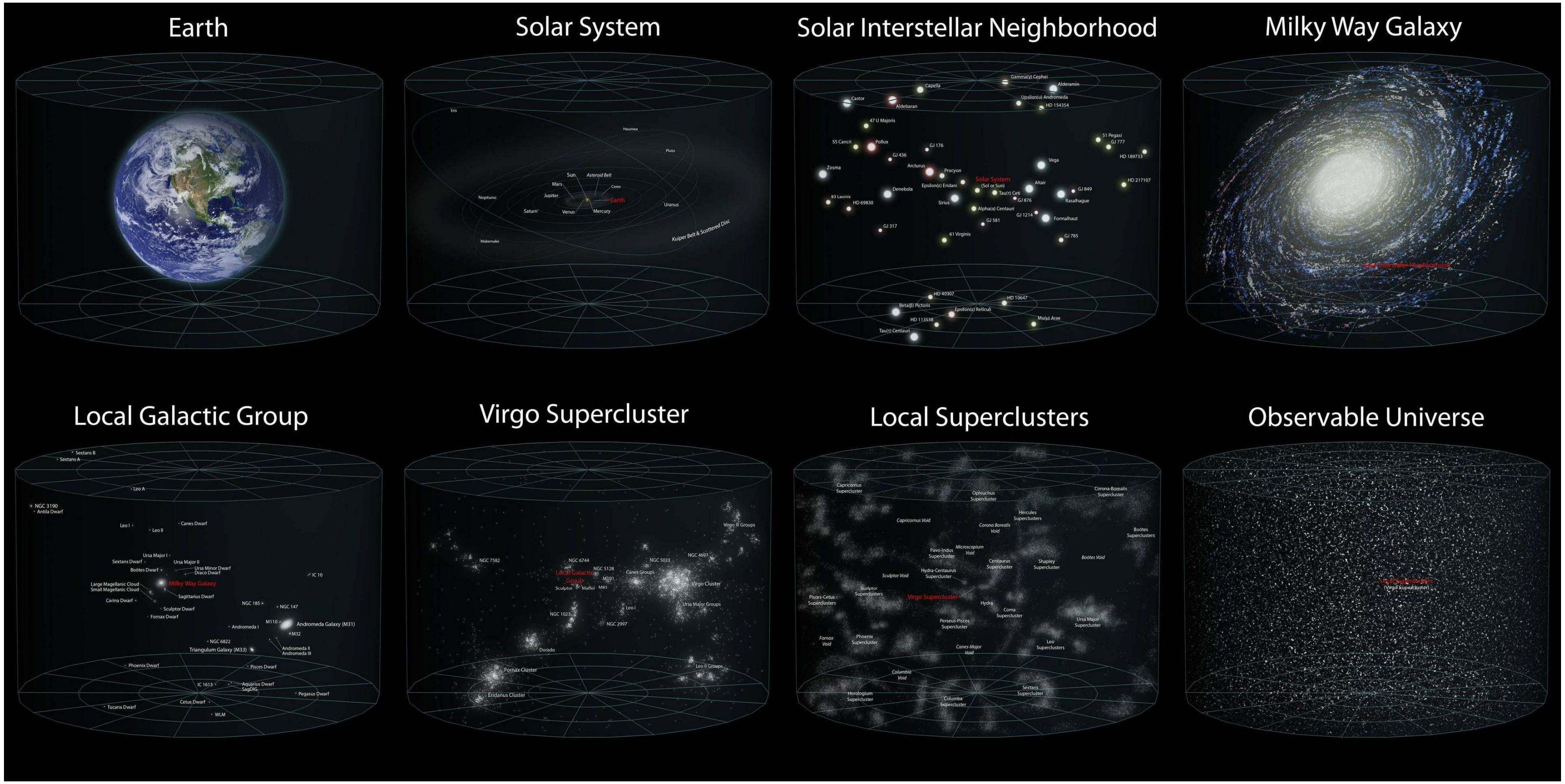
- Most of cosmology can be learned just with basic knowledge of general relativity.
  - Metric $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$
  - Levi-Civita Connection $\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left( \partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right)$
  - Curvature $R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$
  - Ricci tensor & scalar, Einstein tensor $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \quad R = R^\mu{}_\mu, \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$
  - Einstein equation can be derived from the Einstein-Hilbert action.

$$S_{\rm EH} = \int d^4x \sqrt{-g} \left( \frac{M_P^2}{2} R + \mathcal{L}_{\rm M} \right) \quad \Rightarrow \quad G_{\mu\nu} = M_P^{-2}\, T_{\mu\nu}$$

  - Geodesic equation – the path of a freely falling particle $\dfrac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \dfrac{dx^\rho}{d\tau} \dfrac{dx^\sigma}{d\tau} = 0$
  - Symmetry of manifold : Isometry ⇔ Killing vector, Maximally symmetric space

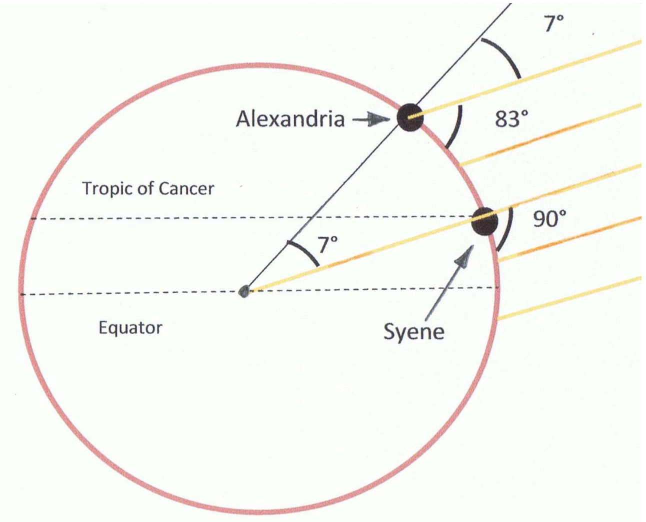
# Observed Universe

# How big is the Universe?



# Size of Earth

- Ancient Greek, including Aristotle, thought that Earth is a sphere.
- In 240 BC, Eratosthenes first estimated the circumference of Earth.

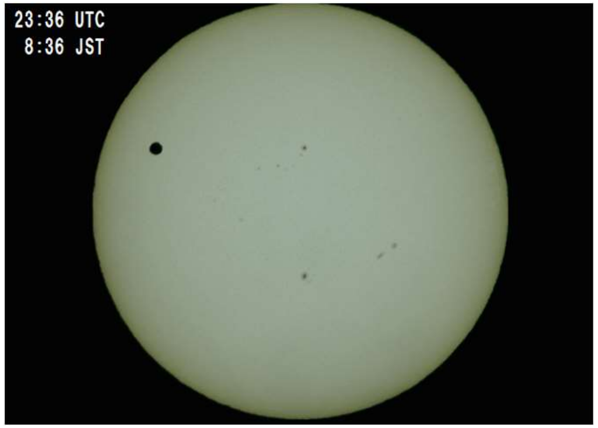



- In Siena, Egypt at noon of summer solstice Sun is right above the head.
- At the same time, in Alexandria which is 5,000 stades distant from Siena, declination angle of shadow is 1/50 of circle.
- So, the circumference of Earth is 5,000 x 50 = 250,000 stades.

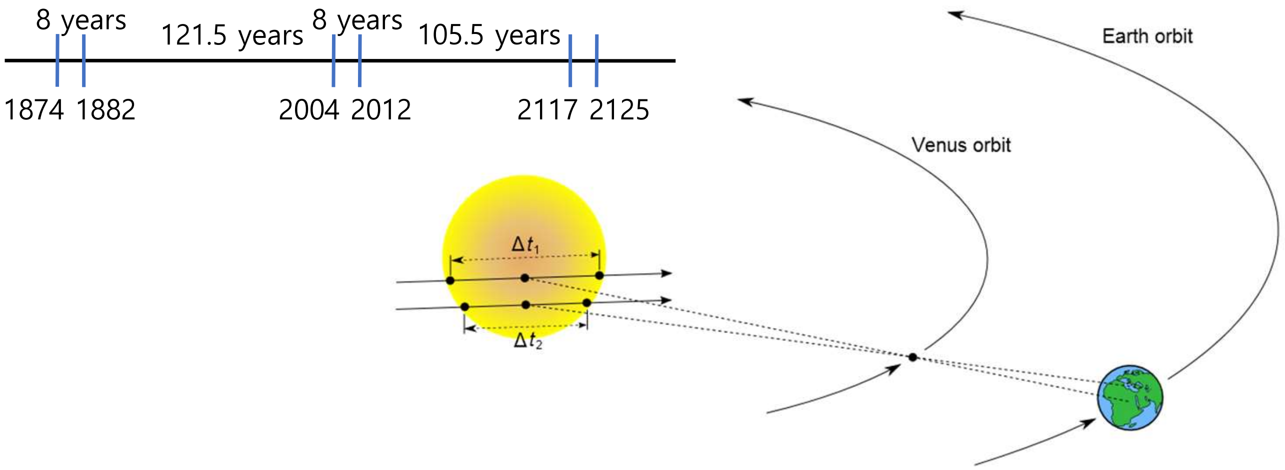
Eratosthenes of Cyrene (276–195 BC)
Greek mathematician, geographer, astronomer,
first used the word geography,
devised latitude and longitude,
estimated the size of Earth.

# Size of the Solar System

- Transit of Venus can be used to measure the distance to Venus, and thus to Sun.
  - In 1627, Johannes Kepler predicted the transit of Venus in 1631, but failed in observation because it cannot be seen in Europe.
  - In 1639, Jeremiah Horrocks succeeded in observation and estimated the distance to Sun to be 95M km. (It's very difficult to measure the angle between two paths, whose size is a few seconds.)
  - In 1716, Edmond Halley suggested measuring the time difference for Venus to transit along two paths. In 1761 and 1769 observations are made and in 1771, Jérôme Lalande reported the distance to be 153M km.

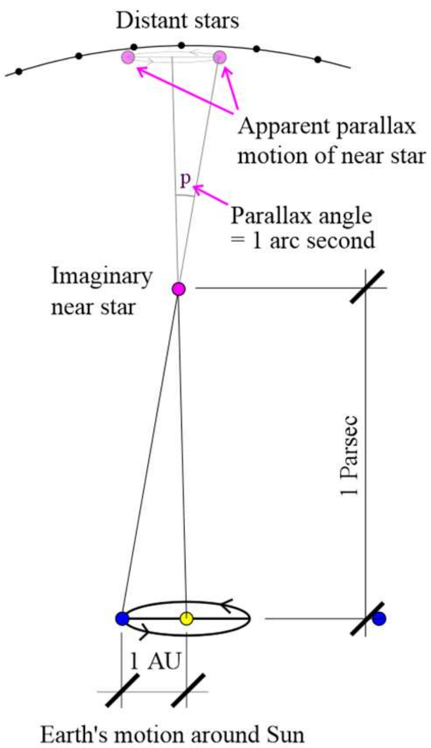



2012 transit of venus (June 5, 2012)

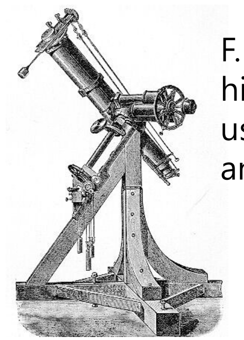


# Distance to Stars

- Annual parallax was seriously searched as a crucial evidence for heliocentrism.




- In 1838, F. Bessel measured successfully the annual parallax of Cygni 61 to be 0.314″. (The current value is 0.286″.)

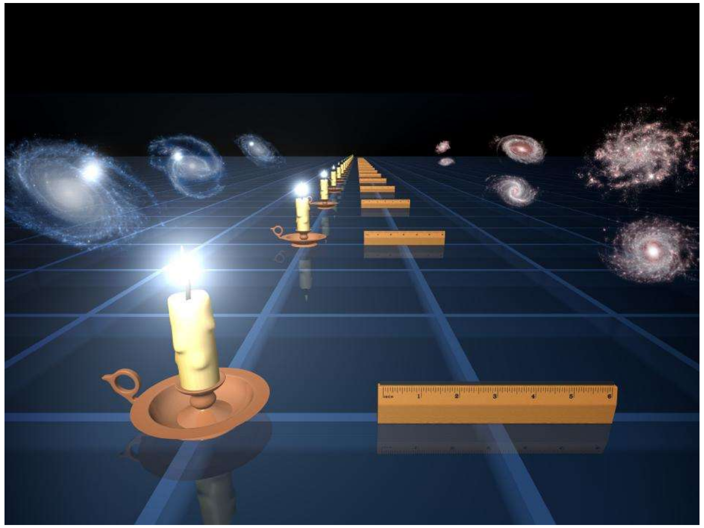
F. Bessel and his heliometer used to measure annual parallax

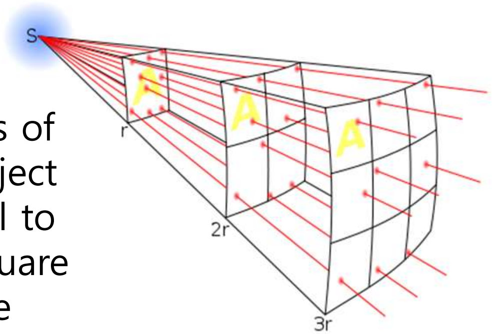
- Annual parallax of 1″ defines 1 parsec.

$$\mathbf{1\,pc = 3.26\,ly = 3.1 \times 10^{16}\,m}$$

  - Angular diameter of the sun = 32′
  - Annual parallax of the nearest star, Proxima Centauri, 0.769″

# Standard Candles and Rulers






The brightness of the distant object is proportional to the inverse square of the distance

- How to measure the distance to farther objects?
- **Luminosity Distance**
  - Brightness of the astronomical objects

$$F = \frac{L}{4\pi r^2} \quad \Rightarrow \quad d_L^2 \equiv \frac{L}{4\pi F}$$

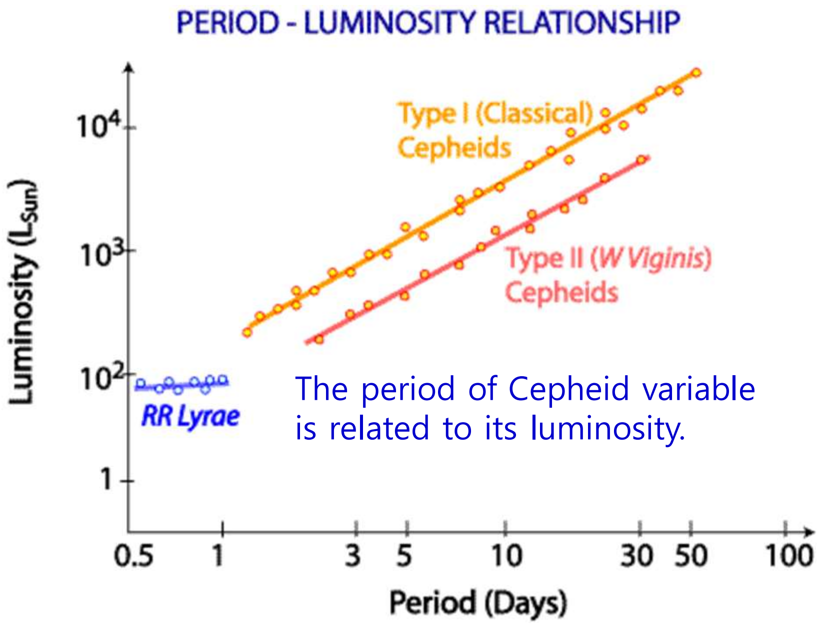
  - objects with known luminosity.
  - brighter objects for larger distance
- **Angular Diameter Distance**
  - Angular diameter of the object

$$\theta = \frac{l}{r} \quad \Rightarrow \quad d_A = \frac{l}{\theta}$$

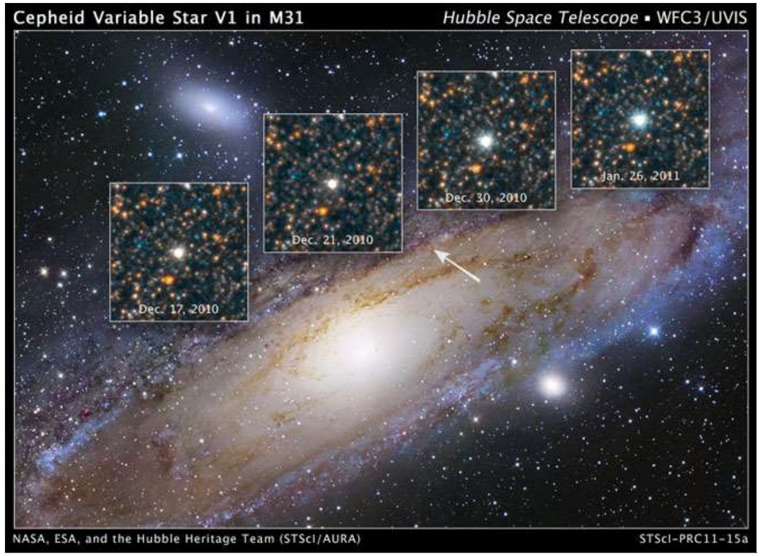
  - objects with known size.
  - larger objects for larger distance

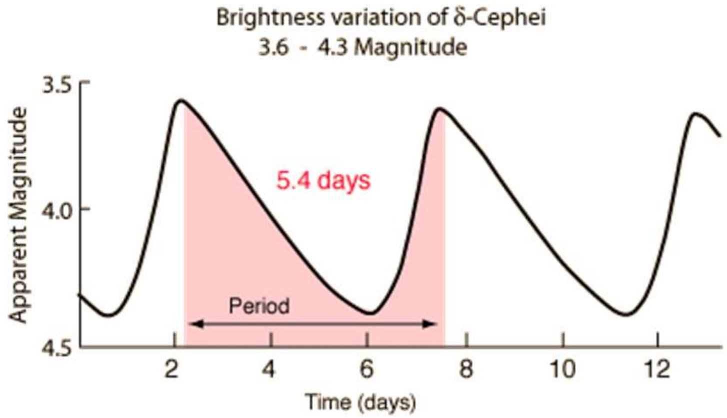
# Cepheid Variables

- John Goodricke, 1784
  firstly discovered the δ-Cepheid variable
- **Henrietta Leavitt, 1908**
  discovered the **period-luminosity relation**




The period of Cepheid variable is related to its luminosity.



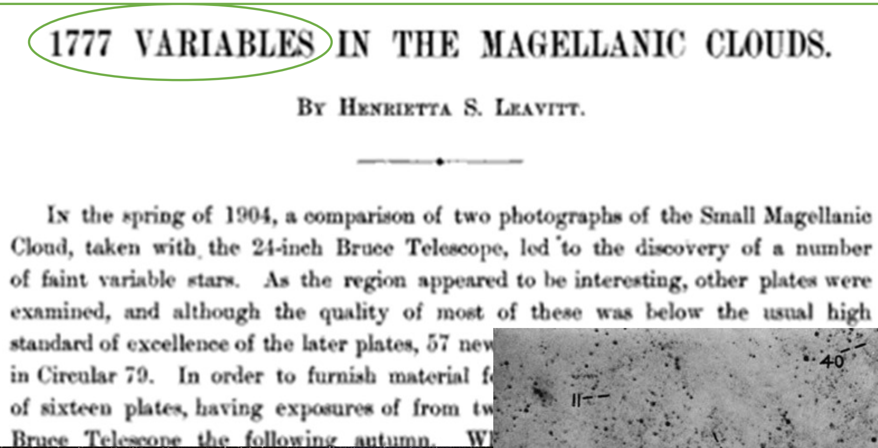



# Henrietta Leavitt


Henrietta Swan Leavitt

Pickering's Harem

In 1908-1912, **Henrietta Leavitt** studied the variables in Magellanic Clouds and found that **Cepheid variables** can be **standard candles**.

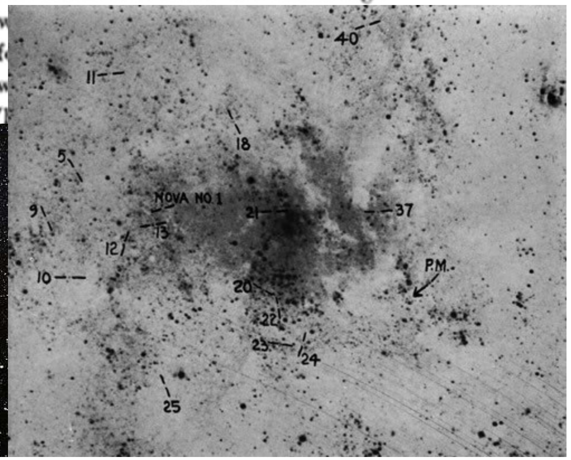


1777 VARIABLES IN THE MAGELLANIC CLOUDS.

BY HENRIETTA S. LEAVITT.

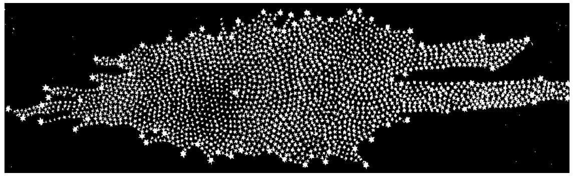
IN the spring of 1904, a comparison of two photographs of the Small Magellanic Cloud, taken with the 24-inch Bruce Telescope, led to the discovery of a number of faint variable stars. As the region appeared to be interesting, other plates were examined, and although the quality of most of these was below the usual high standard of excellence of the later plates, 57 new
in Circular 79. In order to furnish material f
of sixteen plates, having exposures of from tw
Bruce Telescope the following autumn. W

Large and Small
Magellanic Clouds

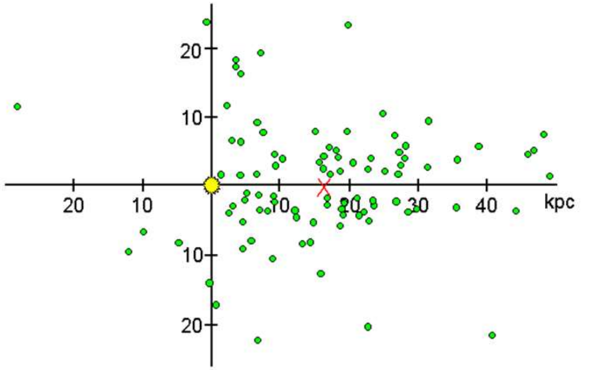


# Size of our Galaxy

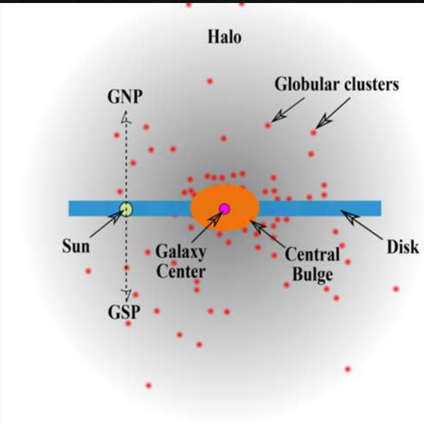
- Milky Way
  - Galileo – using telescope, confirmed that Milky Way is an aggregation of faint stars.
  - Kant – identified the disk shape of star distribution in Milky Way



William Herschel's drawing of star distribution (1785). He assumed that all stars have the same luminosity, so that their distance can be known from their brightness. The sun is located at the center and the star distribution has the disk shape.




Harlow Shapley, using Cepheid variables, obtained the distribution of globular clusters in our galaxy (1915).
The yellow circle is the position of the sun and the X mark is the center of distribution.

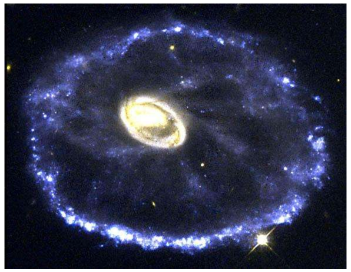



Current view of our galaxy. Stars are mainly distributed in the disk and the sun is 8.5 kpc distant from the galactic center.

# Discovery of Outer Galaxies

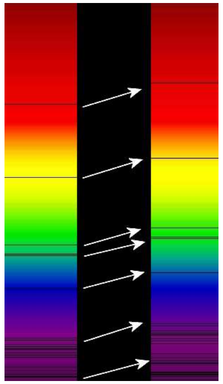
- Shapley-Curtis debate (1920s) – Identity of spiral nebulae
  Aggregations of stars inside our galaxy versus Galaxies outside our galaxy
- Edwin Hubble measured the distance to Andromeda nebula (1925) by finding Cepheid variables, which is much larger than the size of our galaxy, proving the existence of outer galaxies.

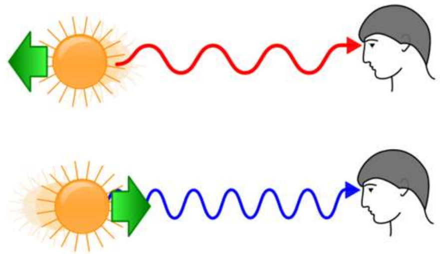
Andromeda Galaxy (M31), a big spiral galaxy,
nearest (70 Mpc) to our galaxy.



# Discovery of Expanding Universe

- **Red Shift**
  - Vesto Slipher discovered the red shift of nebulae (1912)
    Absorption spectra from distant galaxies are red shifted.
  - Interpretation (Doppler shift)
    Distant galaxies are receding from us.





- **Hubble's Law**
  - Edwin Hubble discovered (1929)
    that Red shifts are
    proportional to Distances.
  - Interpretation
    Space of the universe
    is expanding
    and there is no center
    in the expansion.

$$\vec{v} = H_0\,\vec{r}$$

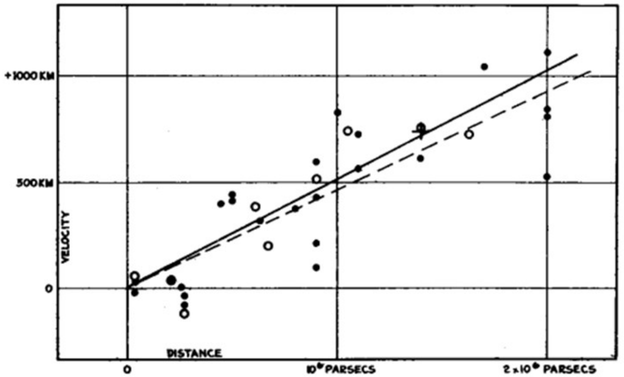
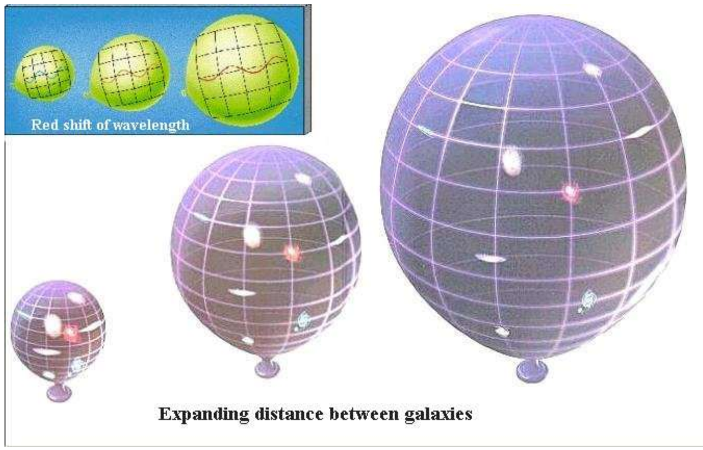



FIGURE 1
Velocity-Distance Relation among Extra-Galactic Nebulae.

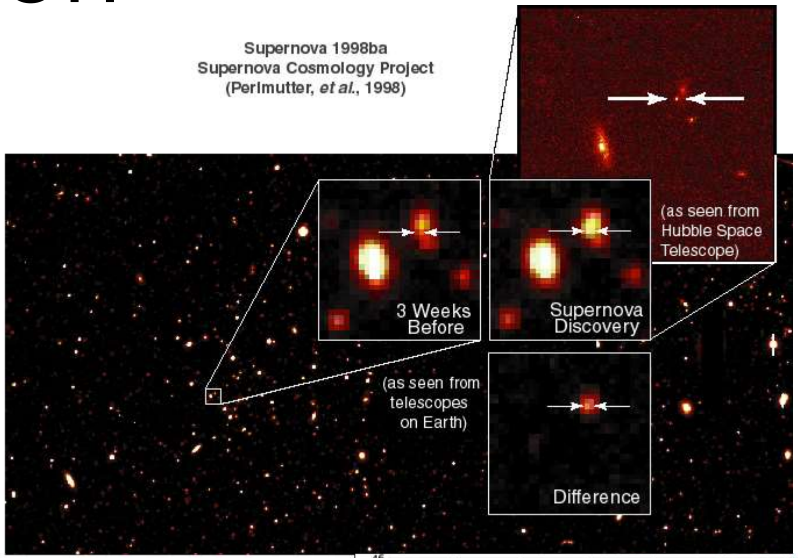


# Observation 1 – Expansion

- Luminosity distance – Red shift relation
  - Luminosity distance – Standard candles
  - Red shift (velocity) – Spectroscopy

- Cepheid variables
  - **H. Leavitt** (1912) – Period–Luminosity relation of Cepheid variables
  - **E. Hubble** (1929) – Hubble's law, Discovery of **the expansion of the universe**

- Super Novae Type Ia
  - Luminosity – Light curve calibration (1990s)
  - Two teams SCP/HZSST (1998) – Discovery of **the accelerating expansion**
  - **Dark Energy** hypothesis



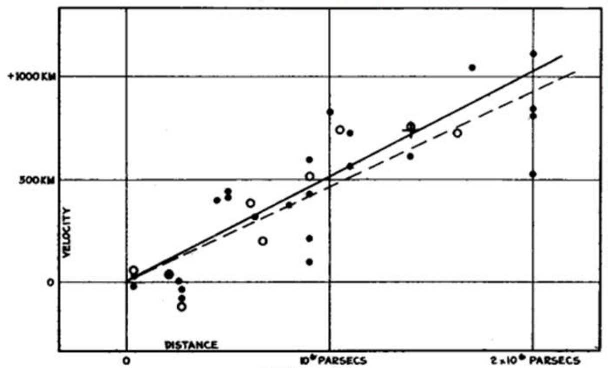
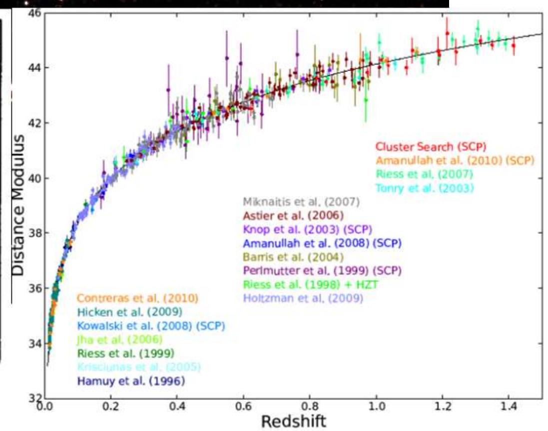
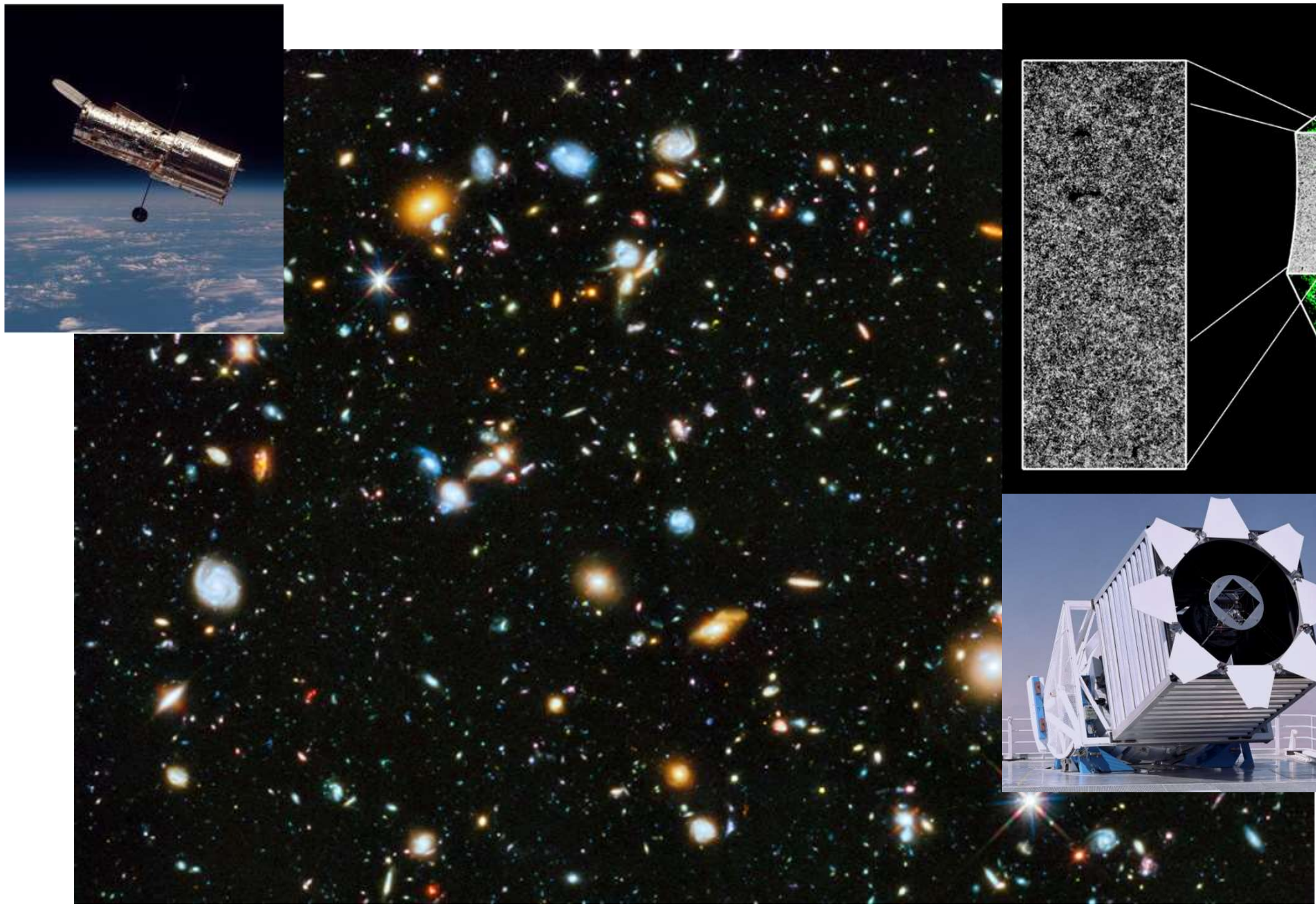
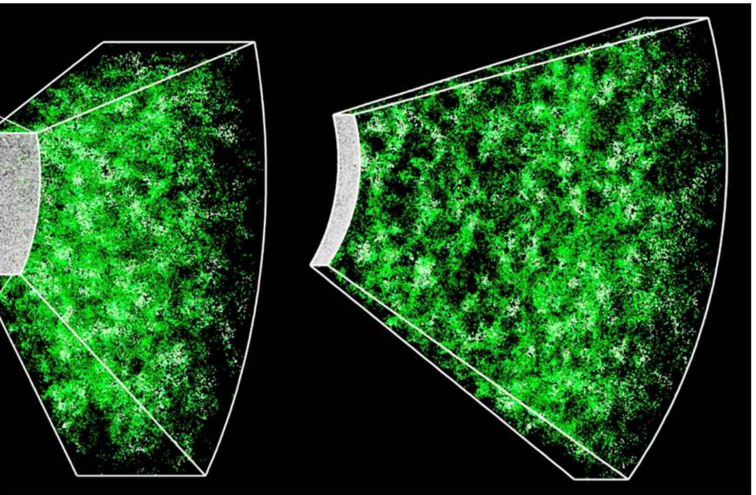



FIGURE 1
Velocity-Distance Relation among Extra-Galactic Nebulae.



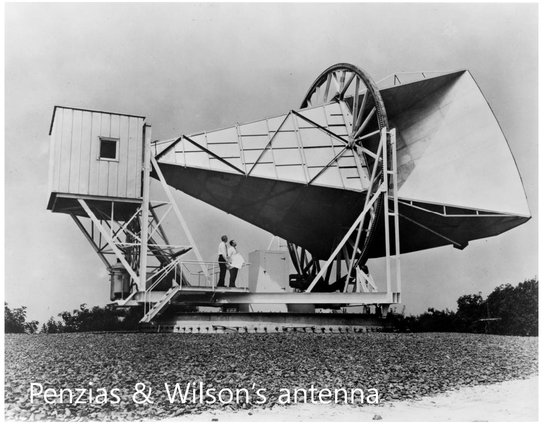
# Observation 2 – Galaxy Distribution

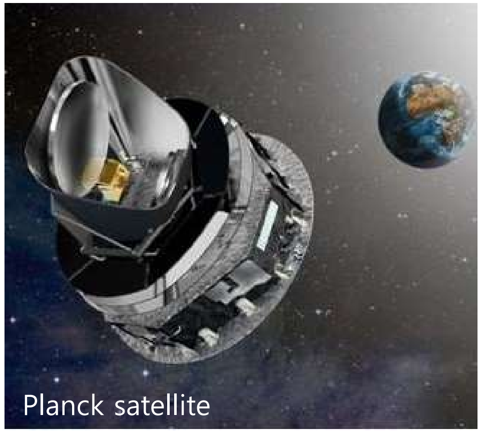


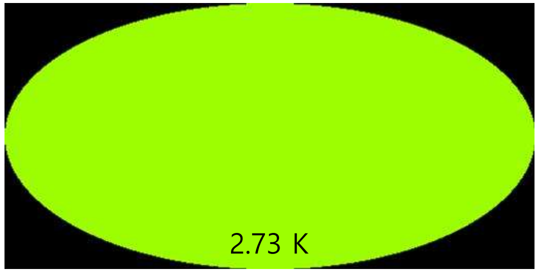


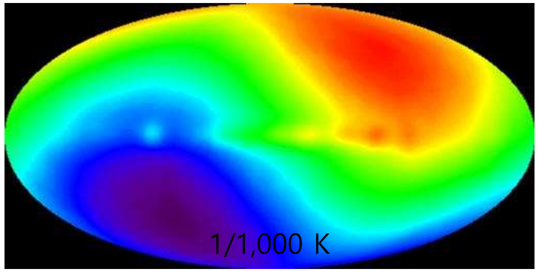
- Galaxy surveys
  – 3D map of the universe
  - Large scales – Homogeneity
  - Small scales – Structures
  - Remnants in structures – Primordial density perturbations, Baryon acoustic oscillations

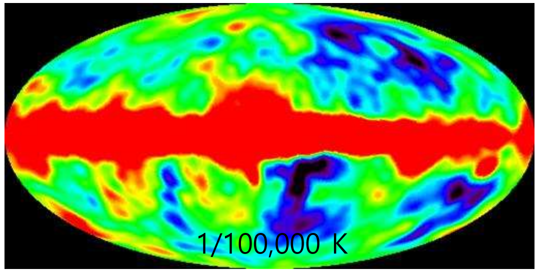
# Observation 3 – CMB Anisotropies

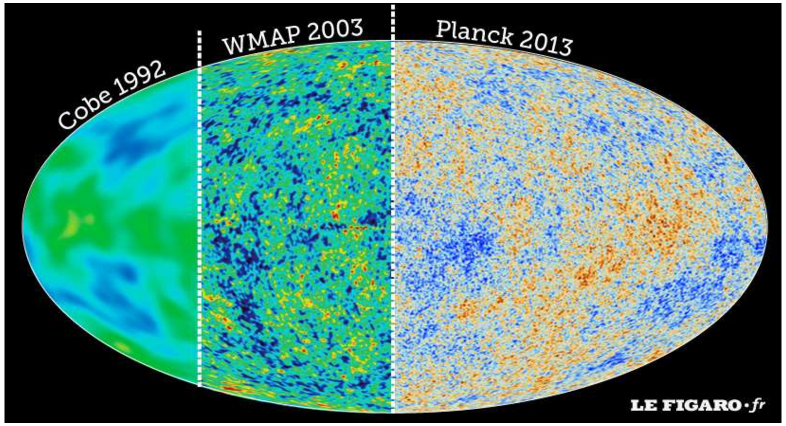


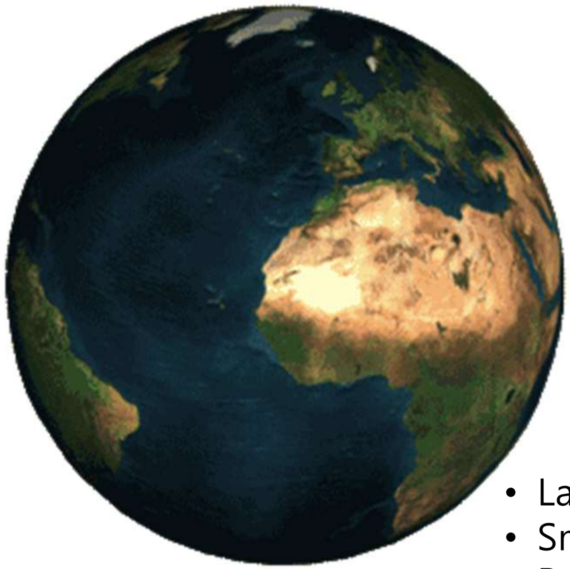

- **Discovery of CMB**
  - Black body spectrum with T=2.7K
- **Discovery of CMB Anisotropies**
  - Temperature perturbations: **$\delta T/T \sim 10^{-5}$**
  - Baryon, Dark Matter, Dark Energy
  - Spatial Geometry: Flat
- **Cosmology became precision science !**

# Shape of the universe

- **Assume a spherical cow in vacuum.**

- **Einstein's universe – Cosmological principle**

# Shape of the universe



- Large scales – Ball / Sphere
- Small scales – Surface structures
- Rotating
- Subject to the sun

- How is its size set?
- Why is it a sphere?
- Why is it rotating?
- How are surface structures formed?

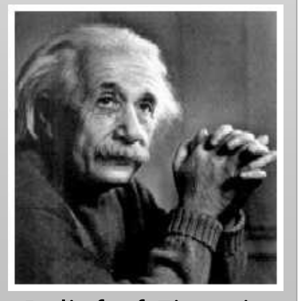
- Large scales – Homogeneous & Isotropic
- Small scales – Cosmic structures
- Expanding

- How large is the universe?
- Why is it homogeneous & isotropic?
- Why is it expanding?
- How are cosmic structures formed?

# Expanding Universe

# FLRW Universe

- **Two important observational facts about the universe**
  - The distribution of matter (galaxies) and radiation (CMB) in the observable universe is homogeneous and isotropic.
  - The universe is expanding now.
- **Cosmological principle**
  - The universe is pretty much the same everywhere.



Belief of Einstein

- **Friedmann-Lemaitre-Robertson-Walker (FLRW) metric**
  - Our local Hubble volume during Hubble time

~ **spacetime with homogeneous and isotropic spatial sections**

$$M = \mathbf{R} \times \Sigma$$

Time

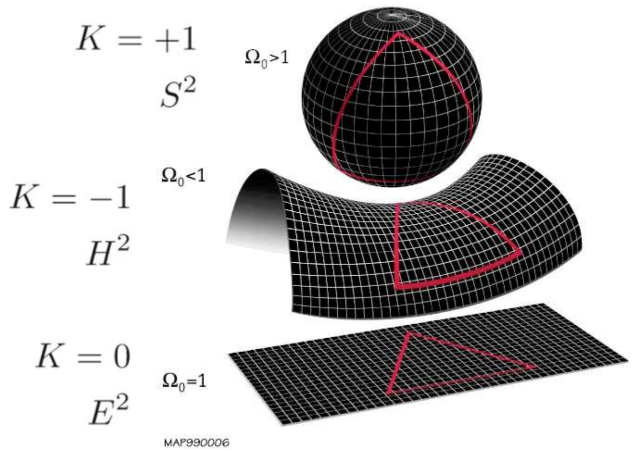
3D Space, maximally symmetric

$$ds^2 = -dt^2 + a(t)^2 \overbrace{\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]}^{\text{Maximally symmetric 3D space}}$$

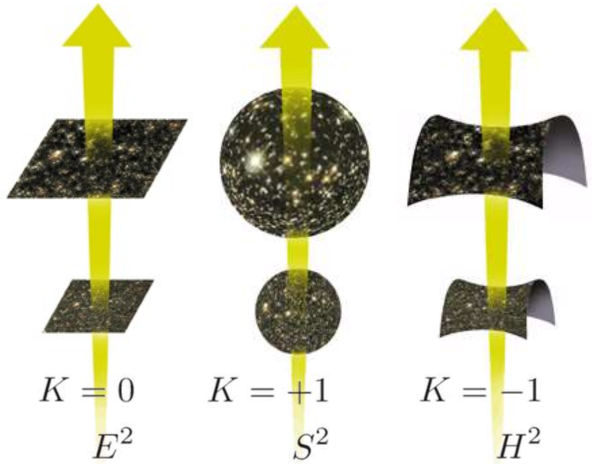
**Cosmic Time**

**Scale Factor**
the only dynamical variable of RW metric

**Comoving Coordinates**
fixed on the expanding space




Examples of maximally symmetric space in 2D




Change of Scale factor

# Kinematics of the expanding universe

- **Features of expanding space**
  - Momentum of a particle is red shifted as the space expands.
  - In the expanding space, measuring distance is a little bit tricky.
    - Comoving distance – (fixed) coordinate distance
    - Physical distance – comoving distance times scale factor
    - Luminosity distance – measured by light intensity
    - Angular diameter distance – measured by angular size
  - Hubble's law – For small red shift, red shift is proportional to distance.
  - Growth of the horizon (Visible universe) is different from the non-expanding universe.

# Red shift

- **Free (free-falling) particle - Geodesic in RW metric** $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$

  - Energy-momentum vector of a particle : $p^\mu = m\frac{dx^\mu}{d\tau} = (E, \vec{p})$ $\qquad dt = \gamma\, d\tau = \frac{E}{m}\, d\tau$
  - 0-comptonent : $E\frac{dE}{dt} = -\Gamma^0_{ij}p^ip^j = -\delta_{ij}a\dot{a}p^ip^j$ $\qquad \frac{1}{|\vec{p}|}\frac{d|\vec{p}|}{dt} + \frac{\dot{a}}{a} = 0 \;\Rightarrow\; |\vec{p}| \propto \frac{1}{a}$

    Momentum is red-shifted as the scale factor increases.
  - For light, $|\vec{p}| = \frac{2\pi}{\lambda} \quad \Rightarrow \quad \boldsymbol{\frac{\lambda_0}{\lambda(t)} = \frac{a_0}{a(t)} \equiv 1 + z(t)}$

    **red shift parameter**
  - Red shift parameter z can be used to parameterize the time, instead of the cosmic time t or the scale factor a(t).
  - Red shift in the light from far distant galaxies is actually not due to Doppler effect, but due to momentum red shift caused by the expansion of space.
  - Red shift in the light from near galaxies is a mixture of Doppler effect and momentum red shift, and we cannot distinguish between them.
  - Due to momentum red shift, the temperature ($\propto$ the average kinetic energy) of hot idea gas (consisting of free particles) cools down as the space expands.

# Hubble's Law

- Luminosity distance :
  Energy conservation requires that
  the flux decreases by distance square.

$$F = \frac{L}{4\pi d^2} \quad \Rightarrow \quad d_L^2 \equiv \frac{L}{4\pi F}$$

- Effect of expansion : $F = \dfrac{L}{4\pi(a_0 r(z))^2} \dfrac{1}{(1+z)^2}$

  - Red shift of light
  - Dilation of arrival time

Comoving distance to the light source

$$\int_0^r \frac{dr'}{\sqrt{1-Kr'^2}} = \int_t^{t_0} \frac{dt'}{a(t')}$$

$$r(z) \equiv f(z),\ \sin f(z),\ \sinh f(z) \quad \text{for } K = 0,\ +1,\ -1, \text{ repectively}$$

$$f(z) = \int_t^{t_0} \frac{dt'}{a(t')} = \int_0^z \frac{dz'}{a_0 H(z')}$$

For small z, $H(z) = H_0 + H_0' z + \cdots$

- Luminosity distance – Red shift relation : $d_L = a_0 r(z)\,(1+z)$

$$\boldsymbol{H_0 d_L = z + \frac{1}{2}(1-q_0)z^2 + \cdots}$$

**Hubble's law**

Deceleration parameter $q_0 = -\dfrac{a_0 \ddot{a}_0}{\dot{a}_0^2}$

# Comoving Horizon

- Total comoving distance light $(ds^2 = 0)$ have traveled since t=0

$$\eta(t) \equiv \int_0^{r_H} \frac{dr}{\sqrt{1-Kr^2}} = \int_0^t \frac{dt'}{a(t')}$$

- No information could have propagated further than this.
  ⇒ The size of the universe we can see at present ⇒ **comoving horizon**

- Physical distance to the horizon $d_H(t) = \int_0^{r_H} \sqrt{g_{rr}}\, dr = a(t)\eta$

- Comparison to non-expanding universe : $\eta_{\mathrm{NE}}(t) = d_{\mathrm{H,NE}}(t) = t$
  For $a(t) \propto t^\alpha \quad (0 < \alpha < 1)$, comoving horizon grows slower and physical horizon grows faster.

$$\eta_{\mathrm{E}}(t) = \int_0^t \frac{dt'}{a(t')} = \frac{t^{1-\alpha}}{1-\alpha} \qquad d_{\mathrm{H,E}}(t) = a(t)\int_0^t \frac{dt'}{a(t')} = \frac{t}{1-\alpha}$$

# Dynamics of the expanding universe

- How is the evolution of the universe determined?
- Einstein equation for FLRW metric

$$G_{\mu\nu} = M_P^{-2}\, T_{\mu\nu}$$

- SE tensor must have this form to be consistent with FLRW metric.

- Geometry of our universe

$$ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - Kr^2} + r^2 d\Omega \right]$$

**Scale Factor**

- Matter distribution

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho(t) & 0 \\ 0 & p(t)\delta_{ij} \end{pmatrix}$$

**Energy density** **Pressure**

- **Friedmann equation**

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3M_P^2} \sum_i \rho_i \qquad \frac{\ddot{a}}{a} = -\frac{1}{6M_P^2} \sum_i (\rho_i + 3p_i)$$

$$H(t) \equiv \frac{\dot{a}}{a}$$ **Hubble parameter** expansion rate

- Positive energy density makes the universe expand or shrink.

- Pressure also gravitates.
- The combination ρ+3p makes the expansion decelerate or accelerate.

# Matter Content

- Evolution of the scale factor is determined by the matter content.

Spacetime Geometry ↔ Matter Distribution
**scale factor change** **species and amount**

- **Species** – **Equation of state** ( relation of energy density(ρ) and pressure(p) )
  - Simplest case

$$p = p(\rho) = w\,\rho$$

Conservation equation

$$\rho = \rho_0\,(a/a_0)^{-3(1+w)}$$

| Name | | Radiation | Matter | Vacuum E |
|---|---|---|---|---|
| Eq. of state | $p = w\rho$ | $1/3$ | $0$ | $-1$ |
| Energy density | $\rho \propto a^{-3(1+w)}$ | $a^{-4}$ | $a^{-3}$ | constant |
| Scale Factor (K=0) | $a \propto t^{2/3(1+w)}$ | $t^{1/2}$ | $t^{2/3}$ | $e^{Ht}$ |

- **Amount** – **Density parameter** ( ratio of energy density to the critical density )

$$\boldsymbol{\Omega_i = \frac{\rho_i}{\rho_c}}$$

- The critical density is determined by the Hubble constant.
- The present value is roughly - 6 protons per 1m$^3$

$$\rho_c = 3M_P^2 H_0^2 = 1.9h^2 \times 10^{-26}\,\mathrm{kg/m^3}$$

# Solving Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3M_P^2}\sum_i \rho_i = \frac{1}{3M_P^2}\sum_i \rho_{i0}\left(\frac{a}{a_0}\right)^{-3(1+w_i)}$$

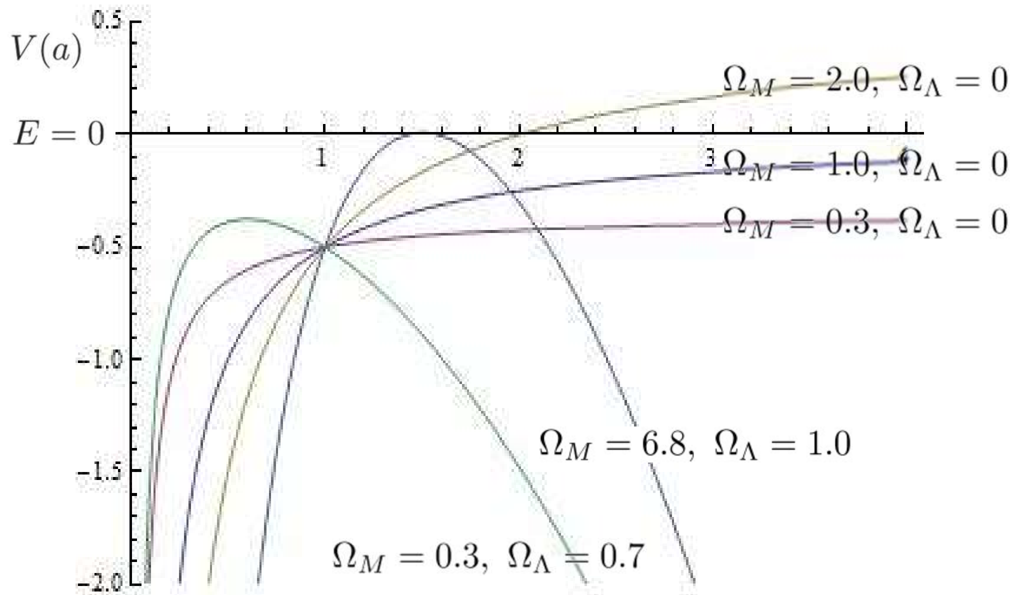
$$\frac{1}{2}\dot{a}^2 + V(a) = 0$$

$$\Omega_0 \equiv \textstyle\sum_i \Omega_i$$

$$\Omega_K \equiv \Omega_0 - 1$$

$$V(a) = \frac{1}{2}H_0^2\left[\Omega_K - \sum_i \Omega_i a^{-1-3w_i}\right]$$

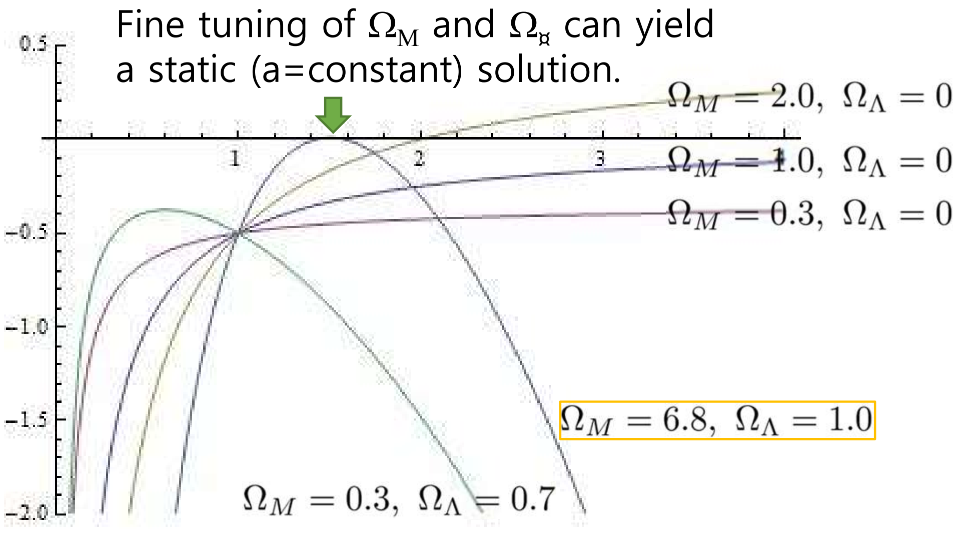
⇒ the motion of a particle with E=0
under the potential V(a)

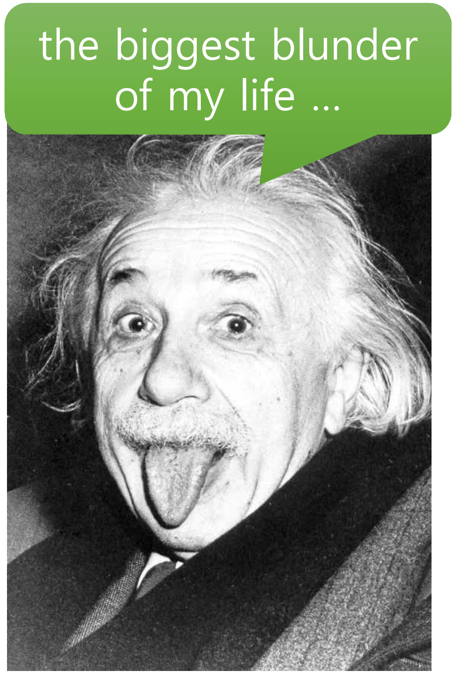


# Einstein's biggest blunder

- Einstein introduced the cosmological constant to obtain the static universe. (1917)

$$\frac{1}{2}\dot{a}^2 + V(a) = 0, \quad V(a) = \frac{1}{2}H_0^2\left[\Omega_K - \sum_i \Omega_i a^{-1-3w_i}\right]$$

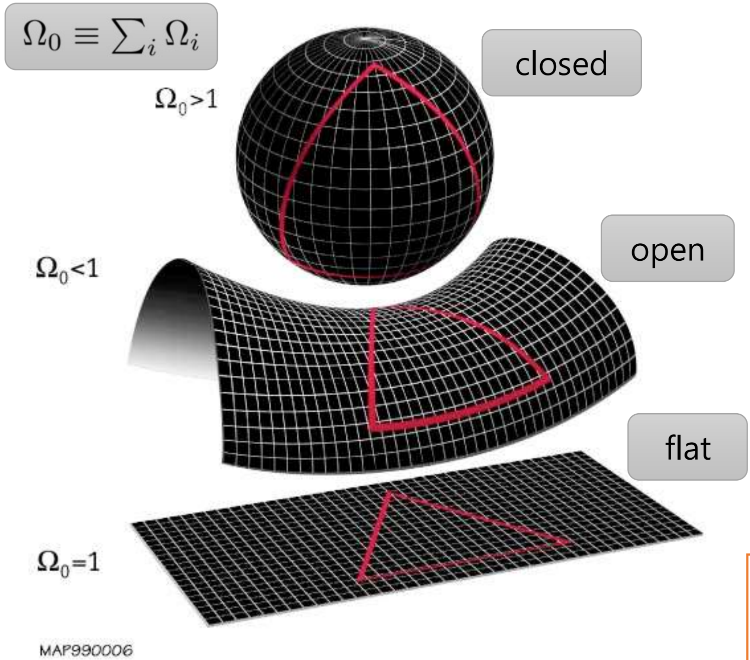


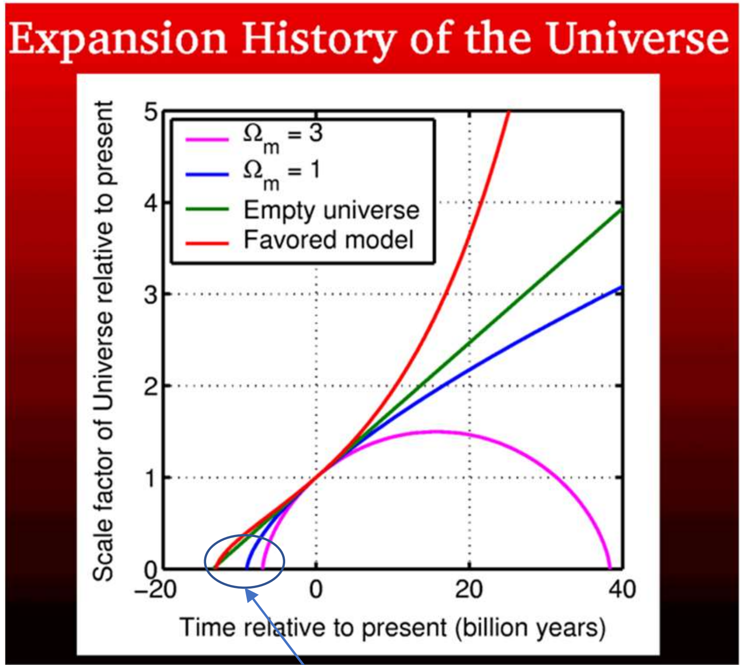




- Gave up the static universe after Hubble's discovery of expansion (1929)
- Resurrection of CC to explain the accelerating expansion (1998)

# Expansion History

- Expansion history of the universe depends on the species and amounts of matter in the universe.

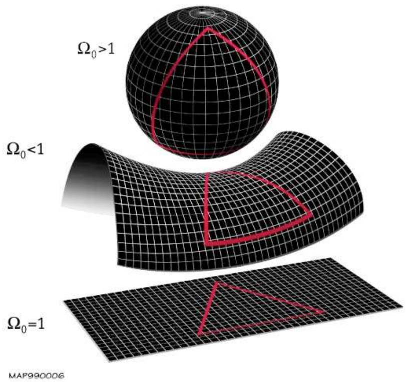



**Big Bang**

**Our universe has the beginning.**
If we trace back the expansion history, we meet a singular (infinite energy density) point of a=0 in finite time.

# Expansion History




- Comoving distance depends on geometry.

$$r_1 = f_K(z) \equiv \begin{cases} \sin f(z), & \text{for } K=+1 \ (\Omega>1), \\ f(z), & \text{for } K=0 \ (\Omega=1), \\ \sinh f(z), & \text{for } K=-1 \ (\Omega<1), \end{cases}$$

$$f(z) = \int_0^z \frac{dz'}{a_0 H(z')} = \frac{1}{a_0 H_0} \int_0^z \frac{dz'}{\left[\Omega_K (1+z')^2 + \sum_i \Omega_i (1+z')^{3(1+w_i)}\right]^{1/2}}$$

- Luminosity distance – Red shift relation : $d_L = a_0 r(z)\,(1+z)$

$$\boldsymbol{H_0 d_L = z} + \boldsymbol{\frac{1}{2}(1-q_0)z^2} + \cdots$$

Distance modulus is frequently used instead of lumosity distance.
$\mu_0 = m - M = 5\log\left(d_L/10\,\mathrm{pc}\right)$

**Hubble's law, Hubble Cons.**

$$H_0 = \frac{\dot{a}_0}{a_0} = 73.8\,(\mathrm{km/s})/\mathrm{Mpc}$$

$$H_0^{-1} = 13.8\,\mathrm{Gy} = 4230\,\mathrm{Mpc}$$

Determine the age and the size of the universe

**Deceleration parameter**

$$q_0 = -\frac{a_0 \ddot{a}_0}{\dot{a}_0^2} = \frac{1}{2}\sum_i \underline{\Omega_i (1+3w_i)}$$

Determined by species and amounts of matter.

For acceleration, matter with $w_i < -\frac{1}{3}$ must dominate.

# Age of the universe

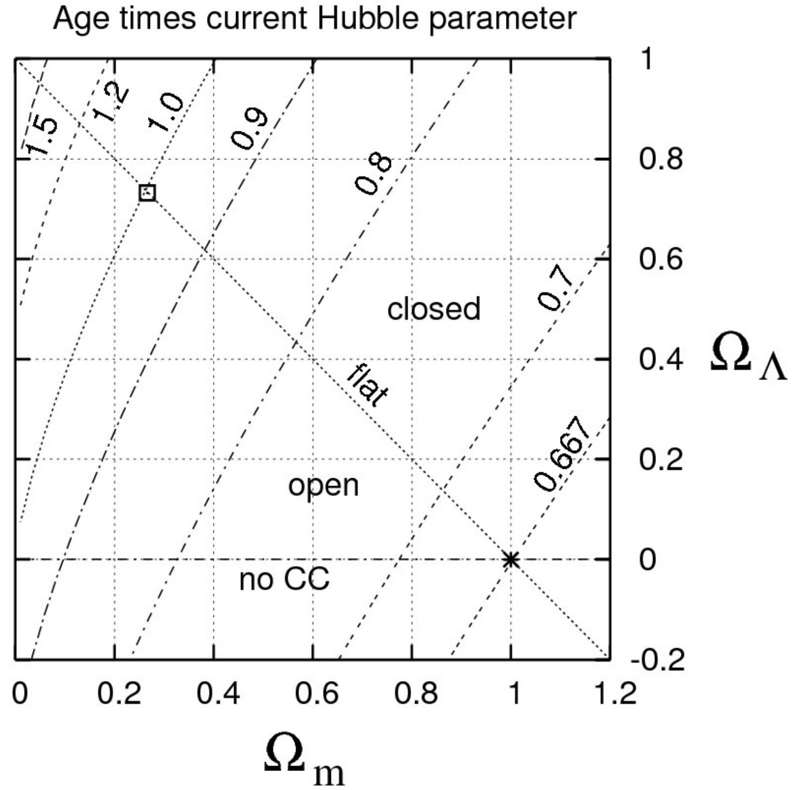
- Age of FLRW universe

$$t_0 = H_0^{-1}\, f(\Omega_i)$$

$$H_0^{-1} = \left(\frac{0.71}{h}\right) \times 13.8\,\mathrm{Gyr}$$ **Hubble Time**

$$f(\Omega_i) = \int_0^1 \left[-\Omega_K + \sum_i \Omega_i x^{-1-3w_i}\right]^{-1/2} dx$$

- Lower bounds on the age of the universe



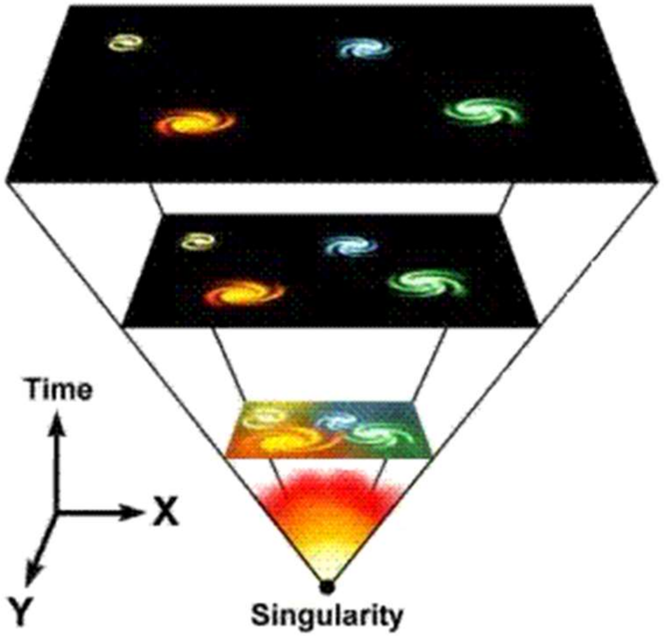
# Consequences of Expansion

- CMB with black body spectrum of 2.73K
  ⇒ Our universe was in **thermal equilibrium** in the past.

- Scale factor and Temperature in thermal equilibrium

$$a(t)\,T(t) = \text{constant.}$$

- Expansion and Temperature
  - small **a** in the past → high **T** in the past.
  - **Hot Big Bang : Our universe started in thermal equilibrium at high temperature.**




- High Temperature (T) ⇔ High Energy (E) ⇔ Short Distance
  (quantum principle)

**To understand the high temperature state of the early universe, we need the knowledge at short distance (high energy, that is particle physics).**

# Particles in thermal equilibrium

- The early universe is filled with **hot ideal gases in thermal equilibrium**.

- Energy density and pressure of ideal gas at temperature T

$$\rho_i(T) = g_i \int \frac{d^3\vec{p}}{(2\pi)^3} f_i(\vec{p}) E(\vec{p})$$

$$p_i(T) = g_i \int \frac{d^3\vec{p}}{(2\pi)^3} f_i(\vec{p}) \frac{\vec{p}^2}{3E(\vec{p})}$$

$$E = \sqrt{|\vec{p}|^2 + m^2}$$

$$f(\vec{p}) = \frac{1}{e^{(E-\mu)/T} \pm 1}$$

\- boson;
+, [ ] fermion
Momentum distribution in thermal equilibrium

- Relativistic, non-degenerate : $T \gg m, \mu$

$$n = \left[\frac{3}{4}\right] \frac{\zeta(3)}{\pi^2} g T^3, \quad \rho = \left[\frac{7}{8}\right] \frac{\pi^2}{30} g T^4, \quad p = \frac{1}{3}\rho$$

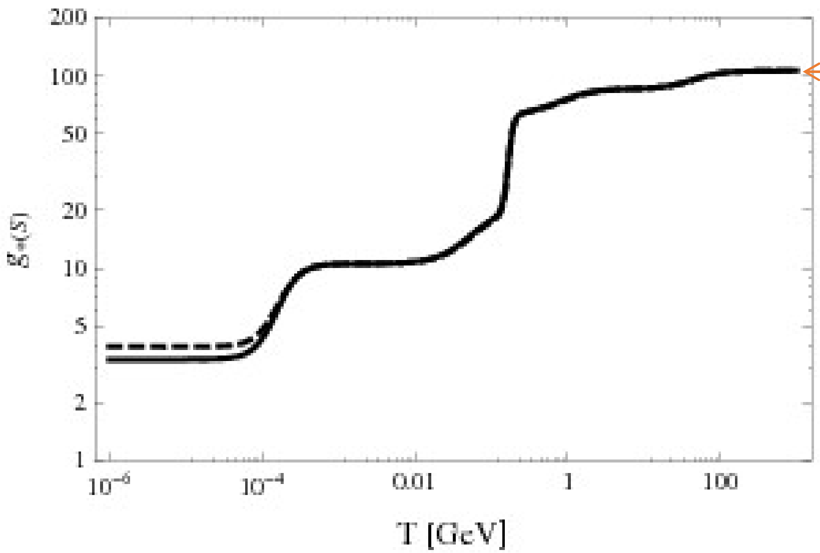
- Non-relativistic : $T \ll m$

$$n = g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m-\mu)/T}, \quad \rho = mn + \frac{3}{2}p, \quad p = nT \ll \rho$$

# Radiation in the early universe

- Total energy density and pressure in the universe
  - In thermal equilibrium, the energy density of non-relativistic species is exponentially smaller than that of relativistic species.

$$\rho_R = \frac{\pi^2}{30} g_*(T)\, T^4, \quad p_R = \frac{1}{3}\rho_R, \quad g_*(T) = \sum_{\text{bosons}} g_b + \frac{7}{8}\sum_{\text{fermions}} g_f$$




- Standard Model : $\frac{427}{4} = 106.75$

F: $\frac{7}{8}(4\times 3\times 6 + 4\times 3 + 2\times 3) = \frac{315}{8}$

B: $2\times 8 + 3\times 3 + 2 + 1 = 28$

# Entropy of the universe

- The entropy in a comoving volume is conserved in thermal equilibrium.
  - 1st law of thermodynamics : $dE = T\,dS - p\,dV$
  - In thermal equilibrium : $\dfrac{dS}{dt} = 0$
  - Entropy density : $s = \dfrac{\rho + p}{T}$

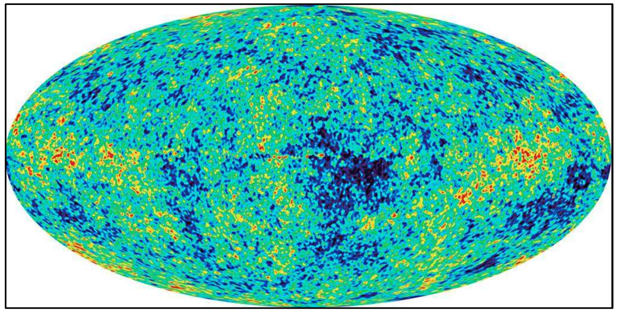
$$E = \rho V \; , \;\; S = sV$$
$$V\,d\rho + \rho\,dV = T(s\,dV + V\,ds) - p\,dV$$
$$\underbrace{d\rho - T\,ds}_{\text{intensive}} = (Ts - \rho - p)\underbrace{\frac{dV}{V}}_{\text{extensive}} = 0$$

- Entropy in the early universe
  - Dominated by relativistic species. Entropy density $s = \sum_i \dfrac{\rho_i + p_i}{T_i} = \dfrac{2\pi^2}{45} g_* T^3$
  - Exact temperature – scale factor relation obtained from entropy conservation $\boldsymbol{T \propto g_*^{-1/3} a^{-1}}$
  - Since $n \propto a^{-3}$ and $s \propto a^{-3}$, $\boldsymbol{Y_i \equiv n_i/s}$ is a convenient quantity for representing the abundance of decoupled species.

# Thermal History

# Remnants of Expansion

- Thermal equilibrium and its breakdown
  - To keep thermal equilibrium, **the reaction rate** must be larger than **the expansion rate**. If thermal equilibrium is kept on, no remnant from the past can be found.
  - As temperature goes down, the reaction rate decreases faster than the expansion rate and thermal equilibrium is broken.
- Breakdowns of equilibrium made the history of the universe !
  - **Baryogenesis, Big Bang Nucleosynthesis, Decouplings of Dark Matter, Neutrinos, Photons**

Cosmology is similar to archeology in the sense that it deduces the past from the remnants.

The expansion of the universe makes the history.

# Out of Equilibrium

- The universe has been very nearly in thermal equilibrium for most of its history.
- Departure from thermal equilibrium might make fossil record of the early universe.
- **Rule of thumb for thermal equilibrium**

$$\textbf{Interaction rate } \boldsymbol{\Gamma}_{\textbf{int}} > \textbf{Expansion rate } \boldsymbol{H}$$

$$\Gamma_{\text{int}}(T) = n(T)\langle\sigma|v|\rangle^{T} \qquad H(T) \approx \frac{T^2}{M_P}$$

- Rough understanding of decoupling of species
  - Interaction mediated by a massive gauge boson

$$\sigma \sim \frac{\alpha^2 s}{m_X^2} \quad \Rightarrow \quad \Gamma_{\text{int}} \sim T^3 \cdot \frac{\alpha^2 T^2}{m_X^4} = \frac{\alpha^2 T^5}{m_X^4}$$

$$T \lesssim \left(\frac{m_X^4}{\alpha^2 M_P}\right)^{1/3} \sim \left(\frac{m_X}{100\,\text{GeV}}\right)^{4/3} \text{MeV} \quad \Rightarrow \quad \text{freeze out}$$

# Boltzmann Equation for Annihilation

- Boltzmann Equation : Rate of abundance change = Rate of production – Rate of elimination

$$\frac{df}{dt} = C[f]$$

- Consider the particle 1 in a process 1 + 2 $\leftrightarrow$ 3 + 4 :

Distribution function and number density

$$n_i(t) = \int \frac{d^3\vec{p}_i}{(2\pi)^3} f_i(\vec{p}_i, t)$$

$$\underbrace{\frac{1}{a^3}\frac{d(n_1 a^3)}{dt}}_{\text{Change in comoving volume}} = \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}$$

$$\times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) |\mathcal{M}(1 + 2 \leftrightarrow 3 + 4)|^2$$

$$\times \{\underbrace{f_3 f_4 (1 \pm f_1)(1 \pm f_2)}_{\substack{\text{Production of 1} \\ 3 + 4 \rightarrow 1 + 2}} - \underbrace{f_1 f_2 (1 \pm f_3)(1 \pm f_4)}_{\substack{\text{Elimination of 1} \\ 1 + 2 \rightarrow 3 + 4}}\}$$

Scattering amplitude
**Particle physics** enters here.
CP(or T) symmetry assumed

- Simplifying assumptions
  - Kinetic equilibrium – Rapid elastic scattering ➡ $f(\vec{p},t) = \dfrac{1}{e^{(E(\vec{p})-\mu(t))/T(t)} \pm 1}$
  - Annihilation in equilibrium : $\mu(t)$ ⟶ chemical potential

    $f(\vec{p},t)$ : described by chemical potential (and temperature)
  - Low temperature approximation : $T \ll E - \mu$ ➡ $f \approx e^{-(E-\mu(t))/T}, \quad 1 + f \approx 1$
  - Change of variables
    chemical potential → number density
    Ordinary differential equation for $n_i(t)$

$$\mu_i(t) \;\to\; n_i(t) = g_i e^{\mu_i(t)/T} \int \frac{d^3\vec{p}_i}{(2\pi)^3} e^{-E_i/T}$$
$$= e^{\mu_i(t)/T} n_i^{(0)}$$

$$f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4)$$
$$\approx e^{-(E_1+E_2)/T} \left( e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T} \right) = e^{-(E_1+E_2)/T} \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\}$$

- Define the thermally averaged cross section

$$\langle \sigma v \rangle \equiv \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4} e^{-(E_1+E_2)/T}$$
$$\times (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)\,|\mathcal{M}(1+2\leftrightarrow 3+4)|^2$$

- Simplified Boltzmann equation

$$\underbrace{\frac{1}{a^3}\frac{d(n_1 a^3)}{dt}}_{\sim \frac{n_1}{t_H} \sim n_1 H} = \underbrace{n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle}_{n_1 n_2 \langle \sigma v \rangle \sim n_1 \Gamma_{\rm int}} \left( \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right)$$

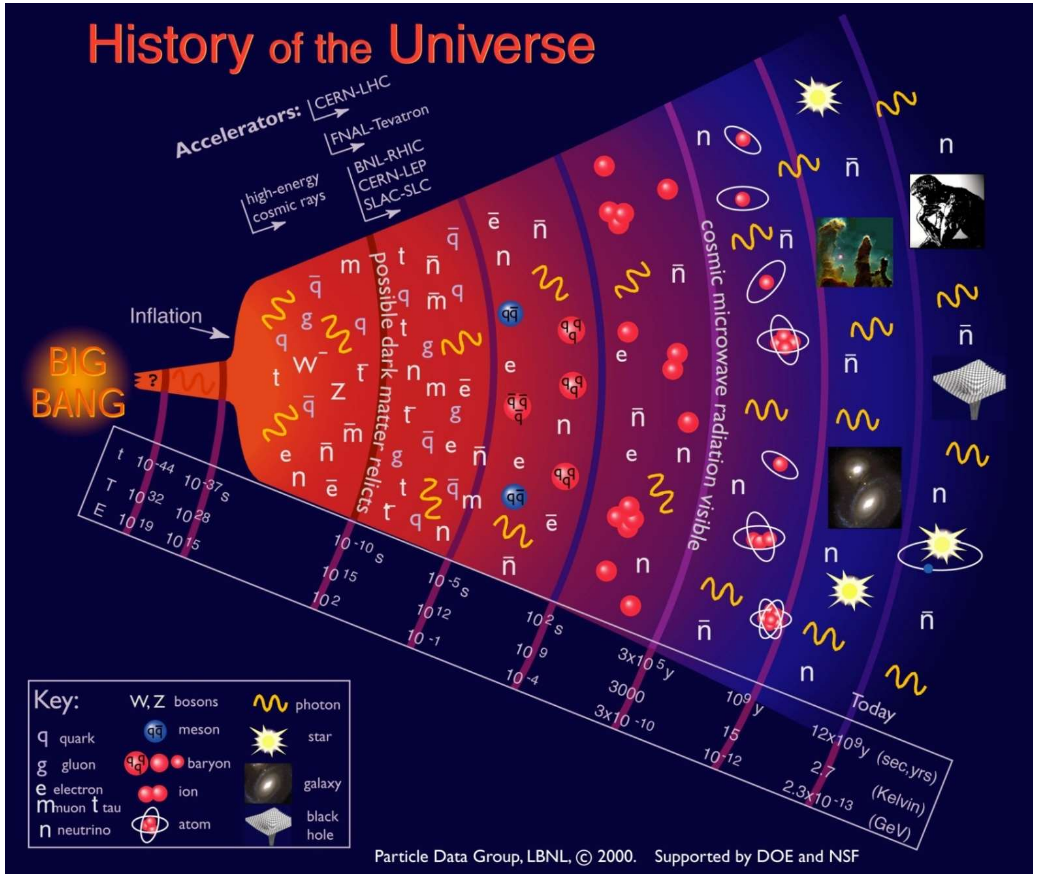
$$H \ll \Gamma_{\rm int} \quad \Rightarrow \quad \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} = 0$$

Chemical equilibrium

# Thermal History of the Universe

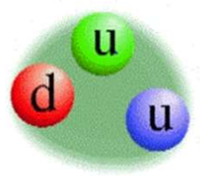
- Radiation Domination
  - Heating / Thermalization
  - Initial temperature

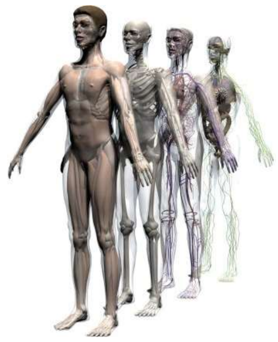
- Baryogenesis
- Neutrino decoupling
- Big Bang Nucleosynthesis
- Photon decoupling



# Baryon Asymmetry

- Matter content of the universe – **Baryons (Stars) and Radiation (CMB)**
  - Matter forming our body : Baryons (protons, neutron) and leptons (electron)
  - Stars, Planets, Dust, Gas, … (Most baryons are in intergalactic gases.)



- Baryon Asymmetry of the universe
  - SM of particle physics is very symmetric in baryon and anti-baryon.
  - The universe is dominated by baryons, with little anti-baryon.

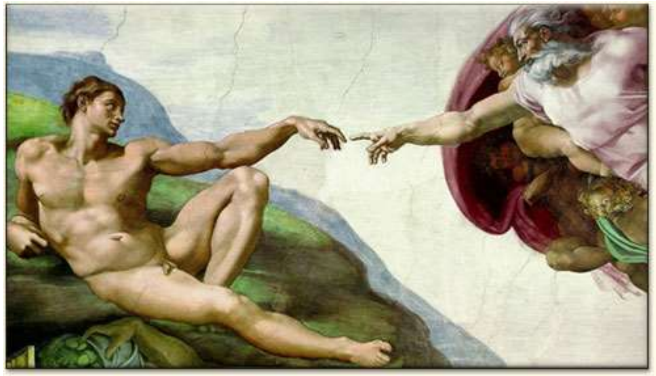
Observed : $\dfrac{n_B - n_{\bar{B}}}{n_B + n_{\bar{B}}} \approx 1$ SM prediction : $\dfrac{n_B - n_{\bar{B}}}{n_B + n_{\bar{B}}} \approx 0$

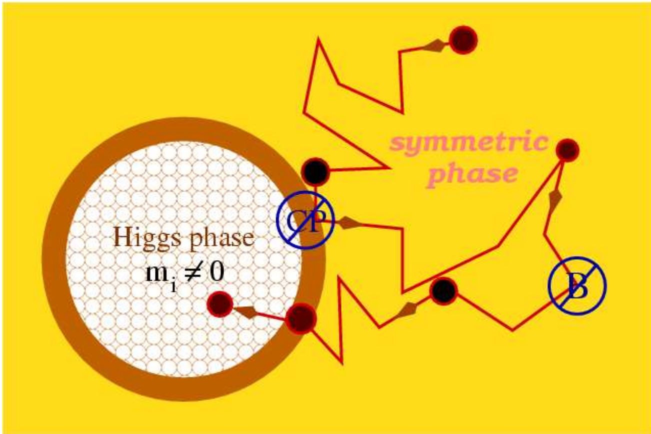
- The amount of baryon in the universe
  - Good agreement between BBN and CMBA

$$\eta = \frac{n_B - n_{\bar{B}}}{n_\gamma} \approx 4 \times 10^{-9}$$ (4 baryons to 1 billion photons)

# Baryogenesis

- Baryogenesis
  - In the beginning,
    the universe was supposed to be baryon symmetric.
  - Baryon asymmetry was produced in the early universe
    through baryogenesis (non-equilibrium) process.
  - Sakharov conditions for baryogenesis
    1. B violation
    2. C & CP violation
    3. Out-of-equilibrium






- Standard Model cannot make sufficient baryon asymmetry.
  - SM satisfies all three conditions, but …
  - CP violation is too small and PT is not strong enough.

# Baryogenesis

- GUT baryogenesis
  - Out-of-equilibrium decay

- Supersymmetry and Affleck-Dine baryogenesis
  - Complex scalar field dynamics

- Standard Model and the Sphaleron
  - B-L conversion : conserve B-L, but not B+L

- Leptogenesis
  - Neutrino masses requires the extension of SM. For example, the sea-saw mechanism.
  - Lepton asymmetry can be generated in the extended lepton sector.
  - Sphaleron effect can turn lepton asymmetry into baryon asymmetry.

# Neutrino Decoupling

- Thermal equilibrium maintained by the weak interaction is broken around T ~ 1 MeV.
- Species in equilibrium around T ~ 1 MeV :
  - baryon : proton, neutron (baryon asymmetry, non-relativistic)
  - lepton : electron, positron, 3 types of neutrinos
- weak interaction

$$e^+ + e^- \leftrightarrow \nu_e + \bar{\nu}_e \qquad \Gamma(T) = n(T)\langle \sigma v \rangle_T \sim T^5/M_W^4$$

- Neutrino decoupling
  - decoupling temperature $T_{\rm dec} \sim 1\,\mathrm{MeV}$
  - relic abundance (hot relic)

$$Y_\nu = n_\nu/s = g_\nu/g_*(T_{\rm dec}) = 21/43$$

$$\Omega_\nu h^2 = 1.68 \times 10^{-5}\ \text{(massless)}$$

$$\Omega_\nu h^2 = \sigma_{\nu_i} m_{\nu_i}/94\,\mathrm{eV}\ \text{(massive)}$$

- temperature difference between photons and neutrinos
  - Below T=$m_e$=0.52 MeV, $e^+$, $e^-$ annihilate and dump energy (entropy) only to photons, and thus photons cools slower than neutrinos.

$$T_\gamma/T_\nu = (4/11)^{1/3} = 0.71$$

# Proton-Neutron Freeze-out

- Baryon number density is fixed by baryon asymmetry.
- Neutron – Proton equilibrium is maintained by weak interaction.

$$n + e^+ \leftrightarrow p + \bar{\nu}_e \,, \quad n + \nu_e \leftrightarrow p + e^- \,, \quad n \leftrightarrow p + e^- + \bar{\nu}_e$$

- When temperature goes down below the neutron-proton mass difference, the neutron-proton equilibrium shift to proton.
  - neutron-proton mass difference $\Delta m = m_n - m_p = 1.3\,\mathrm{MeV}$
  - equilibrium neutron-proton ratio $n/p = e^{-\Delta m/T}$

- Below 1 MeV, neutron-proton conversion freeze out.
  - freeze-out temperature $T_f \approx 0.75\,\mathrm{MeV}$
  - neutron-proton ratio frozen $(n/p)_f = e^{-\Delta m/T_f} \approx 0.18$
  - neutron decay slowly reduces the neutron-proton ratio, reaching 0.13 at the beginning of big bang nucleosynthesis (t=200 s, T=0.07 MeV).

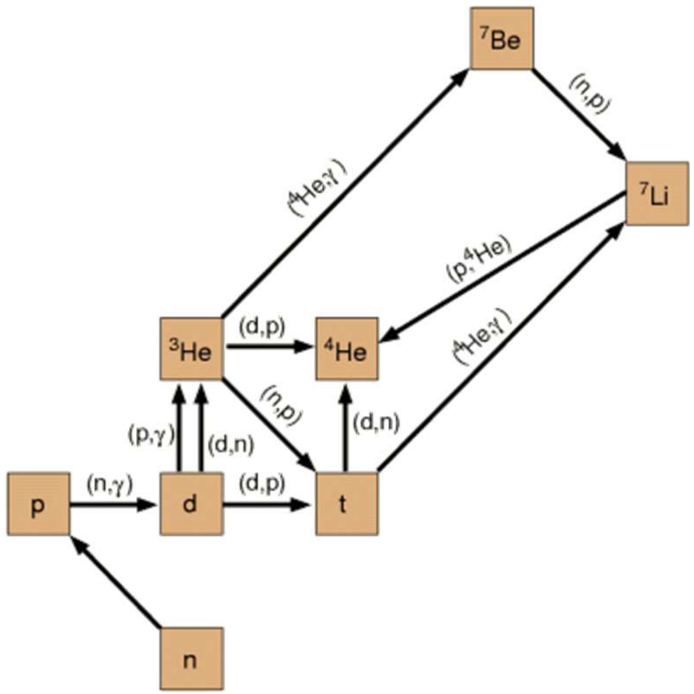
# Big Bang Nucleosynthesis

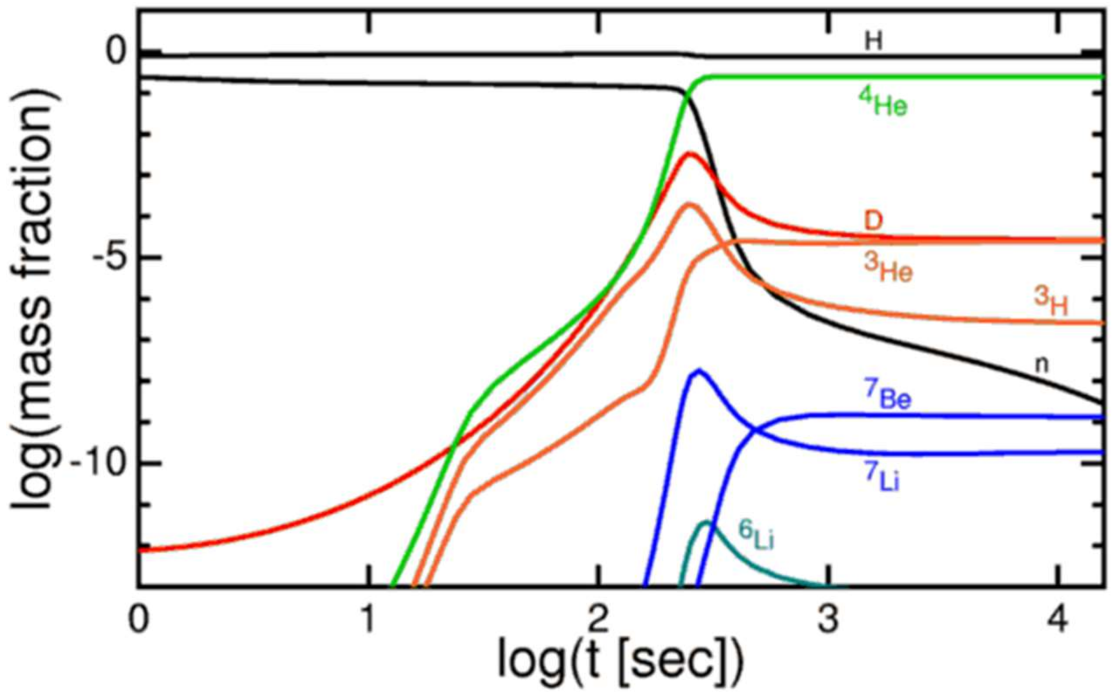
- As the universe cools down, light nuclei are synthesized from protons and neutrons. (Heavy nuclei are produced in the process of star evolution.)
- BBN is one of supporting evidences of Big Bang,
  by explaining very well the ratios of light nuclei in our universe.
- The ratios depends on the amount of baryon and the expansion rate at the time of BBN. BBN is a good probe of baryon amount.

- Universe at T ~ 1 MeV
  - Species in equilibrium : $\text{(photons) } \boldsymbol{\gamma}, \text{ (leptons) } \boldsymbol{e^+}, \ \boldsymbol{e^-}, \text{ (baryons) } \boldsymbol{p}, \ \boldsymbol{n}$
  - Species decoupled : $\text{(neutrinos) } \boldsymbol{\nu_e}, \ \boldsymbol{\nu_\mu}, \ \boldsymbol{\nu_\tau}$
  - Initial baryon asymmetry : $\eta_b \equiv \dfrac{n_b}{n_\gamma} = 5.5 \times 10^{-10} \left( \dfrac{\Omega_b h^2}{0.020} \right)$

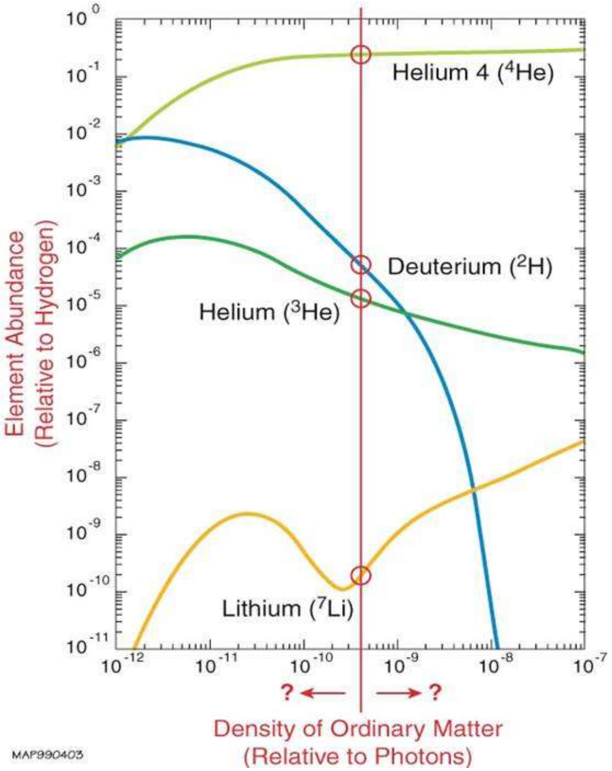
- Where the baryons end up?
  - Nuclear binding energies are of order MeV,
    but the nucleosyntheis is delayed until T ~ 0.1 MeV by the effect of small $\eta_b$.
  - If thermal equilibrium is kept through out,
    the nuclear state with the lowest energy per baryon (iron nucleus) will dominates.
  - BBN produced no elements heavier than beryllium due to a bottleneck:
    the absence of a stable nucleus with 8 or 5 nucleons.

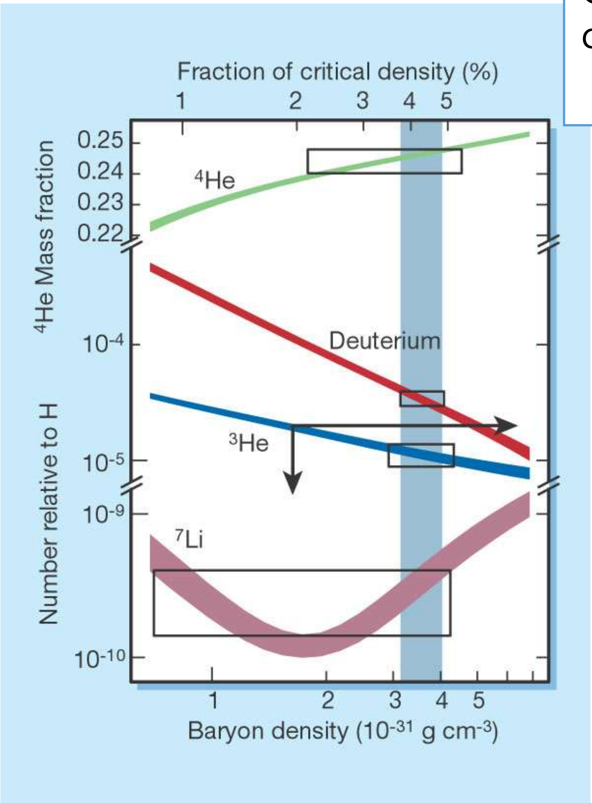
- Numerical solution of Boltzmann equations (BBN code)



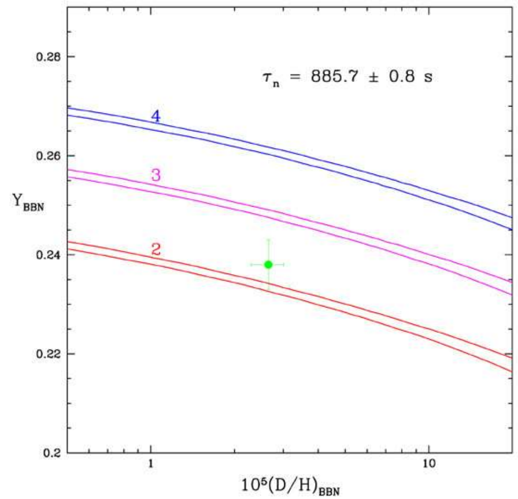
- Comparison with observations




Observed abundances of light nuclei constrain the baryon density to be

$$\eta_b \approx 4 \times 10^{-9}, \quad \Omega_b \approx 0.05$$




The number of neutrino species affects the energy density at BBN, which can change the neutron-proton ratio, and thereby the helium fraction. $N_\nu \sim 3$

# Photon Decoupling

- Sketch of Photon (CMB) Decoupling
  - Thermal equilibrium between protons, electron, hydrogen atoms, and photons (about 300,000 years after big bang)

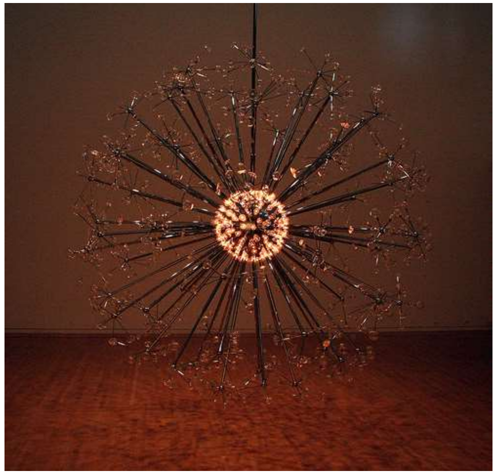
$$p + e^- \quad \leftrightarrow \quad H + \gamma \qquad (E_{\rm bind} = 13.6 \text{ eV})$$
$$e^- + \gamma \quad \leftrightarrow \quad e^- + \gamma \qquad (\text{Thomson scattering})$$

  - Below the temperature T ~ $E_{bind}$ :

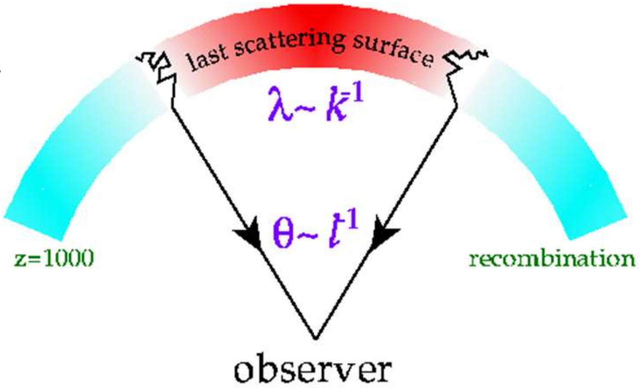
$$H \text{ is preferred} \rightarrow \text{Reduction of } e^- \rightarrow \text{Decoupling of } \gamma$$

  - Decoupling occurs in relatively short time (z~1100) and CMB we see today comes from the last scattering surface.

Formation of H-atom is called recombination.
(By historical reason, ‘re’ is wrongly attached.)



The Last Scattering Surface, an art installation at the Henry Art Gallery on the University of Washington campus in Seattle



# Photon Decoupling – Details

- Particle species at temperature T ~ eV : $\underline{\gamma,\ e^-,\ p}$ $\qquad$ $\underline{\nu_e,\ \nu_\mu,\ \nu_\tau}$

Tightly coupled $\qquad$ decoupled

Compton scattering $e^- + \gamma \leftrightarrow e^- + \gamma$

Coulomb scattering $e^- + p \leftrightarrow H + \gamma$

- Evolution of the free electron fraction : $X_e \equiv \dfrac{n_e}{n_e + n_H} = \dfrac{n_p}{n_p + n_H}$

  - If $e^- + p \leftrightarrow H + \gamma$ remains in equilibrium

$$\frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} \Rightarrow \frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H}\left[\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-[m_e + m_p - m_H]/T}\right] \approx 10^9 \left(\frac{m_e}{2\pi T}\right)^{3/2} e^{-\epsilon_0/T}$$

$$\approx n_b = \eta_b n_\gamma \sim 10^{-9} T^3$$

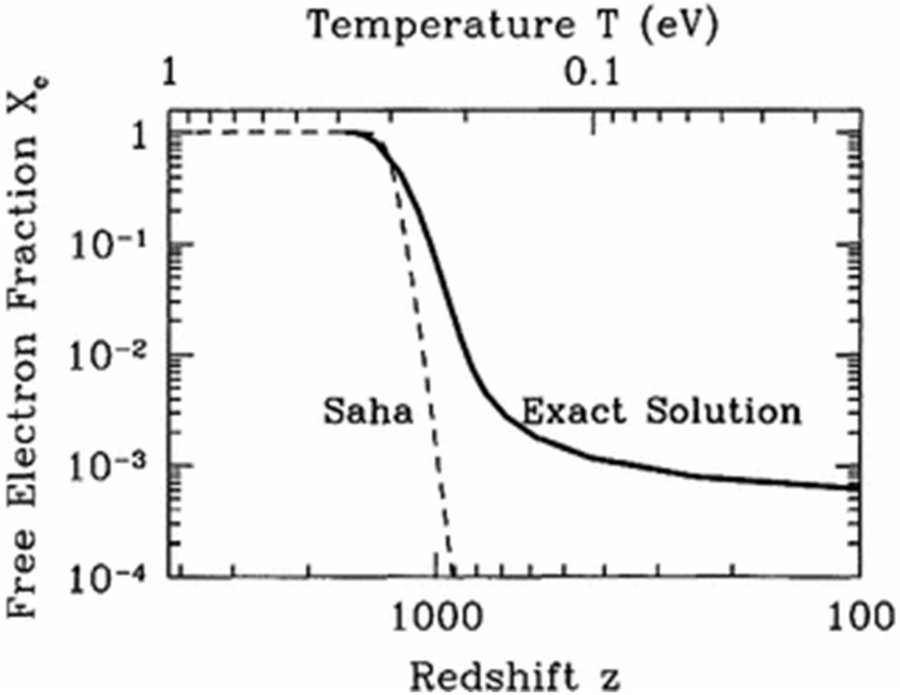
$X_e \approx 1 \text{ at } T \sim \epsilon_0$ $\qquad$ As $X_e \to 0,$ out of equilibrium.

- Out of equilibrium – Boltzmann equation

$$\frac{dX_e}{dt} = \left[(1 - X_e)\beta - X_e^2 n_b \alpha^{(2)}\right]$$

ionization rate $\beta = \langle \sigma v \rangle \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T}$

recombination rate
(to n=2 state) $\alpha^{(2)} \equiv \langle \sigma v \rangle_{n=2}$



- Decoupling of photon occurs when  Compton scattering rate ~ Expansion rate

$$n_e \sigma_{\mathrm{T}} = X_e n_b \sigma_{\mathrm{T}} = 7.5 \times 10^{-30}\,\mathrm{cm}^{-1}\, X_e \Omega_b h^2 a^{-3}$$

$$\frac{n_e \sigma_{\mathrm{T}}}{H} = 113 X_e \left(\frac{\Omega_b h^2}{0.02}\right) \left(\frac{0.15}{\Omega_m h^2}\right)^{1/2} \left(\frac{1+z}{1000}\right)^{3/2} \left[1 + \frac{1+z}{3600}\,\frac{0.15}{\Omega_m h^2}\right]^{-1/2}$$

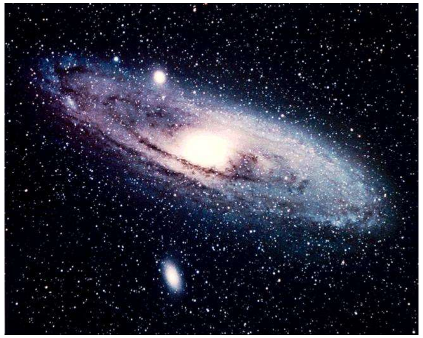
Decoupling of photons occurs during recombination ( $X_e \lesssim 10^{-2}$ )

- Reionization

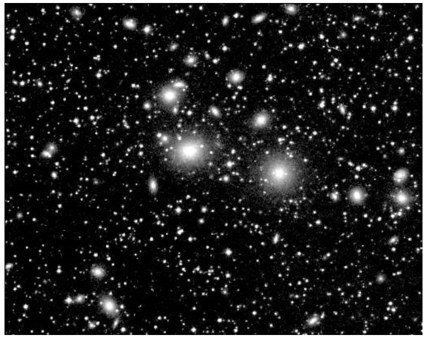
# Inhomogeneity

# Inhomogeneity

- On very large scale, the universe is very homogeneous.
- On smaller scales, there is inhomogeneity, seen as stars, galaxies, clusters.
- Without **inhomogeneity,** we cannot explain **our existence** itself.



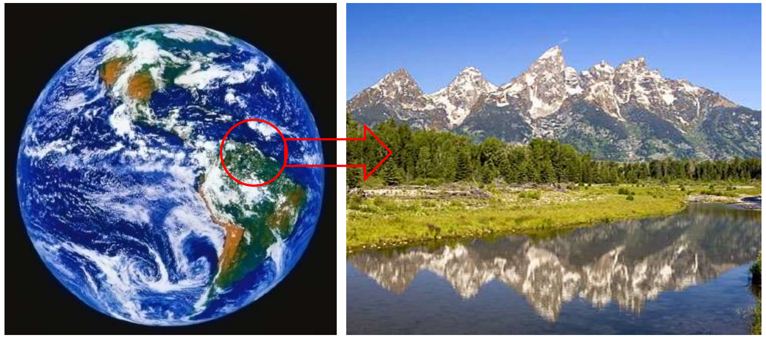
Galaxy: Andromeda



Supercluster: Perseus

Measure of inhomogeneity – **density contrast**

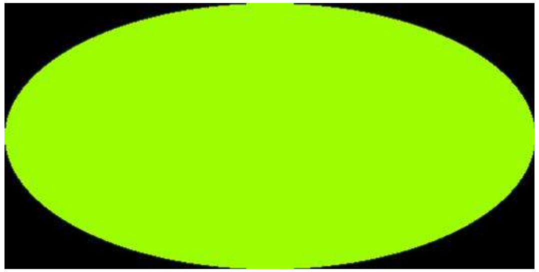
$$\frac{\delta\rho}{\rho} \sim 10^{-5} - 10^{5}$$ (depending on length scale)



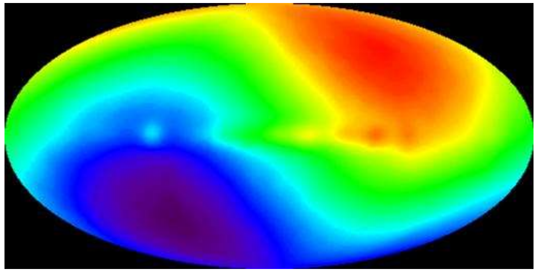
- Analogy with Earth
  - On large scale, Earth is a nearly perfect sphere.
  - On small scales, we see surface fluctuations, such as mountains, valleys, trenches.
  - Measure of surface fluctuations : $\frac{\delta R}{R} \sim 10^{-3}$
  - What makes Earth a sphere?
    What creates surface fluctuations?

# CMB Anisotropies

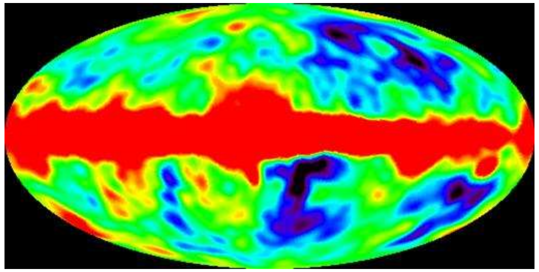
- Is there any other evidence or probe for primordial density perturbation?
- CMB Anisotropies (CMBA)
  - CMB has **temperature fluctuations of order $10^{-5}$**. $\boldsymbol{\delta T/T \sim 10^{-5}}$
- Origin of CMBA – depending on scale
  - Gravitational potential due to **density perturbation of CDM** $\delta\rho \longrightarrow \delta\Phi \longrightarrow \delta T$
  - **Baryon Acoustic Oscillation** – Oscillation of strongly coupled baryon-photon plasma
- COBE Observations (1992) of CMBA



2.73 K



1/1,000 K



1/100,000 K (COBE)

# Structure Formation

- **Understanding the formation of large scale structures**
  - Basic Ideas : Small primordial density perturbations grow to form large scale structures.

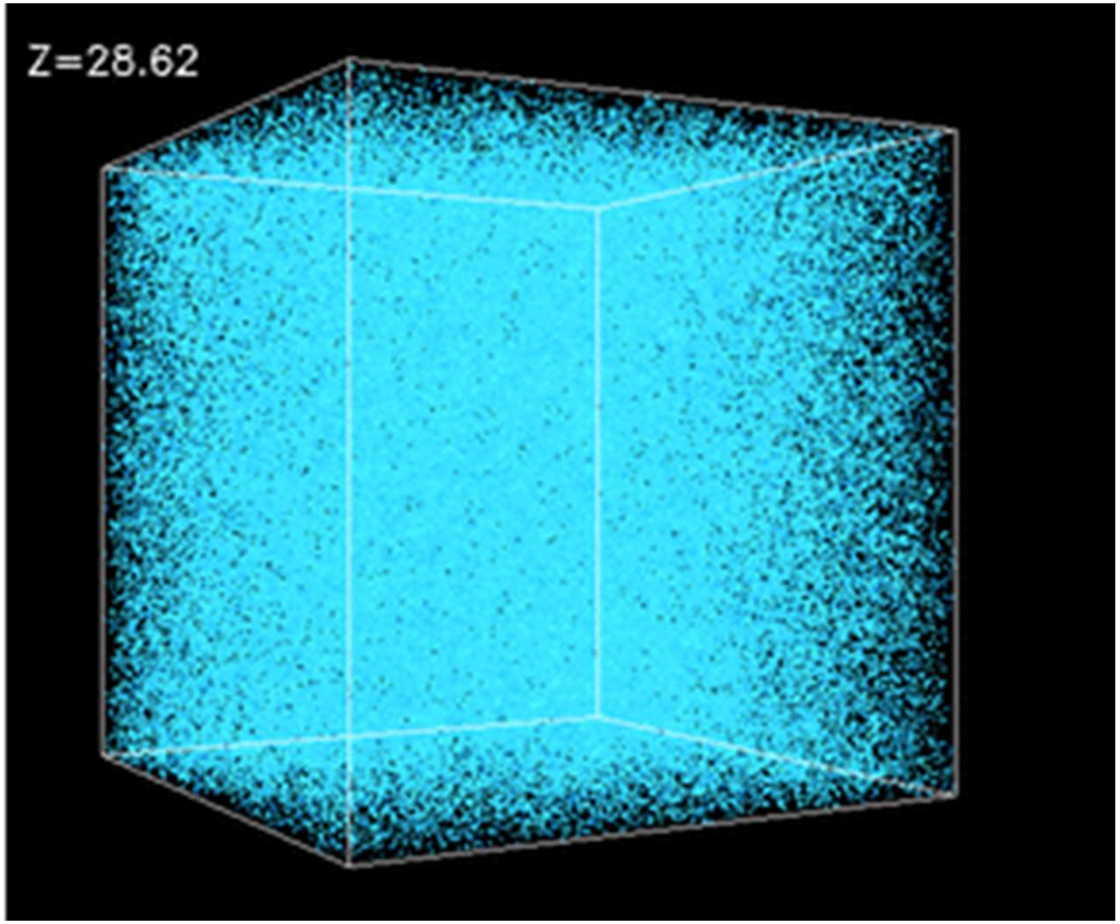
**Action of gravity**

Initial density perturbation → Evolution → Large scale structures

Origin of initial density perturbation?
- Inflation
- Topological defects

Comparison with observation
- Galaxy distribution
- CMB anisotropies

- **Observed facts**
  - The size of initial density perturbation is about $10^{-5}$.
  - Cold Dark Matter must be there.
    Structure can grows after matter domination. If matter consists solely of baryons, structures can grows only after photon decoupling and there is not enough time for structures to grow.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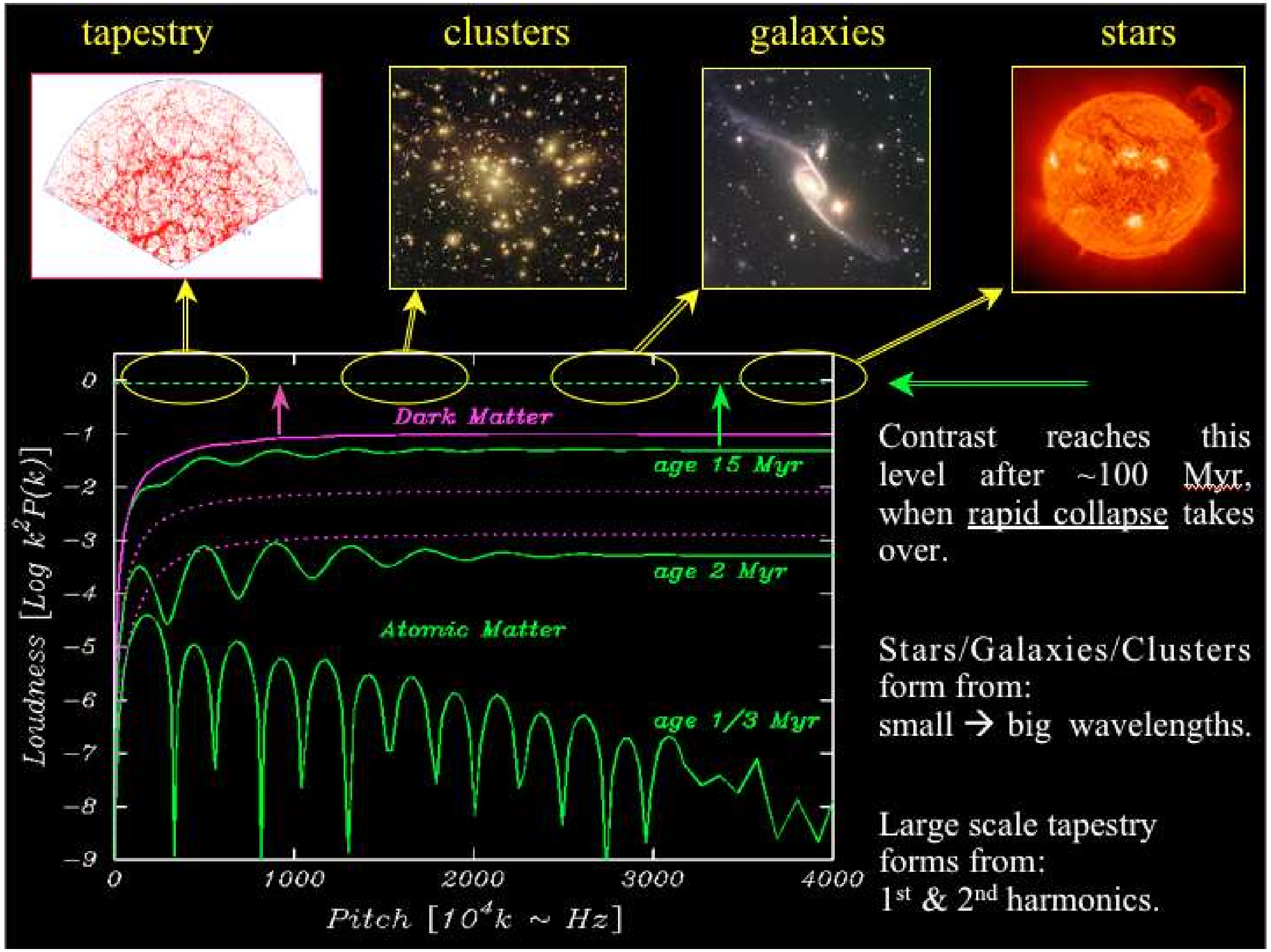
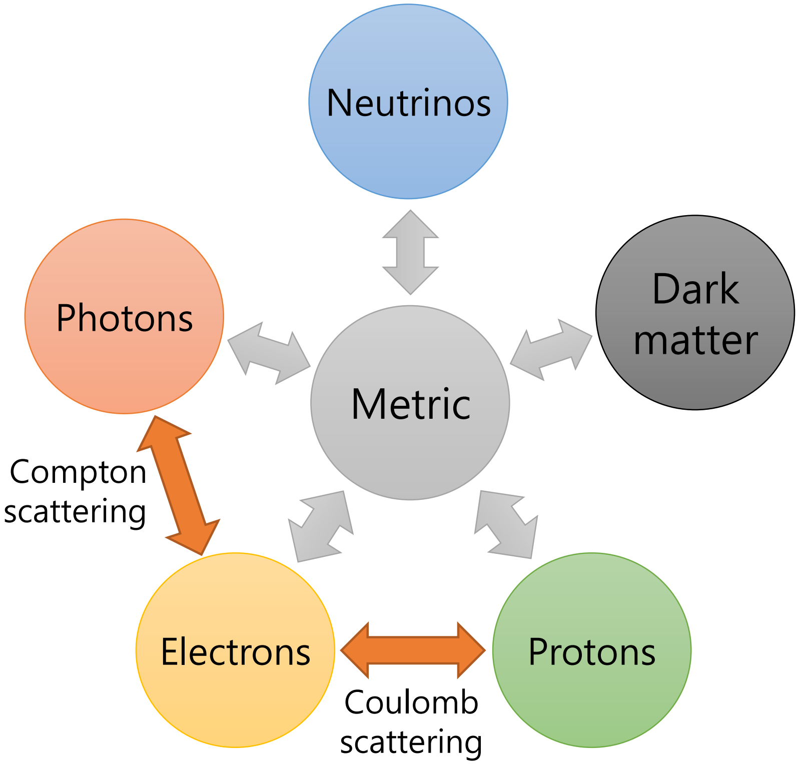

tapestry
clusters
galaxies
stars
Dark Matter
age 15 Myr
age 2 Myr
Atomic Matter
age 1/3 Myr
Loudness [Log k²P(k)]
0
−1
−2
−3
−4
−5
−6
−7
−8
−9
0
1000
2000
3000
4000
Pitch [10⁴k ~ Hz]
Contrast reaches this level after ~100 Myr, when rapid collapse takes over.
Stars/Galaxies/Clusters form from: small → big wavelengths.
Large scale tapestry forms from: 1st & 2nd harmonics.

# Evolution of Inhomogeneity

- Basic equations
  - Spacetime Dynamics (Einstein equation) $G^{(0)}_{\mu\nu} + \delta G_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta T_{\mu\nu}$
  - Matter Dynamics (Botzmann equation) $\frac{d}{dt}\left(f^{(0)} + \delta f\right) = C[f^{(0)} + \delta f]$
- Small perturbation – Linear Approximation $\delta f \ll f^{(0)} \quad \rightarrow \quad \delta T_{\mu\nu} \ll T^{(0)}_{\mu\nu} \quad \leftrightarrow \quad \delta G_{\mu\nu} \ll G^{(0)}_{\mu\nu}$
- Convenient independent variable – comoving horizon $t \quad \rightarrow \quad \eta(t) = \int^t \frac{dt'}{a(t')}$
- Decomposition of perturbations – scalar, vector, tensor
  - In linear approximation, decomposed perturbations evolve independently of each other.
  - Vector perturbation decays away.
  - Tensor perturbation – gravitational waves

# Evolution of Inhomogeneity




- Linear and Nonlinear regime
  - Linear regime – linear perturbation equations
  - Nonlinear regime – Numerical simulations
- Linear regime – Perturbations
  - Scalar perturbations - 9 perturbation variables
  - Tensor perturbations
- Initial conditions
  - At early time, all modes are super-horizon and all variables depend on the gravitational potential $\Phi$.
  - Types of perturbation

$$\delta = -\frac{3}{2}\Phi + C$$

C=0 : **Adiabatic**

C ≠0 : **Isocurvature**

  - What produce $\Phi$ initially?

# Linear Perturbation Equations

- Scalar perturbations – Coupled first order differential eqs. for 9 scalar perturbation variables
  - Photons $\Theta, \Theta_P$
  - Neutrinos $N$
  - Cold Dark Matter $\delta, v$
  - Baryon $\delta_b, v_b$
  - Metric $\Phi, \Psi$

$$n_{\rm dm}(\vec{x},t) = n_{\rm dm}^{(0)}\left[1+\delta(\vec{x},t)\right] = \int \frac{d^3\vec{p}}{(2\pi)^3} f_{\rm dm}(\vec{x},\vec{p},t)$$

$$\vec{v}(\vec{x},t) = \frac{1}{n_{\rm dm}} \int \frac{d^3\vec{p}}{(2\pi)^3} f_{\rm dm}(\vec{x},\vec{p},t)\frac{\vec{p}}{E}$$

$$f_\gamma(\vec{x},\vec{p},t) = \left[e^{p/T(t)[1+\Theta(\vec{x},\vec{p},t)]}-1\right]^{-1}$$

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 + \Theta + \mu v_b - \tfrac{1}{2}P_2(\mu)\Pi\right]$$

$$\dot{\Theta}_P + ik\mu\Theta_P = -\dot{\tau}\left[-\Theta_P + \tfrac{1}{2}\left(1-P_2(\mu)\right)\Pi\right]$$

$$\dot{\delta} + ikv = -3\dot{\Phi}, \quad \dot{v} + \frac{\dot{a}}{a}v = -ik\Psi$$

$$\dot{\delta}_b + ikv_b = -3\dot{\Phi}, \quad \dot{v}_b + \frac{\dot{a}}{a}v_b = -ik\Psi + \frac{\dot{\tau}}{R}\left[v_b + 3i\Theta\right]$$

$$\dot{N} + ik\mu N = -\dot{\Phi} - ik\mu\Psi$$

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi G a^2\left[\rho_{\rm dm}\delta + \rho_{\rm b}\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu N_0\right]$$

$$k^2\left(\Phi+\Psi\right) = -32\pi G a^2\left[\rho_\gamma\Theta_2 + \rho_\nu N_2\right]$$

$\mu \equiv \hat{k}\cdot\hat{p}_\gamma$ $\quad\quad$ $\dot{\tau} = -n_e\,\sigma_{\rm T}\,a$ optical depth

$\Pi \equiv \Theta_2 - \Theta_{P2} + \Theta_{P0}$ $\quad\quad$ $R \equiv \dfrac{3\rho_b^{(0)}}{4\rho_\gamma^{(0)}}$

# Initial Conditions

- Despite 9 variables, initial conditions are set at super-horizon state where the equations are much simplified. We need only to specify the metric perturbation and the integration constant.

$$k\eta \ll 1 \ : \quad \Phi,\ C \quad \rightarrow \ \Theta = N = \frac{1}{2}\Phi,\ \ \delta = \delta_b = \frac{3}{2}\Phi + C,\ \ v = v_b = 0,\ \ \Psi = -\Phi$$

  - Adiabatic perturbation : $\Phi \neq 0,\ C = 0$
  - Isocurvature perturbation : $\Phi = 0,\ C \neq 0$

- Power spectrum $\langle \Phi(\vec{k}) \Phi^*(\vec{k}') \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')\, P_\Phi(k)$

$$P_\Phi(k) = A_s k^{n_s - 4} = \frac{50\pi^2}{9k^3} \left(\frac{k}{H_0}\right)^{n_s - 1} \delta_H^2 \left(\frac{\Omega_m}{D_1(a=1)}\right)^2$$

  - Two input parameters - Amplitude and spectral index : $A_s\,(\delta_H),\ n_s$
  - Harrison-Zeldovich (scale-invariant) spectrum : $n_s = 1$
  - Most inflation models : $n_s \approx 1$

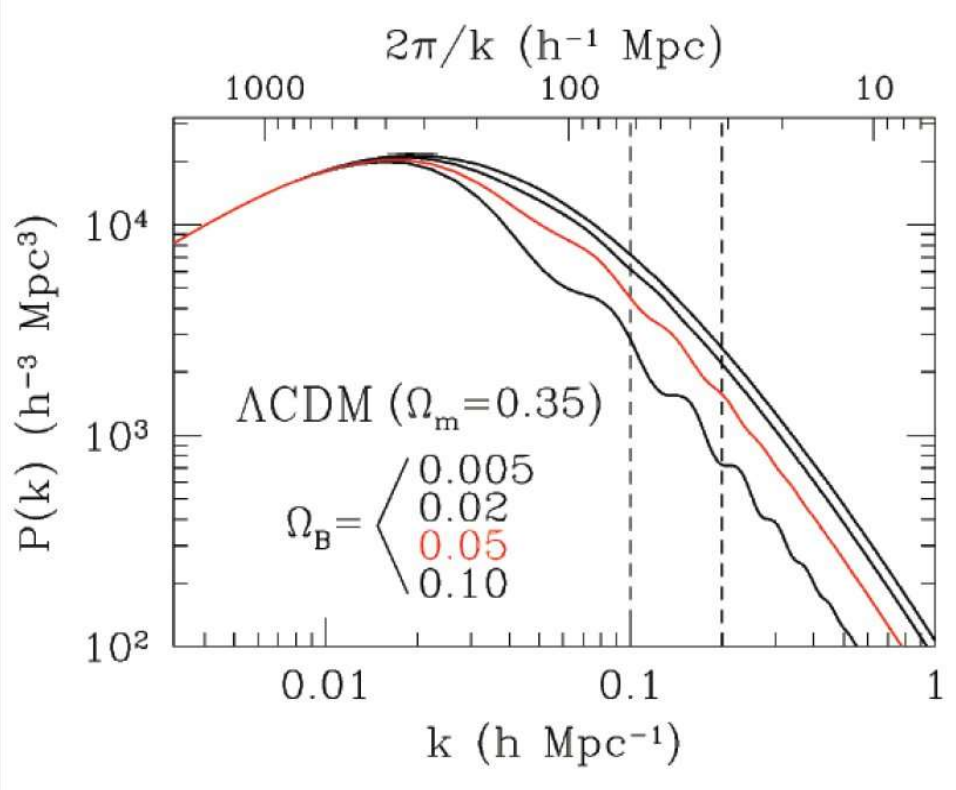
# Growth of Matter Perturbations

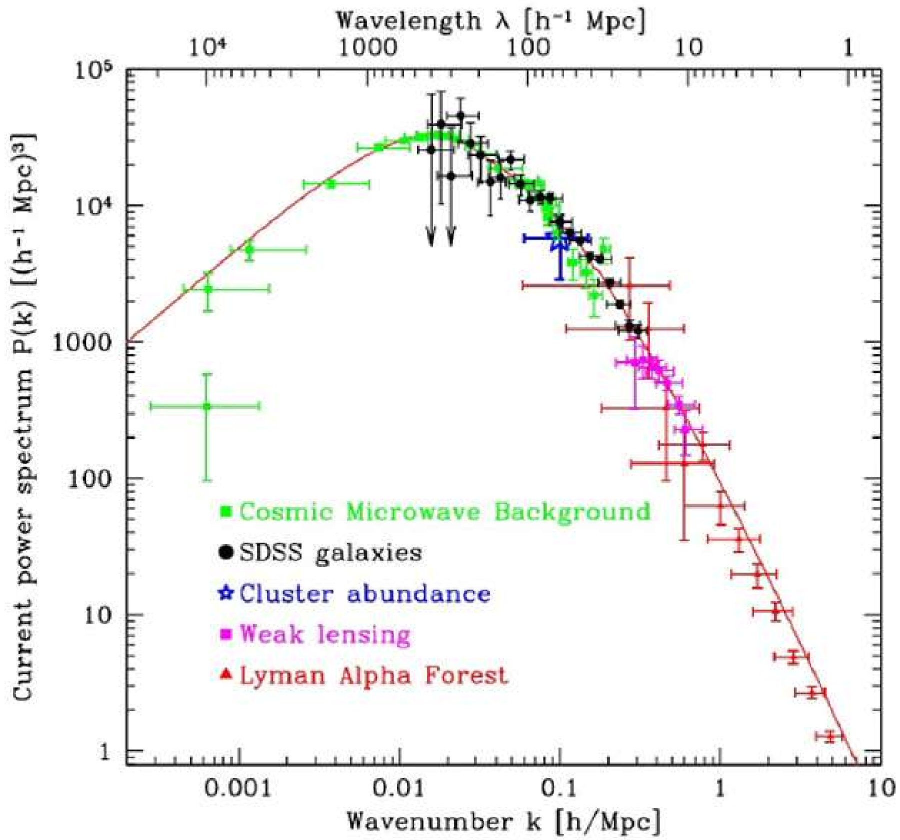
- Gravitational instability – Matter accumulates in initially overdense region.
- Equation governing overdensities in simplified form

$$\ddot{\boldsymbol{\delta}} + [\ \underbrace{\text{Pressure}}_{\substack{\text{Random thermal motion} \\ \text{causing loss, oscillation}}} - \underbrace{\text{Gravity}}_{\substack{\text{Gravity increase overdensity} \\ \text{causing exponential growth}}}\ ]\,\boldsymbol{\delta} = \mathbf{0}$$

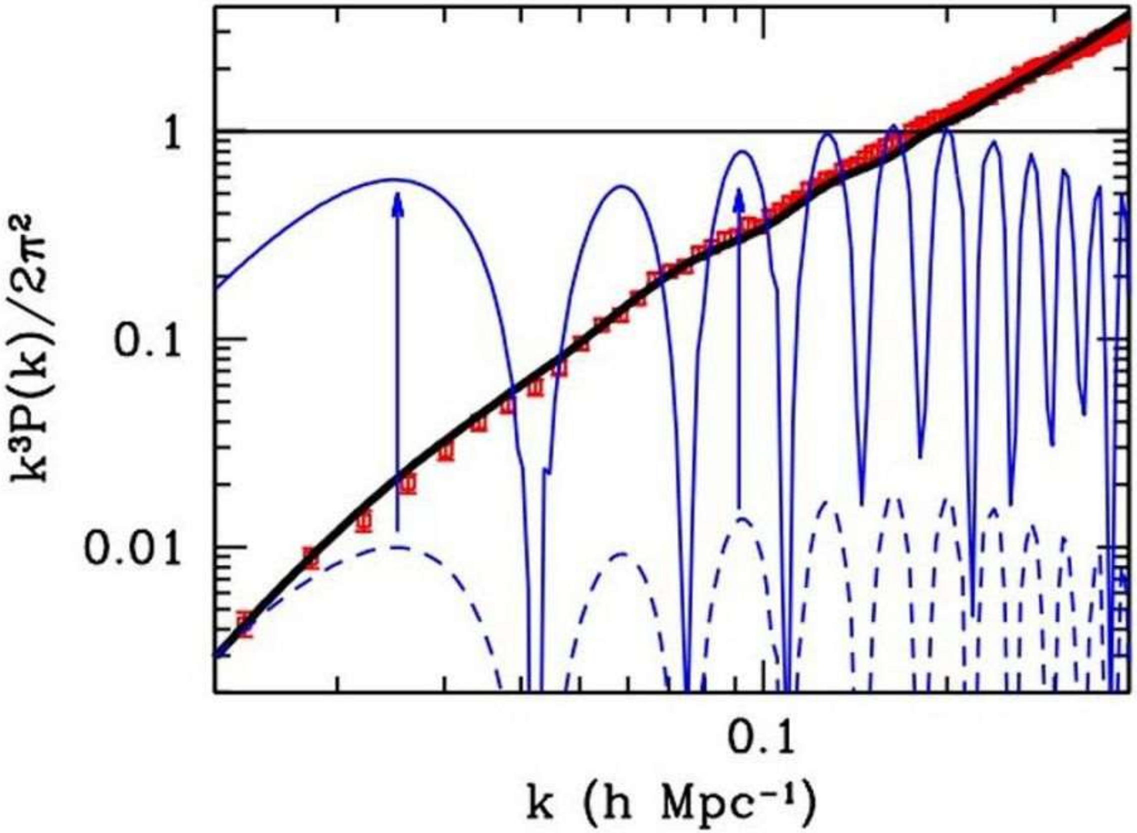
- 3 stages of evolution
  - Early on – all modes are outside the horizon
  - Intermediate times – each mode falls inside the horizon and the universe evolves from RD to MD.
  - Late times – all the modes evolve identically.
- Comparison with observations
  - Galaxy distribution – Matter power spectrum
  - CMB anisotropies – CMB power spectrum

# Matter Power Spectrum



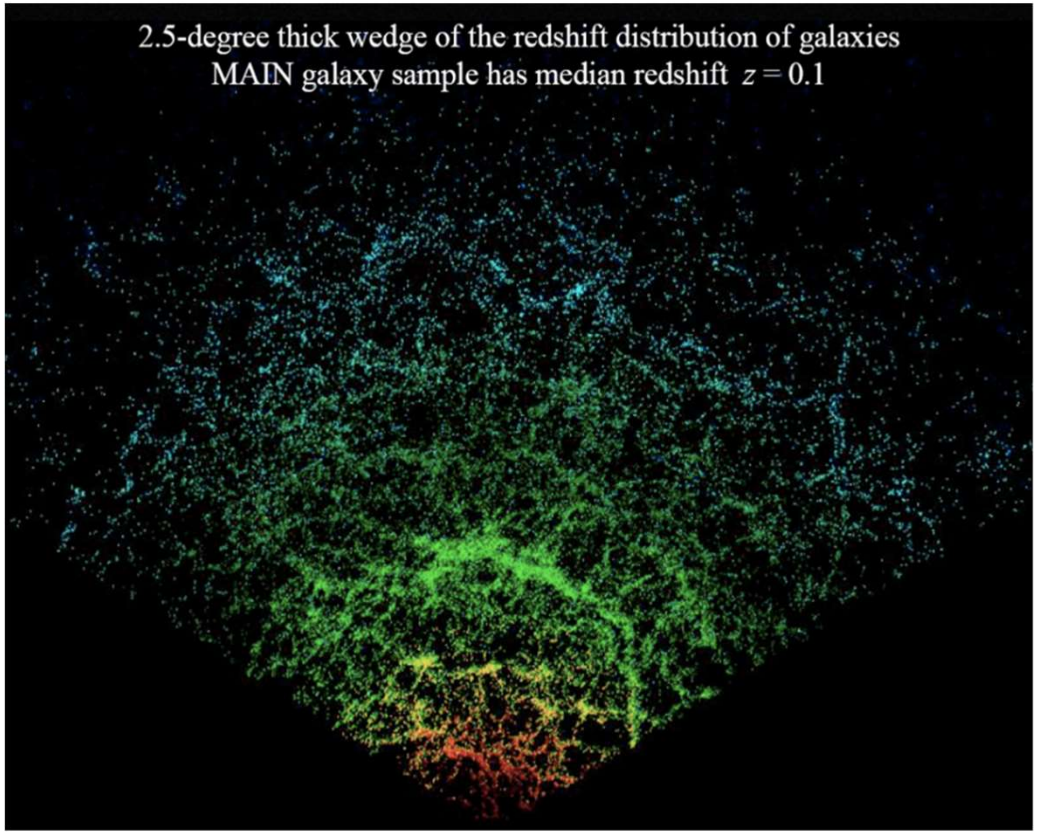
# Dark Matter

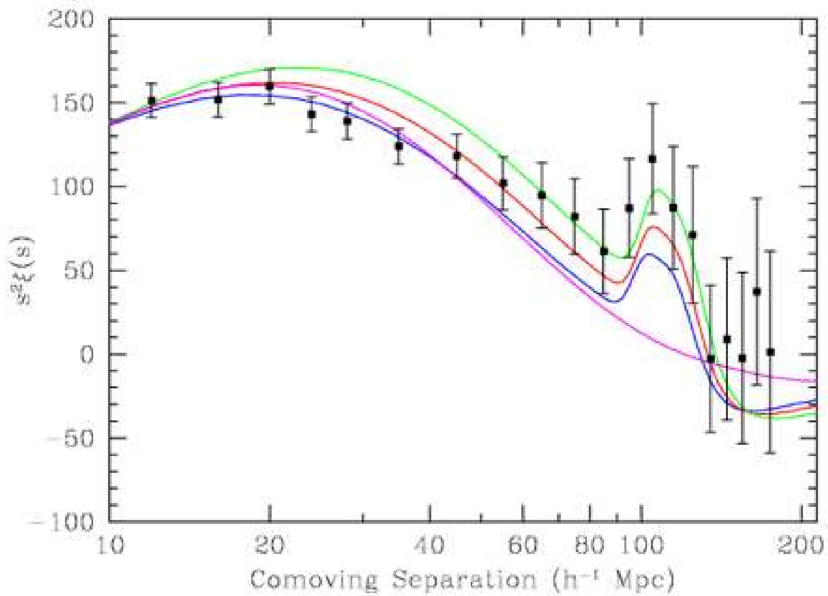



S. Dodelson,
from http://arxiv.org/abs/1112.1320

The data points from our observed galaxies (red points) and the predictions from a cosmology with dark matter (black line) line up incredibly well. The blue lines, with and without modifications to gravity, cannot reproduce this observation without dark matter.

# Baryon Acoustic Oscillations




- Dependence on $\Omega_{DM}$ (beginning of MD era)
- Effect of BAO

# CMB Anisotropies

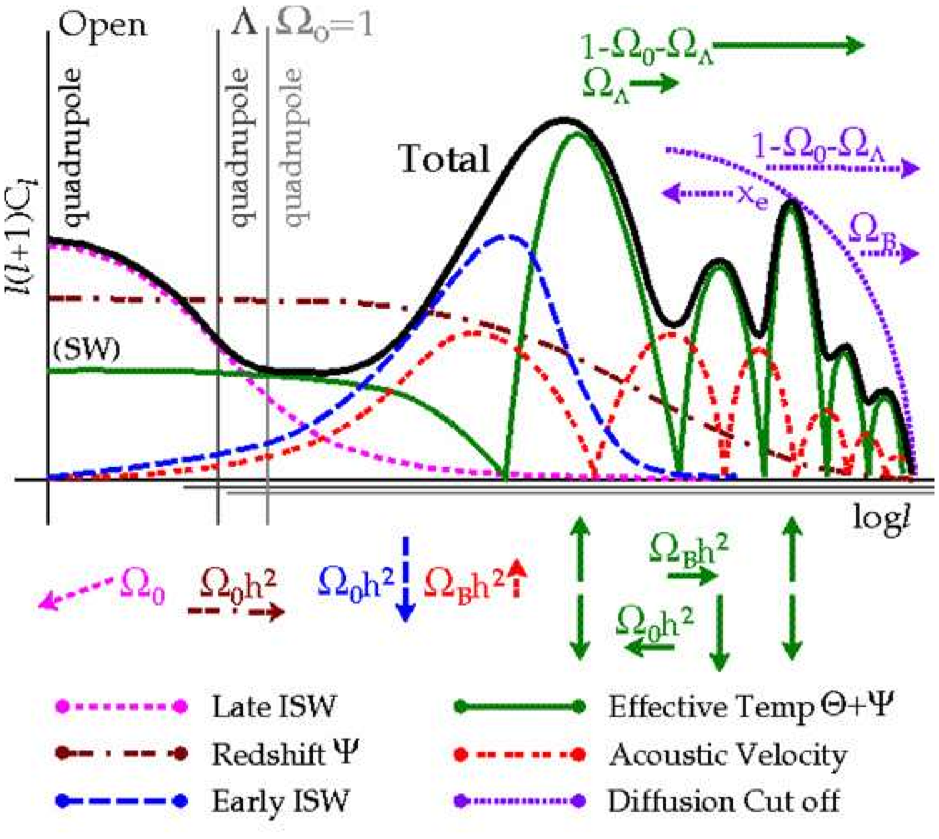
- Temperature fluctuations – Multipole moment expansion $\Theta(\mu, \eta) = \sum_{l=0}^{\infty} \Theta_l(\eta) P_l(\mu)$

- Before recombination – strongly coupled to baryons
  → Plasma oscillation (Baryon Acoustic Oscillation) $\ddot{\Theta} + k^2 c_s^2 \Theta = F$

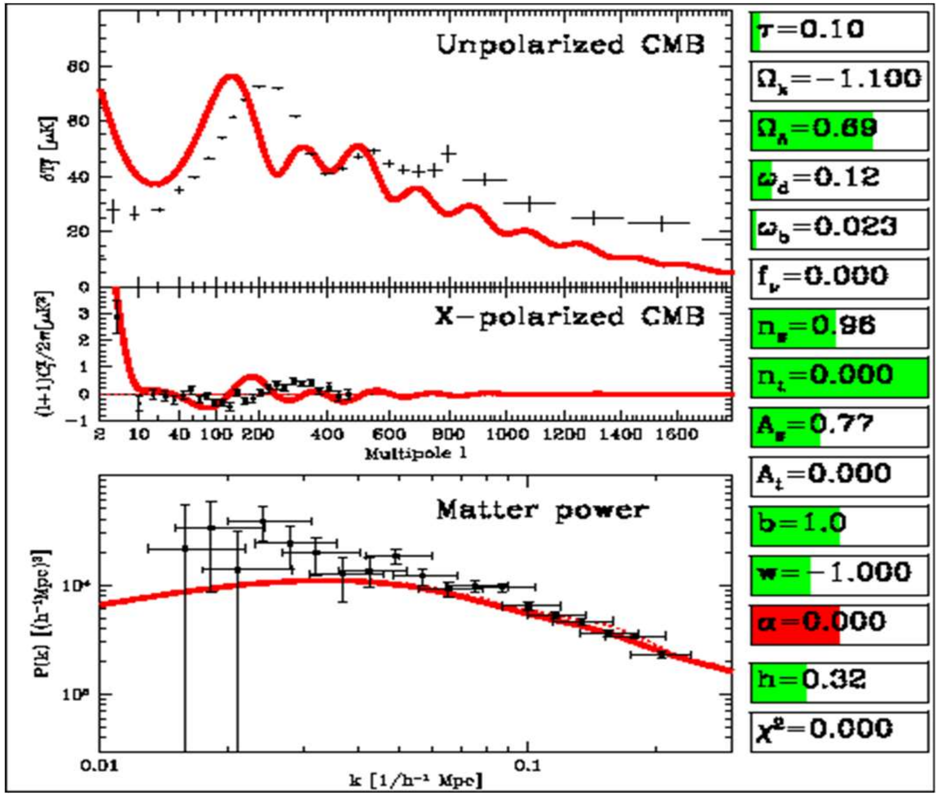
  - Sound speed and sound horizon $c_s^2 = \dfrac{1}{3(1 + R(\eta))}$ where $R = \dfrac{3\rho_b}{4\rho_\gamma}$ $\quad r_s(\eta) = \int_0^\eta d\eta' \, c_s(\eta')$

- After recombination – free propagation

# CMBA Spectrum

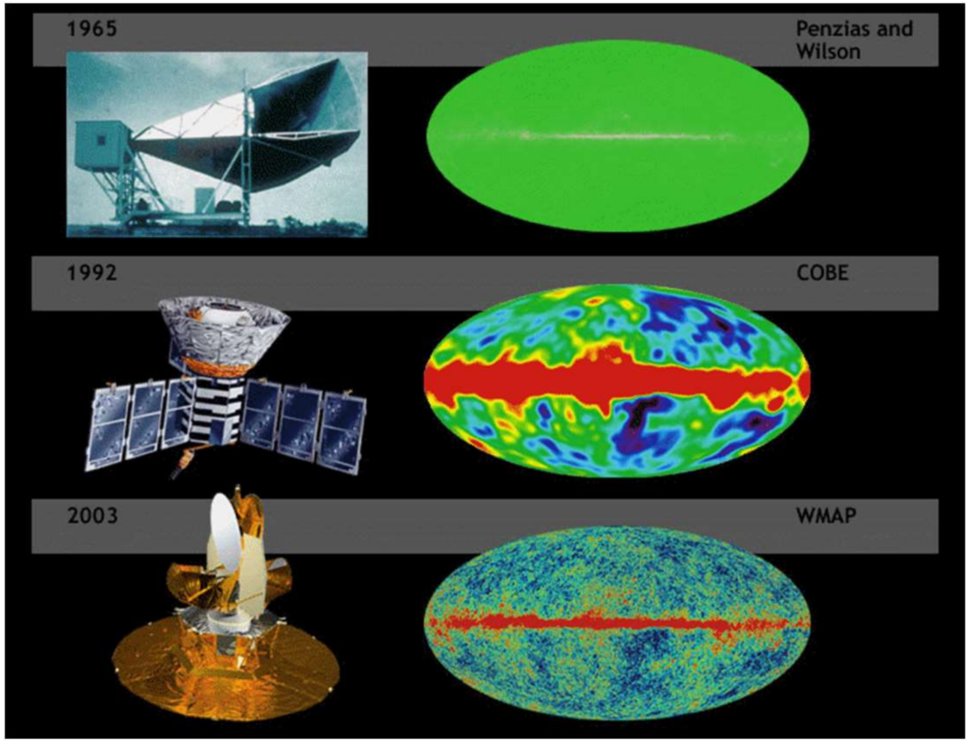



Hu, Sugiyama, & Silk (1995)

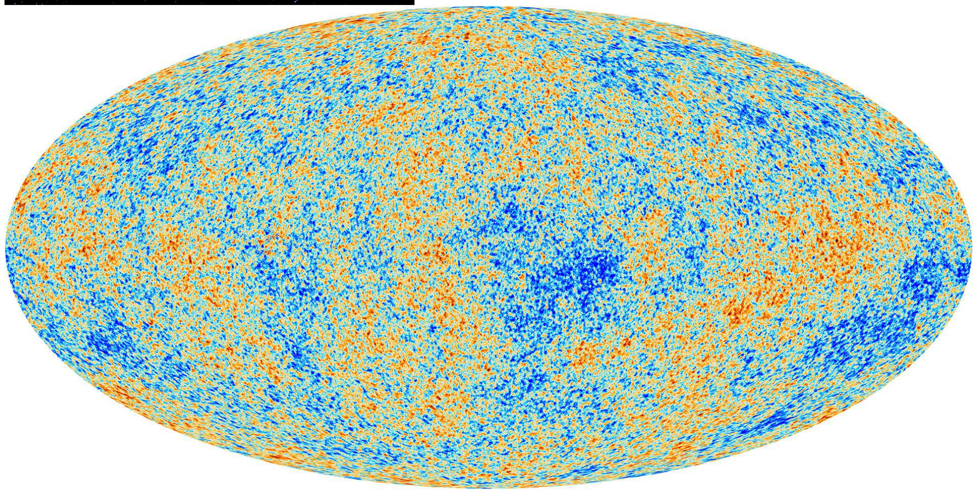



CMBA contain much information about our universe.

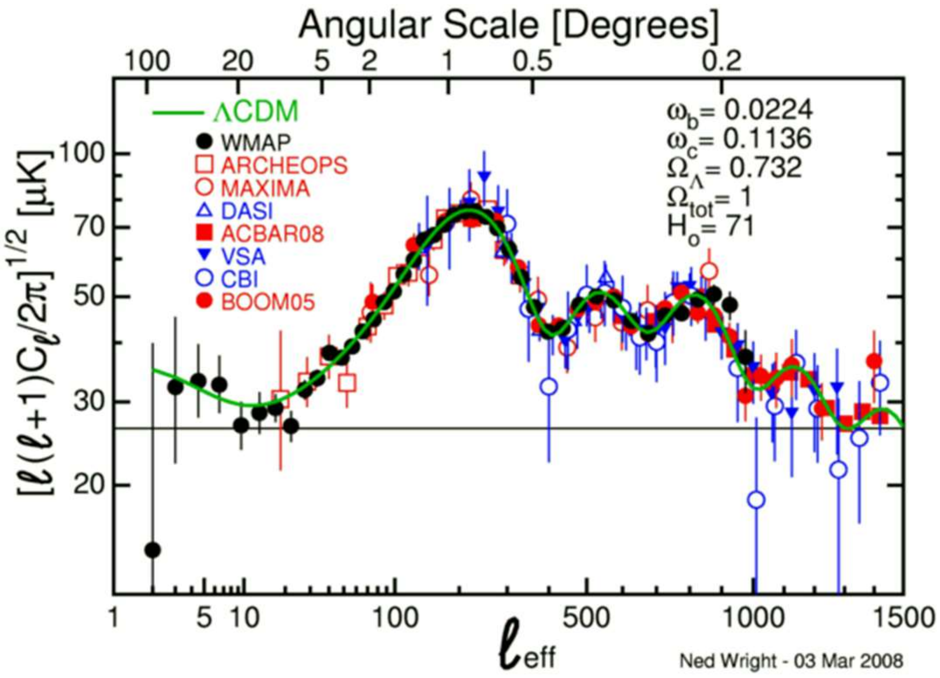
# Observations of CMBA

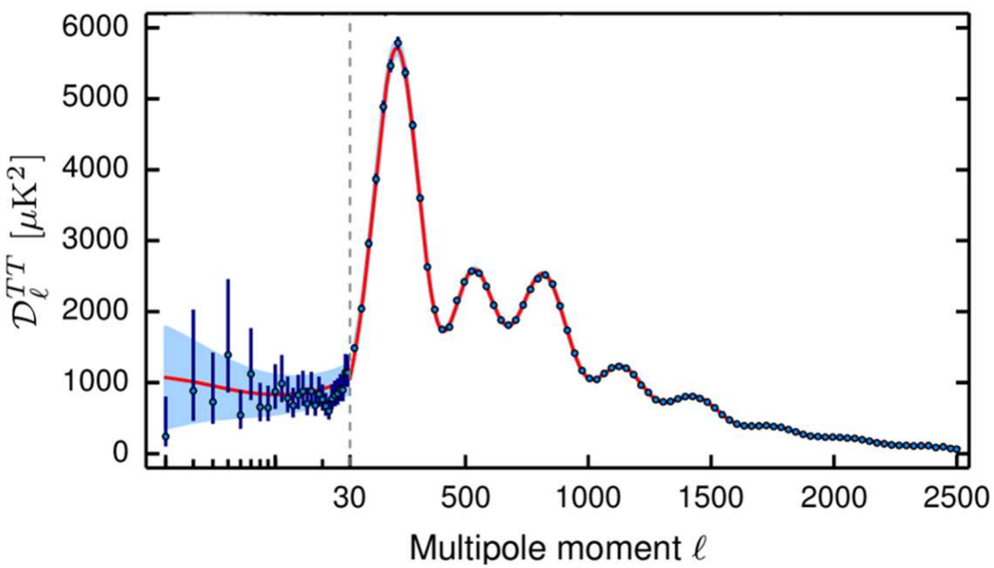






# Observed CMBA spectrum




Planck 2015 results. Constraints on inflation
Planck Collaboration, arXiv:1502.02114

# Cosmic Parameters

- Expansion parameters

  $H_0,\ q_0$

- Density parameters

  $\Omega_0,\ \Omega_b,\ \Omega_{\rm cdm},\ \Omega_\Lambda,\ \Omega_\gamma,\ \Omega_\nu$

- Density perturbation parameters

  $\Delta^2_R,\ n_s \qquad \dfrac{dn_s}{d\ln k},\ r,\ n_T$

- Cosmic environment parameters

  $\tau$

| Parameter | Combined |
|---|---|
| $\Omega_b h^2$ . . . . . . . . . . . . . . | $0.02233 \pm 0.00015$ |
| $\Omega_c h^2$ . . . . . . . . . . . . . . | $0.1198 \pm 0.0012$ |
| $100\theta_{\rm MC}$ . . . . . . . . . . . | $1.04089 \pm 0.00031$ |
| $\tau$ . . . . . . . . . . . . . . . . . | $0.0540 \pm 0.0074$ |
| $\ln(10^{10}A_s)$ . . . . . . . . . . | $3.043 \pm 0.014$ |
| $n_s$ . . . . . . . . . . . . . . . . | $0.9652 \pm 0.0042$ |
| $\Omega_m h^2$ . . . . . . . . . . . . | $0.1428 \pm 0.0011$ |
| $H_0$ [ km s$^{-1}$Mpc$^{-1}$] . . . | $67.37 \pm 0.54$ |
| $\Omega_m$ . . . . . . . . . . . . . . . | $0.3147 \pm 0.0074$ |
| Age [Gyr] . . . . . . . . . | $13.801 \pm 0.024$ |
| $\sigma_8$ . . . . . . . . . . . . . . . . | $0.8101 \pm 0.0061$ |
| $S_8 \equiv \sigma_8(\Omega_m/0.3)^{0.5}$ . . | $0.830 \pm 0.013$ |
| $z_{\rm re}$ . . . . . . . . . . . . . . . | $7.64 \pm 0.74$ |
| $100\theta_*$ . . . . . . . . . . . . | $1.04108 \pm 0.00031$ |
| $r_{\rm drag}$ [Mpc] . . . . . . . . . | $147.18 \pm 0.29$ |

# What we learn from inhomogeneity

1. Existence of primordial perturbations is inferred from CMBA.

$$\frac{\delta\rho}{\rho} \sim \frac{\delta T}{T} \sim 10^{-5}$$

2. Primordial power spectrum is close to scale invariance.

$$n_s \approx 1$$

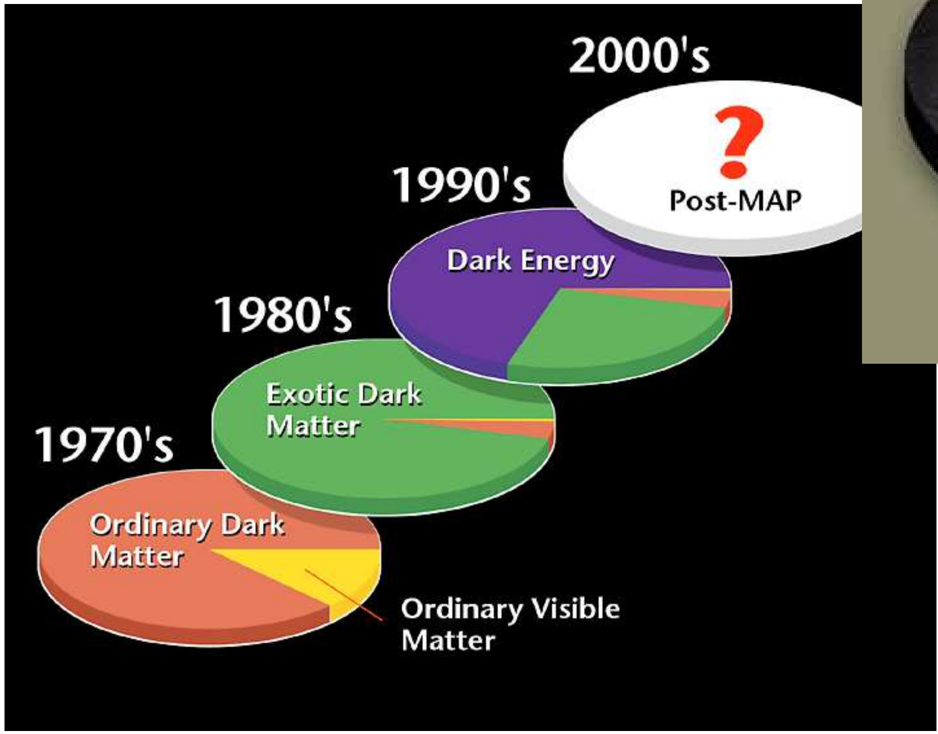
3. Existence of CDM is required to get the cosmic structures.

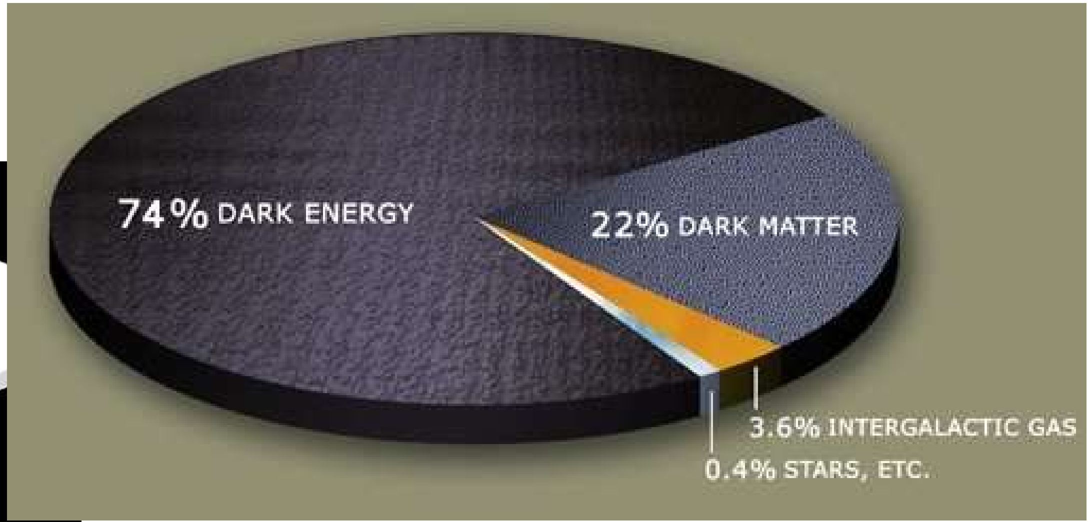
$$\Omega_b \approx 0.5, \quad \Omega_{\rm cdm} \approx 0.25$$

4. Space is flat, inferred from the peaks of BAO.

$$\Omega_0 = 1 \qquad\qquad \Omega_\Lambda \approx 0.7$$

# Cosmic Inventory




We get precise composition of the energy density of the universe from many observations.

**Standard Cosmology – $\Lambda$CDM model**

# In the Beginning …

# When and How did the Big Bang begin?

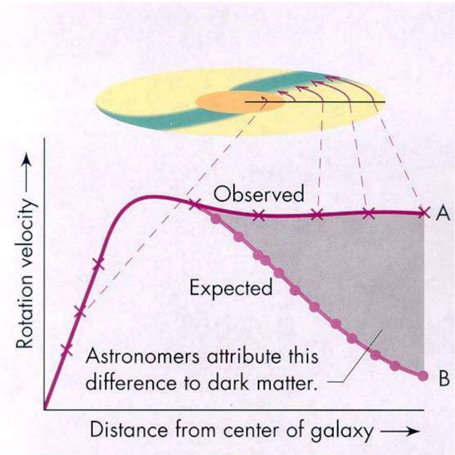
- Short comings of the big bang universe
- When and how was **baryon asymmetry** made?
- What is **dark matter** and How was it created?
- What is **dark energy**?
- Why is our universe so flat and so homogenous?
  - The **flatness problem**
  - the **horizon problem**
  - Scale Factor versus Horizon size – The current Hubble volume was not causally connected in the past.
- How was the **initial density perturbations** created?
  - Density perturbations at large scales which were not causally connected in the past cannot be created.


**All problems are related to the initial state of the big bang universe.**

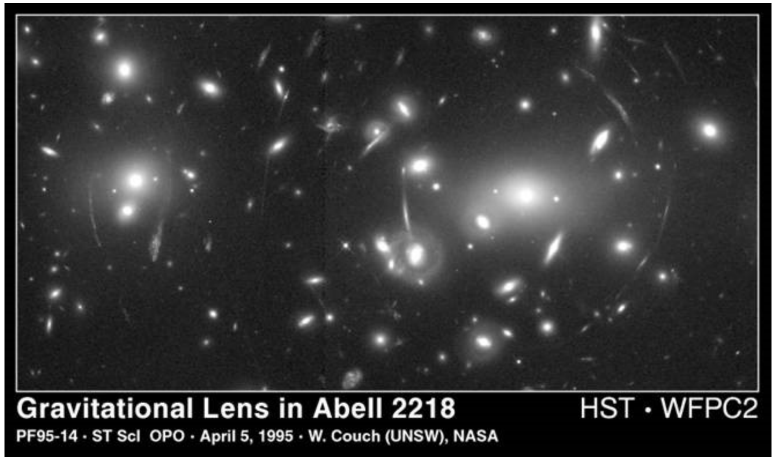
**What gives the solution, cosmology or particle physics?**

# Dark Matter

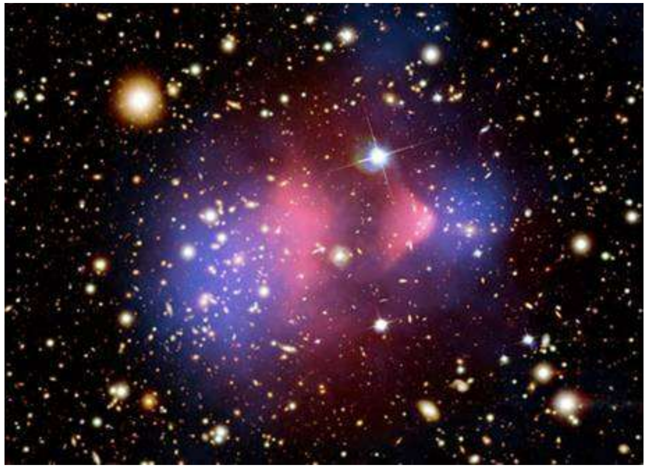
- To explain the observed LSS, dark matter is a necessity. $\Omega_{\mathrm{CDM}} \approx 0.25$
- Evidences of dark matter at various scales
  - Motion of galaxy clusters, Rotation curves of galaxies
  - Gravitational lens, mismatch in baryon and matter distribution




Rotation curve of galaxy shows the existence of dark matter outside the visible disk of galaxy.




Gravitational lensing effect reveals the existence of matter unseen between far galaxies and us.



Colliding clusters : Baryon(red, X-ray) and dark matter (blue, gravitational lensing) reside separately. Galaxies follow the dark matter distribution.

- Required properties of dark matter
  - Darkness (Transparency) – Shedding no light, weakly interacting with ordinary particles.
  - Matter – Pressureless at the time of structure formation (MD epoch)
  - Amount and Stability – The required amount must survive until now.

- Dark Matter Candidates
  - WIMP – LSP (Neutralino)
  - Very light scalar – Axion
  - Exotic – Gravitino, LKK, ...

- Dark matter search
  - Direct search : DM – Nucleon scattering
  - Indirect search : Annihilation or Decay products of DM

# Dark Matter – WIMP

- **Generic WIMP** scenario

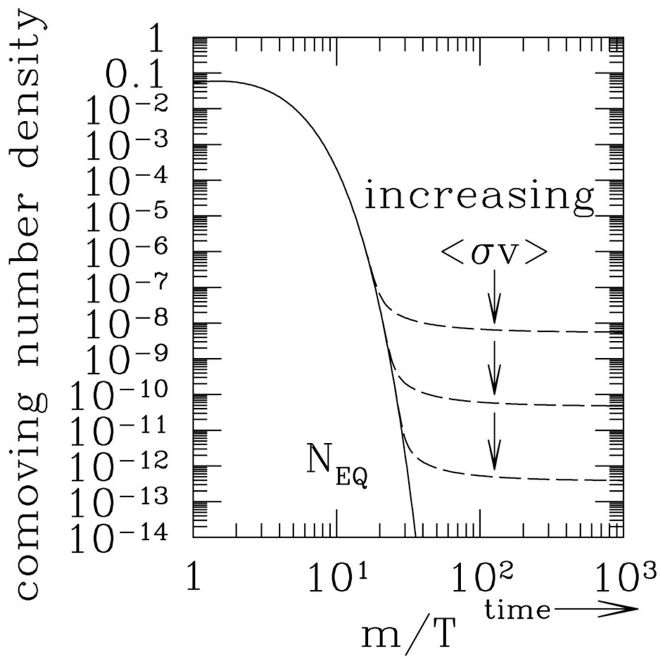
heavy particle $X + \bar{X} \underset{\text{weak interaction}}{\longleftrightarrow} \ell + \bar{\ell}$ light particle, tightly coupled to cosmic plasma (in equilibrium)

- Boltzmann eq. for X

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = \langle\sigma v\rangle \left(n_X^{(0)} - n_X^2\right)$$

$$\Omega_X = \left(\frac{4\pi^3 G g_*(m)}{45}\right)^{1/2} \frac{x_f T_0^3}{30\langle\sigma v\rangle \rho_c}$$

$$= 0.3h^{-2}\left(\frac{x_f}{10}\right)\left(\frac{g_*(m)}{100}\right)^{1/2} \frac{10^{-39}\,\mathrm{cm}^2}{\langle\sigma v\rangle}$$




- No explicit mass dependence.
- Relic abundance is mainly determined by cross section.
  – WIMP miracle

# Dark Matter – Axion

- **Axion**
  - Strong CP problem – Non-trivial vacuum structure of QCD makes $\theta_s F\tilde{F}$ observable, which breaks CP symmetry. Neutron EDM constrain $\bar{\theta} = \theta_s + \arg\det M_q < 10^{-10}$
  - Spontaneously broken PQ symmetry dynamically relaxes $\bar{\theta}$ to zero.
  - QCD instantons break PQ symmetry and gives the axion (Goldstone boson) a small mass $m_a = 6\,\mu\mathrm{eV}\,(10^{12}\,\mathrm{GeV}/f_a)$.

- **Coherent oscillation of scalar field**
  - massive scalar field in expanding universe : $\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0$
  - friction-dominated ( $H \gg m$ ) : $\phi \approx \phi_0$ (constant)
  - mass-dominated ( $H \ll m$ ) : oscillation about the minimum, matter-like (condensate)

- **Relic density of axion**
  - Setting of initial misalignment
  - Relic density

$$\Omega_a h^2 = 0.7 \left(\frac{f_a}{10^{12}\,\mathrm{GeV}}\right)^{7/6} \left(\frac{\bar{\theta}_i}{\pi}\right)^2$$

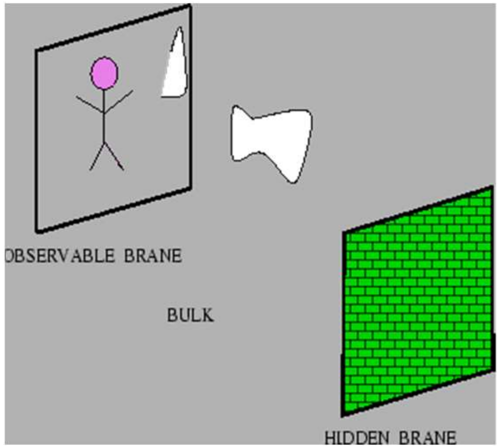
# Why is the expansion accelerating?

- **Option 1 – The energy density is dominated by Dark Energy.**
  - What is dark energy?
    - Negative pressure ($w < -\frac{1}{3}$ ) accelerates the expansion.
    - No interaction with ordinary matter (other than gravity)
  - Candidates for dark energy
    - Vacuum energy (Cosmological constant)
    - Slow-rolling scalar field (quintessence)

- **Option 2 – Gravity deviates from GR at scales larger than galaxies.**
  - DGP model (5D brane-world model)

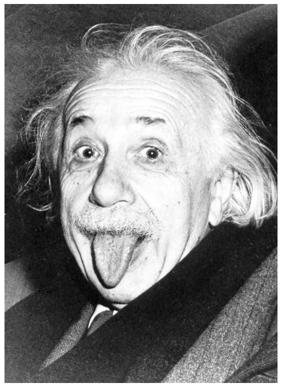
$$S_{\rm DGP} = \underbrace{-\frac{M_5^2}{2}\int d^5x\sqrt{-g}\,R_5}_{\text{5D gravity}} \underbrace{-\frac{M_P^2}{2}\int d^4x\sqrt{-h}\,R_4}_{\text{Magic piece of DGP}} + \underbrace{\int d^4x\sqrt{-h}\,\mathcal{L}_{\rm M}}_{\text{Matter living on 4D-brane}} + S_{\rm GH}$$

Gives acceleration, but with many other troubles …

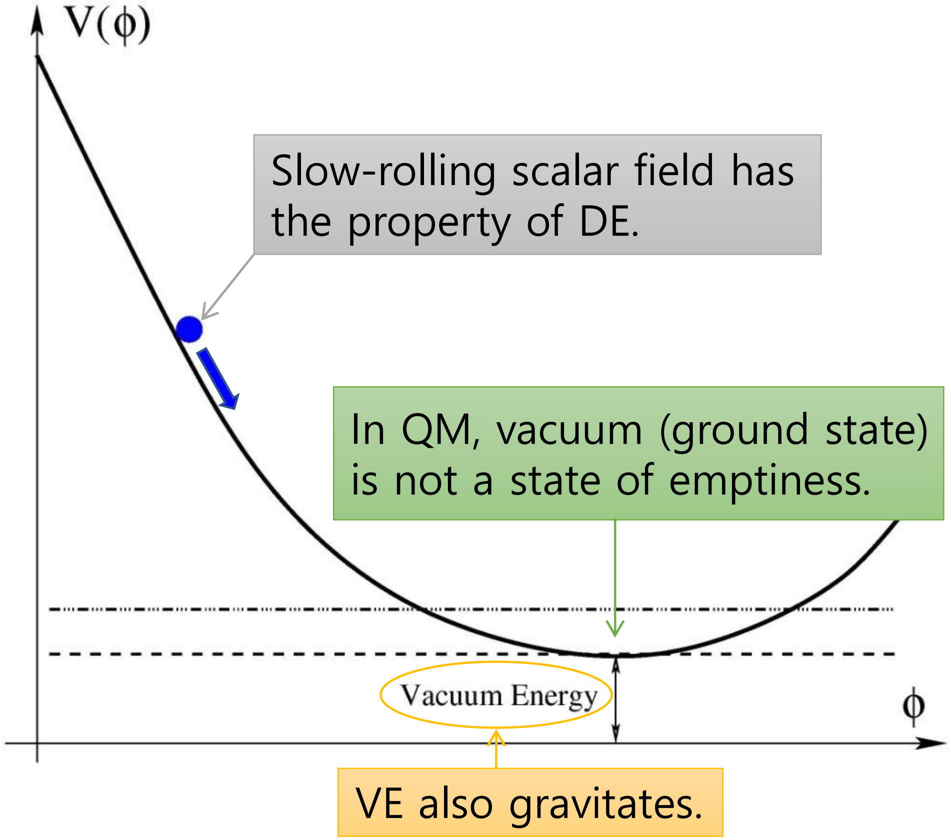


# Dark Energy

- **Cosmological constant (Vacuum energy density)**
  - Vacuum energy density also gravitates.
  - It has negative pressure.
- **Dynamical model**
  - Assume that V.E is set to 0 by some reason.
  - Then Slow-rolling scalar field can DE.







# Cosmological Constant

- Can we calculate the vacuum energy density?
  - QFT : VED is the sum of zero-point energy and subject to renormalization.

$$\rho_{\text{vac}} = \sum_{\text{all fields}} (-1)^F g_i \underbrace{\int \frac{d^3\vec{k}}{(2\pi)^3}}_{\text{sum over all modes}} \underbrace{\frac{1}{2}\sqrt{k^2+m^2}}_{\text{zero-point energy of each mode}} = \sum_{\text{all fields}} (-1)^F \, g_i \, \frac{k_{\text{max}}^4}{16\pi^2}$$

Fermions contribute negative.

$\infty$, diverge.

**Energy cutoff**
The highest energy at which QFT holds

- **Cosmological constant problem**

$$\rho_{\text{observed}} \sim \rho_{\text{crit}} \quad \Rightarrow \quad k_{\text{max}} = 0.01\,\text{eV}$$

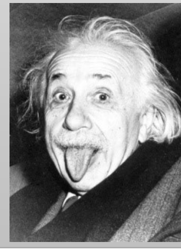
QFT prediction $k_{\text{max}} \sim M_P \approx 10^{19}\,\text{GeV}$

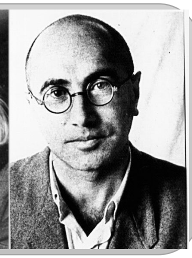
$$\frac{\rho_{\text{expected}}}{\rho_{\text{observed}}} \approx 10^{120}\ !$$

most serious naturalness problem

- **Need for Quantum Gravity ?**

  - Einstein 1917 – Introduced the cosmological constant to get a static universe from GR
  - Zeldovich 1968 – Identified the cosmological constant as the vacuum energy density and raised the cosmological constant problem





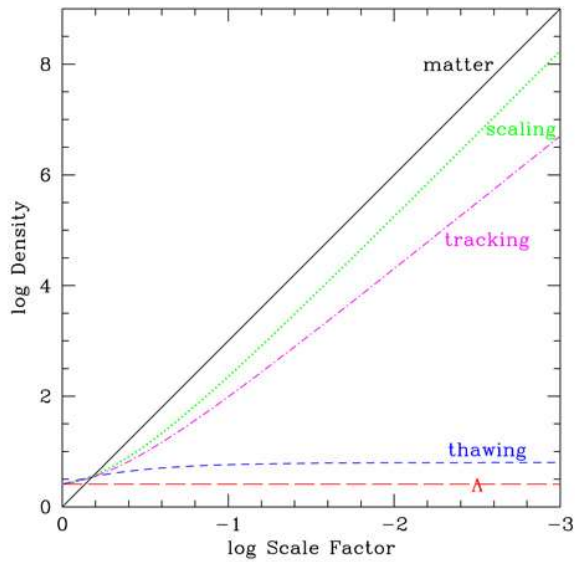
# Slow-Rolling Scalar

- Dynamics of homogeneous scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \quad \Rightarrow \quad \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad w(\phi) = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}$$

  - Dark Energy-like behavior : $\frac{1}{2}\dot{\phi}^2 \ll V(\phi) \quad \Rightarrow \quad w \approx -1$
  - Matter-like behavior : $\langle\frac{1}{2}\dot{\phi}^2\rangle \approx \langle V(\phi)\rangle \quad \Rightarrow \quad w \approx 0$




- Merits of slow-rolling scalar field
  - Equation of state that varies in time
  - Possibility to explain the present ratio of DE and DM densities (coincidence problem).
- Troubles of the slow-rolling scalar field as DE
  - The question why V.E.D is zero is still remained.
  - The mass of scalar is extremely small. ( $m_\phi = V''(\phi)^{1/2} \sim 3H_0 \approx 10^{-33}\,\mathrm{eV}$ )

# Flatness Problem

- Physical radius of curvature $R_{\rm cur} = \dfrac{H^{-1}}{|\Omega - 1|^{1/2}}$
- $\Omega$ as a function of $a$ $\quad \Omega - 1 = \dfrac{k}{H^2 a^2} \propto \dfrac{1}{\rho a^2} \propto \begin{cases} a, & \text{MD} \\ a^2, & \text{RD} \end{cases}$

$$t = t_{\rm BBN} \sim 1\,{\rm s} \;:\; |\Omega - 1| \lesssim 10^{-16},\; R_{\rm cur} \gtrsim 10^{8} H^{-1}$$
$$t = t_{\rm P} \sim 10^{-43}\,{\rm s} \;:\; |\Omega - 1| \lesssim 10^{-60},\; R_{\rm cur} \gtrsim 10^{30} H^{-1}$$

- Big bang universe requires **a very special initial condition**.
- If $\Omega \sim 1$ and $R_{\rm cur} \sim H^{-1}$ at Planck time,

  $k > 0$ : the universe re-collapse within few $\times 10^{-43}\,{\rm s}$

  $k < 0$ : temperature 3K reached at $t \approx 10^{-11}\,{\rm s}$

  The natural time scale for cosmology is $10^{-43}\,{\rm s}$, while the age of universe is $10^{60} \times 10^{-43}\,{\rm s}$ .

# Horizon Problem

- Comoving horizon grows during RD and MD eras.
  This means that the particle horizon grows faster than the scale factor.

$$\eta = \int_0^t \frac{dt'}{a(t')} \propto \begin{cases} a^{1/2}, & \text{MD} \\ a, & \text{RD} \end{cases} \qquad a \propto \begin{cases} t^{2/3}, & \text{MD} \\ t^{1/2}, & \text{RD} \end{cases} \qquad d_H \propto t$$

- **Horizon problem**
  - **Large-Scale Smoothness Problem**
    CMB we see today is very close to isotropy ( $\delta T/T \approx 10^{-5}$ ). How can this be?
    The largest scales observed today have entered the horizon just recently, long after decoupling. Microscopic causal physics cannot make it !
  - **Small-Scale Inhomogeneity Problem**
    Where does the density perturbation ( $\delta\rho/\rho \approx \delta T/T \approx 10^{-5}$ ) come from?
    E.g. $(\delta\rho/\rho)_{\lambda_{\rm galaxy}}$ – The galaxy scale was outside the horizon in the past.

- The entropy within a horizon volume

$$S_{\mathrm{H}} = s \cdot \frac{4\pi}{3} d_{\mathrm{H}}^3 \approx \begin{cases} 0.05 g_*^{-1/2} (M_P/T)^3, & \mathrm{RD} \\ 3 \times 10^{87} (\Omega_0 h^2)^{-3/2} (1+z)^{-3/2}, & \mathrm{MD} \end{cases}$$

$$S_{\mathrm{H}}(t = t_0) = 10^{88} \;\leftarrow\; 10^5 \text{ Hubble volumes at recombination}$$

$$S_{\mathrm{H}}(t = t_{\mathrm{rec}}) = 10^{83}$$

- Monopole problem
  - Phase transition in the early universe can leave topological defects.
  - Among topological defects, string is not harmful,
    but domain walls and monopoles can over-close the universe.
  - Many GUTs predict the existence of magnetic monopoles, which must be avoided in cosmology.

# Inflation

- What is inflation and how does it occur ?
  - Epoch of accelerating expansion, preceding RD epoch
  - Scalar field slowly rolling along the (nearly flat) potential

- What are good things of inflation ?
  - Inflation can make the universe flat and homogeneous.

    If the scale factor grows by more than $e^{60}$, flatness and horizon problems are solved.
  - Inflation generates the density perturbations. (Inflation explains our existence.)

    **Quantum fluctuations** $\delta\phi$ $\rightarrow$ **Density perturbations** $\delta\rho$

- Transition from Inflation to Hot Big Bang
  - Oscillation and decay of scalar field $\rightarrow$ (Re)Heating $\rightarrow$ Big Bang universe (RD epoch)

- When and how did inflation begin ?
  - Endless questions again.... ?

- **Basic ideas**
  - An early epoch of accelerating expansion solves the horizon and flatness problems.
  - Matter having large negative pressure is needed, which can be realized by a scalar field.

- **How to solve the horizon problem**
  - Comoving horizon : $\eta = \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da'}{a'} \frac{1}{a'H(a')}$

    Comoving Hubble radius $H^{-1}/a$
    = the distance over which particles can travel in one expansion time

    $r > \eta$ : never have communicated $\qquad$ $r > H^{-1}/a$ : cannot communicate now

  - It is possible to have $\eta \gg H^{-1}/a\big|_{t_0}$ : $\qquad H^{-1}/a\big|_{\mathrm{early}} \gg H^{-1}/a\big|_{\mathrm{now}}$

    That is, $\eta$ get contribution mostly from early epoch.
    In RD or MD, 1/aH increase with time, so the latter epoch contributions dominate.

- In the early epoch, the comoving Hubble radius decreased.
  1/aH must decrease ⇒ aH must increase.

$$\frac{d}{dt}(aH) = \frac{d^2}{dt^2}\,a > 0$$ accelerating expansion, **inflation**

- Quantitative understanding : Suppose the energy scale of inflation ~ $10^{15}$ GeV.

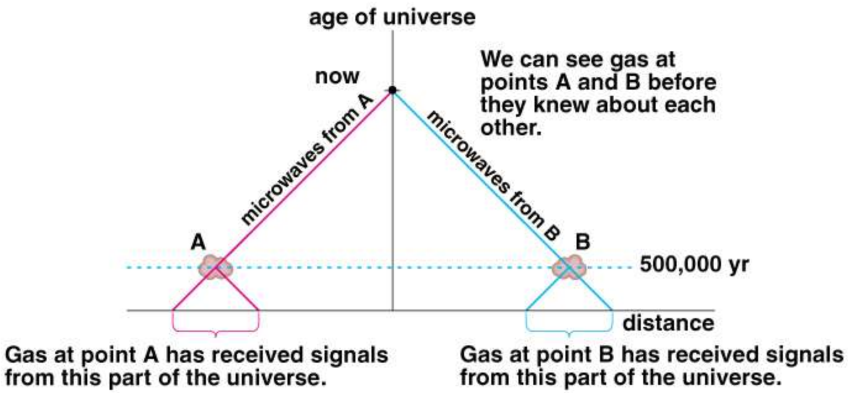
$$(aH)^{-1}\big|_{T\approx 10^{15}\ \mathrm{GeV}} = 10^{-28}\,(aH)^{-1}\big|_{T=T_0}$$

During inflation, the comoving Hubble radius had to decrease by, at least, 28 orders of magnitude.

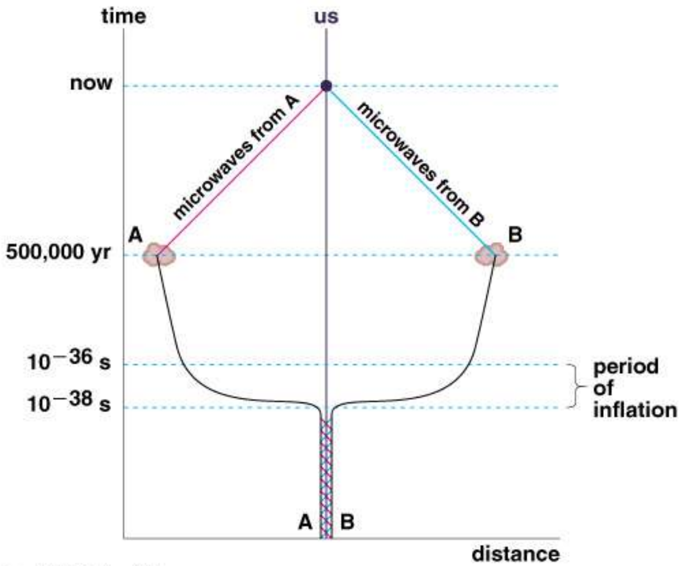
Most common way to construct a model – H ~ constant

$$H = \frac{\dot{a}}{a} = \mathrm{const.} \;\; \Rightarrow \;\; a(t) = a_e e^{H(t-t_e)}$$

$$(aH)^{-1} \propto e^{-Ht}, \quad 10^{28} \approx e^{64}$$ More than 60 e-folds are needed.


age of universe
now
We can see gas at points A and B before they knew about each other.
microwaves from A
microwaves from B
A
B
500,000 yr
distance
Gas at point A has received signals from this part of the universe.
Gas at point B has received signals from this part of the universe.




time
us
now
microwaves from A
microwaves from B
A
B
500,000 yr
$10^{-36}$ s
$10^{-38}$ s
period of inflation
A
B
distance



- **Negative pressure** is required for accelerating expansion.

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_P^2}(\rho + 3p) > 0 \quad \Rightarrow \quad p < -\frac{1}{3}\rho < 0$$

- **Implementation** using a scalar field $\phi(x^\mu)$ with Lagrangian $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$

Energy-momentum tensor
$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}$ $\Rightarrow$ $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$

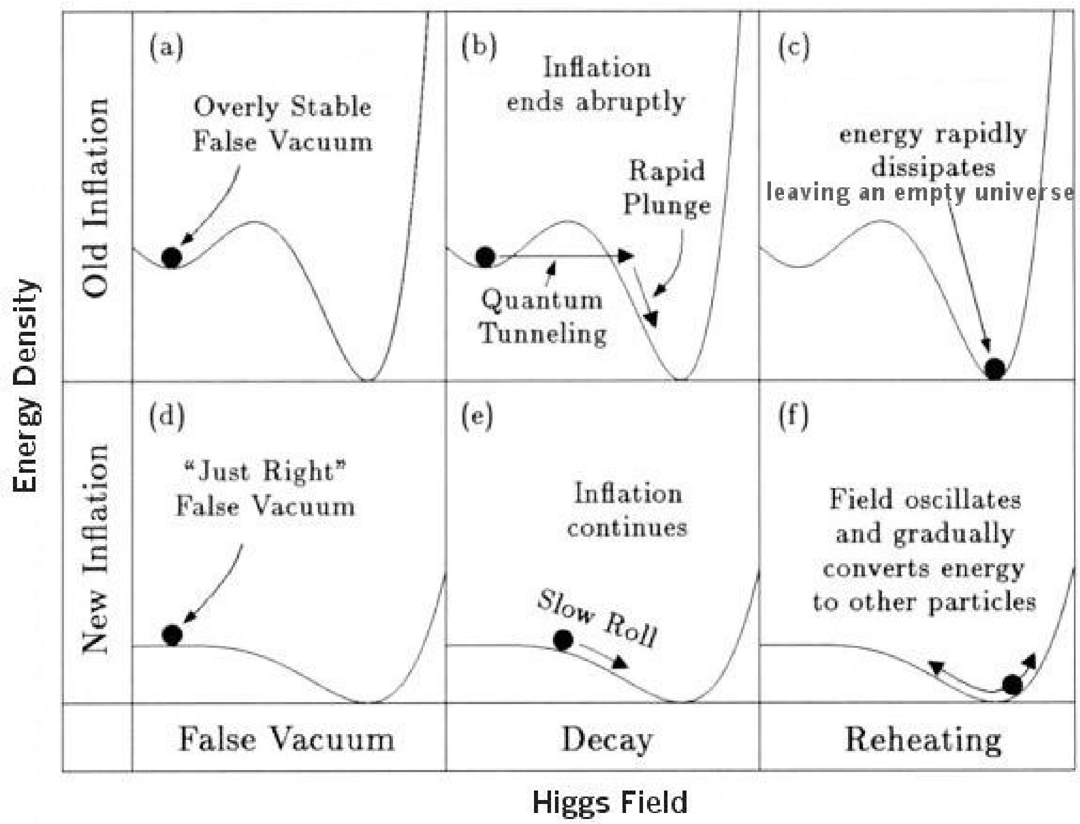
Negative pressure : potential energy dominates over kinetic energy, $V(\phi) > \frac{1}{2}\dot{\phi}^2$

- Scalar field trapped in a false vacuum
- Scalar field slow-rolling toward its true vacuum

# Old and New Inflation

- A. Guth (Old inflation, 1980)
  - On-set – Trapping at false vacuum due to thermal phase transition
  - Exit – Decay to true vacuum via bubble nucleation
  - Barrier between false and true vacuum – Graceful exit problem

- A. Linde (New inflation, 1980)
  - On-set – Trapping at false vacuum due to thermal phase transition
  - Exit – Rolling down to true vacuum
  - Flat potential and Quantum fluctuations – Eternal inflation



# Slow-roll Inflation

- Evolution of the universe dominated by a homogeneous scalar field

$$H^2 = \frac{1}{3M_P^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \qquad \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

- Slow-rolling : dynamics dominated by the friction, $\ddot{\phi} \ll 3H\dot{\phi}$ , $H \approx \text{constant (slow-varying)}$
- Consistency requires two **slow-roll parameters** are small.

$$\epsilon = \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2, \quad \eta = M_P^2\left(\frac{V''}{V}\right) \qquad \epsilon \equiv \frac{d}{dt}H^{-1}, \quad \eta = \frac{\ddot{\phi}}{H\dot{\phi}}$$

- On-set : Chaotic Inflation, … ??
- Exit : Break-down of slow-roll condition near the potential minimum

# Generation of Gravitational Wave

- **Inflation generates gravitational waves (tensor perturbations).**
- Tensor perturbations ($h_+$, $h_\times$) satisfy the linear equation $\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} + k^2 h = 0$
- Quantization of tensor perturbations
  - Introduce the field having mass dimension $\tilde{h} \equiv \frac{M_P}{\sqrt{2}}\, a\, h\ , \quad \ddot{\tilde{h}} + \left(k^2 - \frac{\ddot{a}}{a}\right)\tilde{h} = 0$
  - Annihilation and creation operator $\hat{\tilde{h}}(\vec{k},\eta) = v(k,\eta)\,\hat{a}_{\vec{k}} + v^*(k,\eta)\,\hat{a}^\dagger_{\vec{k}}\ , \quad \ddot{v} + \left(k^2 - \frac{\ddot{a}}{a}\right)v = 0$
  - Quantization $[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = \delta(\vec{k} - \vec{k}') \qquad \langle \hat{\tilde{h}}^\dagger(\vec{k},\eta)\hat{\tilde{h}}(\vec{k}',\eta)\rangle = |v(k,\eta)|^2\ (2\pi)^3\delta^3(\vec{k} - \vec{k}')$
  - Vacuum fluctuation $\langle \hat{h}^\dagger(\vec{k},\eta)\hat{h}(\vec{k}',\eta)\rangle = \frac{2}{M_P^2 a^2}\ |v(k,\eta)|^2\ (2\pi)^3\delta^3(\vec{k} - \vec{k}')$
  - Power spectrum $P_h(k) = \frac{2}{M_P^2\, a^2}\ |v(k,\eta)|^2$

- During inflation : $\frac{\ddot{a}}{a} \approx \frac{2}{\eta^2}$ $\quad \ddot{v} + \left(k^2 - \frac{2}{\eta^2}\right) v = 0$ $\quad v = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)$

$$\boxed{\begin{array}{l} \text{For } k\eta \gg 1 \text{ (sub-horizon)}, \quad h \propto \frac{v}{a} \propto \frac{1}{a} \\ \text{For } k\eta \ll 1 \text{ (super-horizon)}, \quad h \propto \frac{v}{a} \propto H \end{array}}$$

- After horizon crossing ( $k\eta \sim 1, \text{ or } aH \sim k$ )
  Power spectrum approaches to constant

$$P_h(k) = \left. \frac{H^2}{M_P^2\, k^3} \right|_{aH\sim k}$$

- Inflation produces gravitons (gravitational waves).

# Generation of Density Perturbation

- **Inflation generates density perturbations.**
- Quantum fluctuations of the inflaton satisfy the same linear equation as the gravitational waves.

$$\phi(\vec{x},t) = \phi^{(0)}(t) + \delta\phi(\vec{x},t)$$

$$\ddot{\delta\phi} + 2aH\dot{\delta\phi} + (k^2 + a^2V'')\delta\phi = 0$$

negligible during slow-roll inflation

- Power spectrum

$$P_{\delta\phi}(k) = \left.\frac{H^2}{2\,k^3}\right|_{aH\sim k} \qquad \text{cf. } P_h(k) = \left.\frac{H^2}{M_P^2\,k^3}\right|_{aH\sim k}$$

- Perturbation spectrum of $\Psi$
  - Why are we justified in neglecting $\Psi$ until horizon crossing?
  - How do the perturbations get transferred from $\delta\phi$ to $\Psi$?

  - Curvature perturbation – conserved for super-horizon mode
    - For sub-horizon and just-left-horizon modes, $\Psi$ is negligible.
    - Post-inflation, perturbation shared between $\delta\phi$ and $\Psi$.

- Curvature perturbation – conserved for super-horizon mode
  - For sub-horizon and just-left-horizon modes, $\Psi$ is negligible.
  - Post-inflation, perturbation shared between $\delta T^{0}{}_{i}$ and $\Psi$.

$$\zeta \equiv -\frac{ik_i \delta T^0{}_i H}{k^2(\rho+p)} - \Psi$$

$$\zeta\Big|_{\text{horizon crossing}} = -\frac{aH\delta\phi}{\dot{\phi}^{(0)}}\ , \quad \zeta\Big|_{\text{post inflation}} = -\frac{3}{2}\Psi \quad \Rightarrow \quad \Psi\Big|_{\text{post inflation}} = \frac{2}{3} aH \frac{\delta\phi}{\dot{\phi}^{(0)}}\Bigg|_{\text{horizon crossing}}$$

$$P_\Psi = \frac{4}{9}\left(\frac{aH}{\dot{\phi}^{(0)}}\right)^2 P_{\delta\phi}\Bigg|_{aH=k} = \frac{2}{9k^3}\left(\frac{aH^2}{\dot{\phi}^{(0)}}\right)^2\Bigg|_{aH=k} = \frac{H^2}{9\epsilon\, M_P^2 k^3}\Bigg|_{aH=k} = \frac{16\pi}{9M_P^2 k^3}\left(\frac{H^2 V^2}{V'^2}\right)\Bigg|_{aH=k}$$

# Density Perturbation in Slow-roll Inflation

- **Density perturbation in slow-roll inflation**
  - Slow-roll parameters : $\epsilon = \frac{M_P^2 V'^2}{2V^2}, \quad \eta = \frac{M_P^2 V''}{V}, \quad \xi = \frac{M_P^4 V' V'''}{V^2}$
  - Spectral index : $n_s - 1 = 2\eta - 6\epsilon$
  - Running of spectral index : $n_s' \equiv dn_s/d\ln k = 16\epsilon\eta - 24\epsilon^2 - 2\xi$
  - Tensor to scalar ratio : $r = 16\epsilon$
  - Number of e-folds : $N(\phi) = \int_t^{t_{\rm end}} H\, dt = \frac{1}{M_P^2}\int_{\phi_{\rm end}}^{\phi} \frac{V(\phi)}{V'(\phi)} d\phi$

# Chaotic Inflation

- Single scalar field with a potential $V(\phi) = \Lambda^4 \left(\frac{\phi}{M_P}\right)^p$
- Slow-roll (condition) is achieved for (trans-Planckian field value) $\phi \gtrsim \phi_{\rm end} \sim M_P$ $\quad \epsilon = \frac{p}{4N_*},\ \eta = \frac{p-1}{2N_*}$
- Primordial density perturbation requires (fine tuning of parameter) $\frac{\Lambda}{M_P} \approx 10^{-2} \epsilon^{1/4}$
- Spectral index and its running : $n_s - 1 = -\frac{2+p}{2N_*}$ $\quad n_s' = -\frac{2+p}{2N_*^2}$
- Tensor to scalar ratio : $r = \frac{4p}{N_*}$
- Setting of initial condition for inflation – thermal fluctuation?

# Hybrid Inflation

- Two scalar fields

$$\begin{aligned} \phi \quad &\rightarrow \quad \text{Slow-roll inflation} \\ \psi \quad &\rightarrow \quad \text{To end inflation} \end{aligned}$$

- Illustrative example : $V(\phi,\psi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\lambda'\psi^2\phi^2 + \frac{1}{4}\lambda(M^2-\psi^2)^2$

  - For $\phi > \phi_c = \lambda M^2/\lambda'$, $V$ has the minimum at $\psi = 0$.

$$V_{\text{eff}}(\phi) = V(\phi, \psi = 0) = V_0 + \frac{1}{2}m^2\phi^2 \quad \Rightarrow \quad \text{Slow-roll Inflation}$$

  - For $\phi < \phi_c$, $V$ has the minimum at $\psi = \pm M$. $\quad \Rightarrow \quad$ End of Inflation

# Higgs Inflation

- Higgs field **non-minimally coupled to gravity**

$$S = \int d^4x\sqrt{-g}\left(-\frac{M_P^2}{2}R - \xi H^\dagger H R + \mathcal{L}_{\mathrm{SM}}\right) \quad \Rightarrow \quad \int d^4x\sqrt{-g}\left(-\frac{M_P^2 + \xi\phi^2}{2}R + \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{\lambda}{4}\phi^4\right)$$

- Jordan frame ⇒ Einstein frame by conformal transformation

$$g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu}, \quad \Omega = 1 + \xi\phi^2/M_P^2$$

$$S = \int d^4x\sqrt{-\tilde{g}}\left(-\frac{M_P^2}{2}\tilde{R} + \frac{1+(6\xi+2)\xi\phi^2/M_P^2}{2(1+\xi\phi^2/M_P^2)^2}\left(\partial_\mu\phi\right)^2 - \frac{\lambda\phi^4}{4(1+\xi\phi^2/M_P^2)^2}\right)$$

- Canonical kinetic term by field redefinition

$$\frac{d\chi}{d\phi} = \frac{\sqrt{1+(6\xi+1)\xi\phi^2/M_P^2}}{1+\xi\phi^2/M_P^2} \qquad S = \int d^4x\sqrt{-\tilde{g}}\left(-\frac{M_P^2}{2}\tilde{R} + \frac{1}{2}\left(\partial_\mu\chi\right)^2 - \frac{\lambda\phi(\chi)^4}{4(1+\xi\phi(\chi)^2/M_P^2)^2}\right)$$

- Slow-roll condition is satisfied for $\phi > \phi_e \approx M_P/\sqrt{\xi}$

- Primordial density perturbation requires $\xi \approx 47,000\sqrt{\lambda}$

# $R^2$ Inflation

- R² **(curvature-squared) inflation – Starobinsky model**
- Einstein gravity + **R² term**

$$\boldsymbol{S = \int d^4x\sqrt{-g}\left(-\frac{M_P^2}{2}R + \alpha R^2\right)}$$

- Does quantum gravity induce more derivatives?
- $R^2$ term introduces an additional degree of freedom, because it contains more derivatives
- Conformal transformation : $\tilde{g}_{\mu\nu} = [1 + 2\alpha R]\, g_{\mu\nu}$
- Field redefinition :

$$\psi = \sqrt{3/2}\ln(1 + 2\alpha R) \qquad S = \int d^4x\sqrt{-\tilde{g}}\left(-\frac{M_P^2}{2}\tilde{R} + (\tilde{\partial}\psi)^2 - \frac{1}{4\alpha}(1 - e^{\sqrt{2/3}\psi})^2\right)$$

- Slow-roll condition is satisfied for …
- Primordial density perturbation requires …

# (Re)Heating

- **(Re)Heating, or Thermalization**
  - Inflation cools down the universe, almost to temperature zero.
  - After the end of inflation, hot thermal radiations needs to be produced, starting the hot big bang universe.
  - Because the nature of the inflation is not known, this process is still poorly understood.
  - Energy source of – Large potential energy of the inflaton field
  - **Inflaton decay** Inflaton decay into relativisitic (standard model) particles during it oscillates around the potential minimum.

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi \dot{\phi} + m^2\phi = 0$$ During oscillation, $\langle \dot{\phi}^2 \rangle = \langle m\phi^2 \rangle$, pressureless matter

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi \qquad \dot{\rho_R} + 4H\rho_R = \Gamma_\phi \rho_\phi \qquad H^2 = \frac{1}{3M_P^2}(\rho_\phi + \rho_R) \qquad \boldsymbol{T_R \approx 0.2\sqrt{M_P \Gamma_\phi}}$$

- **Parametric resonance** Particles can be produced more efficiently through parametric resonance.

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + m_\chi^2 + g^2\phi^2\right)\chi_k = 0$$

Oscillation of the inflaton field may cause parametric resonance.