Boundary stratification of the loop momentum amplituhedron

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Work in progress with L. Ferro and J. Stalknecht

Introduction

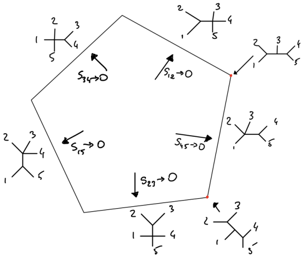
- Positive geometries provide us with an excellent tool to study **properties of scattering amplitudes** in various theories: $\mathcal{N}=4$ sYM, ABJM, ϕ^3, etc
- Some of these properties are **difficult/impossible to capture** using any other available methods
- In this talk I will focus on the singularity stratifications of **tree-level amplitudes** and **loop-level amplitude integrands** for $\mathcal{N}=4$ sYM

  $\rightarrow$ for a parallel story for ABJM – see Jonah's talk

Why positive geometries?

- **Positive geometries** encode the singularity structure of amplitudes

  E.g. Associahedron for the ϕ^3 adjoint scalar theory:

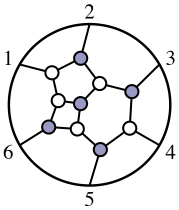


- The boundary elements encode all possible factorisations of amplitudes
- Works for **scalar particles** – what about **gluons**, etc. ?

  $\to$ More intricate structure of singularities

From polytopes to Grassmannians

- For theories with gluons: polytopes $\rightarrow$ curvy geometries: Graßmannians
- Positive Graßmannian $G_+(k,n)$ provides a prototype example of curvy positive geometry
- Boundaries of $G_+(k,n)$ labelled by on-shell diagrams:



- Not all on-shell diagrams can be mapped to manifestly tree-level processes
- **Amplituhedra** combine a particular collection of positive Grassmannian cells into a single object encoding tree-level amplitudes

Why momentum amplituhedron?

- **Amplituhedron** $\mathcal{A}_{n,k'}$ – defined in the momentum twistor space – encodes singularities of the polygonal Wilson loops

$$\Phi_Z : \quad G_+(k-2,n) \quad \to \quad G(4,n)$$
$$\tilde{C} \qquad\qquad\qquad z$$

[Arkani-Hamed,Trnka]

- **Momentum Amplituhedron** $\mathcal{M}_{n,k}$ – defined in the spinor helicity space – captures all singularities of the tree scattering amplitude

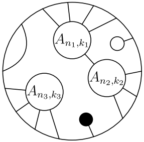
$$\Phi_{\Lambda,\tilde{\Lambda}} : \quad G_+(k,n) \quad \to \quad G(2,n) \quad \times \quad G(2,n)$$
$$C \qquad\qquad \lambda \qquad\qquad \tilde{\lambda}$$

[Damgaard, Ferro, TL, Parisi]

Boundary stratification of $\mathcal{M}_{n,k}$ known for all multiplicities n and helicity sectors k

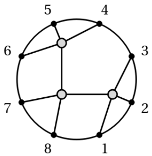
Tree-level momentum amplituhedron – stratification

$\mathcal{N} = 4$ sYM [Ferro, TL, Moerman]



- All possible tree-level physical processes in $\mathcal{N} = 4$ sYM
- Labelled by Graßmannian forests [Moerman, Williams]

ABJM [TL, Moerman, Stalknecht]



- All possible tree-level physical processes in ABJM
- Labelled by orthogonal Graßmannian forests

Beyond tree level

- Definition of the **Loop Amplituhedron** known for some time now [Arkani-Hamed,Trnka]

$$\tilde{\Phi}_Z : \quad \underset{\tilde{C}}{G_+(k-2,n)} \times \underset{D_l}{G(2,n)^l} \to \underset{z}{G(4,n)} \times \underset{(AB)_l}{G(2,4)^l}$$

- Momentum space version – **Loop Momentum Amplituhedron** – introduced recently [Ferro, TL]

$$\tilde{\Phi}_{\Lambda,\tilde{\Lambda}} : \quad \underset{C}{G_+(k,n)} \times \underset{D_l}{G(2,n)^l} \to \underset{\lambda}{G(2,n)} \times \underset{\tilde{\lambda}}{G(2,n)} \times \underset{\ell_l}{GL(2)^l}$$

where

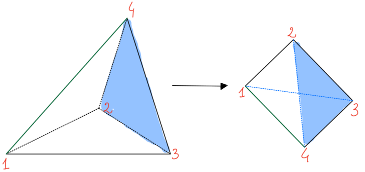
$$\ell = \lambda_A\tilde{\lambda}_B - \lambda_B\tilde{\lambda}_A = \left(\sum_i d_{Ai}\lambda_i\right)\left(\sum_{j<i} d_{Bi}\frac{\langle ij\rangle}{\langle AB\rangle}\tilde{\lambda}_j\right) - \left(\sum_i d_{Bi}\lambda_i\right)\left(\sum_{j<i} d_{Ai}\frac{\langle ij\rangle}{\langle AB\rangle}\tilde{\lambda}_j\right)$$

Can we classify all boundaries of this geometry?

$\to$ a very interesting combinatorial problem!

Two types of behaviour for positive geometries

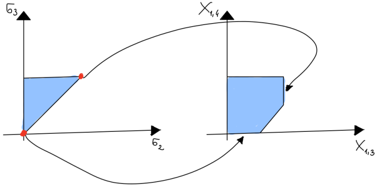
Projection



$$Y = CZ$$

→ dimensions of the images are **lower** than the dimensions of preimages

Blow-up



$$X_{1,3} = \frac{\sigma_2}{\sigma_3}(1+\sigma_3)$$

$$X_{1,4} = \frac{1}{1-\sigma_2}(2\sigma_3 - \sigma_2 - \sigma_2\sigma_3)$$

→ dimensions of the images are **higher** than the dimensions of preimages

Loop momentum amplituhedron combines these two behaviours!

Blow-ups for the loop momentum amplituhedron

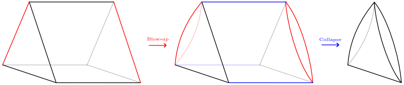
$$\ell = \left(\sum_i d_{Ai}\lambda_i\right)\left(\sum_{j<i} d_{Bi}\frac{\langle ij\rangle}{\langle AB\rangle}\tilde{\lambda}_j\right) - \left(\sum_i d_{Bi}\lambda_i\right)\left(\sum_{j<i} d_{Ai}\frac{\langle ij\rangle}{\langle AB\rangle}\tilde{\lambda}_j\right)$$

- The denominator $\langle AB\rangle$ can **vanish** for some lower-dimensional cells $(C,D) \in G_+(k,n) \times G(2,n)$
  $\to$ one needs to approach the singular cells from all possible directions in the positive Grassmannian
- **Blowing-up** points in **projective spaces** is a well-understood problem in algebraic geometry: take a singular point S and replace it by a copy of a projective space $\mathbb{P}^n$
- We need to blow up points in **Grassmannian spaces**
  $\to$ less understood

Loop momentum amplituhedron – boundary stratification

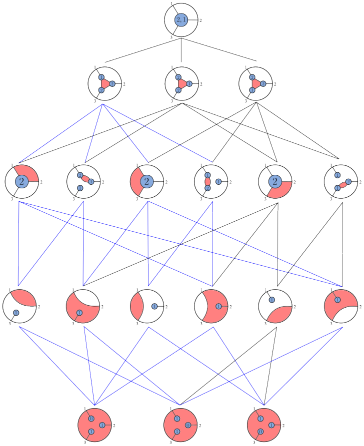
Examples: three points, one loop

- The geometry is 4-dimensional (2 tree dimensions + 2 loop dimensions) $\rightarrow$ loop integration produces 0
- Action of the map $\tilde{\Phi}_{\Lambda,\tilde{\Lambda}}$ producing 3-dimensional faces



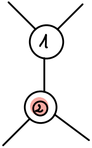
- For 3-particle amplitudes, this result generalizes to any loop order since loop geometries are independent from each other

Boundary poset



Loop momentum amplituhedron - boundary stratification

Examples: four points, one loop



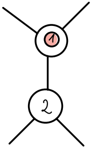
$$C = \begin{pmatrix} 1 & \alpha_3 & \alpha_2 & 0 \\ 0 & 0 & 1 & \alpha_1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & \beta_2 + \beta_4 & \beta_2\beta_3 & 0 \\ 0 & 1 & \beta_3 & \beta_1 \end{pmatrix}$$

$$\dim C + \dim D = 3 + 4 = 7$$

$$\dim \tilde{\Phi}(C, D) = 5$$

regular image



$$C = \begin{pmatrix} 1 & \alpha_3 & \alpha_2 & 0 \\ 0 & 0 & 1 & \alpha_1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

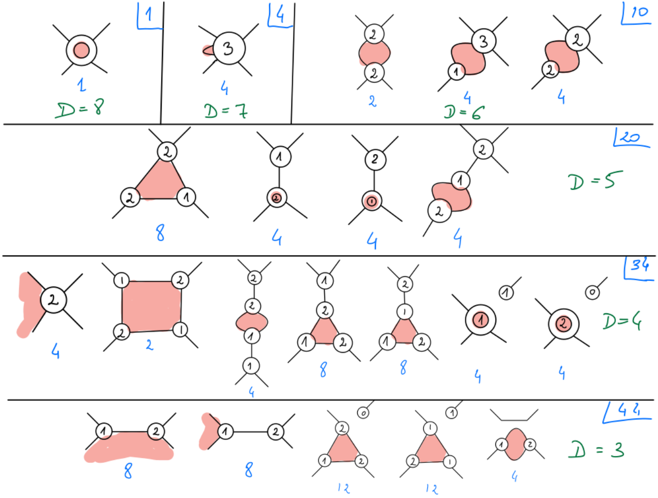
$$\dim C + \dim D = 3 + 0 = 3$$

$$\dim \tilde{\Phi}(C, D) = 5$$

blow-up

Loop momentum amplituhedron - boundary stratification

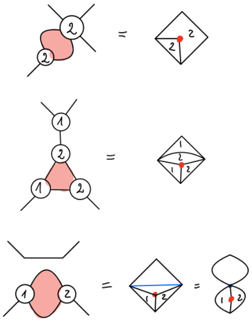
The complete boundary stratification for 4 points at one loop:



$$\chi = 1 - 4 + 10 - 20 + 34 - 44 + 42 - 24 + 6 = 1$$

Dual graph representation

- The diagrams on the previous slides provide a generalization of the Graßmannian trees to loops
- Rigorous definition of diagrams – work in progress
- Interesting observation: dual graphs



- The dual graphs are related to triangulations/dissections of one-punctured polygon (with possible deformations similar to deformations of a moduli space of points on a disk)
- Likely to lead to a better way of enumerating the boundaries

Generalizations

- Beyond 4 points $\rightarrow$ MHV one loop

  $\rightarrow$ we believe (and have some evidence) that the geometry is "nice"

- Beyond MHV $\rightarrow$ 6-points NMHV

  $\rightarrow$ the combinatorics of dual graphs easily generalizes to other helicity sectors

- Beyond one loop $\rightarrow$ 2-loop 4 points

  $\rightarrow$ not so nice since already the domain itself is not a ball

  $\rightarrow$ there is some non-planar behaviour

- All loop/all helicity combinatorics

  $\rightarrow$ stay tuned

Conclusions and Outlook

Conclusions

- We have started an exploration of the newly defined geometry relevant for loops directly in the momentum space
- Boundary stratification naturally labelled by a generalization of Grassmannian forests, leading to an interesting combinatorial problem

Outlook

- Can we understand better the all-loop “deepest cuts” of amplitudes from the geometry of the loop momentum amplituhedron?

  $\rightarrow$ For the loop amplituhedron there are ambiguities in defining them. Can momentum space fix this issue?
- What about geometries for non-planar amplitudes?

  $\rightarrow$ Since the momentum in our construction is defined globally, one could access the non-planar sector by changing the domain of the function $\Phi_{\Lambda,\tilde{\Lambda}}$