Recursion Relations for One-Loop Goldstone Boson Amplitudes

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Based on: [2206.04694] CB, Karol Kampf, Jaroslav Trnka

Non-Linear Sigma Model: Lagrangian description

- Theory of Goldstone Bosons arising from symmetry breaking $SU(N)\times SU(N) \to SU(N)$. Leading-order $\mathcal{O}(p^2)$ Lagrangian:

$$\mathcal{L}_2 = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^{-1}\rangle, \qquad \text{where } U(x) = \sum_{k=0}^{\infty} u_k \Big(\frac{i\sqrt{2}}{F}\phi(x)\Big)^k .$$

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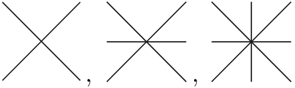
- NLSM is a non-renormalizable theory in $d = 4$. Lagrangian gives rise to even-point vertices, starting with four points:

$\mathcal{L}_2 \subseteq \Big\{$



$, \dots \Big\}.$

Tree-level amplitudes in the NLSM I

- At tree-level decompose n-point amplitude $\mathcal{A}_n$ in terms of **partial** or **ordered** amplitudes A_n:

$$\mathcal{A}_n = \sum_{\sigma \in S_{n-1}} \frac{\langle t^{a_1} t^{a_{\sigma(2)}} \dots t^{a_{\sigma(n)}} \rangle}{(2F^2)^{n/2-1}} A_n(p_1, \dots, p_{\sigma(n)})$$

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- Example: partial amplitudes at **4**- and **6**-points:

$$A_4 = s_{1,2} + s_{2,3}, \quad A_6 = -\frac{1}{2} \frac{(s_{1,2} + s_{2,3})(s_{4,5} + s_{5,6})}{s_{1,3}} + s_{1,2} + \text{cyc.}$$

- Generally:
  $\rightarrow A_n$ are simple rational functions of kinematic invariants,

$$s_{i,j} = (p_i + p_{i+1} + \cdots + p_j)^2$$

  $\rightarrow A_n$ are independent of Lagrangian parameters u_k.

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- **(II) Adler Zero:** Amplitudes vanish when one of the external momenta goes soft:

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- **(I)** and **(II)** are sufficient to recursively compute the tree-level S-Matrix. $\leftrightarrow$ A_n are **fixed uniquely** by **(I)** and **(II)**.

One-loop amplitudes and integrands in the NLSM I

- Analogous to tree-level, define ordered 1-loop amplitude

$$\mathcal{A}_n^{1-\text{loop}} = N \sum_{\sigma \in S_{n-1}} \frac{\langle t^{a_1} t^{a_{\sigma(2)}} \dots t^{a_{\sigma(n)}} \rangle}{(2F^2)^{n/2}} A_n^{1-\text{loop}}(p_1, \dots, p_{\sigma(n)}) + \dots,$$

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- Why integrands?
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- **Apply tree-level philosophy**: Try to fix the integrand uniquely from knowledge of poles **(I)** and Adler zero **(II)**.

One-loop integrand at four points

- Investigate simplest example: The **4**-point integrand.

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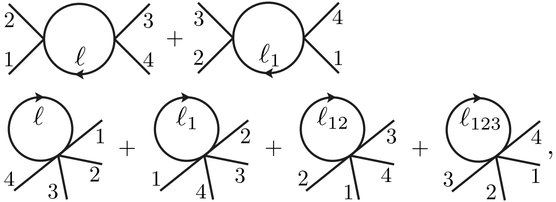
- At $n = 4$ compute Feynman diagrams:

$$\mathcal{I}_4^{\mathrm{FD}} =$$



$$= 16u_3^2 - 32u_5 + \frac{n_4^{\mathrm{FD}}}{\ell^2} + \frac{n_{42}}{\ell^2 \ell_{12}^2} + \text{cyc.}$$

- Numerators are: $n_4^{\mathrm{FD}} = (4u_3 - 1)(2s_{12} + \ell_1^2 + \ell_{123}^2) + (8u_3 - 1)s_{23},$

$$n_{42} = \tfrac{1}{2}(s_{12} + \ell_1^2)(s_{12} + \ell_{123}^2).$$

What about the Adler zero?

- Idea: use freedom in u_3,u_5 to impose Adler zero on ext. lines:

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- Make a more general Ansatz:

$$\mathcal{I}_4^{\mathrm{ans}}(\alpha_i) = \frac{\alpha_0}{4} + \frac{n_4^{\mathrm{ans}}}{\ell^2} + \frac{n_{42}}{\ell^2 \ell_{12}^2} + \text{cyc.}$$

with $n_4^{\mathrm{ans}} = \alpha_1 s_{12} + \alpha_2 s_{23} + \alpha_3 \ell_1^2 + \alpha_4 \ell_{12}^2 + \alpha_5 \ell_{123}^2, \quad \alpha_i$ free.

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- Demanding Adler zero now gives a solution:

$$\alpha_0 = 2,\ \alpha_3 = -1,\ \alpha_4 = 1,\ \alpha_5 = -1.$$

- This defines a two-parameter **soft integrand** $\mathcal{I}_4^{\mathrm{S}} = \mathcal{I}_4^{\mathrm{S}}(\alpha_1, \alpha_2)$.

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- Key object: **B-function** B_6 ($\leftrightarrow$ residue of $\mathcal{I}_4^{\rm S}$ at $\ell^2 = 0$)

$$\begin{aligned} \mathrm{Cut}[\mathcal{I}_4^{\rm S}(\alpha_1, \alpha_2)]_{\ell^2=0} &\equiv -B_6(\ell, p_1, p_2, p_3, p_4; \alpha_1, \alpha_2) \\ &= \alpha_1 s_{12} + \alpha_2 s_{23} - \ell_1^2 + \ell_{12}^2 - \ell_{123}^2 + \frac{2n_{42}}{\ell_{12}^2}. \end{aligned}$$

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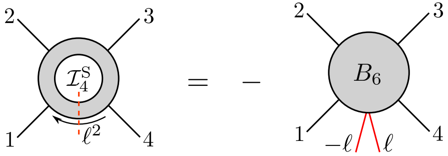

More on the B-function B_6

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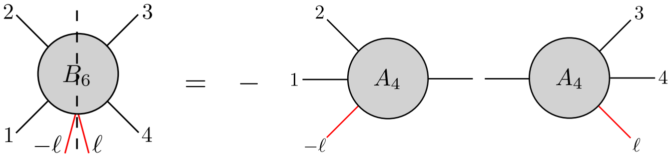

and soft limits:

$$\lim_{p_2 \to 0} B_6 = \lim_{p_3 \to 0} B_6 = 0.$$

The latter are necessary conditions implied by the Adler zero of $\mathcal{I}_4^{\mathrm{S}}$.

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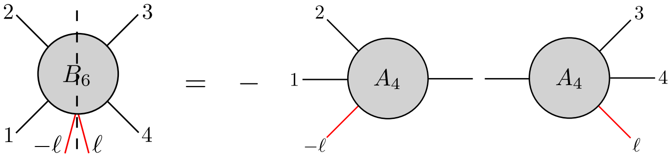

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- **Key observation:** B_6 can be **fixed uniquely** by requirement

$$\lim_{\ell \to 0} B_6 \stackrel{!}{=} 0 \quad \Rightarrow \quad \alpha_1 = -2,\ \alpha_2 = 0,$$

- This implies a **unique** soft integrand $\mathcal{I}_4^{\mathrm{S}} = \mathcal{I}_4^{\mathrm{S}}(-2, 0)$.

Stepping back and enjoying the view

- Explicit expression for B_6

$$B_6 = 2s_{12} + \ell_1^2 - \ell_{12}^2 + \ell_{123}^2 - \frac{(s_{12} + \ell_1^2)(s_{12} + \ell_{123}^2)}{\ell_{12}^2},$$

and the corresponding soft integrand $\mathcal{I}_4^{\mathrm{S}}$

$$\mathcal{I}_4^{\mathrm{S}} = \frac{1}{2} - \frac{2s_{12} + \ell_1^2 - \ell_{12}^2 + \ell_{123}^2}{\ell^2} + \frac{1}{2}\frac{(s_{12} + \ell_1^2)(s_{12} + \ell_{123}^2)}{\ell^2 \ell_{12}^2} + \text{cyc.}$$

- At $n = 4$ points have succeeded in finding a **unique** integrand that has **(I)** a consistent pole structure and **(II)** Adler zero.

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- At $n = 4$ points have succeeded in finding a **unique** integrand that has **(I)** a consistent pole structure and **(II)** Adler zero.

$\rightarrow$ $\mathcal{I}_4^{\mathrm{S}}$ **and** B_6 **can be computed recursively**!

Wrap-Up

- Soft integrand $\mathcal{I}_4^{\rm S}$ differs from Feynman integrand $\mathcal{I}_4^{\rm FD}$ only by terms that vanish upon ℓ-integration:
  $\rightarrow \mathcal{I}_4^{\rm S}$ leads to **known result** for integrated amplitude $A_4^{1-\rm loop}$!

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- Recursion for B_6 and $\mathcal{I}_4^{\mathrm{S}}$ requires **no further input** beyond the 4-point tree-level amplitude A_4. Proceeds in two steps:
  (1) Recursion of B_6. Input: A_4.
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- What can we say for **higher points** $n > 4$?
  $\rightarrow$ A unique 1-loop soft integrand $\mathcal{I}_n^{\mathrm{S}}$ satisfying **(I)** and **(II)** exists for any number of points! This has been explicitly verified up to $n = 8$.

What's next?

- Can this construction be extended to two loops and beyond?
- Are there other exceptional EFTs for which a soft integrand can be found?
- Is there a Lagrangian that can directly compute the soft integrand?

Thank you!

Backups

Double soft integrand

- Amplitudes in the NLSM are known to satisfy a recursion relation in the limit when two external momenta become soft,

$$\lim_{t\to 0} A_n(tp_i, tp_j, \{p_k\}) = \Pi_{i,j} A_{n-2}(\{p_k\}),$$

with $\Pi_{i,j} = \frac{\delta_{j,i+1}}{2}\left(\frac{p_{i+2}\cdot(p_i-p_{i+1})}{p_{i+2}\cdot(p_i+p_{i+1})} - \frac{p_{i-1}\cdot(p_i-p_{i+1})}{p_{i-1}\cdot(p_i+p_{i+1})}\right)$.

- Recall: previously we fixed $\alpha_1{=}-2$ and $\alpha_2{=}0$ in $\mathcal{I}_n^{\mathrm{S}}(\alpha_1, \alpha_2)$ using the extended Adler zero of the B-functions, i.e. $\lim_{\ell\to 0} B_{n+2} = 0$.
- Alternatively we can fix the coefficients by imposing the double soft limit for the integrand

$$\lim_{t\to 0} \mathcal{I}_n(tp_i, , tp_j, \{\ell, p_k\}; \alpha_1, \alpha_2) \overset{!}{=} \Pi_{i,j}\mathcal{I}_{n-2}(\{\ell, p_k\}; \alpha_1, \alpha_2),$$

yielding $\alpha_1{=}-2$ and $\alpha_2{=}1$.

- The resulting double-soft integrand $\mathcal{I}_n^{\mathrm{DS}}$ can also be recursed to all multiplicities. The only difference lies in the respective B-functions.

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- Complex Analysis $\to$ $A_n(z)$ can be uniquely determined from its poles via **residue theorem**.
- Physics input from (I) and (II) crucial to obtain **recursion relation**:

$$A_n(0) = \oint_{z=0} dz \, \frac{A_n(z)}{z} = \cdots = \sum_{I,\pm} \oint_{z_I^\pm} \frac{dz}{z} \frac{A_{n_L}(z) A_{n_R}(z)}{\hat{P}_I^2(z) F_n(z)} + 0$$

This concludes the tree-level story. $\to$ Now move on to 1-loop!

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- Use BCFW-like construction:

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for any function $F_n(z)$ satisfying $F_n(0) = 1$.

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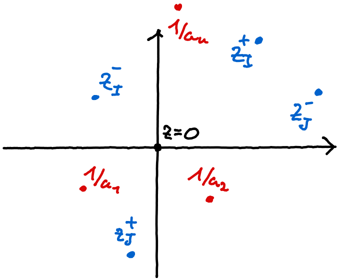
- We will choose $F_n(z) = \prod_{i=1}^{n}(1 - a_i z)$.
- Since $F_n(1/a_i) = 0$, we introduce additional poles at $z = 1/a_i$:



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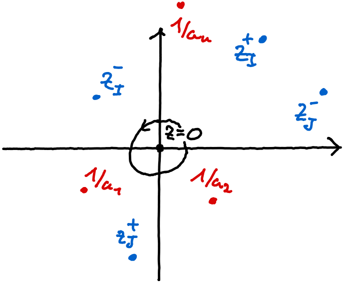

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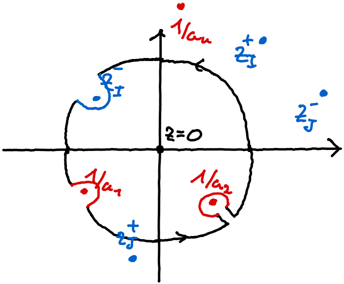

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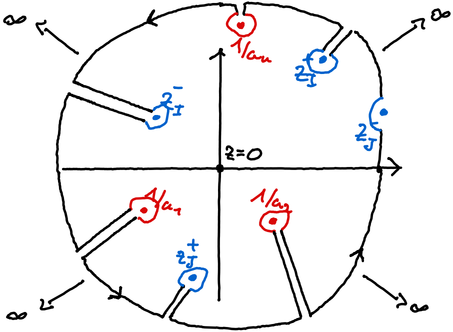

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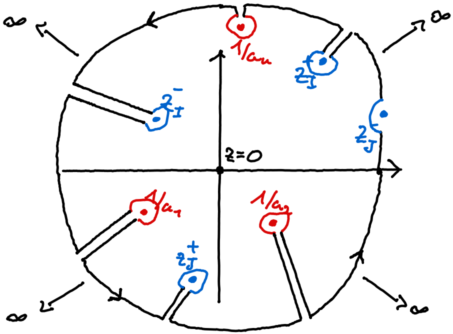

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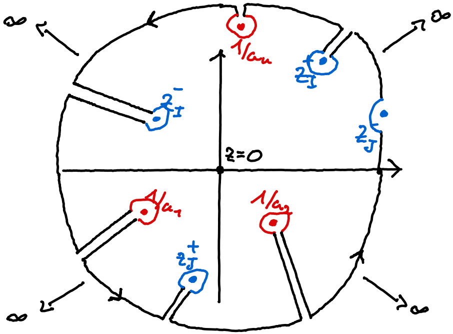

- Pole at infinity is absent: $\frac{A_n(z)}{zF_n(z)} \sim \frac{z^2}{zz^n} \sim \frac{1}{z^{n-1}}$ as $z \to \infty$.

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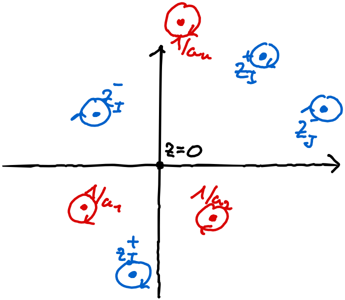

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- (I) Consistent factorization:

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- **(II)** Adler zero:

$$\oint_{1/a_i} \frac{dz}{z} \frac{A_n(z)}{F_n(z)} = \cdots \simeq A_n(1/a_i) = \mathbf{0}.$$

One-loop amplitudes and integrands in the NLSM II

- **Tree-level philosophy**: Would like to determine the integrand from knowledge of poles and Adler zero.

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  - tree-type poles $\sim \frac{1}{s_{i,j}}$, $\qquad s_{i,j} = (p_i + \cdots + p_j)^2$,
  - loop-type poles $\sim \frac{1}{\ell_{1\ldots i}^2}$, $\qquad \ell_{1\ldots i}^2 = (\ell + p_1 + \ldots p_i)^2$.

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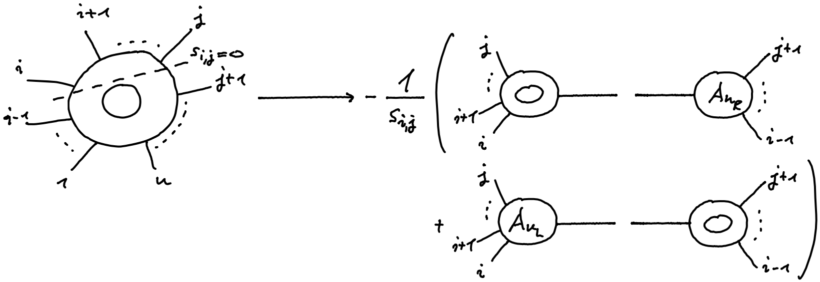
- Investigate simplest example: The 4-point integrand.

Poles and residues I

- Let us formulate a wishlist for the poles and corresponding residues a well-behaved integrand should have.
- On tree-type poles, $s_{i,j} \to 0$, the integrand should factorize as

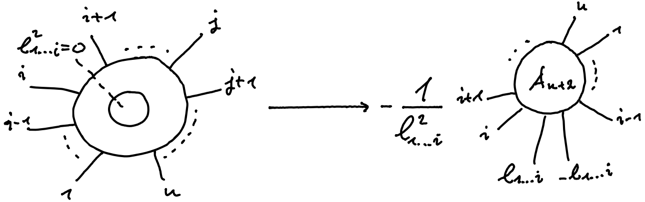
$$\mathcal{I}_n^{1-\text{loop}} \xrightarrow{s_{i,j}=0} -\frac{1}{s_{i,j}} \left(\mathcal{I}_{n_L}^{1-\text{loop}} A_{n_R} + A_{n_L} \mathcal{I}_{n_R}^{1-\text{loop}} \right)$$



Poles and residues II

- On loop-type poles, $\ell_{1\ldots i}^2 \to 0$, the integrand should give the forward limit of a higher-order tree amplitude

$$\mathcal{I}_n^{1-\text{loop}} \xrightarrow{\ell_{1\ldots i}^2=0} -\frac{1}{\ell_{1\ldots i}^2} A_{n+2}(p_1, \ldots, p_{i-1}, -\ell_{1\ldots i}, \ell_{1\ldots i}, \ldots, p_n)$$



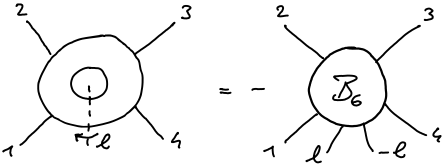
- How many of our wishes actually come true in the NLSM?
  $\to$ Investigate simplest example: The 4-point integrand.

What about the pole structure?

- $\mathcal{I}_4^{\text{ans}}$ does not have any tree-type poles.
- Suffices to look at loop-type poles: Compute the cut at $\ell^2 = 0$:

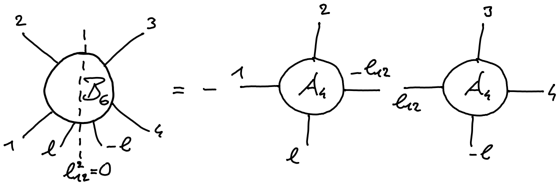
$$\begin{aligned}\text{Cut}[\mathcal{I}_4^{\text{S}}]_{\ell^2=0} &\equiv -B_6(p_1, p_2, p_3, p_4, -\ell, \ell) \\ &= \alpha_1 s_{12} + \alpha_2 s_{23} - \ell_1^2 + \ell_{12}^2 - \ell_{123}^2 + \frac{2n_{42}}{\ell_{12}^2}.\end{aligned}$$

- The **B-function** B_6 is an on-shell function with tree-level structure:



More on the B-function B_6

- B_6 is specified (up to terms $\sim \alpha_1$, α_2) by factorization on $\ell_{12}^2 = 0$:



  and soft limits:

$$\lim_{p_2 \to 0} B_6 = \lim_{p_3 \to 0} B_6 = 0.$$

- Furthermore, B_6 can be *fixed uniquely* by demanding

$$\lim_{\ell \to 0} B_6 = -(\alpha_1 + 2)s_{12} - \alpha_2 s_{23} \overset{!}{=} 0 \quad \Rightarrow \quad \alpha_1 = -2, \alpha_2 = 0,$$

  in turn implying a *unique* soft integrand $\mathcal{I}_4^{\mathrm{S}}(\alpha_1 = -2, \alpha_2 = 0)$.

The B-function B_{n+2}

- At n points $B_{n+2}(p_1, \dots, p_n, -\ell, \ell)$ corresponds to the single cut ($\ell^2 = 0$) of the soft integrand $\mathcal{I}_n^{\mathrm{S}}$.
- It is characterized as the *unique* function satisfying **(I)** consistent factorization on poles $\ell_{1\dots i}^2 = 0$,

$$B_{n+2} \xrightarrow{\ell_{1\dots i}^2=0} -\frac{A_{n_L}(\ell, p_1, \dots, p_i, -\ell_{1\dots i}) A_{n_R}(\ell_{1\dots i}, p_{i+1}, \dots, p_n, -\ell)}{\ell_{1\dots i}^2},$$

- Residues at tree-type poles $s_{i,j} = 0$ are a bit more involved but can also be prescribed.
- At the same time B_{n+2} obeys **(II)** the Adler zero ($i = 2, 3, \dots, n-1$):

$$\lim_{p_i \to 0} B_{n+2} = 0, \qquad \lim_{\ell \to 0} B_{n+2} = 0.$$

- B_{n+2} is soft limit constructible from tree-level amplitudes and lower-point B-functions. This has been verified up to B_{10}.

The soft integrand $\mathcal{I}_n^{\mathrm{S}}$

- The soft integrand $\mathcal{I}_n^{\mathrm{S}}$ has single cuts $(i = 0, \ldots, n-1)$:

$$\mathcal{I}_n^{\mathrm{S}} \xrightarrow{\ell_{1\ldots i}^2=0} -\frac{1}{\ell_{1\ldots i}^2} B_{n+2}(p_1, \ldots, p_i, -\ell_{1\ldots i}, \ell_{1\ldots i}, p_{i+1}, \ldots, p_n).$$

- On tree-type poles $\mathcal{I}_n^{\mathrm{S}}$ factorizes (schematically) as expected

$$\mathcal{I}_n^{\mathrm{S}} \xrightarrow{s_{i,j}=0} -\frac{1}{s_{i,j}} \left(\mathcal{I}_{n_L}^{\mathrm{S}} A_{n_R} + A_{n_L} \mathcal{I}_{n_R}^{\mathrm{S}}\right).$$

- Finally, $\mathcal{I}_n^{\mathrm{S}}$ satisfies the Adler zero on all external lines

$$\lim_{p_i \to 0} \mathcal{I}_n^{\mathrm{S}}(\ell, p_j) = 0.$$

- $\mathcal{I}_n^{\mathrm{S}}$ is soft limit constructible from the B-function B_{n+2} and lower point tree-amps and soft integrands. This has been verified up to $\mathcal{I}_8^{\mathrm{S}}$.

Recursion for B_6 and $\mathcal{I}_4^{\mathrm{S}}$

- For B_6 evaluate contour integral

$$B_6(z=0) = \oint_{z=0} \frac{dz}{z} \frac{B_6(z)}{F_{B_6}(z)} = \dots$$

$$= -\sum_{z_i} \operatorname*{Res}_{z=z_i} \left(\frac{A_4^L(z) A_4^R(z)}{z \hat{\ell}_{12}^2(z) F_{B_6}(z)} \right) - \frac{A_4^L(0) A_4^R(0)}{\ell_{12}^2},$$

- For $\mathcal{I}_4^{\mathrm{S}}$ evaluate contour integral

$$\mathcal{I}_4^{\mathrm{S}}(z=0) = \oint_{z=0} \frac{dz}{z} \frac{\mathcal{I}_4^{\mathrm{S}}(z)}{F_{\mathcal{I}_4}(z)} = \sum_{i=0}^{3} \sum_{\pm} \operatorname*{Res}_{z=z_i^{\pm}} \left(\frac{B_6(z)}{z\, \ell_{1\dots i}^2(z) F_{\mathcal{I}_4}(z)} \right),$$

$\rightarrow$ What can we say for *higher points*? $\rightarrow$ Things work out!