Poles at Infinity in On-shell Diagrams

Prague Spring Amplitudes Workshop

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Motivation

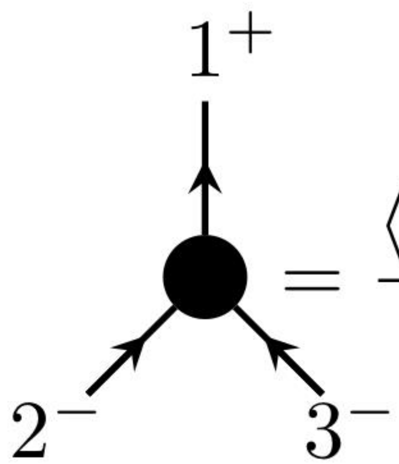
- Unitarity of the S-matrix has been an immensely important concept in the study of scattering amplitudes.
- It does not predict what happens with tree-level amplitudes (or loop integrands) on UV poles when the external momenta (or loop momenta) go to infinity
- Is there a notion of *unitarity at infinity*?
- On-shell diagrams are natural objects to consider (gauge invariance, factorization manifest)

Motivation

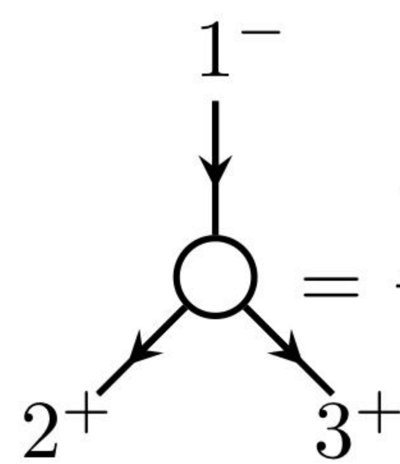
- We will study on-shell diagrams in mainly $\mathcal{N} < 3$ SYM and show that there is a "factorization" property for diagrams with poles at infinity

On-shell diagrams

Consider the fundamental three-point amplitudes for SYM



$$= \frac{\langle 23\rangle^{4-\mathcal{N}}\delta^4(P)\delta^{2\mathcal{N}}(\mathcal{Q})}{\langle 12\rangle\langle 23\rangle\langle 31\rangle},$$



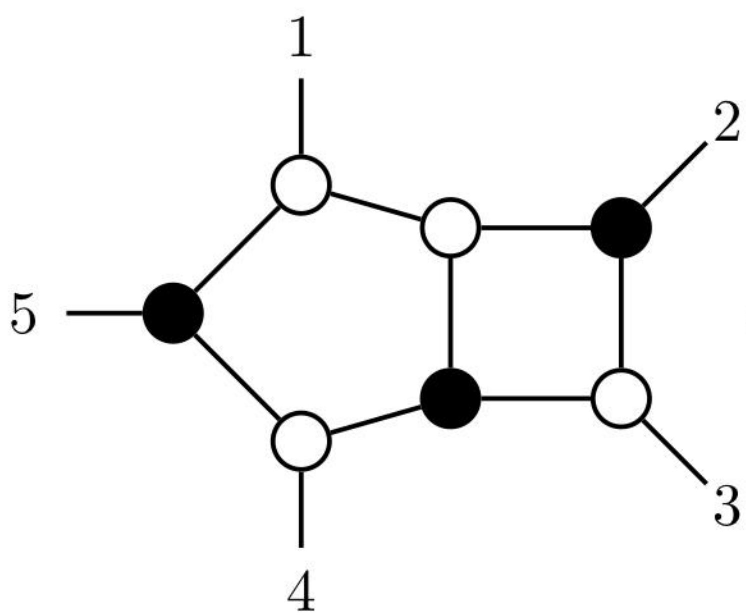
$$= \frac{[23]^{4-\mathcal{N}}\delta^4(P)\delta^{\mathcal{N}}(\widetilde{\mathcal{Q}})}{[12][23][31]}$$

These obey constrained kinematics,

$$\tilde{\lambda}_1 \sim \tilde{\lambda}_2 \sim \tilde{\lambda}_3, \qquad \lambda_1 \sim \lambda_2 \sim \lambda_3 \tag{1}$$

On-shell diagrams

- On-shell diagrams are build by gluing these fundamental three-point vertices together.
- All vertices satisfy momentum conservation.
- Every propagator is on-shell, $p^2 = 0$.

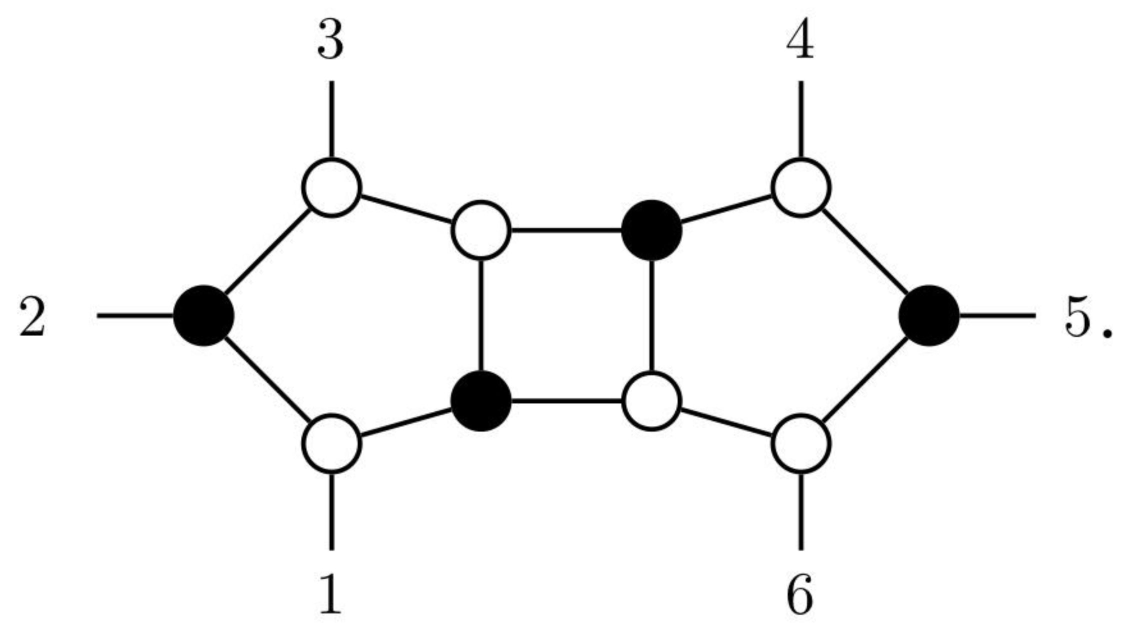


$$= \prod_k \int d^{\mathcal{N}} \widetilde{\eta}_k \int \frac{d^2\lambda_k \, d^2\widetilde{\lambda}_k}{\mathrm{GL}(1)} \left(\prod_j A_3^{(j)} \right) \tag{2}$$

On-shell diagrams

The same diagram represents both

- A term in the BCFW construction of the 6-point MHV tree-level amplitude.
- A maximal cut of 3-loop 6-point MHV amplitude



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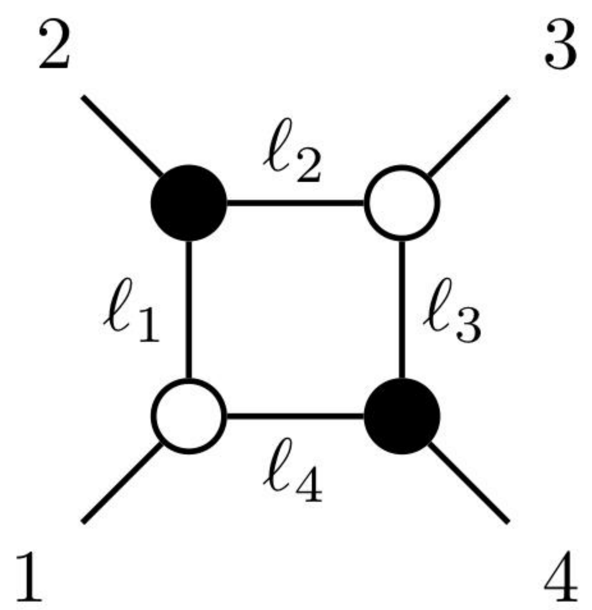
On-shell diagrams

4 point amplitude

- Simplest example



$$\ell_1 = \frac{\langle 23\rangle}{\langle 13\rangle}\lambda_1\widetilde{\lambda}_2, \qquad \ell_3 = \frac{\langle 14\rangle}{\langle 13\rangle}\lambda_3\widetilde{\lambda}_4,$$

$$\ell_2 = \frac{\langle 12\rangle}{\langle 13\rangle}\lambda_3\widetilde{\lambda}_2, \qquad \ell_4 = \frac{\langle 34\rangle}{\langle 13\rangle}\lambda_1\widetilde{\lambda}_4.$$

- Gluing is done by integrating over cut conditions

$$\Omega = \int d^4\widetilde{\eta}_1 \dots d^4\widetilde{\eta}_4 \int \frac{d^2\lambda_{\ell_1} d^2\widetilde{\lambda}_{\ell_1}}{\mathrm{GL}(1)} \dots \frac{d^2\lambda_{\ell_4} d^2\widetilde{\lambda}_{\ell_4}}{\mathrm{GL}(1)}$$

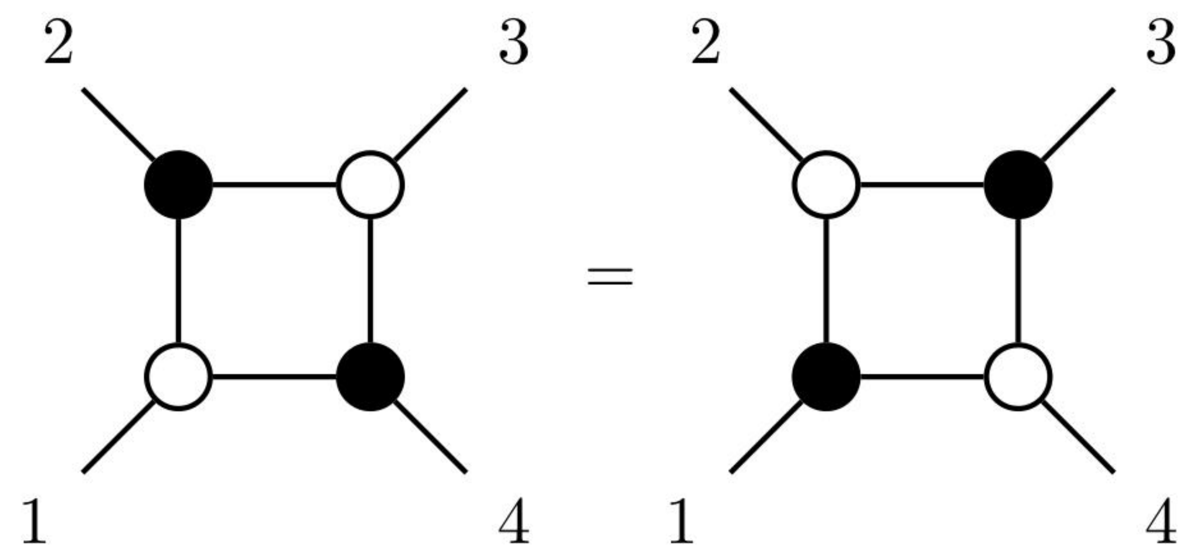
$$\times \left\{ A_3(1,\ell_1,\ell_4) A_3(2,\ell_1,\ell_2) A_3(3,\ell_2,\ell_3) A_3(4,\ell_3,\ell_4) \right\} \qquad (4)$$

$$= \frac{\delta^4(P)\delta^8(\mathcal{Q})}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

On-shell diagrams

- The following moves do not change the on shell function for the diagram – are *identity moves*

Square (only $\mathcal{N} = 4$):



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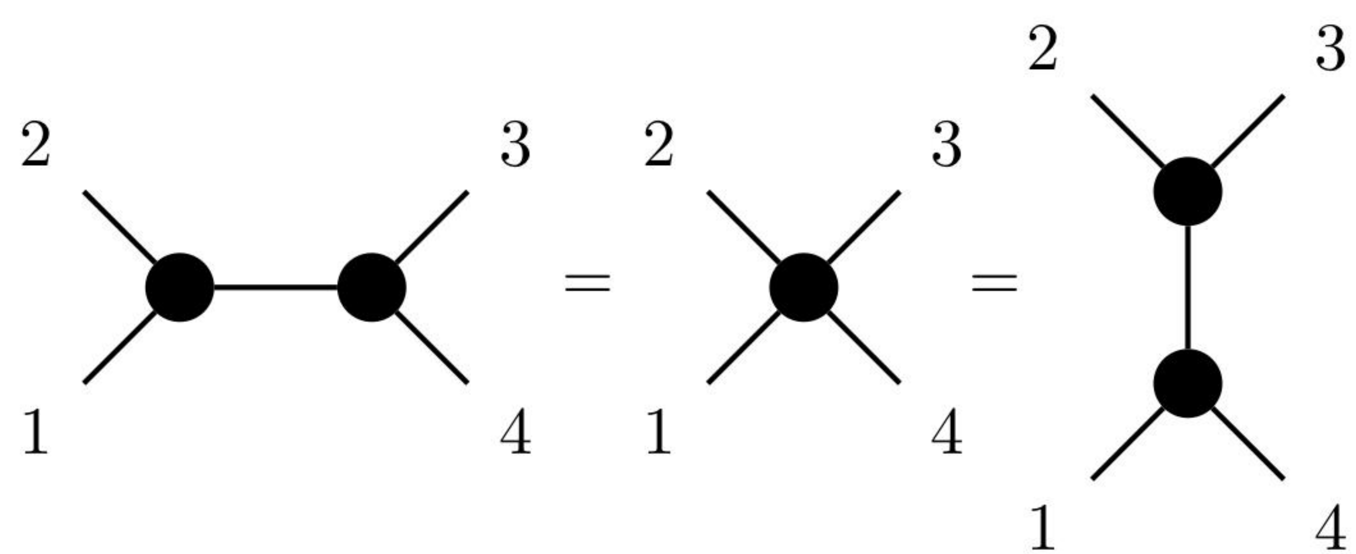
Merge-expand:



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Dual Formulation

- Consider momentum conservation:

$$\delta^4(P) = \delta^4(\lambda \cdot \tilde{\lambda}) = \delta^4(\lambda_1 \tilde{\lambda}_1 + ... + \lambda_n \tilde{\lambda}_n) \tag{7}$$

- Introduce a k-plane in n-dimensions represented by a $(k \times n)-$matrix
- This space is denoted by $G(k, n)$, the *Grassmannian*.
- A point in this space is represented by a $(k \times n)$ matrix, which we refer to as the C-matrix.
- Linearized momentum conservation condition

$$\delta(C \cdot Z) = \delta^{((n-k)\times 2)}(C^{\perp} \cdot \lambda)\delta^{(k\times 2)}(C \cdot \widetilde{\lambda})\delta^{(k\times \mathcal{N})}(C \cdot \widetilde{\eta}) \tag{8}$$

Dual Formulation

- The on-shell diagrams parameterize C in a certain way, by assigning an orientation and *edgevariables* to each diagram.
- Each entry in the C matrix is then given by a product of edge-variables

$$C_{\alpha a} = \sum_{\Gamma_{\alpha \to a}} \prod_j \alpha_j \tag{9}$$

Dual Formulation

- The on-shell function associated with an on-shell diagram in SYM theory is given by

$$\Omega = \int \prod_i \frac{d\alpha_i}{\alpha_i} \delta(C \cdot Z) \mathcal{J}^{\mathcal{N}-4}, \tag{10}$$

- Where the δ-functions let us determine the α's.
- The Jacobian $\mathcal{J}$ is relevant for $\mathcal{N} \neq 4$ and is given by

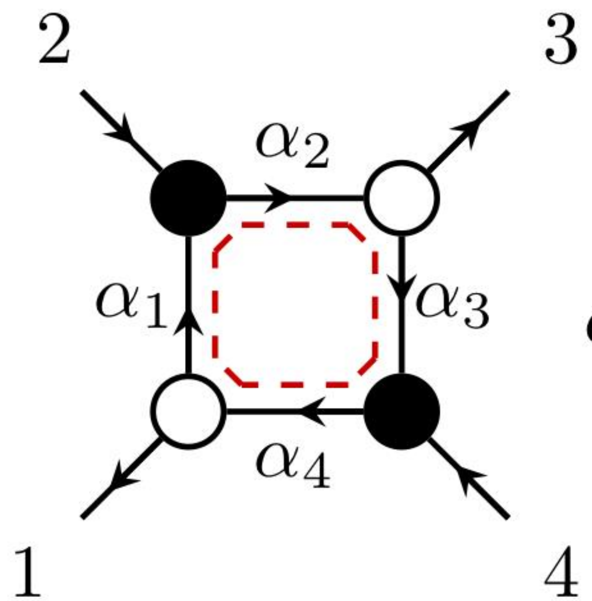
$$\mathcal{J} = 1 + \sum_i f_i + \sum_{\substack{\text{disjoint} \\ \text{pairs } i,j}} f_i f_j + \sum_{\substack{\text{disjoint} \\ \text{pairs } i,j,k}} f_i f_j f_k + \cdots \tag{11}$$

with f_i is a clockwise-oriented product of edge-variables in closed cycles.

UV Pole Structure

$\mathcal{N} = 3$

Let us study the pole structure for $\mathcal{N} = 3$:



$$\alpha_1 = \frac{\langle 23\rangle}{\langle 13\rangle}, \quad \alpha_2 = \frac{\langle 13\rangle}{\langle 12\rangle}, \quad \alpha_3 = \frac{\langle 14\rangle}{\langle 13\rangle}, \quad \alpha_4 = \frac{\langle 13\rangle}{\langle 34\rangle}.$$

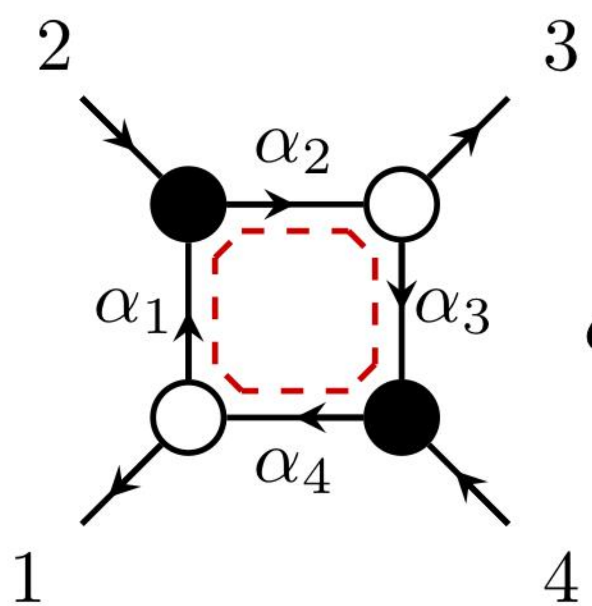
The Jacobian from the internal cycle is

$$\mathcal{J} = 1 - \alpha_1\alpha_2\alpha_3\alpha_4 = \frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 34\rangle} \tag{12}$$

The on-shell form is then given by

$$\Omega = \underbrace{\frac{\langle 24\rangle\, \delta^4(P)\delta^6(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}}_{\Omega_{\text{bare}}} \times \left(\frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle}\right) \tag{13}$$

UV Pole Structure



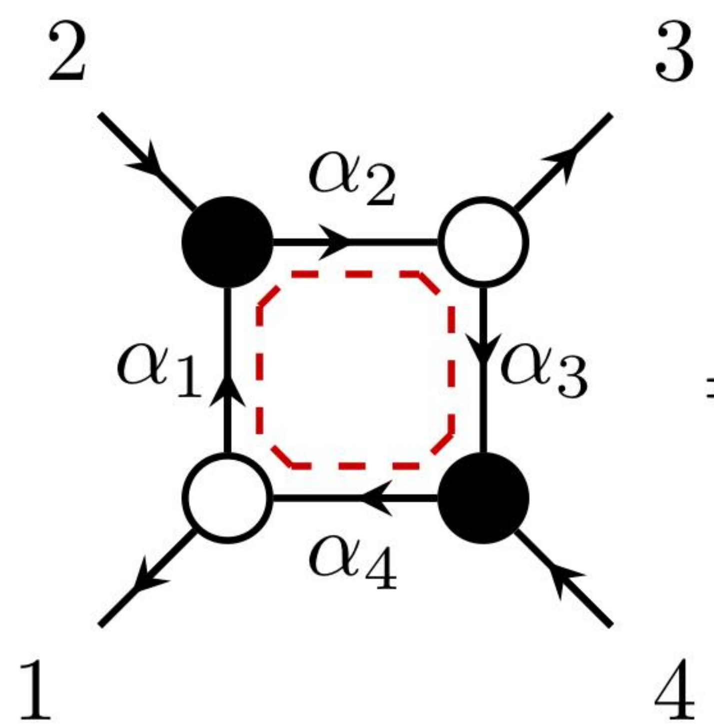
$$\alpha_1 = \frac{\langle 23\rangle}{\langle 13\rangle}, \quad \alpha_2 = \frac{\langle 13\rangle}{\langle 12\rangle}, \quad \alpha_3 = \frac{\langle 14\rangle}{\langle 13\rangle}, \quad \alpha_4 = \frac{\langle 13\rangle}{\langle 34\rangle}.$$

$$\Omega = \underbrace{\frac{\langle 24\rangle\, \delta^4(P)\delta^6(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}}_{\Omega_{\text{bare}}} \left(\frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle}\right) \tag{14}$$

- All other poles correspond to removing edges.
- Jacobian deletes poles that stem from edges that are non-removable
- Introduces pole at infinity $\langle 13\rangle$.

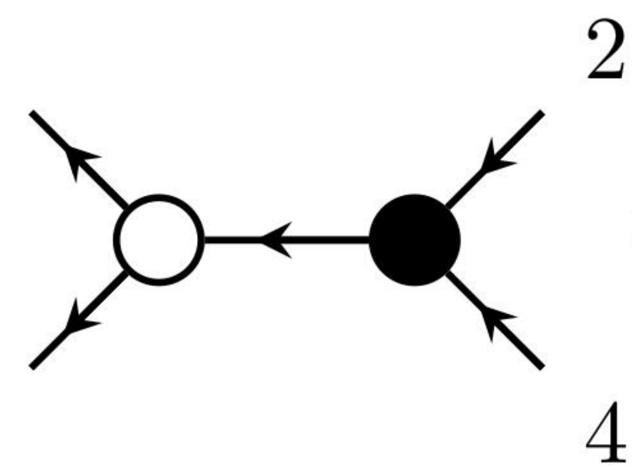
UV Pole Structure

What happens if we sit on the pole at infinity?



$$= \frac{\delta^4(P)\delta^6(Q)}{\langle 23\rangle\langle 14\rangle\langle 13\rangle} \tag{15}$$

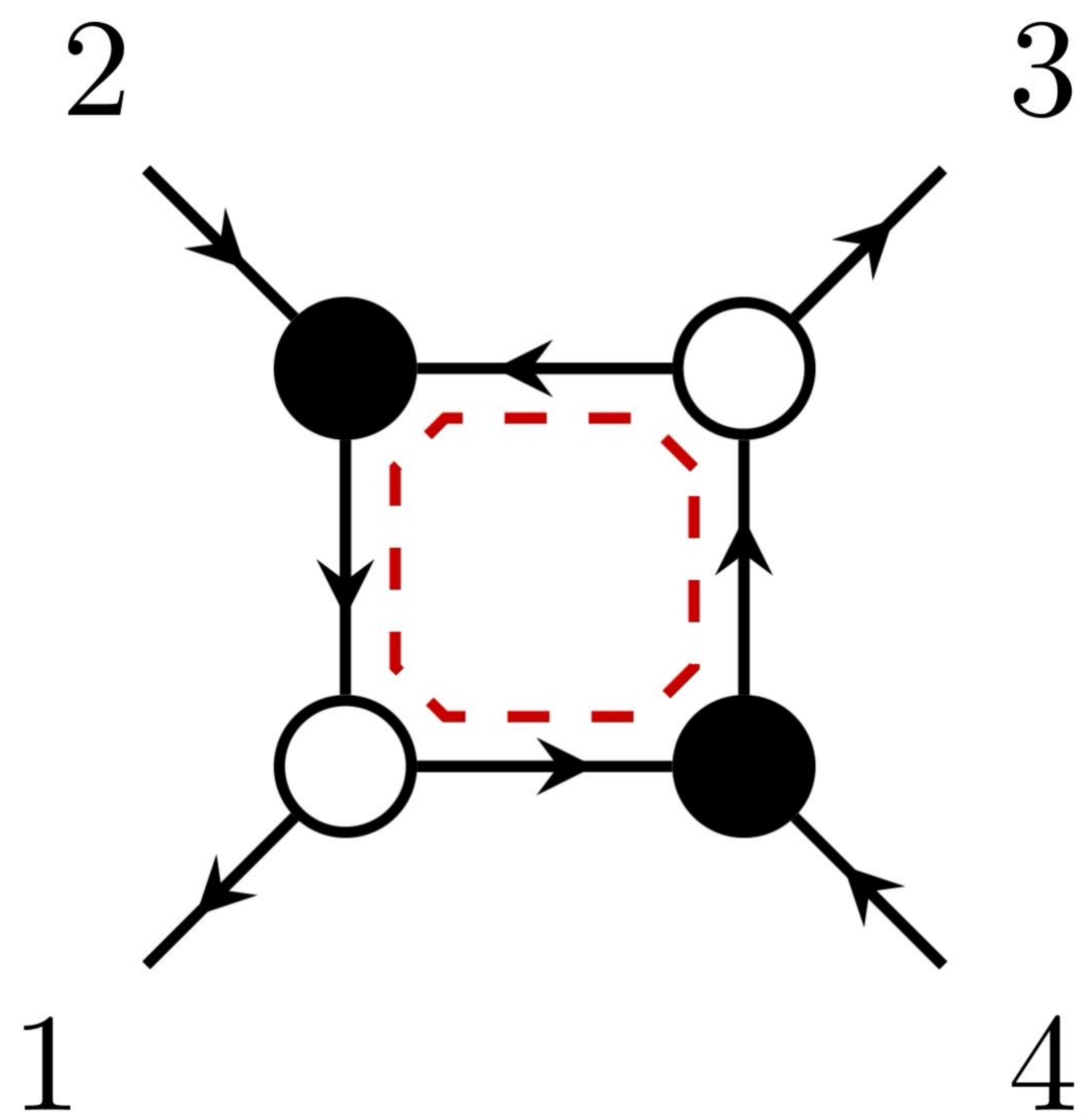
$$\mathrm{Res}(\Omega)_{\langle 13\rangle=0} = \frac{\delta^4(P)\delta^6(Q)\delta(\langle 13\rangle)}{\langle 23\rangle\langle 14\rangle} =$$



$$\equiv \Omega_{UV}$$

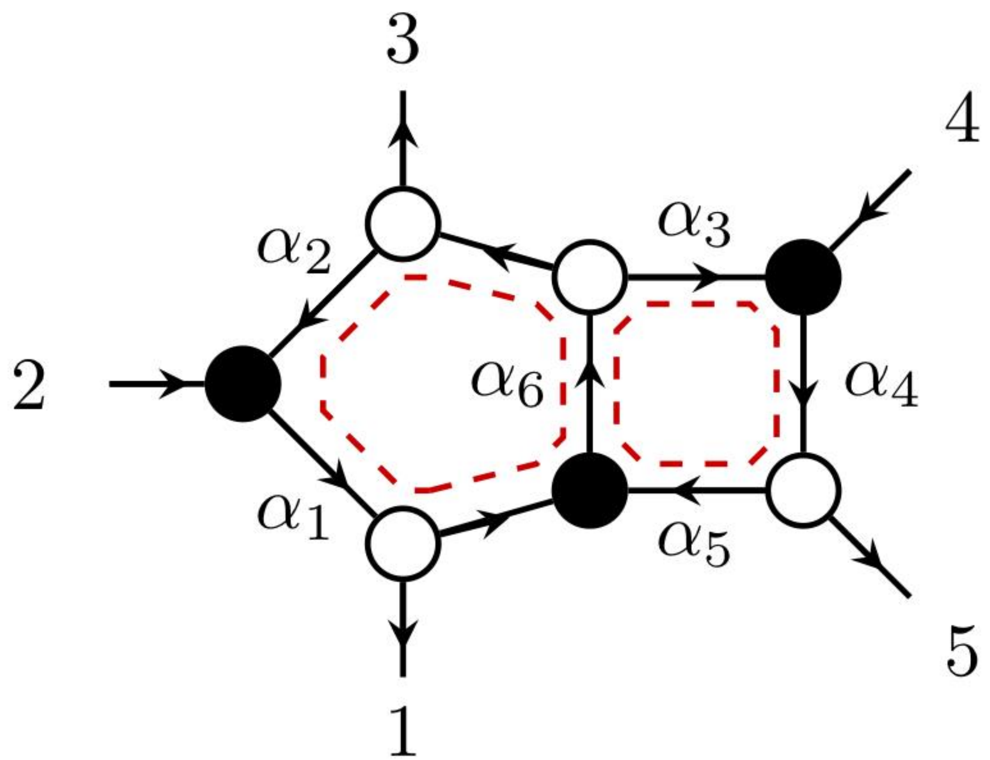
UV Pole Structure

Video showing schematics:



UV Pole Structure

Other example



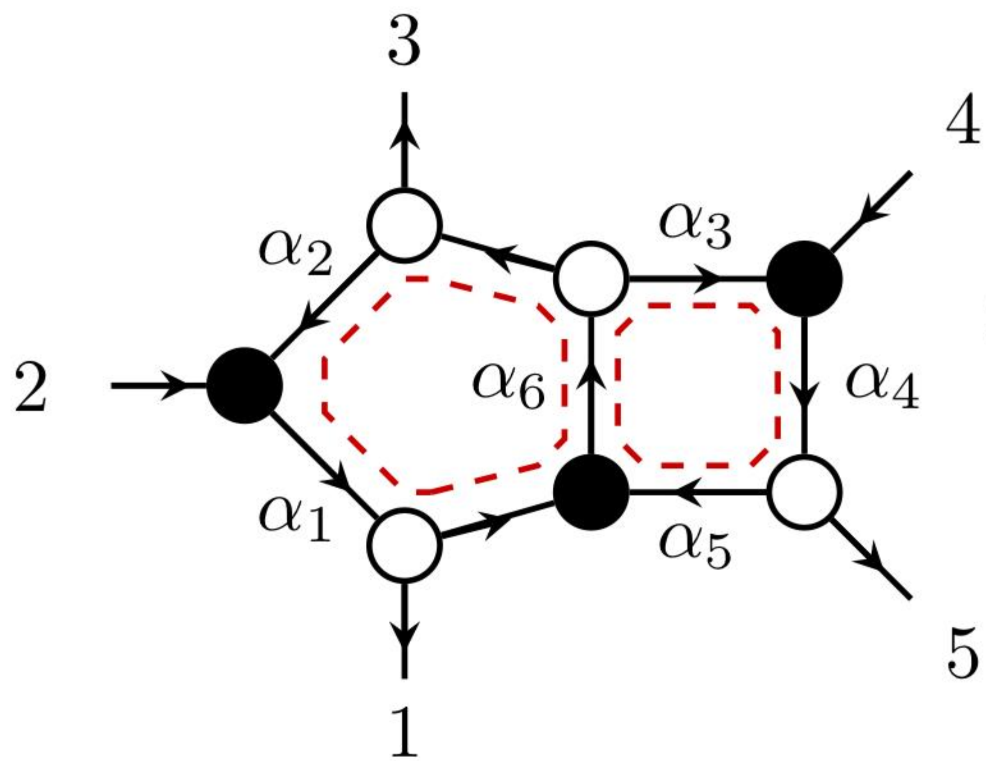
$$\alpha_1 = \frac{\langle 13\rangle}{\langle 23\rangle},\ \alpha_2 = \frac{\langle 12\rangle}{\langle 13\rangle},\ \alpha_3 = \frac{\langle 45\rangle}{\langle 35\rangle},$$
$$\alpha_4 = \frac{\langle 35\rangle}{\langle 34\rangle},\ \alpha_5 = \frac{\langle 13\rangle}{\langle 35\rangle},\ \alpha_6 = \frac{\langle 35\rangle}{\langle 15\rangle}. \tag{16}$$

The on-shell function is then equal to

$$\Omega = \frac{\delta^4(P)\delta^6(Q)}{\langle 12\rangle\langle 45\rangle\langle 13\rangle\langle 35\rangle} \tag{17}$$

UV Pole Structure

Blow up each loop individually

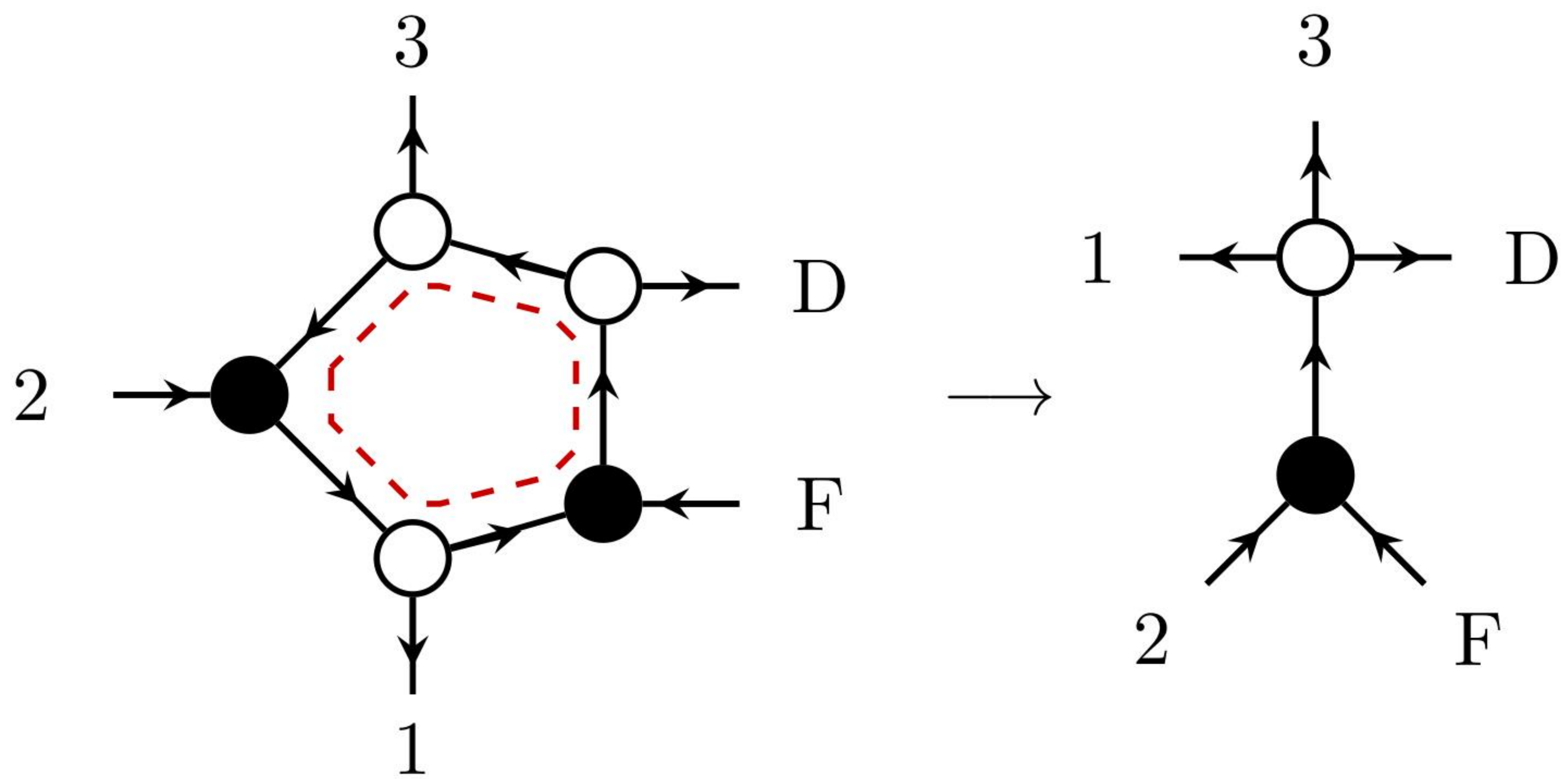


$$\Omega = \frac{\delta^4(P)\delta^6(Q)}{\langle 12\rangle\langle 45\rangle\langle 13\rangle\langle 35\rangle} \tag{18}$$

We know how the box blows up already

UV Pole Structure

The pentagon on the other hand,

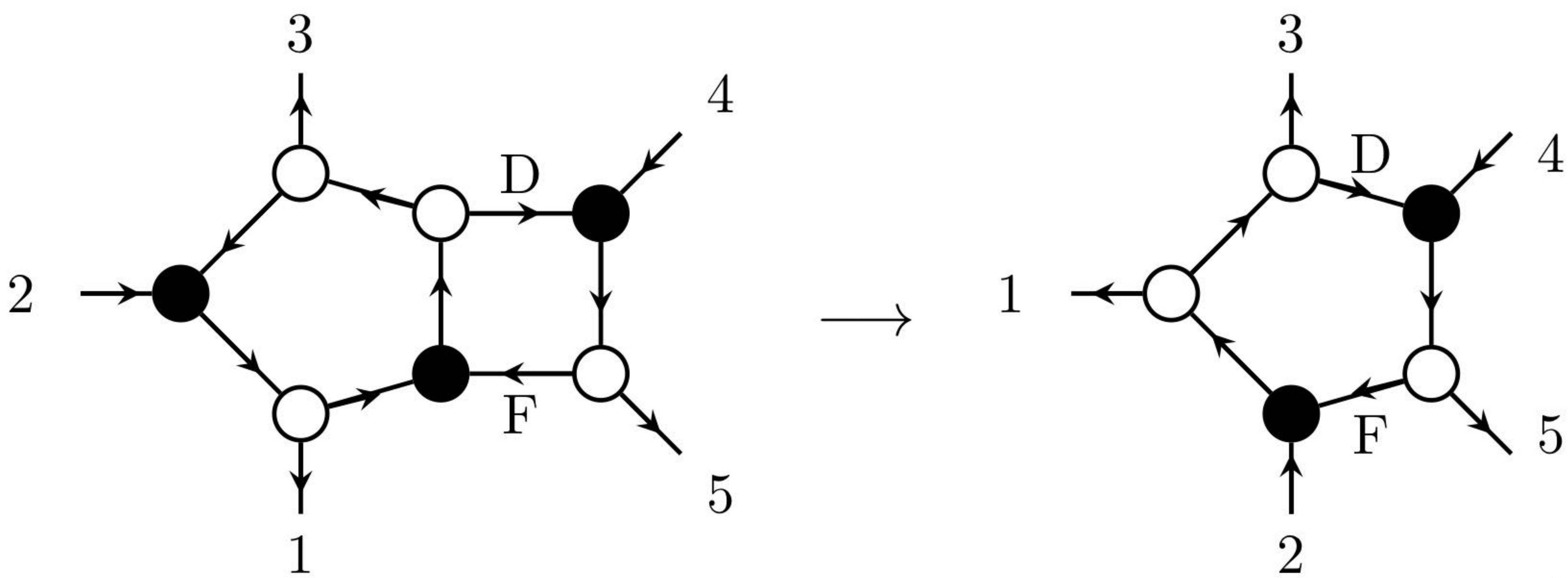


(19)

where the 4-point vertex can be further expanded as a chain of 3-point vertices.

UV Pole Structure

Gluing back with the right box we get a pentagon diagram,



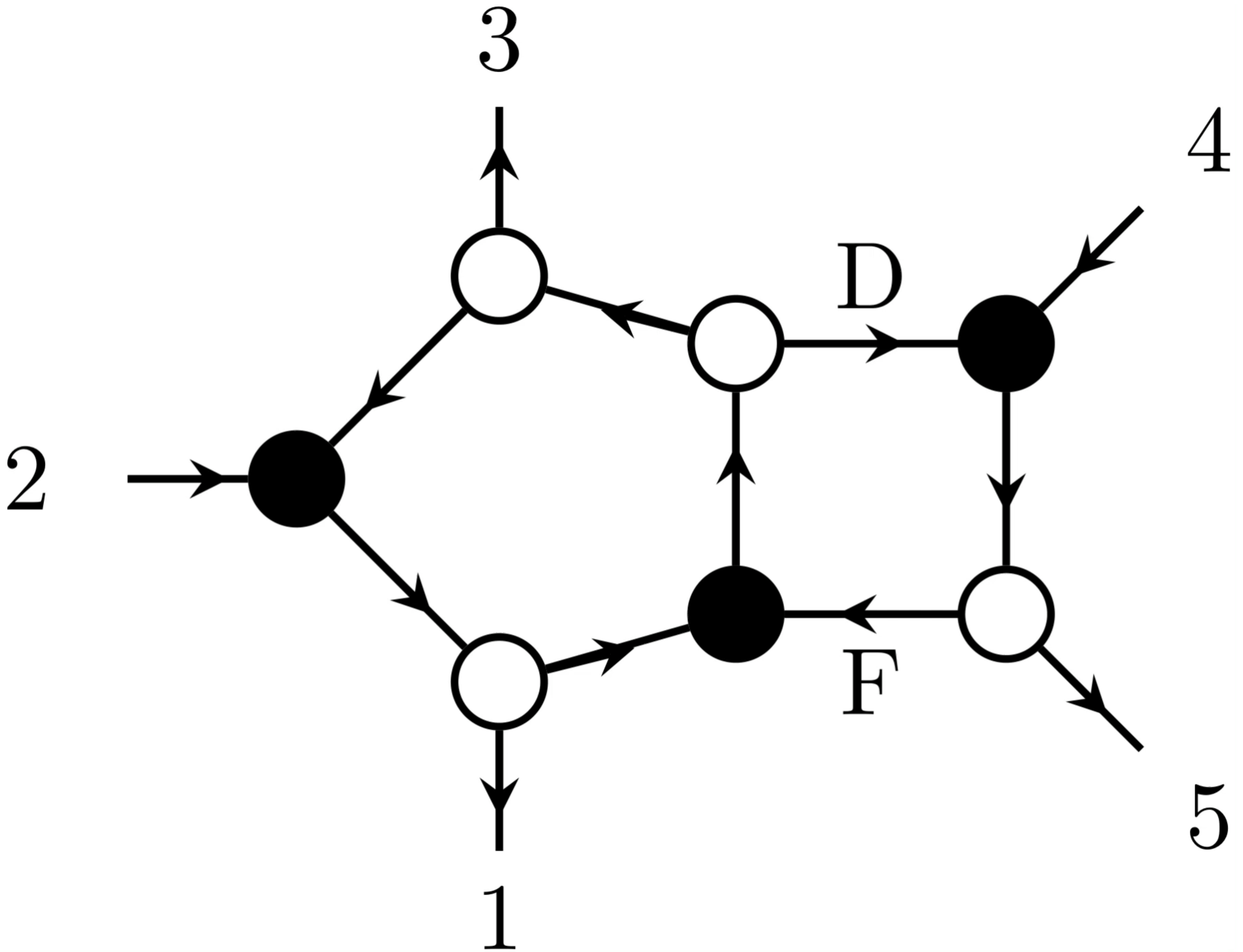
(20)

where

$$\Omega = \frac{\delta^4(P)\delta^6(Q)}{\langle 12\rangle\langle 45\rangle\langle 13\rangle\langle 35\rangle} \quad \longrightarrow \quad \Omega_{UV} = \frac{\delta^4(P)\delta^6(Q)\delta(\langle 13\rangle)}{\langle 12\rangle\langle 45\rangle\langle 35\rangle} \tag{21}$$

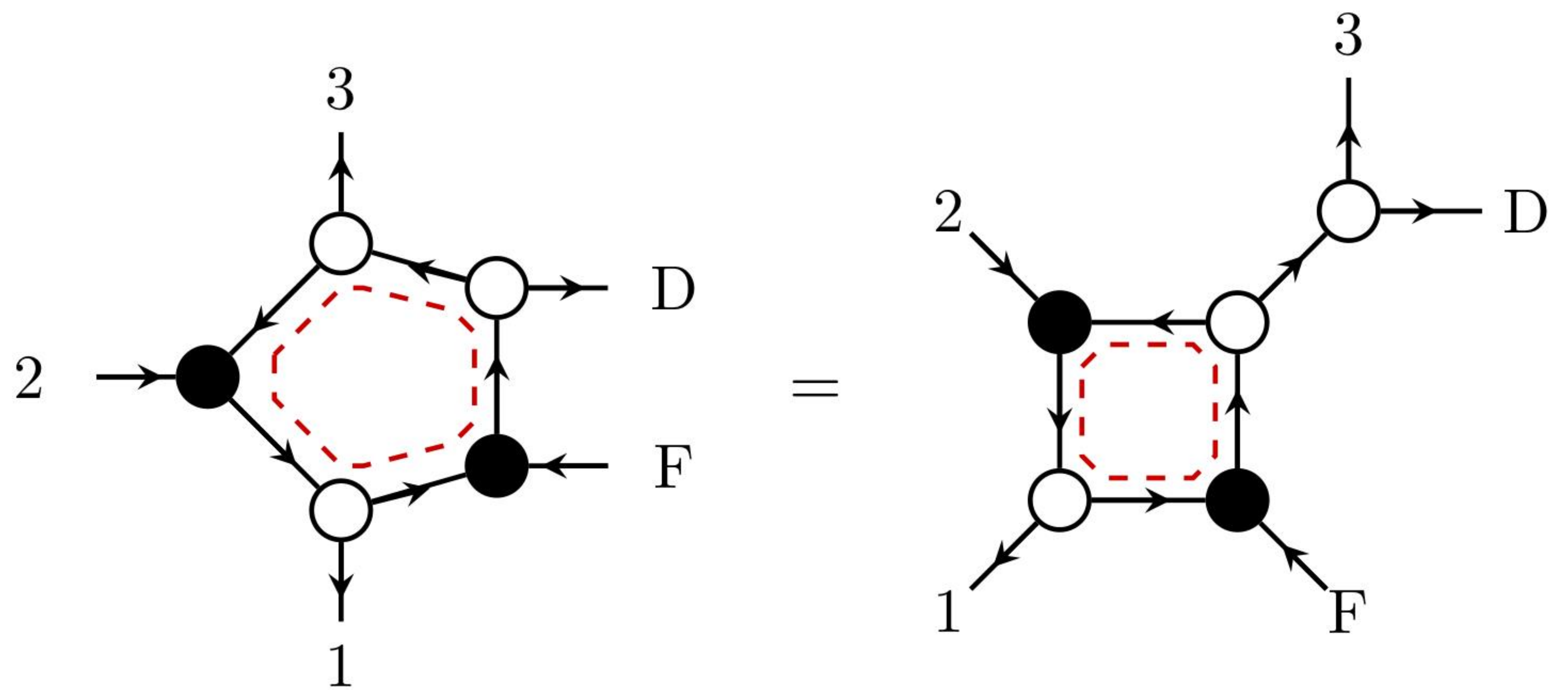
UV Pole Structure

Video showing schematics:



UV Pole Structure

This should not be surprising since the pentagon was really just a box

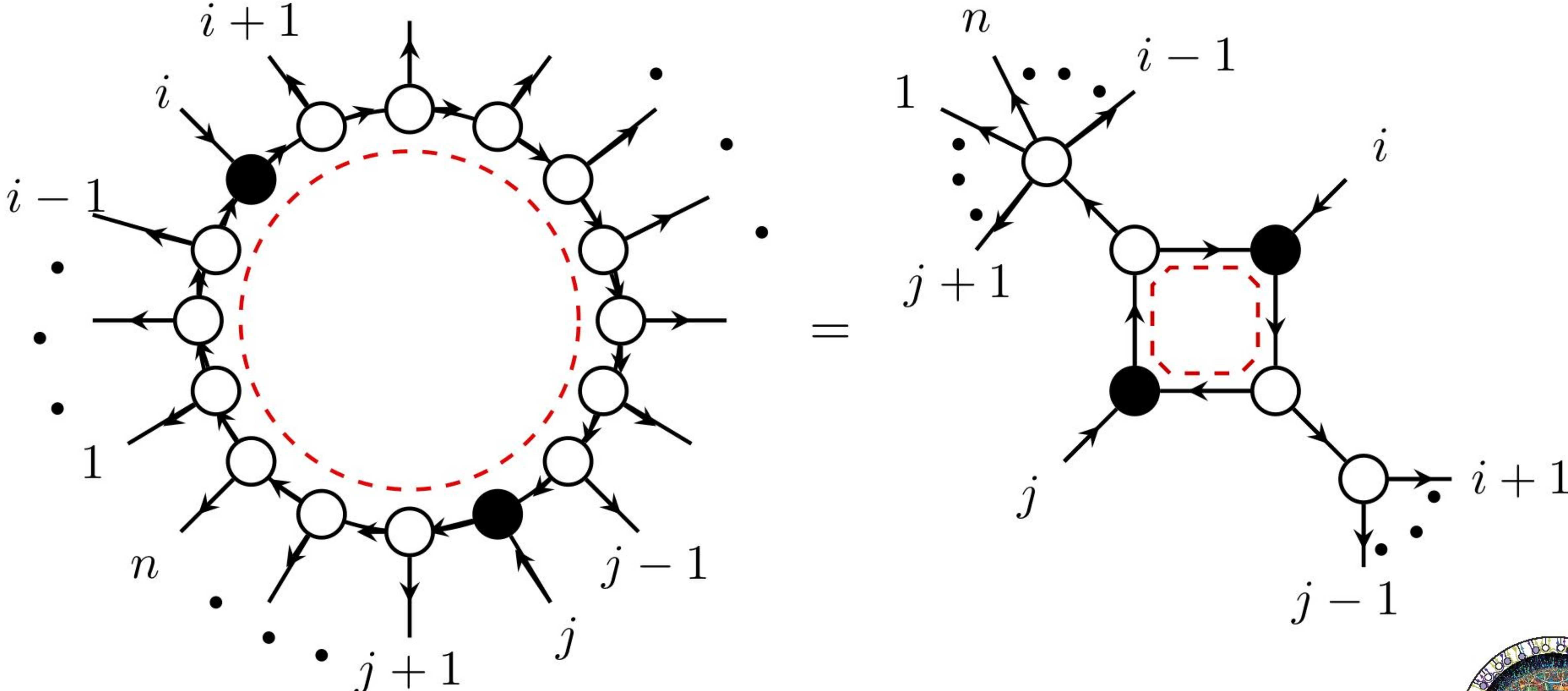


(22)

which we already knew how to do.

UV Pole Structure

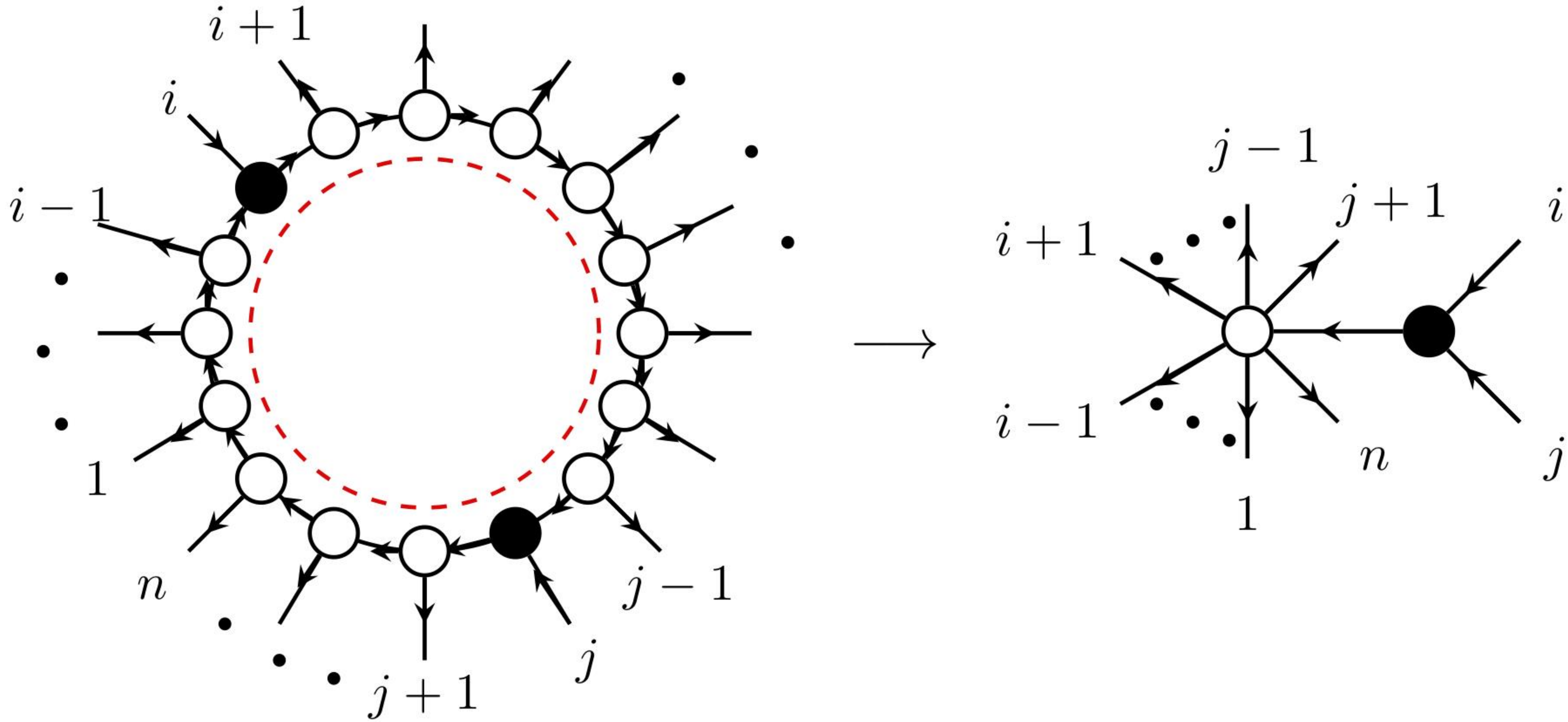
- Let us generalize this to higher points
- Sufficient in planar diagrams to treat n-gons
- These are secretly just boxes!



(23)

UV Pole Structure

The residue is then simple to generalize



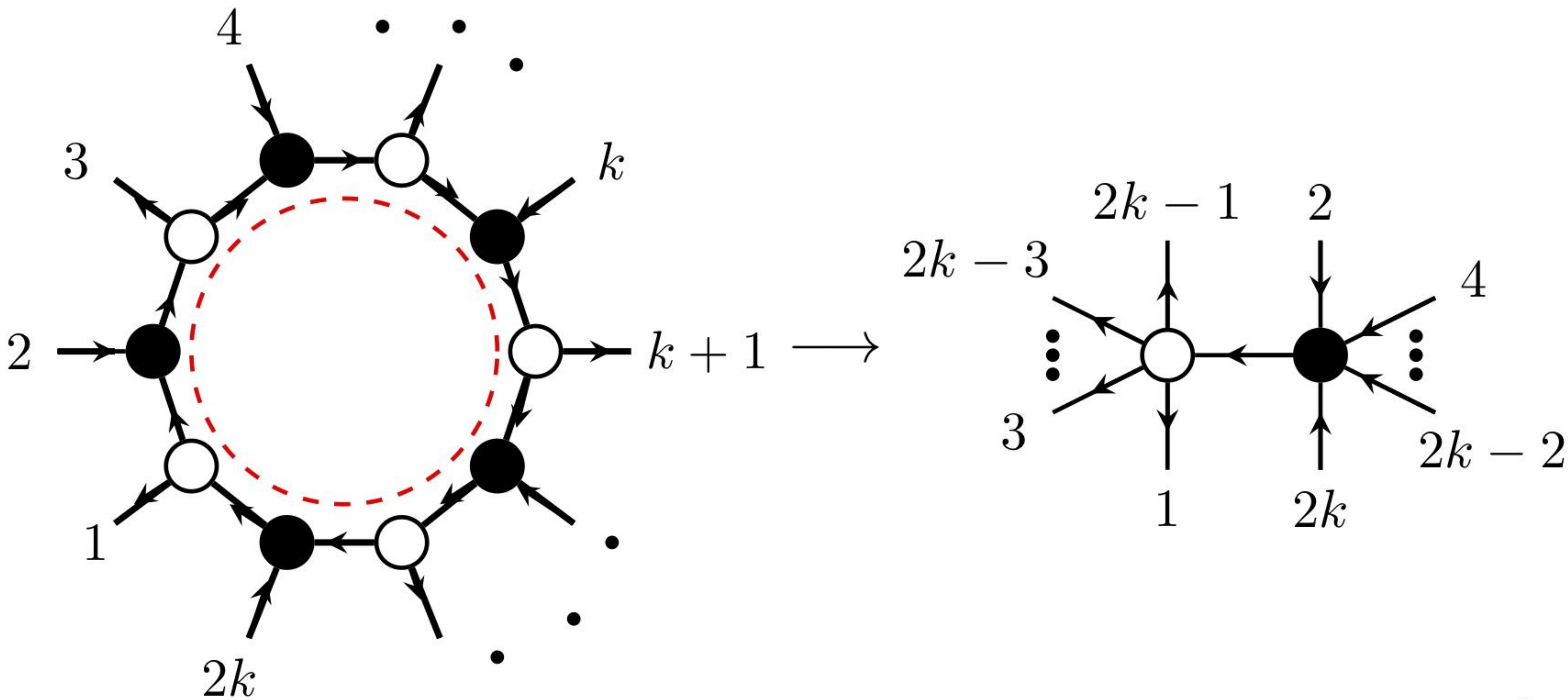
(24)

UV Pole Structure

- These all are MHV diagram with only two black vertices.
- For N^{k}MHV diagrams we have $k-2$ black vertices.
- The expressions are a lot more complicated, but the result is similar
- Find result for n-gon then attach these to remaining diagram.

UV Pole Structure

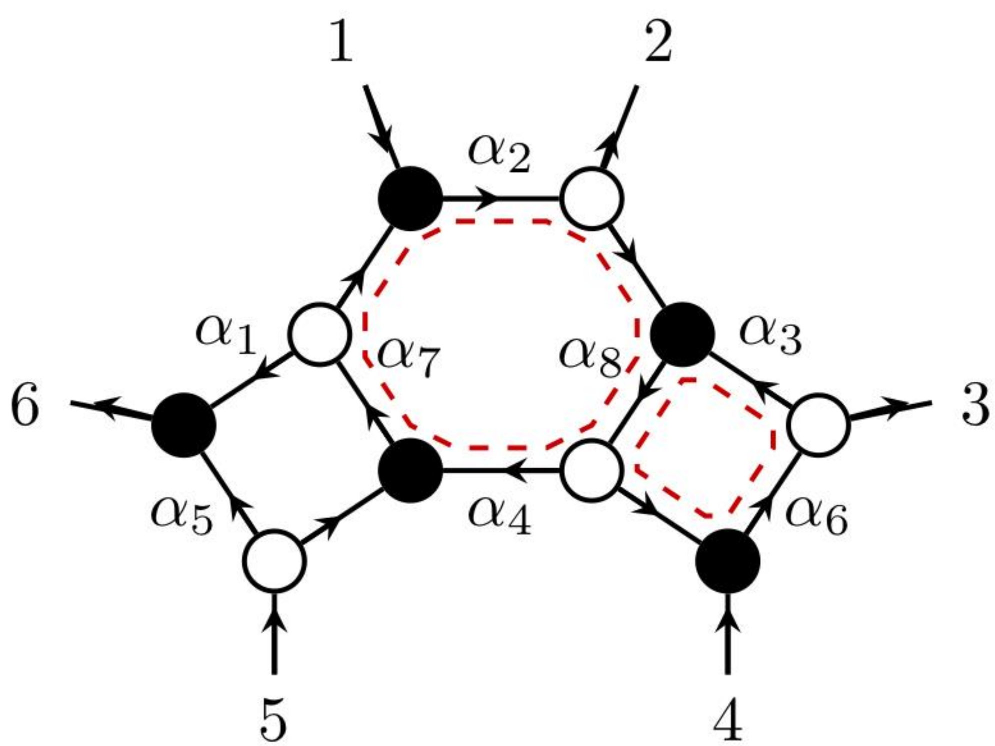
On the UV pole the on-shell diagrams behaves as



(25)

Larger diagrams

Take three-loop six-point NMHV leading singularity diagram

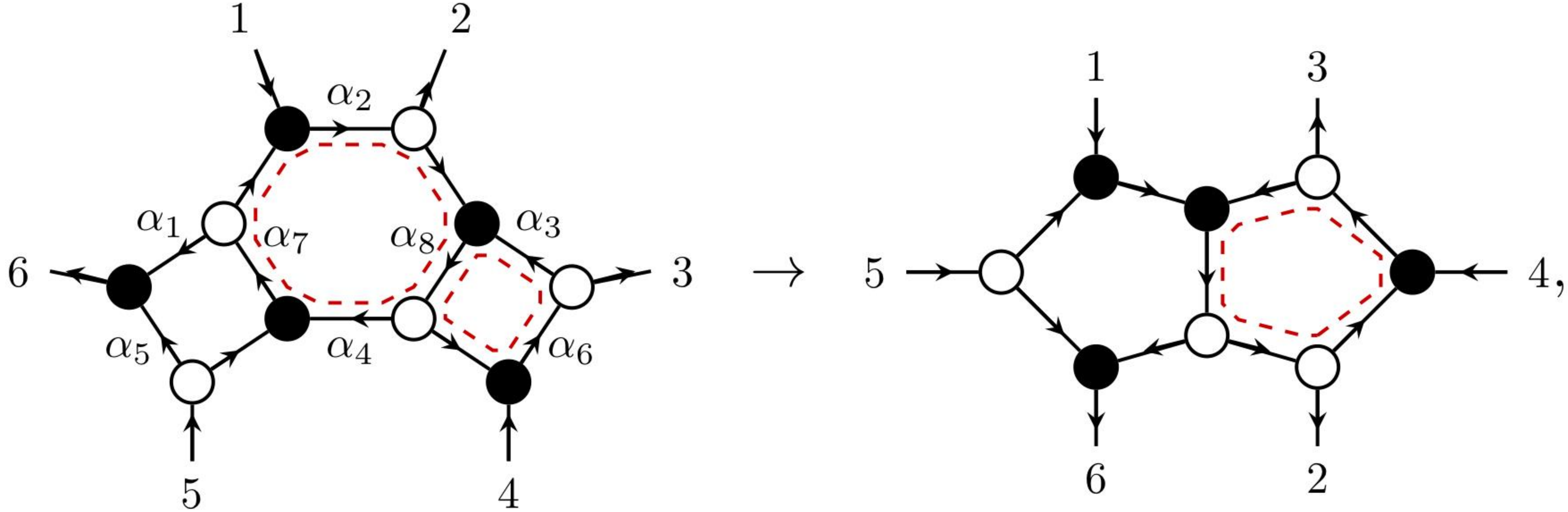


$$= \frac{\delta^6(\mathcal{Q})\delta^4(P)\delta([56]\widetilde{\eta}_1+[61]\widetilde{\eta}_5+[15]\widetilde{\eta}_6)}{\langle 34\rangle\langle 2|3+4|1]\langle 3|5+6|1]\langle 2|1+6|5][56]}$$

- Hexagon pole at $\langle 2|3+4|1] = 0$
- Box pole at $\langle 3|5+6|1]$

Larger diagrams

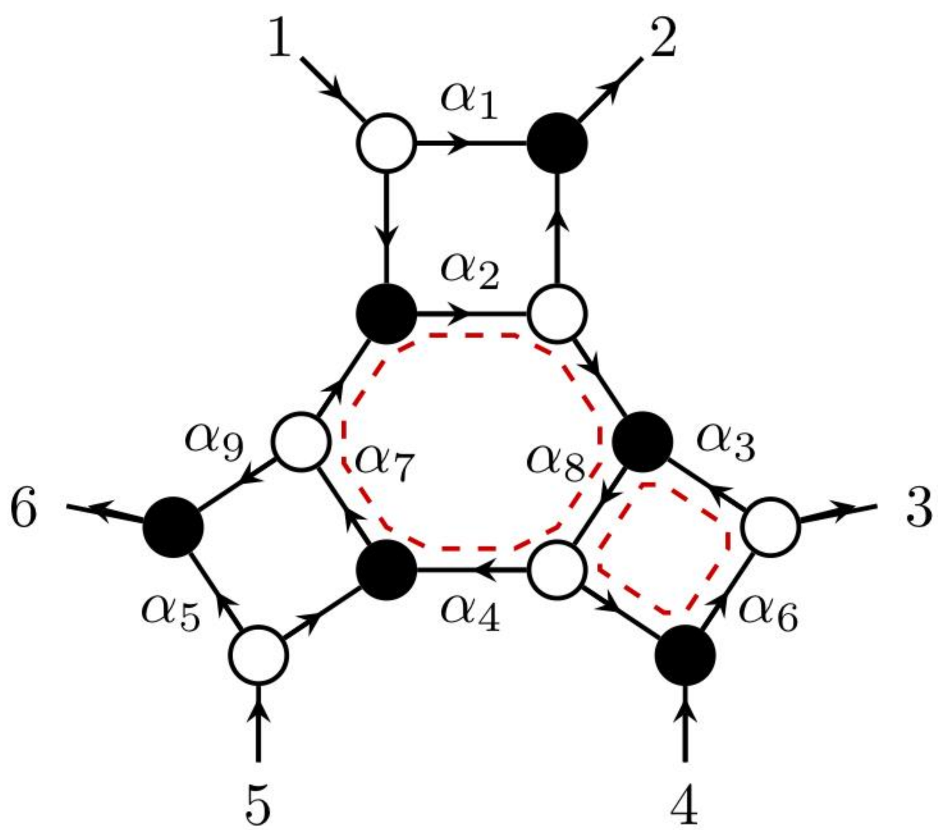
- Contracting the Hexagon into a tree and then attaching to the remaining diagram



- Notice the change in orientation – this is a general feature since for larger diagrams this gives non planar diagram

Larger diagrams

- Consider a more intricate example – the top dimensional cell of $G_+(3,6)$
- This can be obtained by attaching a BCFW bridge to the previous diagram: $\widehat{\widetilde{\lambda}}_1 = \widetilde{\lambda}_1 + \alpha_1 \widetilde{\lambda}_2$ and $\widehat{\lambda}_2 = \lambda_2 - \alpha_1 \lambda_1$
- On shell conditions leave one parameter unfixed

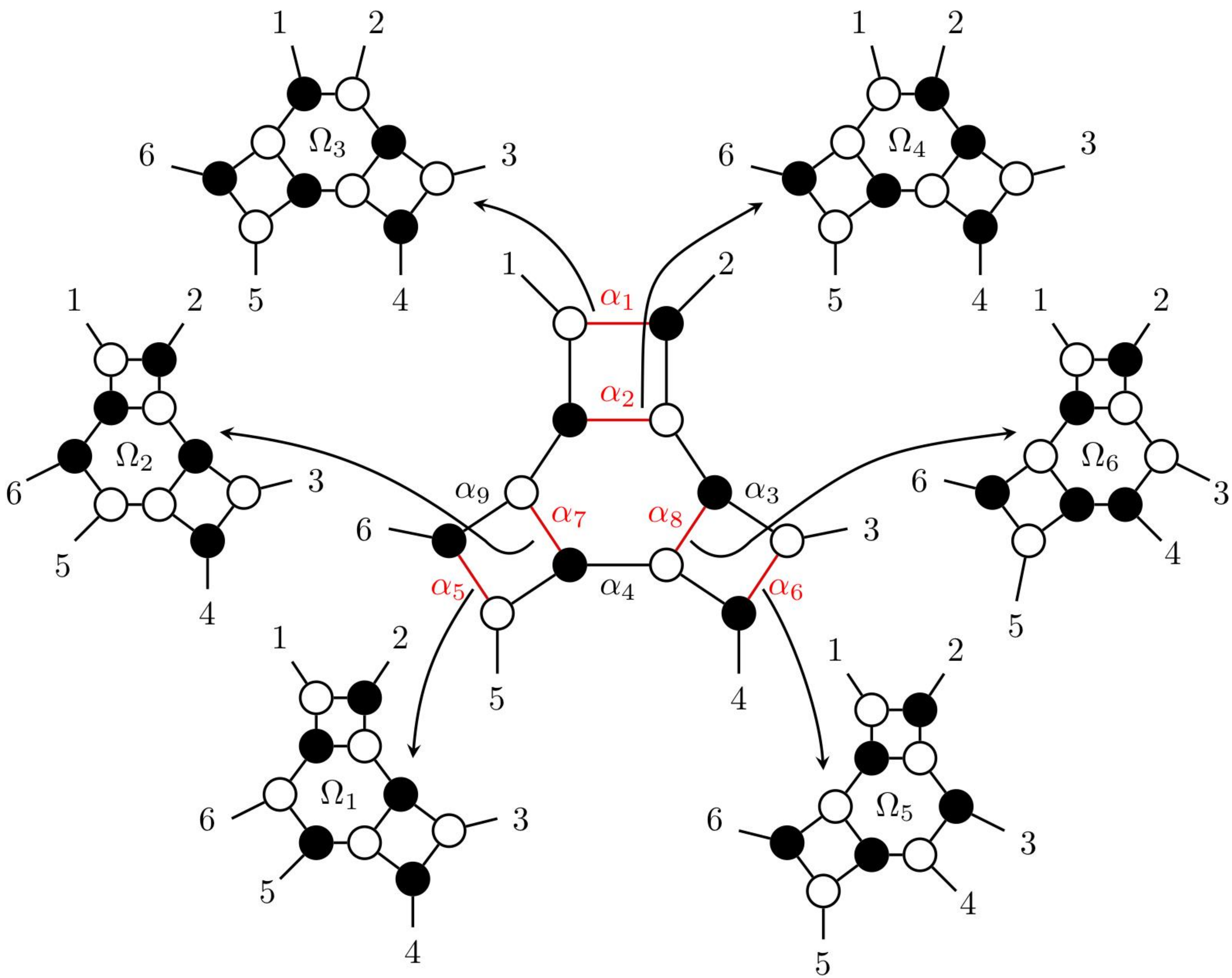


$$= \int \underbrace{\frac{d\alpha_1 \, \langle 4|\widehat{1}+5|6]\delta(\Xi)}{\alpha_1 s_{\widehat{1}56}\langle \widehat{2}3\rangle\langle 34\rangle\langle \widehat{2}|3+4|5]\langle 4|5+6|\widehat{1}][\widehat{1}6][56]}}_{\Omega^{\text{bare}}}$$

$$\times \underbrace{\frac{s_{\widehat{1}56}\langle \widehat{2}3\rangle\langle 4|5+6|\widehat{1}][\widehat{1}6]}{\langle \widehat{2}|5+6|\widehat{1}]\langle 3|5+6|\widehat{1}](\langle 4|\widehat{1}+5|6])}}_{\mathcal{J}^{-1}} \qquad (26)$$

Global Residue Theorem

For $\mathcal{N} = 4$ the pole structure can be illustrated as follows



(27)

GRT

- Where, through the Global Residue Theorem (GRT)

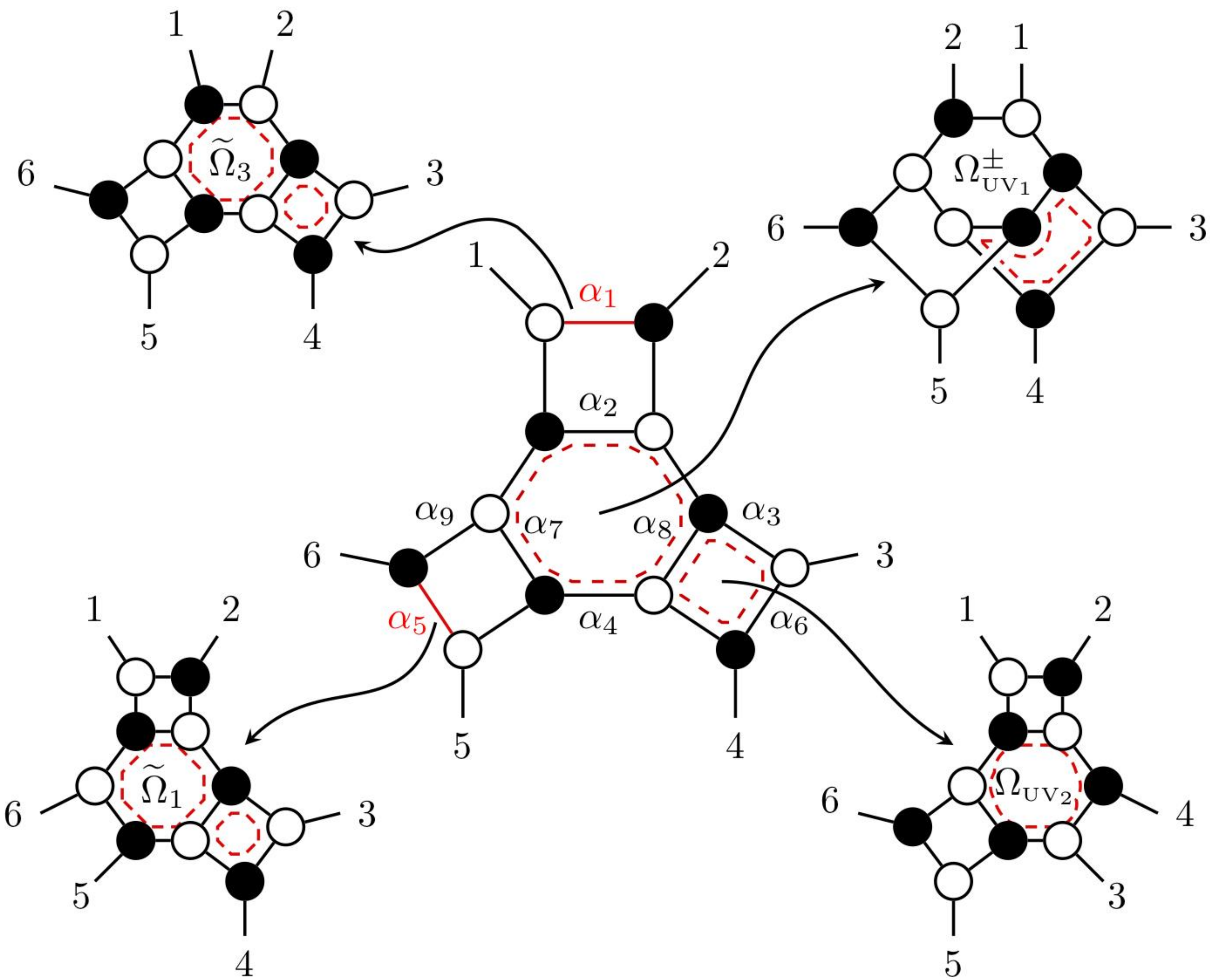
$$\sum_{i=1}^{6} \Omega_i = 0, \quad \text{with} \quad \Omega_1 = \frac{\delta^4(P)\delta^8(\mathcal{Q})\,\delta([34]\widetilde{\eta}_5+[45]\widetilde{\eta}_3+[53]\widetilde{\eta}_4)}{s_{345}[34][45]\langle 16\rangle\langle 12\rangle\langle 2|3+4|5]\langle 6|1+2|3]}$$

- This is directly linked to the six-point NMHV tree-level amplitude

$$\mathcal{A}_{6,3}^{\text{tree}} = \Omega_1 + \Omega_3 + \Omega_5 = -\Omega_2 - \Omega_4 - \Omega_6 \qquad (28)$$

GRT

For our $\mathcal{N} = 3$ example we have a new GRT, where we also explicitly see the non planar structure



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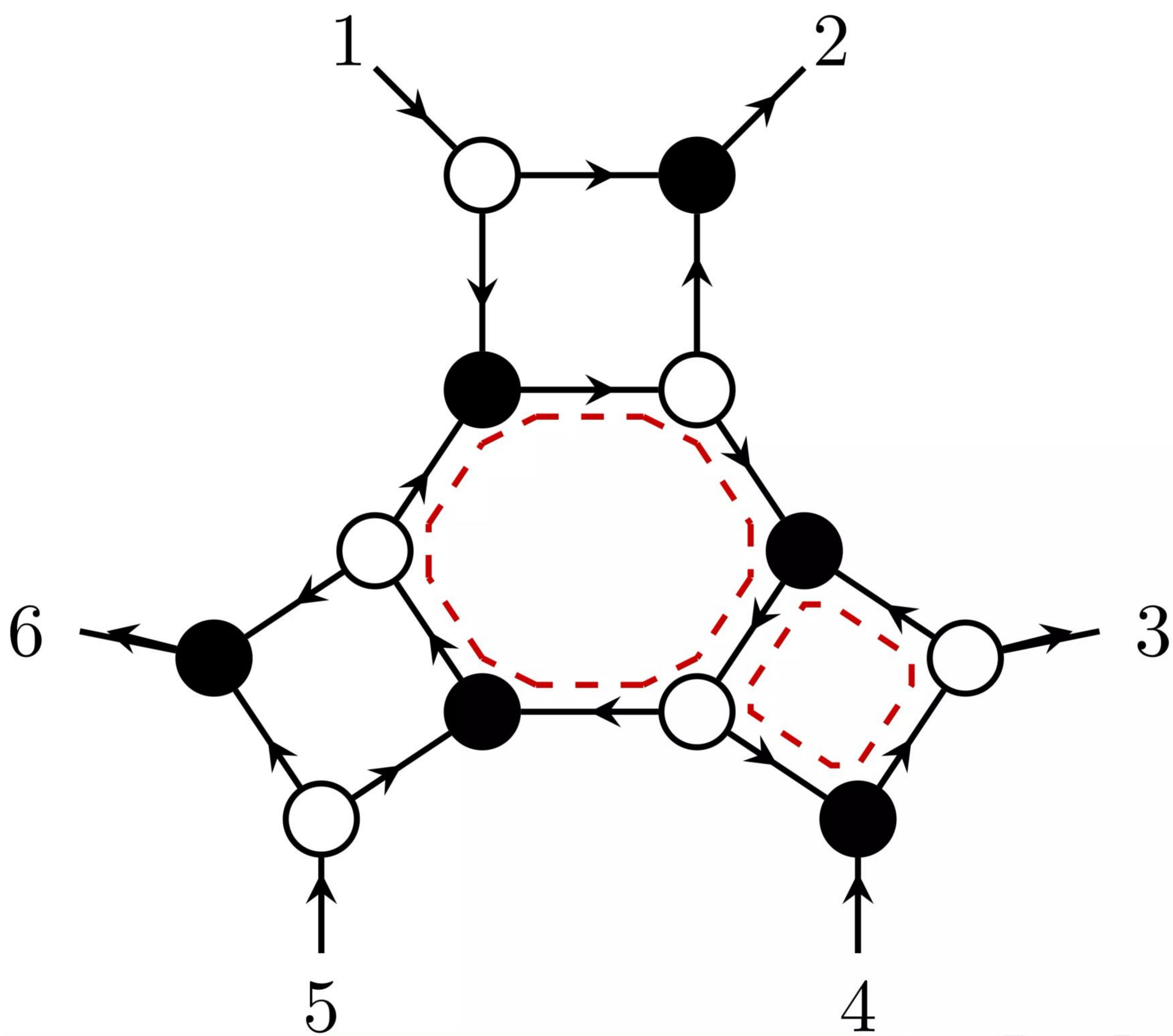
Dual Formulation
$\mathcal{N} \neq 4$

Video showing schematics:



Final points

- We call the procedure a *non-planar twist*
- Planar diagram $\rightarrow$ non planar
- For $\mathcal{N} < 3$ the procedure is the same but one also has to act on the diagram with an *infinity operator* $\mathcal{O}^{\mathcal{N}}$.

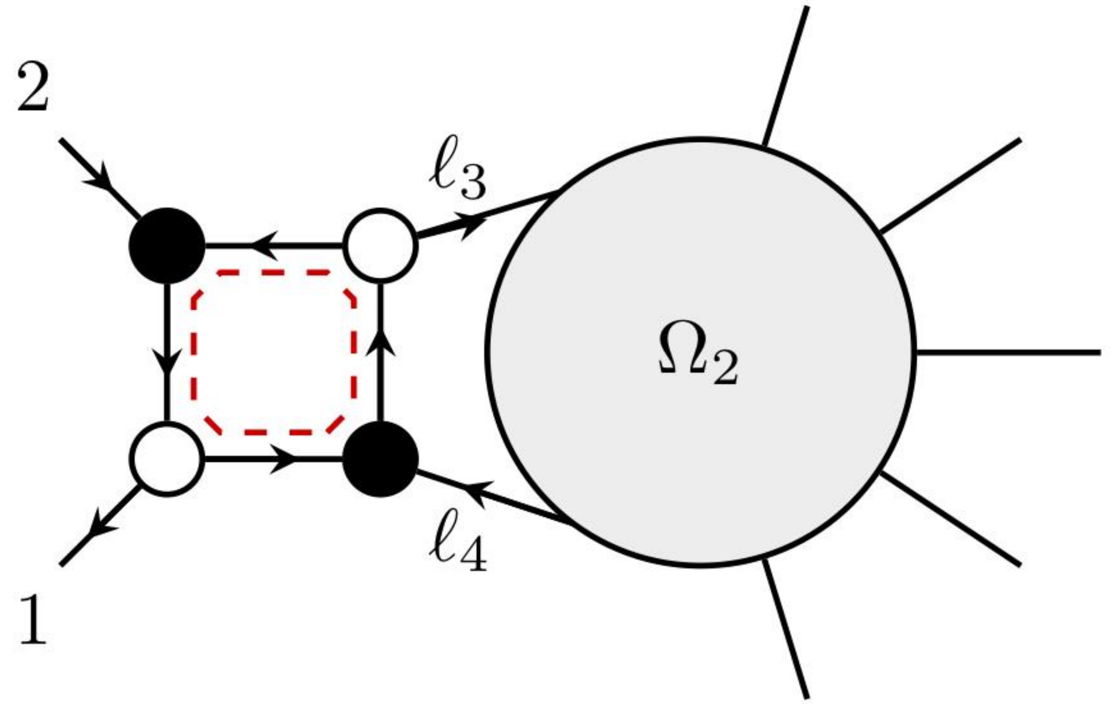
Summary

- On-shell diagrams have UV poles for less $\mathcal{N} < 4$
- For planar diagrams this is achieved by a non-planar twist and infinity operator
- Doesn't work for non-planar, we leave it for future work, this is needed for $\mathcal{N} = 8$ SUGRA

Thank you for your attention!

General $\mathcal{N}$

More general problem



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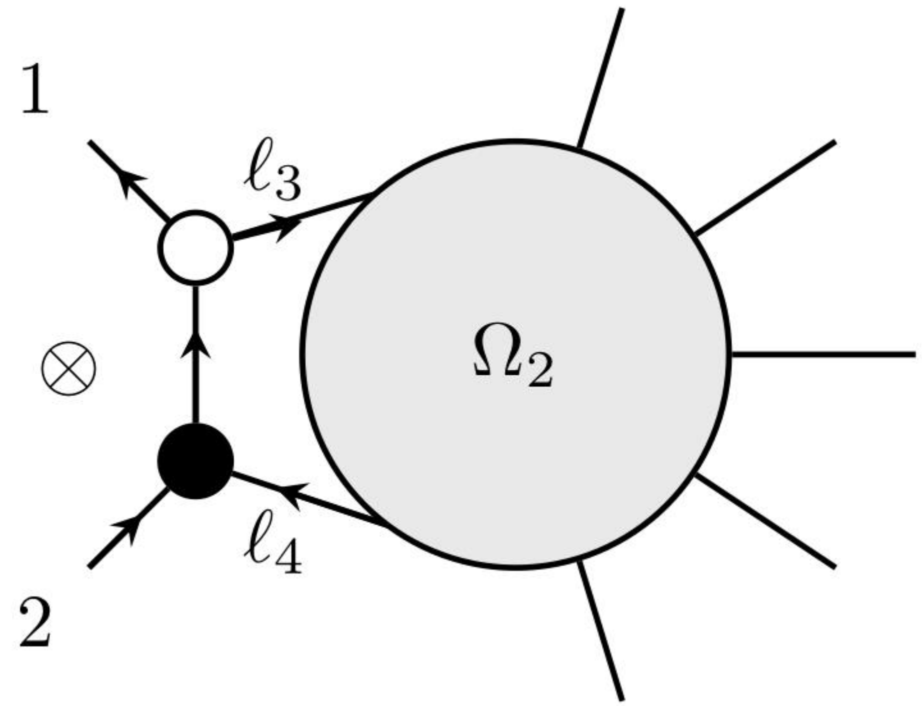
UV pole of this is obtained by acting on the collapsed diagram with a differential operator $\mathcal{O}$

General $\mathcal{N}$

$$\Omega_{\mathrm{UV}} = \mathcal{O}^{\mathcal{N}} \otimes \tag{31}$$



with

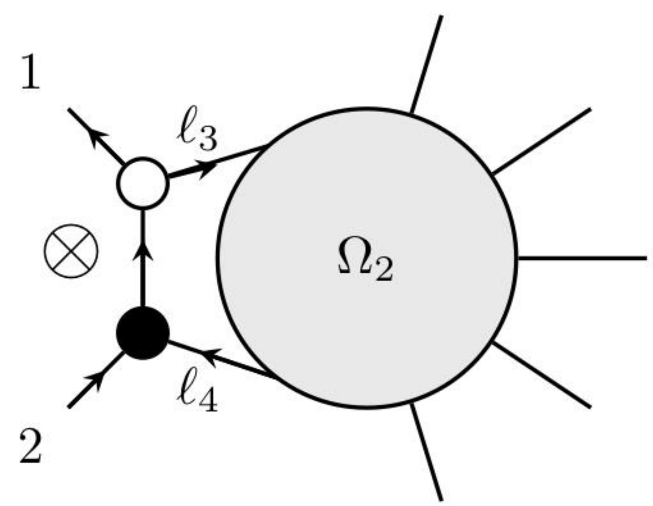
$$\mathcal{O}^{\mathcal{N}} = \frac{1}{(3-\mathcal{N})!}\left(\frac{\langle 2\ell_3\rangle[2\ell_3]}{\langle 12\rangle[1\ell_3]}\left\langle \lambda_2\,\frac{d}{d\lambda_{\ell_3}}\right\rangle\right)^{3-\mathcal{N}} \tag{32}$$

General $\mathcal{N}$

The procedure is as follows

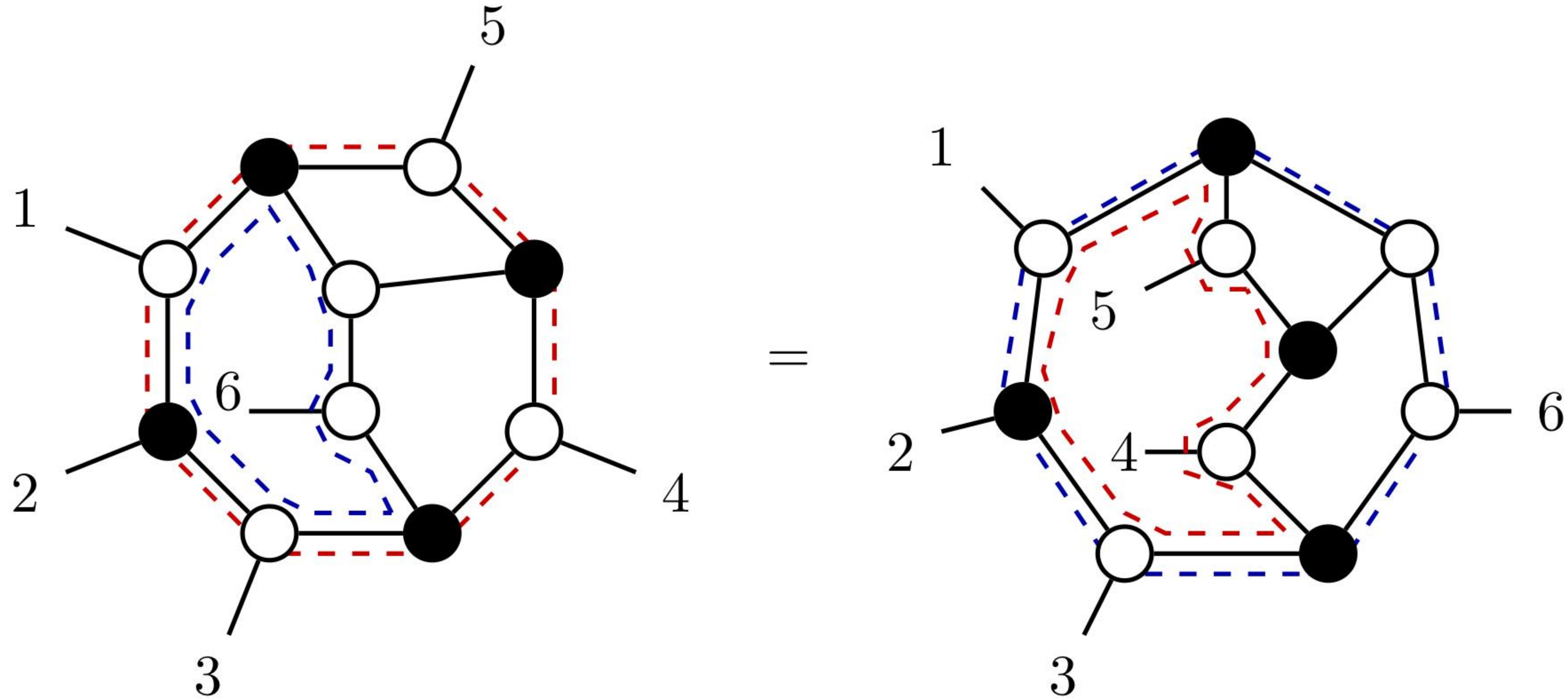
- Calculate the bare on-shell function of the lower-loop on-shell diagram obtained by diagrammatic rules
    - Crucially this relies on leaving the integration over an unfixed leg λ_{ℓ_3} such that one can act with derivative.
    - Also includes an integration over the internal leg ℓ_4 to eliminate the dependencies λ_{ℓ_3} from momentum conservation.
- Take the appropriate number of derivatives with respect to λ_{ℓ_3}

$$\Omega_{\rm UV} = \mathcal{O}^{\mathcal{N}} \otimes \qquad (33)$$



Non-Planar Diagrams

- Can't blow up one loop at a time



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