BCJ Bootstrap for Scalar Fields

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Motivation

- Modern amplitudes methods have been very useful for Effective Field Theories (EFT) (Soft Recursion Relations, color-kinematics, etc)
- Soft bootstrap has been used to explore the space of EFTs satisfying low energy theorems
- If we want to understand corrections to NLSM as an EFT, how can modern methods help?
- BCJ imposes novel constraints

Pions and NLSM

- Chiral perturbation theory is the low energy EFT of QCD

$$SU(N)_L \times SU(N)_R / SU(N)_L \equiv SU(N) \rightarrow \textit{Goldstone Modes}$$

- Leading order term with two derivatives is a non-linear sigma model (NLSM)

$$\mathcal{L} = \frac{F_\pi^2}{4} tr\left[(D_\mu U)(D_\mu U)^\dagger\right] + \dots$$

- Pion amplitudes vanish in the soft limit due to the Lagrangian's shift symmetry
- $\mathcal{L}_{\chi PT}$ admits a perturbative expansion as suppressed high derivative corrections

Soft Bootstrap for NLSM

- SU(N) NLSM tree amplitudes can be written as

$$\mathcal{A}_n^{\mathrm{NLSM}} = \sum_{\sigma} \mathrm{Tr}(T^{a_1} T^{a_2} \ldots T^{a_n})\, A_n^{\mathrm{NLSM}}(1, 2, \ldots, n)$$

- Poles are at $P_{ij}^2 = (p_i + \ldots + p_{j-1})^2 = 0$

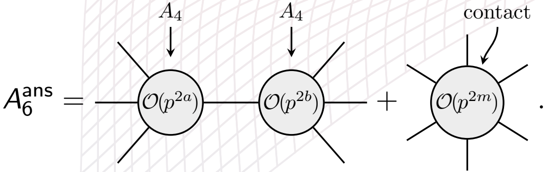
$$A_n \xrightarrow{P^2=0} A_{n_1} \frac{1}{P^2} A_{n-n_1+2}\,.$$

- At 4-point

$$\mathcal{O}(p^{2m}): \quad A_4 \in \{u^{m-a}(s^a + t^a)\}$$

Soft Bootstrap for NLSM

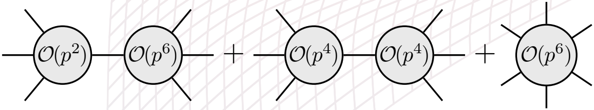
- m independent 4-point amplitudes at order $\mathcal{O}(p^{2m})$
- Soft behavior is automatically satisfied at 4-point, first non-trivial constraints at 6-point



- Must satisfy $a + b = m + 1$

Soft Bootstrap for NLSM

- For example at $\mathcal{O}(p^6)$



Soft Bootstrap for NLSM

- For the leading order $m = 1$, we get a unique solution

$$A_6^{\mathrm{NLSM}} = \left(\frac{s_{13}s_{46}}{s_{123}} + \frac{s_{26}s_{35}}{s_{345}} + \frac{s_{15}s_{24}}{s_{234}}\right) - \frac{1}{2}\sum_{\mathrm{cycl}} s_{13}\,.$$

- At general $\mathcal{O}(p^{2m})$ order we get multiple solutions
- Two important features to note
  - Constraints are placed at fixed $\mathcal{O}(p^{2m})$ order
  - Constraints are placed at fixed multiplicity

BCJ as a Constraint

- At 4-point Bern-Carrasco-Johansson (BCJ) relations are given by

$$sA_4(1,2,3,4) - uA_4(1,3,2,4) = 0\,.$$

- Impose cyclicity and Kleiss-Kuijf (KK) relations

$$A_4(1,2,3,4) + A_4(1,3,4,2) + A_4(1,3,2,4) = 0$$

- From these constraints any 4-point amplitude satisfying BCJ can be expressed as

$$A_4^{\mathrm{BCJ}} = \sum_{m,a,b} \alpha_{a,b}^{(2m)} \left(uF_{2m-4}^{a,b} \right)\,.$$

$$F_{a,b}^{(2m)} \in \left\{ (stu)^a (s^2+t^2+u^2)^b \right\} \quad \text{for } 3a+2b=m.$$

BCJ as a Constraint

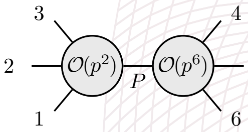
- Going up to 6-point we have

$$s_{12}A_6(123456) + (s_{12}+s_{23})A_6(132456) \\ - (s_{25}+s_{26})A_6(134256) - s_{26}A_6(134526) = 0$$

- Repeat the soft bootstrap procedure, ie fix $\mathcal{O}(p^{2m})$ order and write a local ansatz
- No 4-point $\mathcal{O}(p^4)$ amplitudes that satisfy BCJ, let's take the $\mathcal{O}(p^6)$ example

BCJ as a Constraint

- We have only one factorization diagram at this order



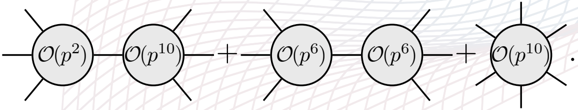
$$= A_4^{(2)}(123)\frac{1}{P^2}A_4^{(6)}(456)]$$

$$A_4^{(2)}(123) \equiv A_4^{(2)}(1,2,3,P) = s_{13},$$
$$A_4^{(6)}(456) \equiv A_4^{(6)}(4,5,6,-P) = \alpha_{0,1}^{(6)} s_{46}(s_{45}^2 + s_{46}^2 + s_{56}^2),$$

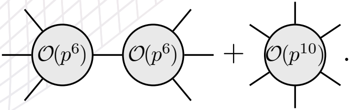
- Imposing BCJ fixes the contact term and we get a unique solution

BCJ as a Constraint

- Novel feature at $\mathcal{O}(p^{10})$ order-multiple factorization diagrams

$A_6^{\mathsf{ans}} =$ 

- Imposing BCJ gives two solutions, the second one fixes the coefficient for the contact term

$A_6^{\mathsf{BCJ,2}} =$ 

BCJ as a Constraint

- Coefficient for the $\mathcal{O}(p^{10})$ contact term related to $\mathcal{O}(p^6)$ 4-pt term coefficient by BCJ
- In Lagrangian language this relates c_2 to c_1^2 in

$$\mathcal{L} = c_1(\partial^6\phi^4) + c_2(\partial^{10}\phi^6) + \dots$$

- Differs from property 1 of the soft bootstrap, we get constraints between different derivative orders

BCJ vs Soft Bootstrap

Table: Soft and BCJ bootstrap results at 4-point

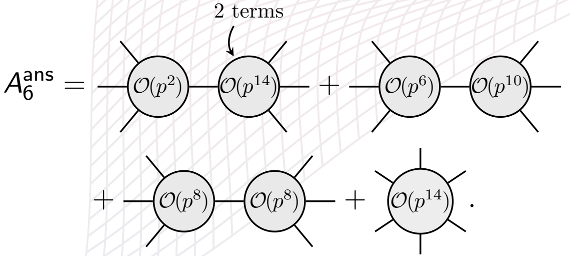
$\mathcal{O}(p^{\#})$	2	4	6	8	10	12	14	16	18
Soft amplitudes	1	2	3	4	5	6	7	8	9
BCJ amplitudes	1	0	1	1	1	1	2	1	2

Table: Soft and BCJ bootstrap results at 6-point

$\mathcal{O}(p^{\#})$	2	4	6	8	10	12	14	16	18
Soft amplitudes	1	2	10	29	83	207	461	945	1819
- Contact terms	0	0	5	22	70	191	434	915	1772
BCJ amplitudes	1	0	1	1	2	4	7	16	36
- Contact terms	0	0	0	0	0	2	4	13	31

BCJ Constraints for 4-Point

- At $\mathcal{O}(p^{14})$ level we start getting relations between 4-point coefficients of different derivative order, differs from property 2 of soft limits



(0)

BCJ Constraints for 4-Point

- The ansatz can be written as

$$A_6^{\text{ans}} = \alpha_{2,0}^{(14)}(\ldots) + \alpha_{0,3}^{(14)}(\ldots) + \alpha_{0,1}^{(6)}\alpha_{0,2}^{(10)}(\ldots) + (\alpha_{1,0}^{(8)})^2(\ldots) + \sum_k \alpha_k^{\text{ct}}(\ldots)$$

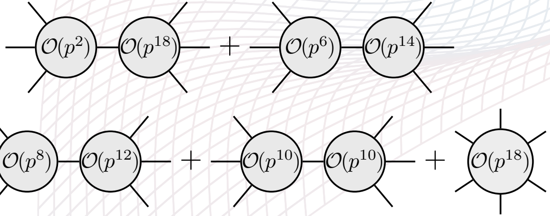
- Imposing BCJ gives 3 solutions instead of 4, giving the relation

$$\alpha_{2,0}^{(14)} - \frac{8}{3}\alpha_{0,3}^{(14)} - \frac{8}{3}\alpha_{0,1}^{(6)}\alpha_{0,2}^{(10)} - \frac{1}{2}(\alpha_{1,0}^{(8)})^2 = 0$$

BCJ Constraints for 4-Point

- Similar constraints for $\mathcal{O}(p^{18})$

$A_6^{\text{ans}} =$



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$$\alpha^{(18)}_{2,1} - 8\alpha^{(18)}_{0,4} + \alpha^{(6)}_{0,1}\alpha^{(14)}_{2,0} - 8\alpha^{(6)}_{0,1}\alpha^{(14)}_{0,3} - \alpha^{(8)}_{1,0}\alpha^{(12)}_{1,1} - 4(\alpha^{(10)}_{0,2})^2 = 0$$

Outlook

- BCJ puts novel constraints on higher derivative scalar amplitudes
- Different derivative orders related to each other, unlike soft limits
- Imposing BCJ at higher point will constrain lower point amplitudes even more
- We know Z-theory satisfies BCJ at every order, is it the only such theory?
- Is there a high derivative BCJ Lagrangian?
- Possible geometric structure underlying BCJ relations?

The End

Thanks for listening!

Z-Theory

- Open string amplitude can be written as

$$A_{string} = SYM \otimes Z\text{-}theory \quad (2)$$

- Z-theory is the part containing the disk integral