Tropical geometry, quantum affine algebras, and scattering amplitudes

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joint work with Nick Early

Grassmannian cluster algebras

- Let $k \leq n \in \mathbb{Z}_{\geq 1}$ and

$$\begin{aligned} \mathrm{Gr}(k,n) &= \{k \text{ dimensional subspaces of } \mathbb{C}^n\} \\ &= \{k \times n \text{ full rank matrices}\}/\text{row operations}. \end{aligned}$$

- A Plücker coordinate $P_{i_1,\ldots,i_k} \in \mathbb{C}[\mathrm{Gr}(k,n)]$ ($i_1 < \cdots < i_k$): for a $k \times n$ matrix $x = (x_{ij})_{k\times n}$, $P_{i_1,\ldots,i_k}(x)$ is the minor of x with 1st, …, kth rows and i_1th, …, i_kth columns.
- Dual canonical basis of $\mathbb{C}[\mathrm{Gr}(k,n)]$ is (CDFL2019)

$$\{\mathrm{ch}(T) : T \in \mathrm{SSYT}(k,[n])\},$$

where $\mathrm{ch}(T)$ is a polynomial in Plücker coordinates and is given by an explicit formula in [CDFL2019], $\mathrm{SSYT}(k,[n])$ is the set of rectangular tableaux with k rows and with entries in $[n]$.

Prime elements in the dual canonical basis

- $\mathrm{ch}(T)$ is called prime if $\mathrm{ch}(T) \neq \mathrm{ch}(T')\mathrm{ch}(T'')$ for any non-trivial tableaux T', T''.
- $\mathbb{C}[\mathrm{Gr}(2,5)]$ has 5 (non-frozen) prime elements $p_{13}, p_{24}, p_{14}, p_{25}, p_{35}$. They are all cluster variables.
- For general $\mathbb{C}[\mathrm{Gr}(k,n)]$, all cluster variables are prime but there are more prime elements than cluster variables.

Prime elements in the dual canonical basis

- How to classify all prime elements in the dual canonical basis of $\mathbb{C}[Gr(k,n)]$? This is a difficult question and it is only known in the case of $k = 2$. An element ch(T) in the dual canonical basis of $\mathbb{C}[Gr(2,n)]$ is prime if and only if T is a one-column tableau, i.e. ch(T) is a Plücker coordinate (Chari-Pressley).
- We will use Newton polytopes to construct prime elements in the dual canonical basis of $\mathbb{C}[Gr(k,n)]$. We conjecture that we can obtain all prime elements.

Prime elements in the dual canonical basis

- Let $\mathcal{T}_{k,n}^{(0)}$ be the set of all one-column tableaux which are obtained by cyclic shifts of the one-column tableau with entries $1, 2, \ldots, k-1, k+1$.
- For $d \geq 0$, we define recursively

$$\mathbf{N}_{k,n}^{(d)} = \mathrm{Newt}\left(\prod_{T \in \mathcal{T}_{k,n}^{(d)}} \mathrm{ch}_T(x_{i,j}) \right),$$

where $\mathcal{T}_{k,n}^{(d+1)}$ is the set of all tableaux which correspond to facets of $\mathbf{N}_{k,n}^{(d)}$, $\mathrm{ch}_T(x_{i,j})$ is the polynomial obtained by evaluating $\mathrm{ch}(T)$ on the web matrix (Speyer and Williams 2005).

From facets of Newton polytopes to tableaux

- The Newton polytope $\mathbf{N}_{k,n}^{(d)}$ can be described using certain equations and inequalities in its H-representation.
- Let F be a facet of the Newton polytope $\mathbf{N}_{k,n}^{(d)}$. The normal vector v_F of F is the coefficient vector in one of the inequalities in the H-representation of $\mathbf{N}_{k,n}^{(d)}$.
- If there is an entry of the vector v_F which is negative, then we add some vectors which are coefficients of the equations in the H-representation of $\mathbf{N}_{k,n}^{(d)}$ to v_F such that the resulting vector v_F' all have non-negative entries.

From facets of Newton polytopes to tableaux

- The vector v'_F can be written as $v'_F = \sum_{i,j} c_{i,j} e_{i,j}$ for some positive integers $c_{i,j}$, where $e_{i,j}$ is the standard basis of $\mathbb{R}^{(k-1)\times(n-k)}$.
- We send the vector $e_{i,j}$ to a fundamental tableau $T_{i,j}$ which is defined to be the one-column tableau with entries $[j, j+k] \setminus \{i+j\}$.
- The tableau T_F corresponding to F is obtained from $\cup_{i,j} T_{i,j}^{\cup c_{i,j}}$ by removing all frozen factors (if any).

Example: Gr(3, 6)

- The web matrix for Gr(3, 6) is

$$M = \begin{bmatrix} 1 & 0 & 0 & x_{1,1}x_{2,1} & x_{1,1}x_{2,12} + x_{1,2}x_{2,2} & x_{1,1}x_{2,123} + x_{1,2}x_{2,23} + x_{1,3}x_{2,3} \\ 0 & 1 & 0 & -x_{2,1} & -x_{2,12} & -x_{2,123} \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

  where we abbreviate for example $x_{2,23} = x_{2,2} + x_{2,3}$.
- Evaluating all Plücker coordinates on M and take their product, we obtain a polynomial p. The Newton polytope $\mathbf{N}_{3,6}^{(1)}$ is the Newton polytope defined by the vertices given by the exponents of monomials of p.

Example: Gr(3, 6)

- The H-representation of $\mathbf{N}_{3,6}^{(1)}$ is given by

$$(0,0,0,1,1,1) \cdot x - 20 = 0,\ (1,1,1,0,0,0) \cdot x - 10 = 0,\ (0,1,1,0,0,0) \cdot x - 4 \geq 0,$$
$$(0,0,1,0,0,0) \cdot x - 1 \geq 0,\ (0,0,0,0,1,1) \cdot x - 11 \geq 0,\ (0,0,0,0,0,1) \cdot x - 4 \geq 0,$$
$$(0,0,1,1,0,0) \cdot x - 6 \geq 0,\ (0,0,0,0,1,0) \cdot x - 4 \geq 0,\ (0,0,0,1,0,0) \cdot x - 4 \geq 0,$$
$$(1,0,0,0,0,0) \cdot x - 1 \geq 0,\ (1,0,0,0,1,0) \cdot x - 6 \geq 0,\ (1,1,0,0,1,1) \cdot x - 16 \geq 0,$$
$$(1,1,0,0,0,0) \cdot x - 4 \geq 0,\ (0,0,0,1,1,0) \cdot x - 11 \geq 0,\ (0,1,0,0,0,0) \cdot x - 1 \geq 0,$$
$$(1,0,0,0,1,1) \cdot x - 14 \geq 0,\ (0,1,0,0,0,1) \cdot x - 6 \geq 0,\ (1,1,0,0,0,1) \cdot x - 11 \geq 0,$$

where $(0,0,0,1,1,1) \cdot x$ is the inner product of the vectors $(0,0,0,1,1,1)$ and x.

Example: Gr(3, 6)

- For the facet F with the normal vector $v_F = (0,1,1,0,0,0)$ in the first line of the above, we have that $v_F = e_{1,2} + e_{1,3}$. The generalized roots $e_{1,2}$, $e_{1,3}$ corresponds to tableaux $\begin{array}{|c|}\hline 2\\\hline 4\\\hline 5\\\hline\end{array}$, $\begin{array}{|c|}\hline 3\\\hline 5\\\hline 6\\\hline\end{array}$ respectively. Removing the frozen factor $\begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline\end{array}$ in $\begin{array}{|c|}\hline 2\\\hline 4\\\hline 5\\\hline\end{array} \cup \begin{array}{|c|}\hline 3\\\hline 5\\\hline 6\\\hline\end{array} = \begin{array}{|c|c|}\hline 2 & 3\\\hline 4 & 5\\\hline 5 & 6\\\hline\end{array}$, we obtain $T_F = \begin{array}{|c|}\hline 2\\\hline 5\\\hline 6\\\hline\end{array}$.

Example: $\mathrm{Gr}(3,6)$

generalized roots	facets, hyperplanes	tableaux	modules
$\gamma_{124} = \alpha_{2,1}$	$(0,0,0,1,0,0)$	[124]	$Y_{1,-1}$
$\gamma_{125} = \alpha_{2,1} + \alpha_{2,2}$	$(0,0,0,1,1,0)$	[125]	$Y_{1,-3}Y_{1,-1}$
$\gamma_{134} = \alpha_{1,1}$	$(1,0,0,0,0,0)$	[134]	$Y_{2,0}$
$\gamma_{135} = \alpha_{1,1} + \alpha_{2,2}$	$(1,0,0,0,1,0)$	[135]	$Y_{1,-3}Y_{2,0}$
$\gamma_{136} = \alpha_{1,1} + \alpha_{2,2} + \alpha_{2,3}$	$(1,0,0,0,1,1)$	[136]	$Y_{1,-5}Y_{1,-3}Y_{2,0}$
$\gamma_{145} = \alpha_{1,1} + \alpha_{1,2}$	$(1,1,0,0,0,0)$	[145]	$Y_{2,-2}Y_{2,0}$
$\gamma_{146} = \alpha_{1,1} + \alpha_{1,2} + \alpha_{2,3}$	$(1,1,0,0,0,1)$	[146]	$Y_{1,-5}Y_{2,-2}Y_{2,0}$
$\gamma_{235} = \alpha_{2,2}$	$(0,0,0,0,1,0)$	[235]	$Y_{1,-3}$
$\gamma_{236} = \alpha_{2,2} + \alpha_{2,3}$	$(0,0,0,0,1,1)$	[236]	$Y_{1,-5}Y_{1,-3}$
$\gamma_{245} = \alpha_{1,2}$	$(0,1,0,0,0,0)$	[245]	$Y_{2,-2}$
$\gamma_{246} = \alpha_{1,2} + \alpha_{2,3}$	$(0,1,0,0,0,1)$	[246]	$Y_{1,-5}Y_{2,-2}$
$\gamma_{256} = \alpha_{1,2} + \alpha_{1,3}$	$(0,1,1,0,0,0)$	[256]	$Y_{2,-4}Y_{2,-2}$
$\gamma_{346} = \alpha_{2,3}$	$(0,0,0,0,0,1)$	[346]	$Y_{1,-5}$
$\gamma_{356} = \alpha_{1,3}$	$(0,0,1,0,0,0)$	[356]	$Y_{2,-4}$
$\gamma_{124} + \gamma_{356} = \alpha_{1,3} + \alpha_{2,1}$	$(0,0,1,1,0,0)$	[[124],[356]]	$Y_{2,-4}Y_{1,-1}$
$\gamma_{145} + \gamma_{236} = \alpha_{1,1} + \alpha_{1,2} + \alpha_{2,2} + \alpha_{2,3}$	$(1,1,0,0,1,1)$	[[135],[246]]	$Y_{1,-5}Y_{2,-2}Y_{1,-3}Y_{2,0}$
$\gamma_{126} = \alpha_{2,1} + \alpha_{2,2} + \alpha_{2,3}$	$(0,0,0,1,1,1)$	[126]	$Y_{1,-5}Y_{1,-3}Y_{1,-1}$
$\gamma_{156} = \alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}$	$(1,1,1,0,0,0)$	[156]	$Y_{2,-4}Y_{2,-2}Y_{2,0}$

Quantum affine algebras

- The results in Grassmannian case correspond to representations of $U_q(\widehat{\mathfrak{sl}_k})$.
- The results in Grassmannian case can be generalized to general quantum affine algebras.

Grassmannian string integrals

Arkani-Hamed, He, and Lam 2019 introduced Grassmannian string integrals:

$$I = (\alpha')^a \int_{\mathbb{R}^a_{>0}} \prod_{i,j} \frac{dx_{ij}}{x_{ij}} \prod_J p_J^{-\alpha' c_J},$$

where the second product runs over all Plücker coordinates p_J, α', c_J are some parameters, $a = (k-1)(n-k-1)$, x_{ij}'s are variables used in the web matrix (Speyer and Williams 2005).

Grassmannian string integrals

In [Early-L. 2023], we generalize the above integral: for every $d \geq 1$, we define

$$\mathbf{I}_{k,n}^{(d)} = (\alpha')^a \int_{\mathbb{R}_{>0}^a} \left(\prod_{(i,j)} \frac{dx_{i,j}}{x_{i,j}} \right) \left(\prod_T \mathrm{ch}_T^{-\alpha' c_T}(x_{i,j}) \right).$$

where the second product is over all tableaux T such that the face $\mathbf{F}_T$ corresponding to T is a facet of the Newton polytope $\mathbf{N}_{k,n}^{(d-1)}$, ch_T is given in [CDFL2019].
We expect that these integrals have applications in physics.

u-variables and u-equations

- Another application to physics is about u-variables and u-equations.
- u-variables are certain rational fractions in Plücker coordinates originally defined by physicists Koba-Nielsen in 1969 in the case of $\mathrm{Gr}(2, n)$.
- Arkani-Hamed, Frost, Plamondon, Salvatori, and Thomas have obtained general formulas for u-variables for categories of representations of quivers with relations.
- In [Early-L. 2023], we give a general formula for u-variables in the case of $\mathrm{Gr}(k, n)$.

Grassmannian cluster categories

- Jensen, King, and Su 2016 gave an additive categorification of $\mathbb{C}[\mathrm{Gr}(k,n)]$ using Cohen-Macaulay modules.
- Denote by $\mathrm{CM}(B_{k,n})$ the category of Cohen-Macaulay $B_{k,n}$-modules. The category $\mathrm{CM}(B_{k,n})$ has an Auslander-Reiten quiver.

Cluster variables, rigid indecomposable modules, real prime modules, tableaux

- Cluster variables in $\mathbb{C}[\mathrm{Gr}(k,n)]$ are in bijection with reachable rigid indecomposable modules in $\mathrm{CM}(B_{k,n})$ [Jensen, King, Su 2016].
- Cluster variables in $\mathbb{C}[\mathrm{Gr}(k,n)]$ are in bijection with reachable prime real modules in $\mathcal{C}_\ell^{\mathfrak{sl}_k}$ [Hernandez-Leclerc 2010, Qin 2017, Kang-Kashiwara-Kim-Oh 2018, Kashiwara-Kim-Oh-Park 2019].
- Cluster variables in $\mathbb{C}[\mathrm{Gr}(k,n)]$ are in bijection with reachable prime real tableaux in $\mathrm{SSYT}(k,[n])$ [Chang-Duan-Fraser-L. 2020].
- We replace the modules at the vertices of the Auslander-Reiten quiver by the corresponding tableaux.

Auslander-Reiten quiver in the case of Gr(3, 6)

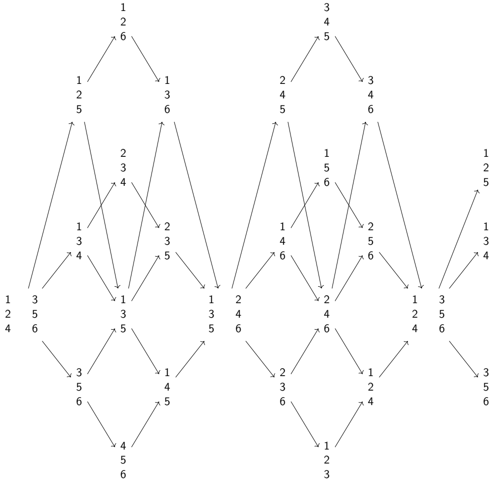


Figure: The Auslander-Reiten quiver for $\mathrm{CM}(B_{3,6})$ with vertices labelled by tableaux.

u-variables in the case of Gr$(3,6)$

The u-variables for Gr$(3,6)$ are

$$u_{126} = \frac{p_{136}}{p_{126}},\ u_{345} = \frac{p_{346}}{p_{345}},\ u_{125} = \frac{p_{126}p_{135}}{p_{125}p_{136}},\ u_{136} = \frac{\mathrm{ch}_{135,246}}{p_{136}p_{245}},\ u_{245} = \frac{p_{345}p_{246}}{p_{245}p_{346}},$$

$$u_{346} = \frac{\mathrm{ch}_{124,356}}{p_{346}p_{125}},\ u_{124,356} = \frac{p_{125}p_{134}p_{356}}{\mathrm{ch}_{124,356}p_{135}},\ u_{134} = \frac{p_{135}p_{234}}{p_{134}p_{235}},\ u_{135} = \frac{p_{136}p_{145}p_{235}}{p_{135}\mathrm{ch}_{135,246}},$$

$$u_{235} = \frac{\mathrm{ch}_{135,246}}{p_{235}p_{146}},\ u_{135,246} = \frac{p_{146}p_{245}p_{236}}{\mathrm{ch}_{135,246}p_{246}},\ u_{146} = \frac{p_{246}p_{156}}{p_{146}p_{256}},\ u_{246} = \frac{p_{346}p_{256}p_{124}}{p_{246}\mathrm{ch}_{124,356}},$$

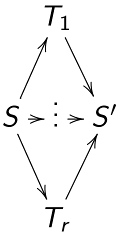
$$u_{256} = \frac{\mathrm{ch}_{124,356}}{p_{256}p_{134}},\ u_{234} = \frac{p_{235}}{p_{234}},\ u_{156} = \frac{p_{256}}{p_{156}},\ u_{356} = \frac{p_{135}p_{456}}{p_{356}p_{145}},\ u_{145} = \frac{\mathrm{ch}_{135,246}}{p_{145}p_{236}},$$

$$u_{236} = \frac{p_{246}p_{123}}{p_{236}p_{124}},\ u_{124} = \frac{\mathrm{ch}_{124,356}}{p_{124}p_{356}},\ u_{456} = \frac{p_{145}}{p_{456}},\ u_{123} = \frac{p_{124}}{p_{123}},$$

where we use $\mathrm{ch}_{T_1,\ldots,T_r}$ to denote ch_T, and T_i's are columns of T. Here $\mathrm{ch}_{124,356} = p_{124}p_{356} - p_{123}p_{456}$, and $\mathrm{ch}_{135,246} = p_{145}p_{236} - p_{123}p_{456}$.

A general formula for u-variables

For every mesh



in the Auslander-Reiten quiver of $\mathrm{CM}(B_{k,n})$, we define the corresponding u-variable as

$$u_S = \frac{\prod_{i=1}^{r} \mathrm{ch}_{T_i}}{\mathrm{ch}_S \mathrm{ch}_{S'}}.$$

u-equations

- We conjecture that there exist unique integers $a_{T,T'}$ such that

$$u_T + \prod_{T' \in \mathrm{PSSYT}_{k,n}} u_{T'}^{a_{T,T'}} = 1,$$

for all $T \in \mathrm{PSSYT}_{k,n}$, $\mathrm{PSSYT}_{k,n}$ is the set of all (non-frozen) prime tableaux in $\mathrm{SSYT}(k,[n])$.

- These equations are called u-equations.
- The following is an example of u-equation in the case of $\mathrm{Gr}(3,6)$:

$$u_{124,356} + u_{135}u_{136}u_{145}u_{146}u_{235}u_{236}u_{245}u_{246}u_{135,246}^2 = 1.$$

Thank you!