The free energy of dense fishnet graphs

Moritz Kade

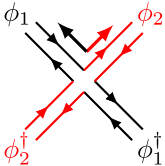
Project together with
M. Staudacher and M. Volk

Prague Spring Amplitudes Workshop
15th May 2023

RTG 2575:
Rethinking
Quantum Field Theory

Bi-Scalar Fishnet Theory [Gürdogan,Kazakov:1512.06704]

$$\mathcal{L} = \frac{\mathrm{N}}{2} \cdot \mathrm{tr}\left[\partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2\right] + \mathrm{N}(4\pi)^2 \xi^2 \cdot \mathrm{tr}\left[\phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2\right]$$



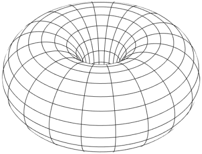
Bi-Scalar Fishnet Theory

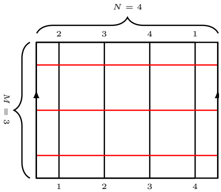
Bi-Scalar Fishnet Theory [Gürdogan,Kazakov:1512.06704]

$$\mathcal{L} = \frac{\mathrm{N}}{2} \cdot \mathrm{tr}\left[\partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2\right] + \mathrm{N}(4\pi)^2 \xi^2 \cdot \mathrm{tr}\left[\phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2\right]$$

Free energy:

$$Z = \sum_{M,N=1}^{\infty} \left(\xi^2\right)^{MN} \sum_{k=0}^{N-1} Z_{MN,k} + \text{counter terms at } M \text{ or } N \leq 2$$

$Z_{20,20,0} =$ 

$Z_{3,4,1} =$ 

$$Z_{MN,k} = \int \left[\prod_{m,n=1}^{M,N} \mathrm{d}^4 x_{m,n} \; \frac{1}{\left(x_{m,n} - x_{m+1,n}\right)^2} \; \frac{1}{\left(x_{m,n} - x_{m,n+1}\right)^2}\right]$$

$$\text{with } \; x_{M+1,n} = x_{1,n+k} \; \text{ and } \; x_{m,N+1} = x_{m,1}$$

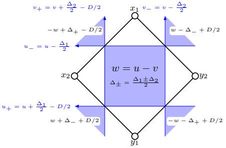
Free energy and R-matrix

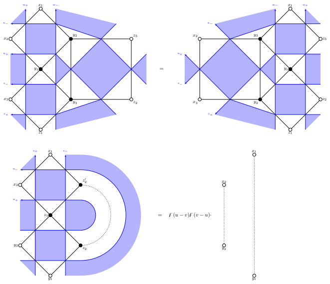
Free energy in thermodynamic limit (i.e. Radius of convergence)
[Zamolodchikov'80]

$$\kappa := \lim_{\substack{M,N\to\infty \\ M/N=1}} \left[Z_{MN,0}\right]^{\frac{1}{MN}}$$

Generalization: R-matrix as lattice face weights [Bazhanov,Kels,Sergeev'16]

[Chicherin,Derkachov,Isaev'12]

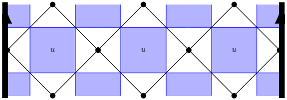
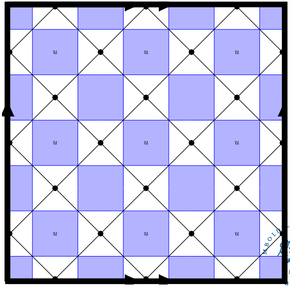




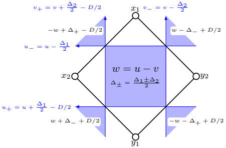
Free energy and R-matrix

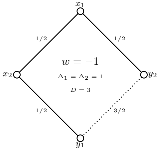
Free energy in thermodynamic limit (i.e. Radius of convergence) [Zamolodchikov'80]

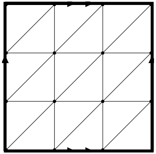
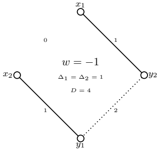
$$\kappa(u) := \lim_{M,N\to\infty} [Z_{MN}(u)]^{\frac{1}{MN}} = \lim_{M,N\to\infty} \operatorname{tr}\left[T_N(u)^M\right]^{\frac{1}{MN}}$$
$$= \lim_{M,N\to\infty} [\lambda_{N,\max}(u)]^{\frac{1}{N}}$$

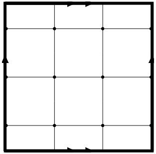
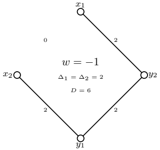
$T_3(u) =$ , $Z_{3,3}(u) =$ 

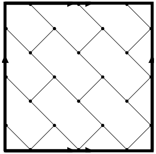
Free energy and R-matrix

<table>
<tr>
<td rowspan="3">


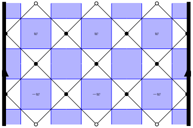
</td>
<td>


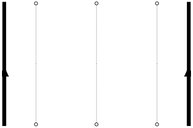
</td>
<td></td>
<td>$\kappa_e = 2\pi^{3/2}\frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{2}{3}\right)}$
ABJM fishnet</td>
</tr>
<tr>
<td>


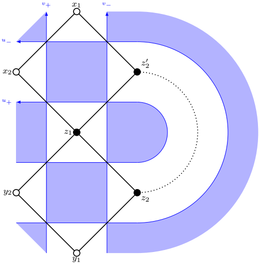
</td>
<td></td>
<td>$\kappa_e = \frac{\pi^{1/2}}{4\sqrt{2}}\Gamma\left(\frac{1}{4}\right)^2$
$\mathcal{N} = 4$ fishnet</td>
</tr>
<tr>
<td>


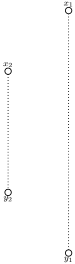
</td>
<td></td>
<td>$\kappa_e = \frac{\pi^{5/2}\Gamma\left(\frac{4}{3}\right)}{2\Gamma\left(\frac{5}{6}\right)}$
?</td>
</tr>
</table>

Inversion Relation

$T_3(w) \circ T_3(-w) =$ 

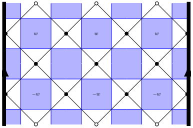
$= [F(w)F(-w)]^3$ 



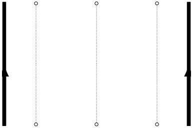
$= \quad F(u-v)F(v-u)\cdot$ 

Inversion Relation

$$T_3(w) \circ T_3(-w) =$$



$$= [F(w)F(-w)]^3$$



$$\Rightarrow T_N(w) \circ T_N(-w) = [F(w)F(-w)]^N \, \mathbb{1}$$

$$\Rightarrow \lambda_{N,\max}(w) \cdot \lambda_{N,\max}(-w) = [F(w)F(-w)]^N$$

$$\Rightarrow \kappa(w)\kappa(-w) = F(w)F(-w)$$

Solving Inversion Relations

Inversion relations:

$$\kappa(w)\kappa(-w) = F(w)F(-w)$$
$$\kappa(w) = \kappa(\frac{D}{2} - w)$$

Solution:

$$\kappa(w) = \pi^{D}\pi^{-2w}\frac{\Gamma\left(w+\Delta_-\right)}{\Gamma\left(\frac{D}{2}-(w+\Delta_-)\right)}\kappa_e\left(\frac{2\pi}{D}\left(w+\Delta_-\right)\right)^2$$
$$\cdot\pi^{-2w}\frac{\Gamma\left(w-\Delta_-\right)}{\Gamma\left(\frac{D}{2}-(w-\Delta_-)\right)}\kappa_e\left(\frac{2\pi}{D}\left(w-\Delta_-\right)\right)^2$$

with

$$\kappa_e(\alpha) = \pi^{D\alpha/2\pi}\frac{\Gamma\left(\frac{D}{2}-\frac{D\alpha}{2\pi}\right)}{\Gamma\left(\frac{D}{2}\right)}\prod_{l=1}^{\infty}\frac{\Gamma\left(Dl+\frac{D}{2}-\frac{D\alpha}{2\pi}\right)\Gamma\left(Dl+\frac{D\alpha}{2\pi}\right)\Gamma\left(Dl-\frac{D}{2}\right)}{\Gamma\left(Dl-\frac{D}{2}+\frac{D\alpha}{2\pi}\right)\Gamma\left(Dl-\frac{D\alpha}{2\pi}\right)\Gamma\left(Dl+\frac{D}{2}\right)}$$

Outlook

- Use fermionic R-matrix to obtain free energy in the thermodynamic limit of the brick-wall model

  [Chicherin,Derkachov,Isaev'12][Chicherin,Kazakov,Loebbert,Müller,Zhong'17]
- Check the result via TBA [Basso,Zhong'18]
- Inhomogeneous transfer matrix to study the effect of the twist
- Apply method of inversion relations to other models (e.g. WZW)

Thanks for your attention!