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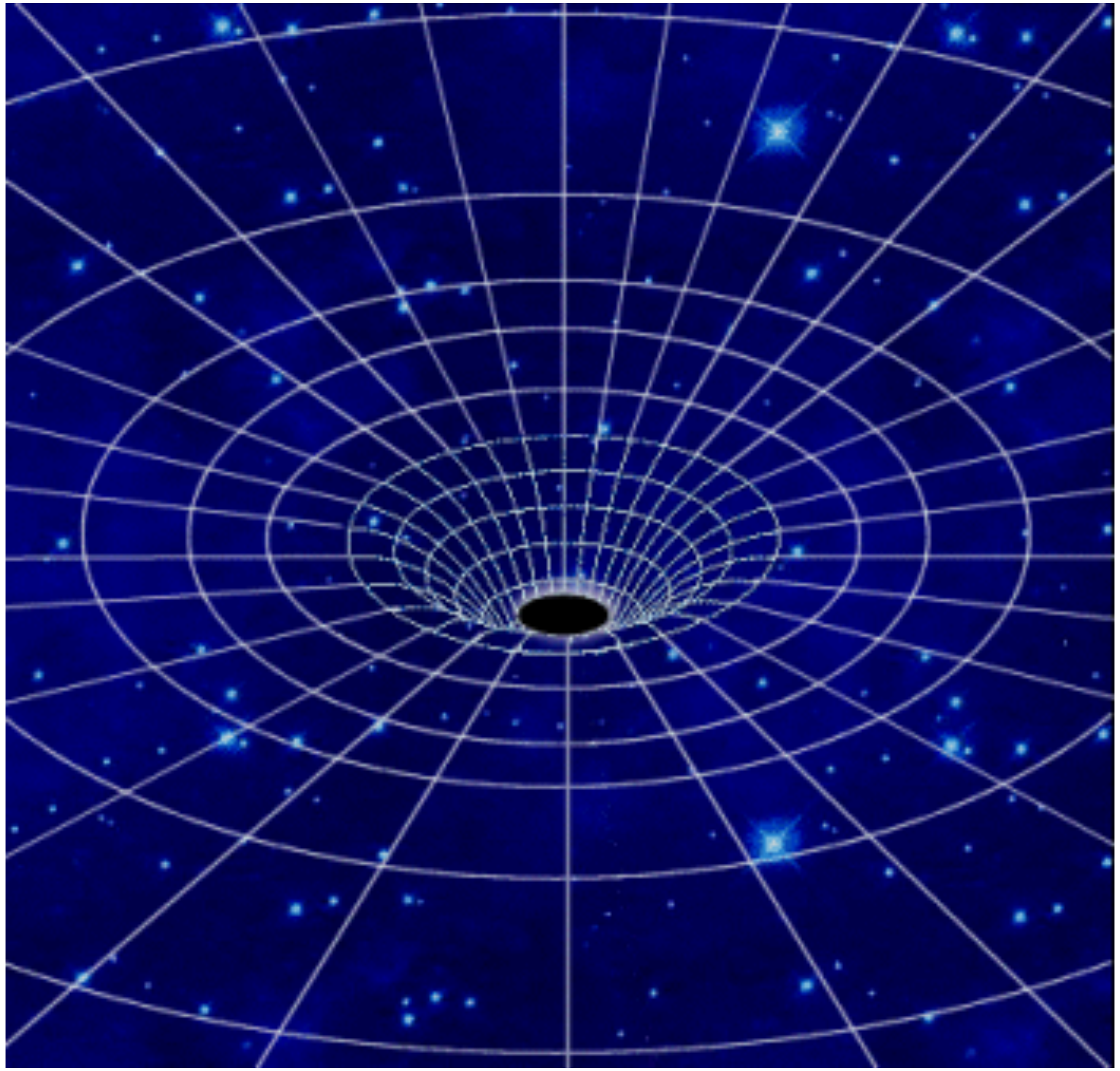
Amplitudes and Black hole merger dynamics

Seminar Prag (May 2023
)

General Relativity

- Einstein's theory presents us with a beautiful theory for gravity. Many exciting questions to study:
  - Extreme gravity
  - Quantum extensions
    - Geometrical description <-> EFT-QFT (flat space) formulation
    - Higher derivative bounds
- Graviton properties/mass etc
- Cosmological models
- Equivalence principle and quantum physics
- Extra dimensions / SUSY
- String theory….



New data - new window

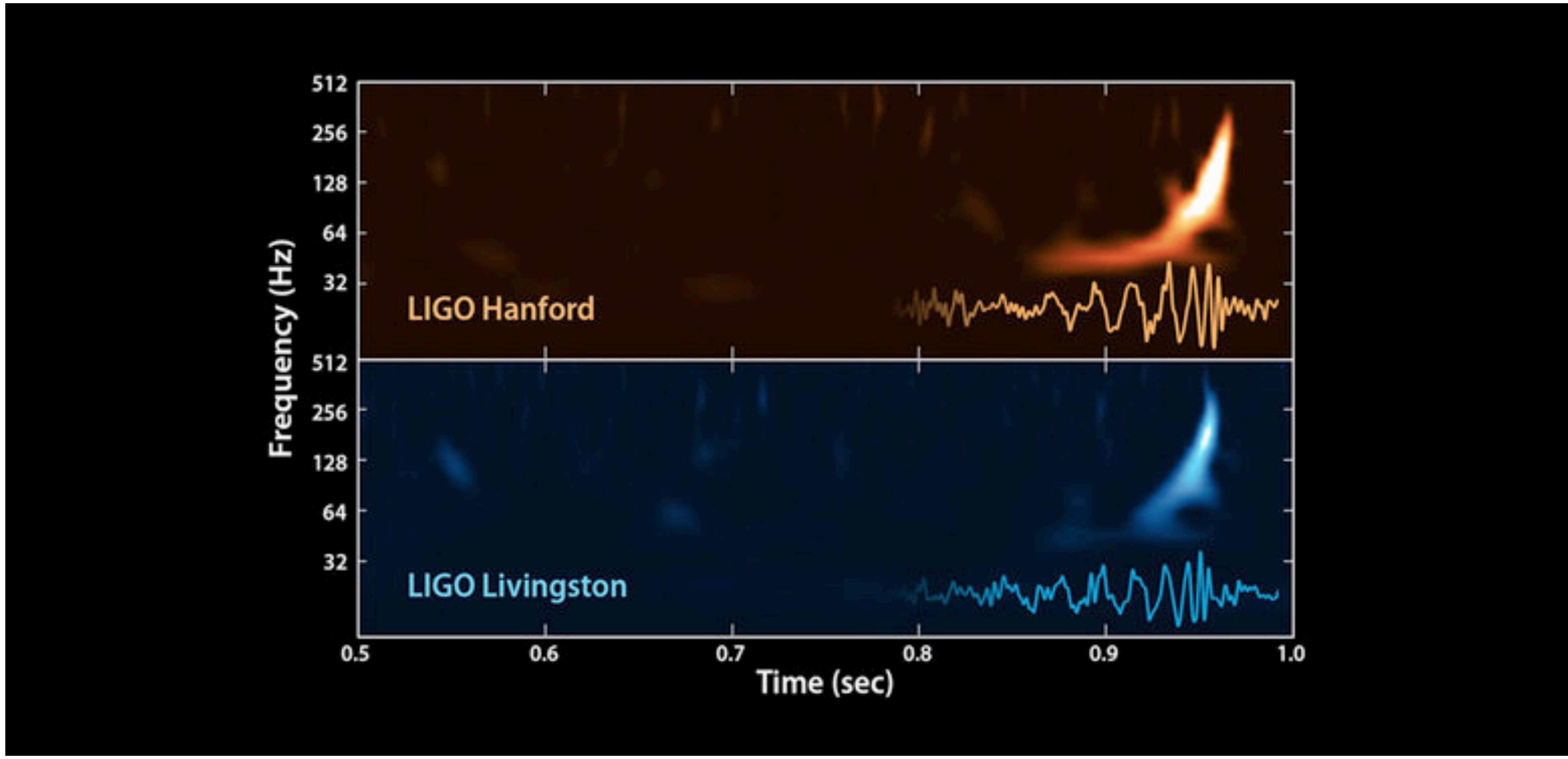
- First direct observation of a binary merger of black holes

- Direct access to gravitational interactions in the most extreme regimes

  - Possibility of complimenting conventional analysis. (See also Jan's talk)

  - A current need for theory to catch up to match observational progress & precision

  - Many Interesting questions to study: Validity of GR/ gravity phenomenology/ new theories?



Amplitudes methods allow refined computation and increased precision!

Key research directions (amplitudes)

Test of general relativity in certain regimes of binary mergers where GR observables are extracted from QFT methods.

Surprise: Classical physics from a relativistic quantized theory of gravitons seems more efficient than directly solving Einstein's field equations!

One key question: Can we formulate a precise extraction of classical gravitational physics from on-shell amplitudes? (with least amount of work)

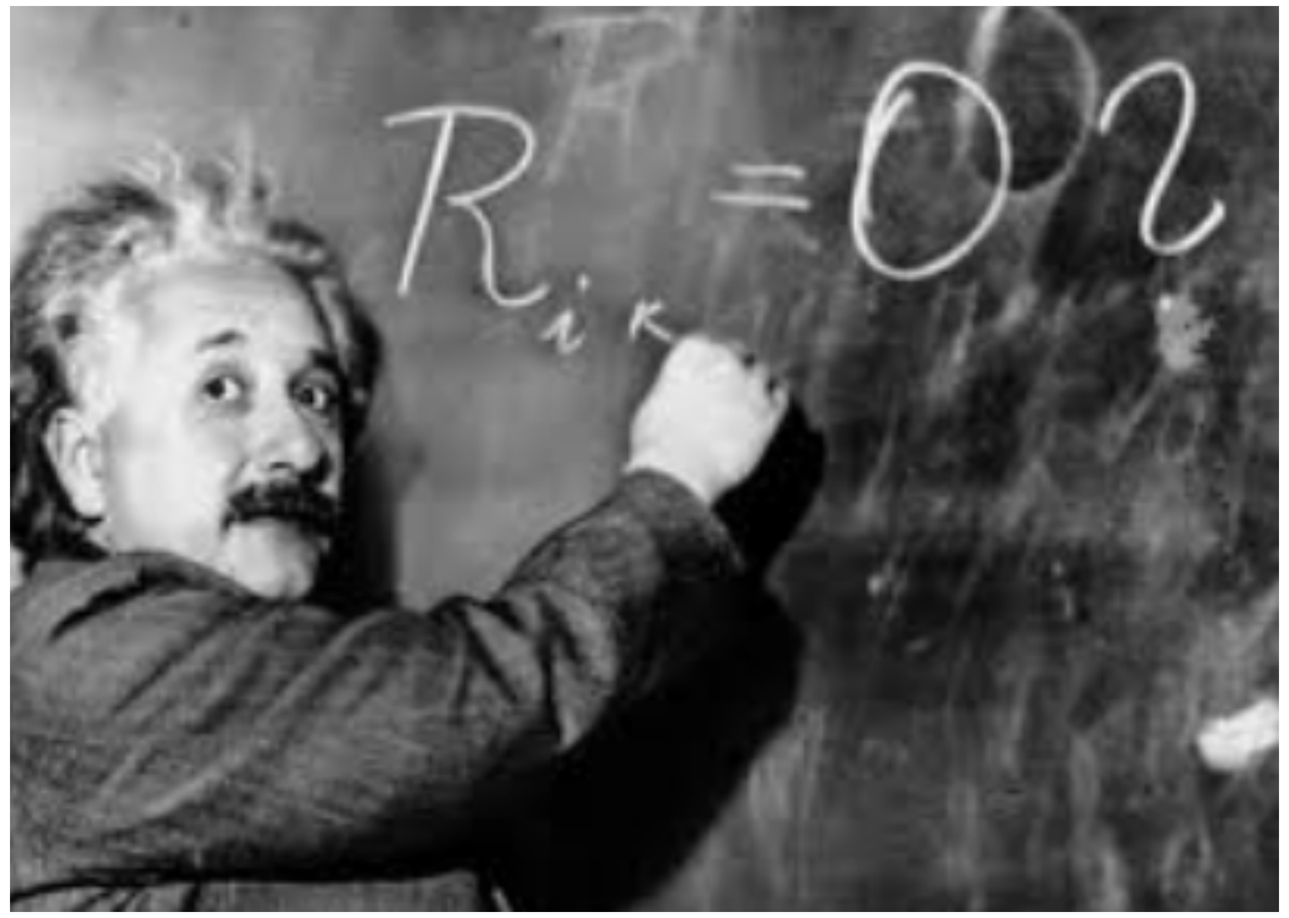
-> Potential: discovery of new physics

-> Faster and more accurate theoretical breakdown of gravitational wave events!

Traditional quantization of gravity

- Known since the 1960ties that a particle version of General Relativity can be derived from the Einstein Hilbert Lagrangian (Feynman, DeWitt)
- Expand Einstein-Hilbert Lagrangian :

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}$$



- Derive vertices as in a particle theory - compute amplitudes as Feynman diagrams! (GW Kovacs and Thorne 1977)

Off-shell QFT methods: not very computationally efficient!

Quantum gravity? An effective field theory

- A modern viewpoint (Weinberg) to view the quantization of general relativity from the viewpoint of effective field theory

$$\mathscr{L}_{\mathrm{EH}} = \sqrt{-g}\left[\frac{R}{16\pi G_N} + \mathscr{L}_{\mathrm{matter}}\right]$$

EFT -gravity　　　EFT -matter

$$\mathscr{L}_{\mathrm{eff\ GR}} = \sqrt{-g}\left[\frac{2R}{16\pi G_N} + R^2 + R_{\mu\nu}^2 + \ldots + \mathscr{L}_{\mathrm{matter}} + \ldots\right]$$

- Consistent quantum gravity at low energies long-distance contributions at one-loop (Donoghue; NEJBB, Donoghue, Holstein)

Advantages: Gravity as an EFT

- Treating general relativity as an effective field theory avoid complications and confusions in quantising gravity -
    - Natural generalisation of Einstein's theory
    - Ideal perturbative setup for QFT analysis of black hole binary mergers
- Universal consequences of underlying fundamental theory
    - Direct connection to low energy phenomenology of string and super-gravity theories
- Classical GR has a huge validity for normal energies
    - GR-EFT is attractive for investigating quantum aspects
    - Using **amplitude methods** we can take advantage of on-shell simplifications

Classical potential

- Problem in scattering theory to relate a scattering loop amplitude M to an interaction potential V.
- In PN we consider non-relativistic quantum mechanics, and this can be generalized to the relativistic case.
- We restrict classical objects that scatter to classical distance scales.

- The Hamiltonian H for the two massive scalars is given by Salpeter eq.

$$H = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + \mathcal{V}(r, p)$$

Classical potential from a Lippmann-Schwinger equation

- Non-relativistic limit, the tree classical potential is simply equal to the amplitude after a Fourier transform:

$$\mathcal{V}(r,p) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r} \mathcal{V}(p,q) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r} \tilde{\mathcal{M}}(p,q)$$

- Extension is given by Lippmann Schwinger eq. (involves iterations/ subtractions)

$$\tilde{\mathcal{M}}(p,p') = \mathcal{V}(p,p') + \int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{V}(p,k)\mathcal{M}(k,p')}{E_p - E_k + i\varepsilon}$$

Classical gravity from quantum theory

- In this context the old-fashioned time-ordered perturbation theory is logical

- In particular we eliminate by hand
  - Annihilation channels
  - Back-tracking diagrams (no intermediate multiparticle states)
  - Anti-particle intermediate states

We will also assume (classical) long-distance scattering (this has the consequence that we can focus on non-analytic contributions -> ideal for unitarity)

(NEJBB, Donoghue, Holstein; Cristofoli, NEJBB, Damgaard, Vanhove)

Spin-less gravity from quantum field theory

- We start with Einstein-Hilbert term

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + g^{\mu\nu} T_{\mu\nu} \right]$$

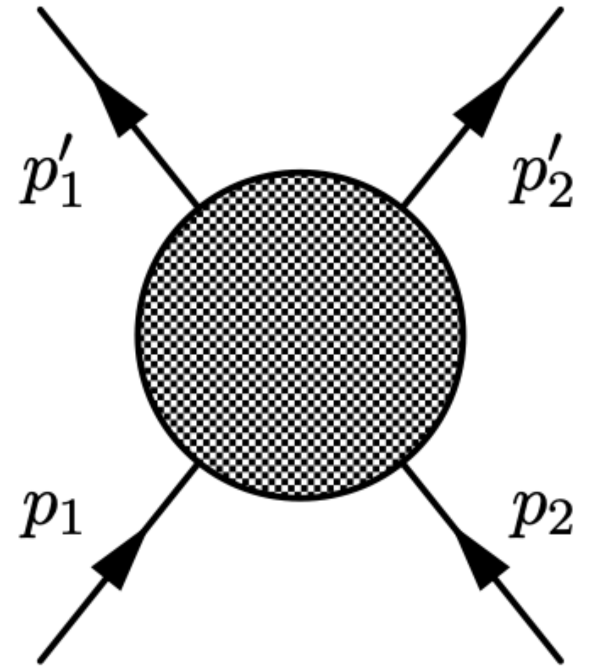
- where the minimal 'energy-momentum' tensor is

$$T_{\mu\nu} \equiv \partial_\mu \varphi \partial_n u \varphi - \frac{\eta_{\mu\nu}}{2} \left[\partial^\alpha \varphi \partial_\alpha \varphi - m^2 \varphi^2 \right]$$

- Consider the 2 -> 2 process from path integral

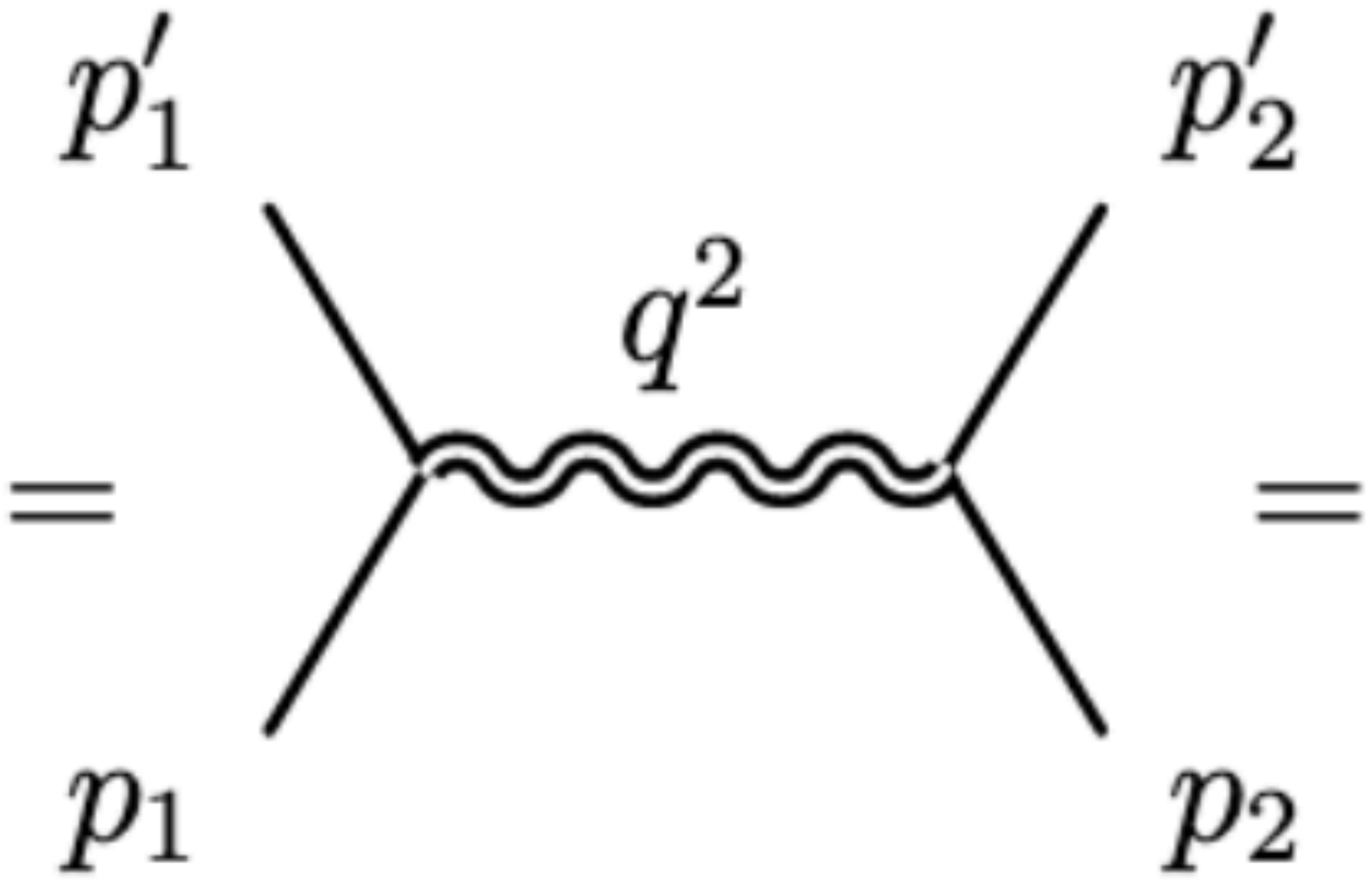
$$\varphi_1(p_1, m_1) + \varphi_2(p_2, m_2) \rightarrow \varphi_1(p_1', m_1) + \varphi_2(p_2', m_2)$$

$$= \sum_{L=0}^{\infty} \mathcal{M}_L(p_1, p_2, p_1', p_2') =$$



Tree level — two ‘heavy’-masses

$$\mathcal{M}_0(\gamma, \underline{q}^2, \hbar) =$$



$$= \hbar \frac{2\pi m_1^2 m_2^2 G_N (2\gamma^2 - 1)}{|\underline{q}|^2} + O(\hbar^0)$$

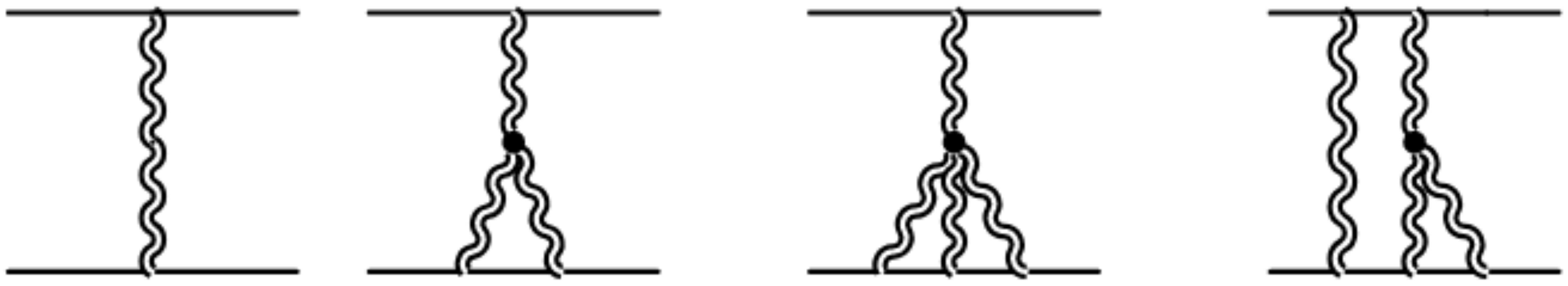
Newton’s law through Fourier transform

$$V(r) = -\frac{G m_1 m_2}{r}$$

- but does **not** give post-Newtonian effects

Classical gravitational scattering from quantum field theory

- Surprise: Non-linear (classical) corrections from loop diagrams!
- Can consider the various exchanges

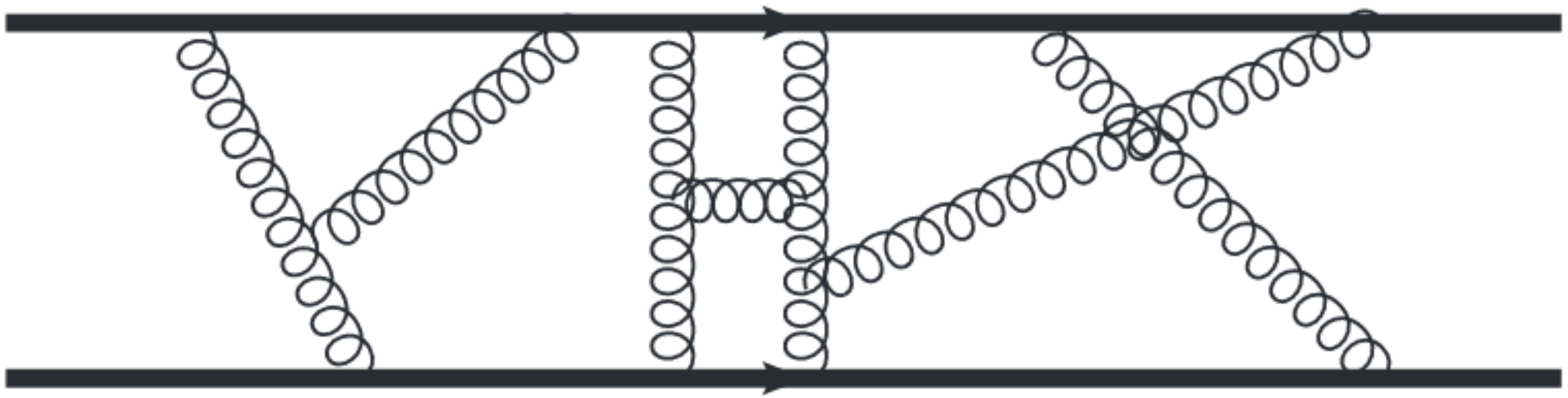


- Define transfer momentum, CM energy

$$q^2 \equiv (p_1 - p_1')^2 \qquad \gamma \equiv \frac{p_1 \cdot p_2}{m_1 m_2}$$

$$\mathcal{E}_{CM}^2 \equiv (p_1 + p_2)^2 \equiv (p_1' + p_2')^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma$$

Classical gravitational scattering from quantum field theory



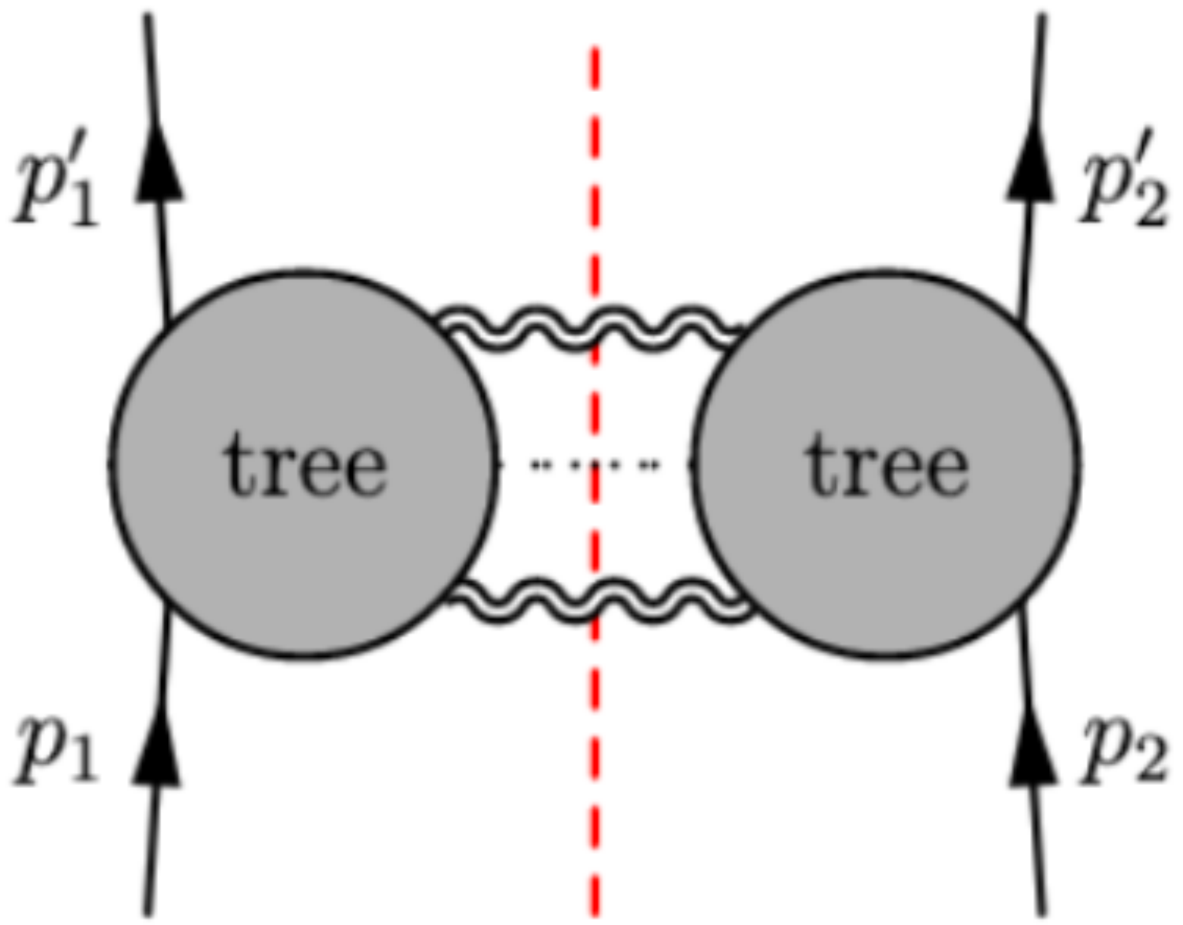
- Classical limit: we keep wave number fixed and take Planck's constant to zero, leads to the following Laurant expansion (quantum / classical / superclassical terms)

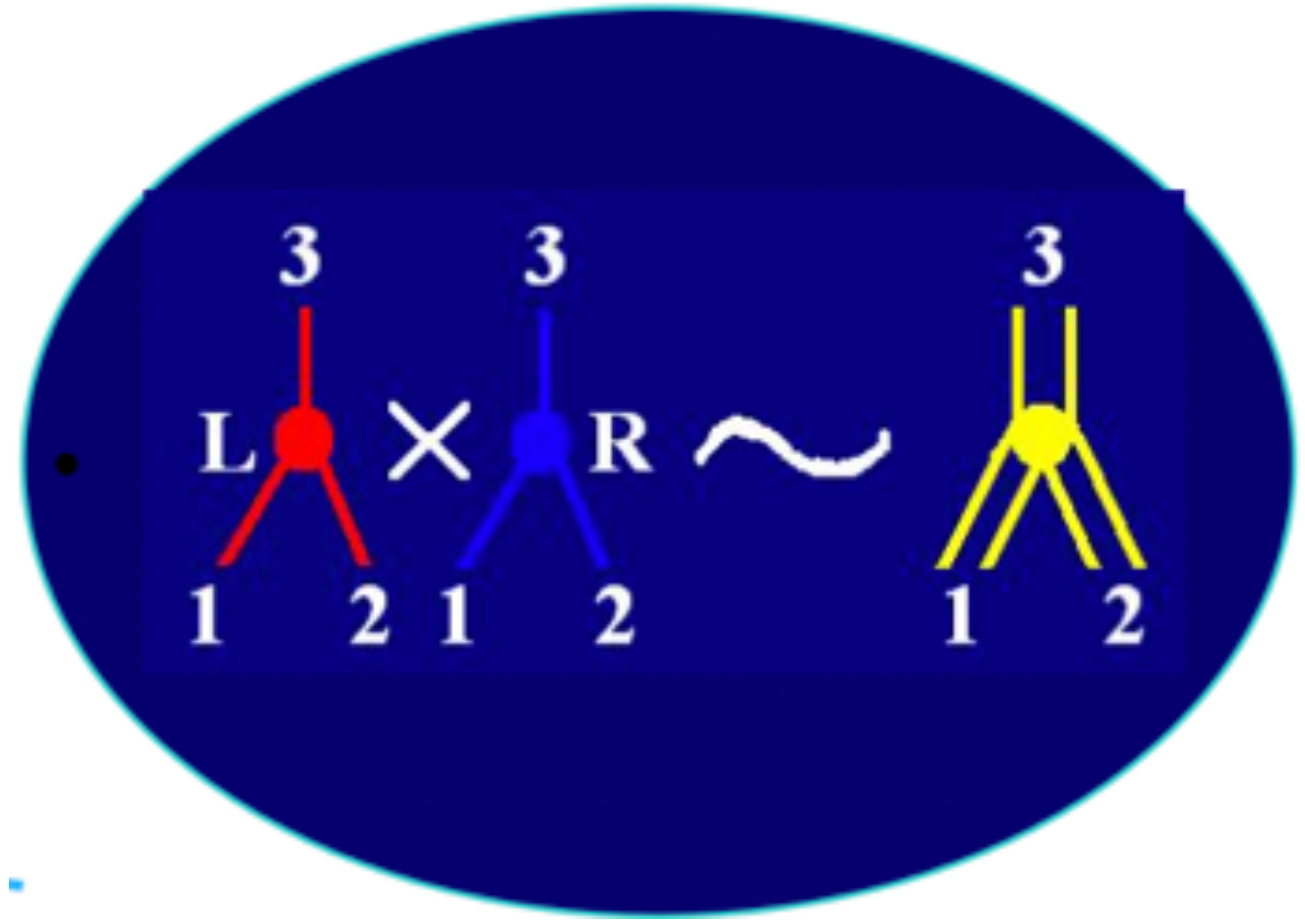
$$\mathcal{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1}|\underline{q}|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar|\underline{q}|^{\frac{L(4-D)}{2}+2-L}} + O(\hbar^0) \qquad 1/\hbar$$

Computations: Loop level

Long range behaviour can be efficiently be captured from the evaluation of unitarity cuts for using on-shell tree amplitudes

$$C_{i,\ldots,j} = \mathrm{Im}_{K_{i,\ldots,j}>0} M^{\text{1-loop}}$$





KLT+on-shell input trees (e.g. Badger et al., Forde, Kosower) recycled from Yang-Mills -> gravity

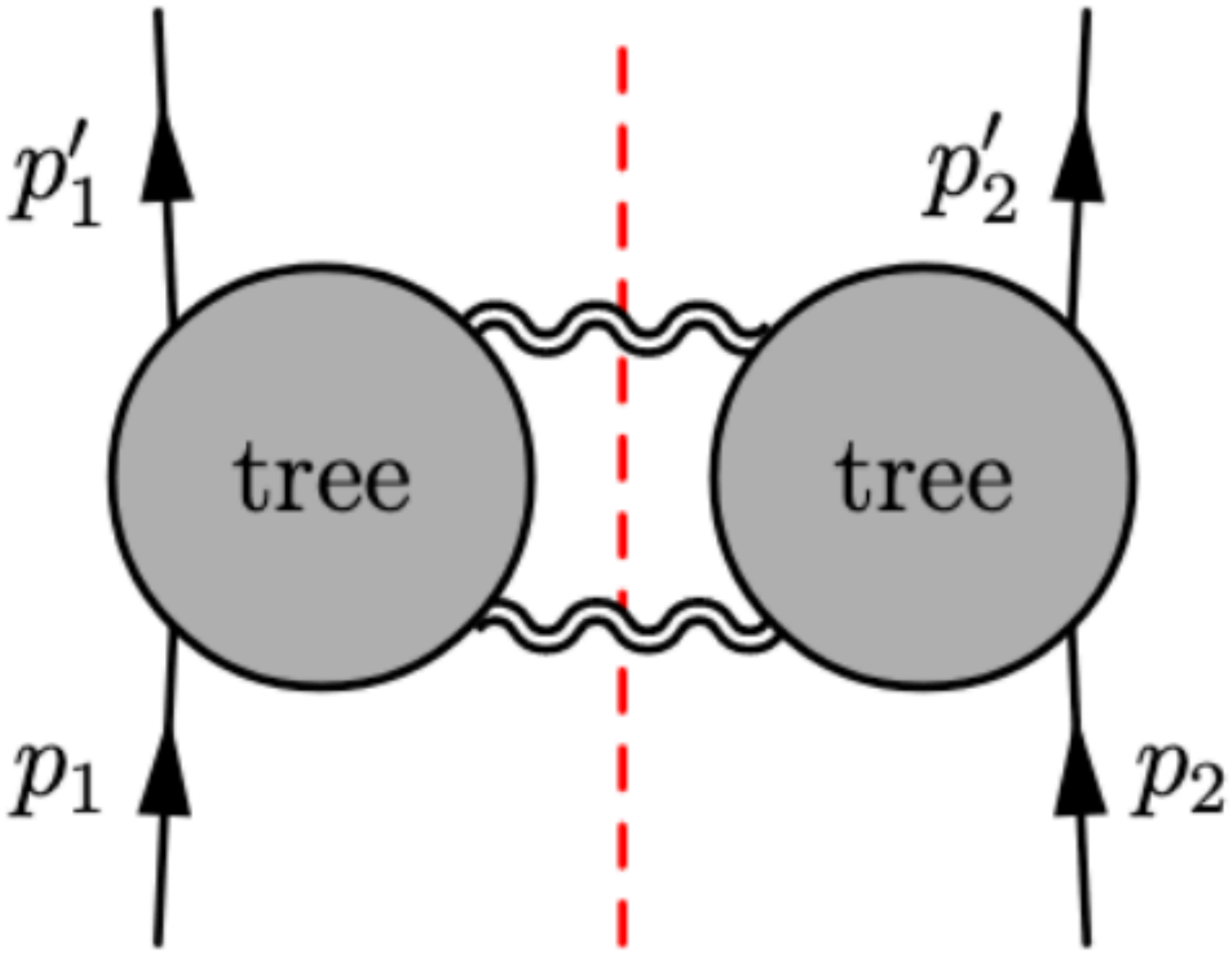
In D-dimensions from CHY (NEJBB, Cristofoli, Damgaard, Gomez; NEJBB, Plante, Vanhove)

Using on-shell amplitude techniques (Neill, Rothstein; NEJBB, Donoghue, Vanhove)

Post-Minkowskian framework and amplitudes

- Post-Minkowskian expansion of Einstein's general theory of relativity has received much recent attention in the amplitude community
- Idea: use scattering amplitude to provide a self-contained framework for deriving the two-body scattering valid in all regimes of energy and employ the computational power of modern amplitude calculations in an expansion in G (Damour)
  - On-shell Integrand construction
  - Multi-loop integration
  - IBP relation reduction (various programs)
    - Modern integration techniques ..
      - Integration regions .. PDE approaches

Example: One-loop amplitude potential



- Reduce to scalar integral basis
- Isolate coefficients

(NEJB, Donoghue, Vanhove)
(See also Cachazo and Guevara; Bern, Cheung Roiban, Shen, Solon, Zeng)

$$\mathcal{M}_1(\gamma, \underline{q}^2, \hbar) = \mathcal{M}_1^{\Box} + \mathcal{M}_1^{\triangleright} + \mathcal{M}_1^{\triangleleft} + \mathcal{M}_1^{\circ}$$

Example: One-loop amplitude potential

The amplitude has a
Laurent expansion

$$\mathcal{M}_1(\gamma, \underline{q}^2, \hbar) = \frac{1}{|\underline{q}|^{4-D}} \left(\frac{\mathcal{M}_1^{(-2)}(\gamma, \underline{q}^2)}{\hbar^2} + \frac{\mathcal{M}_1^{(-1)}(\gamma, \underline{q}^2)}{\hbar} + \mathcal{M}_1^{(0)}(\gamma, \underline{q}^2) + \mathcal{O}(\hbar) \right)$$

$$\mathcal{M}_1^{(-2)}(\gamma, \underline{q}^2) = \mathcal{M}_1^{\Box(-2)}(\gamma, \underline{q}^2),$$

$$\mathcal{M}_1^{(-1)}(\gamma, \underline{q}^2) = \mathcal{M}_1^{\Box(-1)}(\gamma, \underline{q}^2) + \mathcal{M}_1^{\triangleright(-1)}(\gamma, \underline{q}^2) + \mathcal{M}_1^{\triangleleft(-1)}(\gamma, \underline{q}^2),$$

$$\mathcal{M}_1^{(0)}(\gamma, \underline{q}^2) = \mathcal{M}_1^{\Box(0)}(\gamma, \underline{q}^2) + \mathcal{M}_1^{\triangleright(0)}(\gamma, \underline{q}^2) + \mathcal{M}_1^{\triangleleft(0)}(\gamma, \underline{q}^2) + \mathcal{M}_1^{\circ(0)}(\gamma, \underline{q}^2)$$

Order by order in Planck's constant

PM potential one-loop amplitude

$$\mathcal{M}^{1-\text{loop}} = \frac{i16\pi^2 G_N^2}{E_a E_b}\left(c_{\square}\mathcal{I}_{\square} + c_{\bowtie}\mathcal{I}_{\bowtie} + c_{\triangleright}\mathcal{I}_{\triangleright} + c_{\triangleleft}\mathcal{I}_{\triangleleft} + \cdots\right)$$

$$\mathcal{I}_{\square} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+p_1)^2 - m_a^2 + i\varepsilon)((\ell-p_3)^2 - m_b^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+q)^2 + i\varepsilon)}$$

$$\mathcal{I}_{\bowtie} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+p_1)^2 - m_a^2 + i\varepsilon)((\ell+p_4)^2 - m_b^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+q)^2 + i\varepsilon)}$$

$$\mathcal{I}_{\triangleright} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell+q)^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell+p_1)^2 - m_a^2 + i\varepsilon)}$$

$$\mathcal{I}_{\triangleleft} = \int \frac{d^{d+1}\ell}{(2\pi)^{d+1}} \frac{1}{((\ell-q)^2 + i\varepsilon)(\ell^2 + i\varepsilon)((\ell-p_3)^2 - m_b^2 + i\varepsilon)}$$

Putting it all together

Ignore quantum keep only classical pieces

phase

$$\mathcal{M}^{1-\text{loop}} = \frac{i16\pi^2 G_N^2}{E_a E_b}\left(c_{\Box}\mathcal{I}_{\Box} + c_{\bowtie}\mathcal{I}_{\bowtie} + c_{\triangleright}\mathcal{I}_{\triangleright} + c_{\triangleleft}\mathcal{I}_{\triangleleft} + \cdots\right)$$

$$\mathcal{I}_{\Box} = -\frac{i}{16\pi^2|\vec{q}|^2}\left(-\frac{1}{m_a m_b} + \frac{m_a(m_a - m_b)}{3m_a^2 m_b^2} + \frac{i\pi}{|p|E_p}\right)\left(\frac{2}{3-d} - \log|\vec{q}|^2\right) + \cdots$$

$$\mathcal{I}_{\bowtie} = -\frac{i}{16\pi^2|\vec{q}|^2}\left(\frac{1}{m_a m_b} - \frac{m_a(m_a - m_b)}{3m_a^2 m_b^2}\right)\left(\frac{2}{3-d} - \log|\vec{q}|^2\right) + \cdots$$

$$\mathcal{I}_{\triangleright} = -\frac{i}{32 m_a}\frac{1}{|\vec{q}|} + \cdots$$

$$\mathcal{I}_{\triangleleft} = -\frac{i}{32 m_b}\frac{1}{|\vec{q}|} + \cdots$$

$$c_{\Box} = c_{\bowtie} = 16 m_1^4 m_2^4 \frac{(1-(D-2)\sigma^2)^2}{(D-2)^2},$$

$$c_{\triangleright} = \frac{4m_1^4 m_2^2\,(D - 7 + (D(4D-17)+19)\sigma^2)}{(D-2)^2}$$

$$c_{\triangleleft} = \frac{4m_1^2 m_2^4\,(D - 7 + (D(4D-17)+19)\sigma^2)}{(D-2)^2}$$

One-loop

Subtraction important to make contact with classical physics potential

$$\mathcal{M}^{\text{Iterated}} = \frac{i\pi G_N^2}{E_p^3\xi}\frac{4c_1^2}{|\vec{p}|}\frac{(\log|\vec{q}|^2 - \frac{2}{3-d})}{|\vec{q}|^2} + \frac{2\pi^2 G_N^2}{E_p^3\xi^2|\vec{q}|}\left(\frac{c_1^2(\xi-1)}{2E_p^2\xi} - 4c_1 p_1\cdot p_3\right)$$

$$\mathcal{M}^{1-\text{loop}} = \frac{\pi^2 G_N^2}{E_p^2\xi}\left[\frac{1}{2|\vec{q}|}\left(\frac{c_\triangleright}{m_a} + \frac{c_\triangleleft}{m_b}\right) + \frac{i}{E_p}\frac{c_\Box}{|\vec{p}|}\frac{(\frac{2}{3-d} - \log|\vec{q}|^2)}{\pi|\vec{q}|^2}\right]$$

$$V_{2\text{PM}}(p,q) = \mathcal{M}^{1-\text{loop}} + \mathcal{M}^{\text{Iterated}} = \frac{\pi^2 G_N^2}{E_p^2\xi|\vec{q}|}\left[\frac{1}{2}\left(\frac{c_\triangleright}{m_a} + \frac{c_\triangleleft}{m_b}\right) + \frac{2}{E_p\xi}\left(\frac{c_1^2(\xi-1)}{2E_p^2\xi} - 4c_1 p_1\cdot p_3\right)\right]$$

Follows from the Lippmann-Schwinger subtraction. Again same result as from matching (Bern et al), the effect is that singular terms are gone!

Relation to a PM potential

One-loop amplitude after summing all contributions

$$\mathcal{M}^{1-\text{loop}} = \frac{\pi^2 G_N^2}{E_p^2 \xi}\left[\frac{1}{2|\vec{q}|}\left(\frac{c_{\triangleright}}{m_a} + \frac{c_{\triangleleft}}{m_b}\right) + \frac{i}{E_p}\frac{c_{\Box}}{|\vec{p}|}\frac{\left(\frac{2}{3-d} - \log|\vec{q}|^2\right)}{\pi|\vec{q}|^2}\right]$$

(NEJBB, Cristofoli, Damgaard, Vanhove)

Imaginary super-classical/singular ..

How to relate to a classical potential

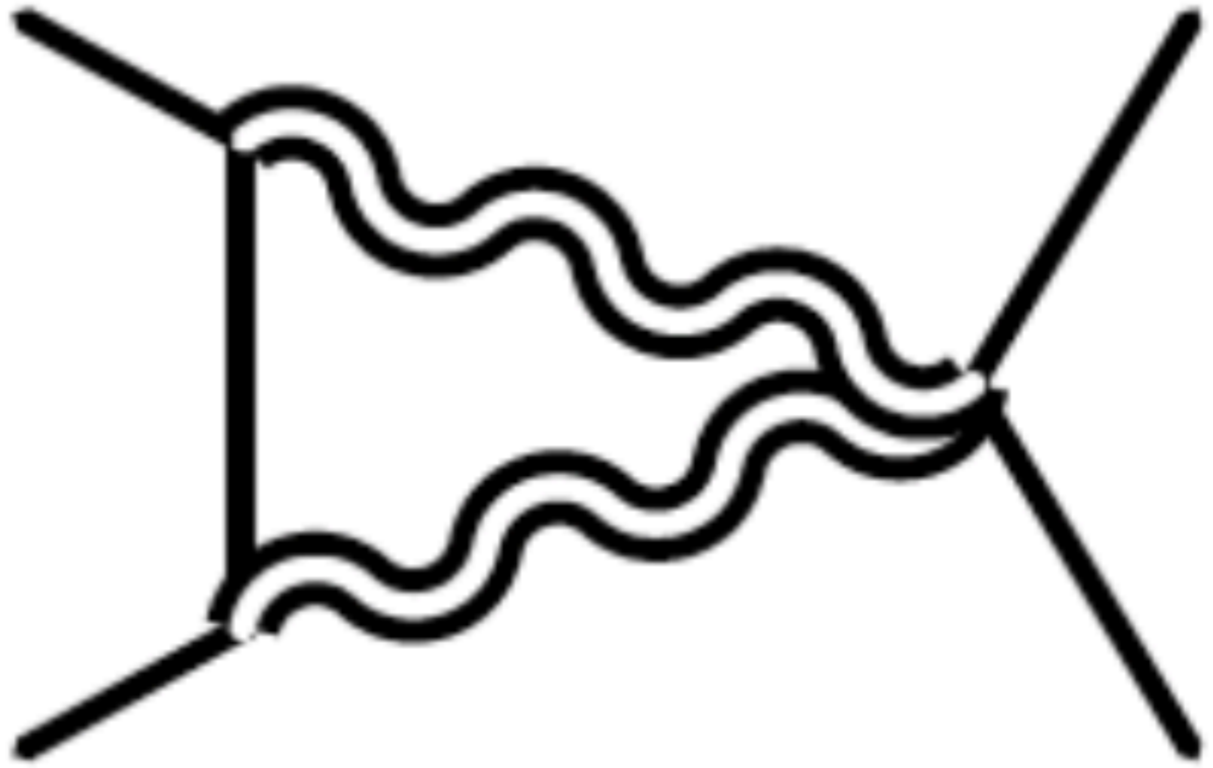
- Choice of coordinates
- Subtraction/Lippmann-Schwinger

Scalar interaction potentials (one-loop)

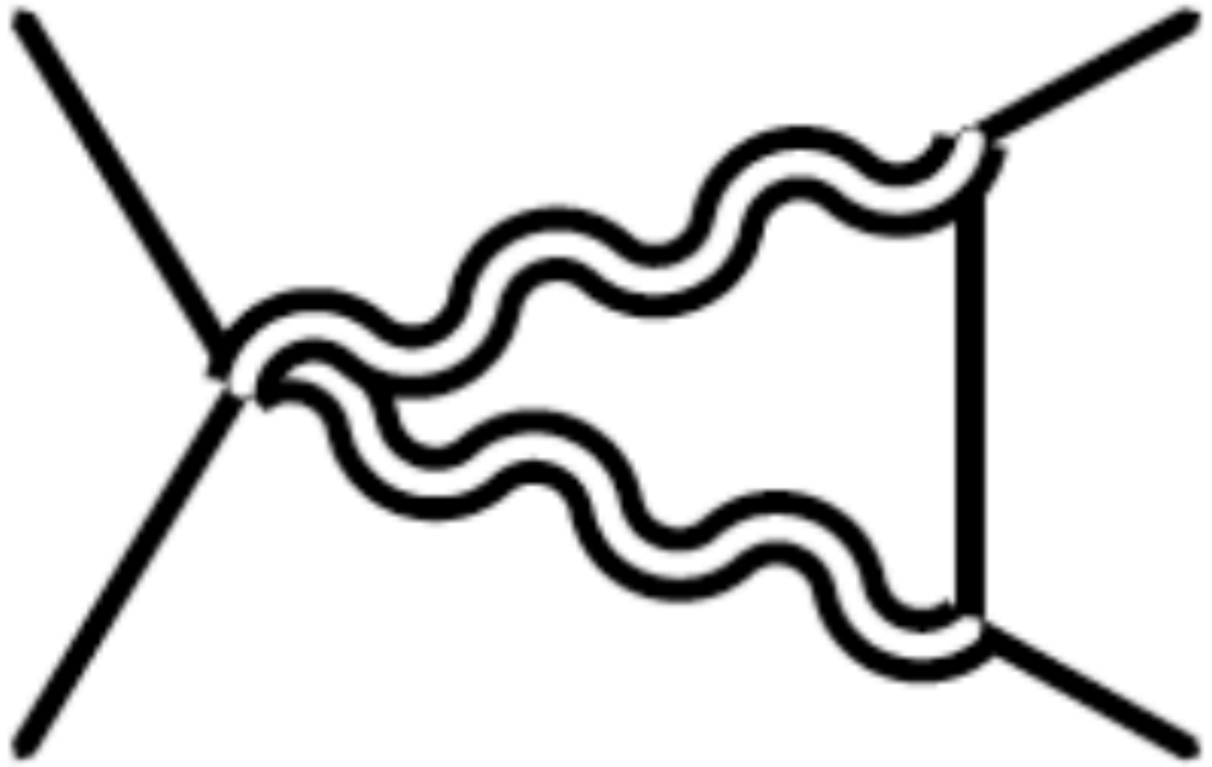
Important ‘empirical’ observation classical part of radial action that for the gravitational Hamiltonian is given by triangle diagrams only rest is cancelled in subtractions

One-loop level

$$\mathcal{M}_2 =$$



$$+$$



$$= -i(8\pi G)^2 \left(\frac{c(m_1, m_2) I_{\triangleright}(p_1, q)}{(q^2 - 4m_1^2)^2} + \frac{c(m_2, m_1) I_{\triangleright}(p_4, -q)}{(q^2 - 4m_2^2)^2} \right)$$

Example: One-loop amplitude potential

It follows that the classical part is

$$\widetilde{\mathcal{M}}_1^{\text{Cl.}}(\gamma, b, \hbar) = \frac{3\pi G_N^2 (m_1 + m_2) m_1 m_2 (5\gamma^2 - 1)}{4b\sqrt{\gamma^2 - 1}\hbar} (\pi b^2 e^{\gamma_E})^{4-D} + \mathcal{O}(4 - D)$$

With quantum correction (important in iterations)

$$\widetilde{\mathcal{M}}_1^{\text{Qt.}}(\gamma, b) = \frac{G_N^2 (\pi b^2 e^{\gamma_E})^{4-D}}{b^2} \Bigg(i \frac{4-D}{2} \frac{(2\gamma^2 - 1)^2 \mathcal{E}_{\text{C.M.}}^2}{(\gamma^2 - 1)^2}$$

$$- \frac{m_1 m_2}{\pi (\gamma^2 - 1)^{\frac{3}{2}}} \Big(\frac{1 - 49\gamma^2 + 18\gamma^4}{15} - \frac{2\gamma(2\gamma^2 - 1)(6\gamma^2 - 7)\,\text{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}} \Big) \Bigg) + \mathcal{O}((4 - D)^2)$$

Lessons from one-loop

- Only part of the amplitude is relevant for deriving observables in General Relativity

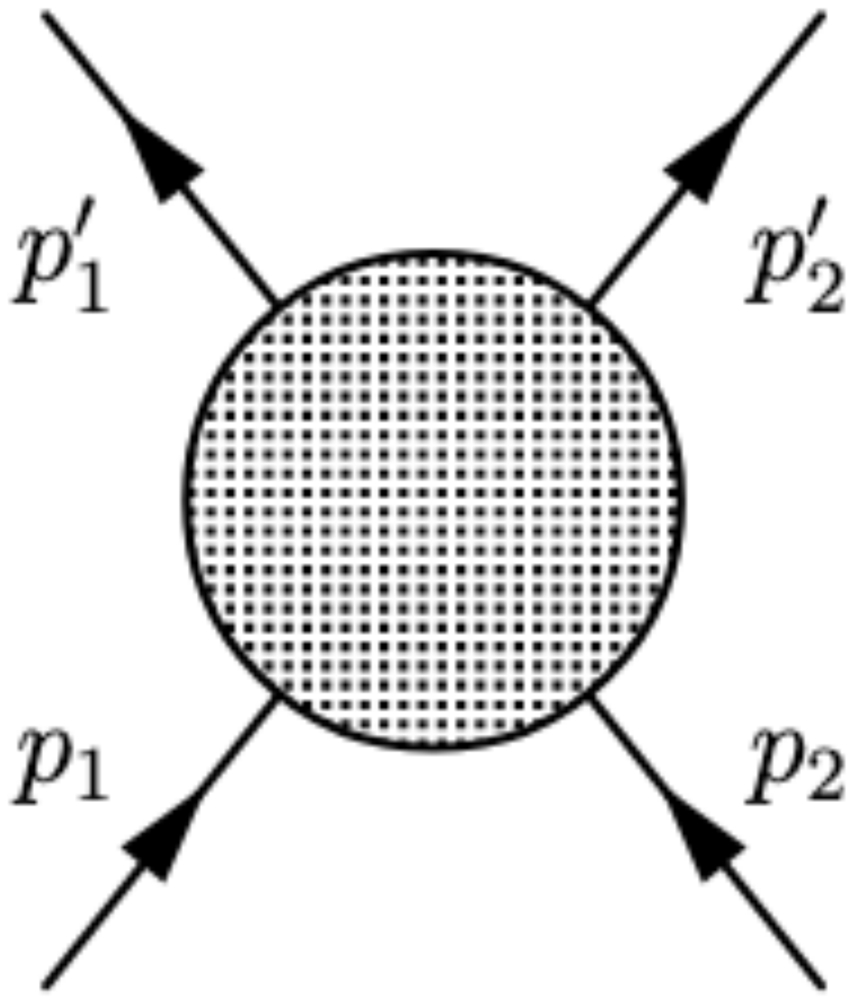
- Part of the amplitude is there to be subtracted for consistency with matching with a Quantum-Mechanical potential

We will now consider what happens at two-loops

Classical gravitational scattering: Generic loop level

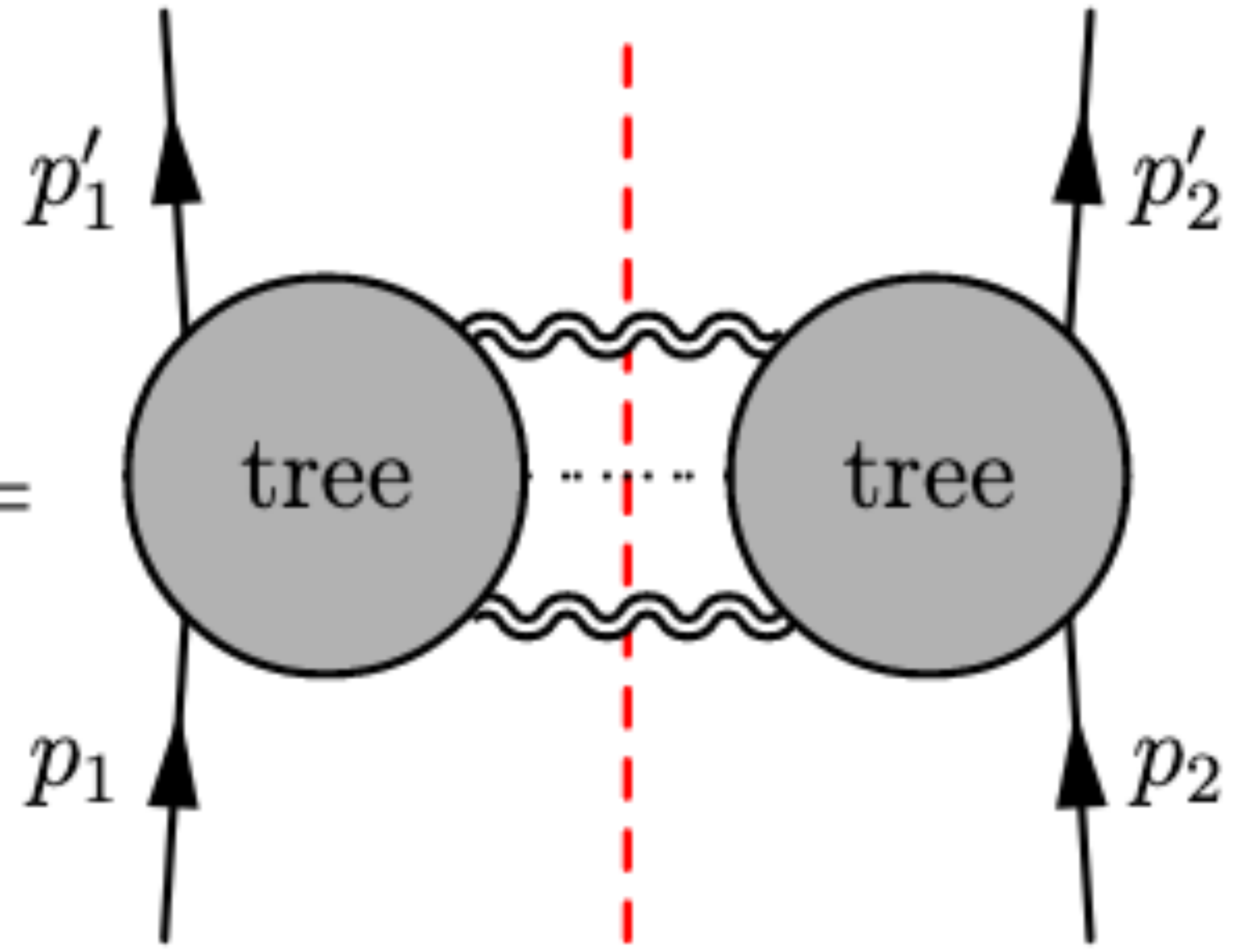
- 1) compute multi-loop cuts and 2) use consistency of the representation in master integrals to generate the full non-analytics pieces of the amplitude (classical and super-classical contributions)

Extraction of integrand similar to QCD
Spinor-helicity and D-dimension
covariant tree
amplitudes can be used in cuts



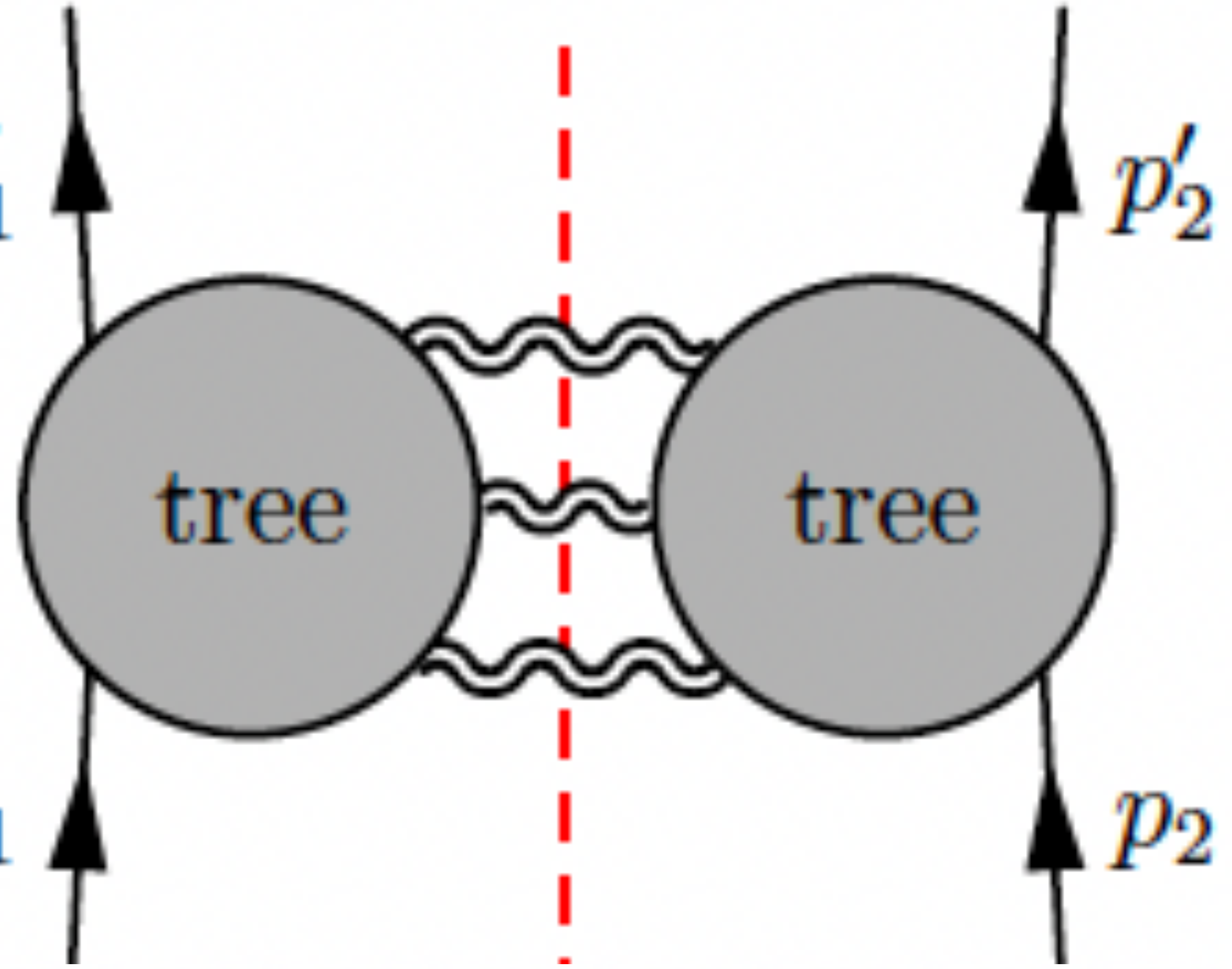
$$= \mathcal{M}(\gamma, q^2) = \sum_{L=0}^{\infty} \mathcal{M}_L(\gamma, q^2)\,.$$

$$\mathcal{M}^{\text{cut}}_{L+1}(\gamma, q^2) =$$



Example: Einstein gravity at two-loop order

$$\mathcal{M}^2_{3-\mathrm{cut}}(\sigma, q^2) =$$



$$\mathcal{M}_2^{3-\mathrm{cut}}(\sigma, q^2) = \int \frac{d^D l_1 d^D l_2 d^D l_3}{(2\pi)^{3D}} (2\pi)^D \delta^{(D)}(l_1 + l_2 + l_3 + q) \frac{i^3}{l_1^2 l_2^2 l_3^2}$$

$$\times \frac{1}{3!} \sum_{\substack{\mathrm{Perm}(l_1,l_2,l_3) \\ \lambda_1=\pm,\lambda_2=\pm,\lambda_3=\pm}} \mathcal{M}_0(p_1, p_1', l_1^{\lambda_1}, l_2^{\lambda_2}, l_3^{\lambda_3})(\mathcal{M}_0(p_2, p_2', -l_1^{\lambda_1}, -l_2^{\lambda_2}, -l_3^{\lambda_3}))^*$$

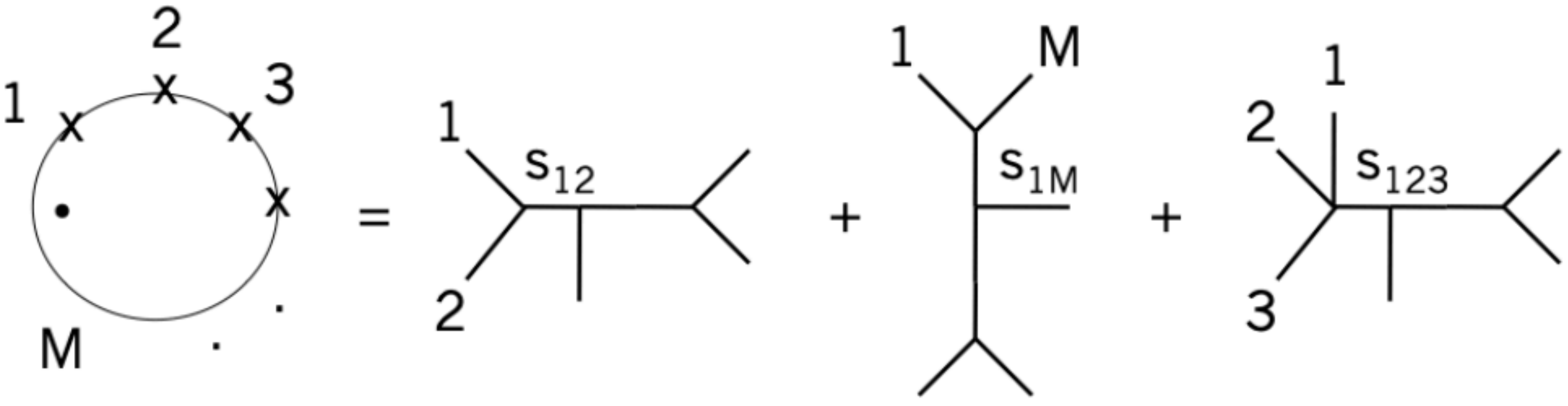
Little detour: 'Stringy' inspiration for efficient trees

Different form for amplitude

String theory
add channels up..

<->

Feynman diagrams
sums separate
kinematic poles



Example : Compact massive trees

Find 'stringy' structure in the scattering equation prescription (CHY)

(NEJB, Damgaard, Tourkine, Vanhove)

$$A_{n-2}(1,\{2,\ldots,n-1\},n) = \int \frac{\prod_{i=1}^n dz_i}{\text{vol}(\text{SL}(2,\mathbb{C}))} \prod_{i=1}^{n} \delta'\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{k_i\cdot k_j}{z_{ij}}\right) \frac{1}{z_{12}\cdots z_{n-1\,n}}$$
$$\times \sum_{\beta\in\mathfrak{S}_{n-2}} \frac{N_{n-2}(1,\beta(2,\ldots,n-1),n)}{z_{1\beta(2)}z_{\beta(2)\beta(3)}\cdots z_{\beta(n-1)\,n}},$$

We can generate gravity amplitudes in the following way

$$M_{n-2}^{\text{tree}}(1,2,\ldots,n) = i \sum_{\beta\in\mathfrak{S}_{n-2}} N_{n-2}(1,\beta(2,\cdots,n-1),n) A_{n-2}(1,\beta(2,\ldots,n-1),n)$$

Advantage that all poles are simple — no spurious poles!

Compact massive tree amplitudes

CHY formalism leads to the following very compact amplitudes

$$M_1^{\rm tree}(p,\ell_2,-p') = i\,N_1(p,\ell_2,-p')A_1(p,\ell_2,-p') = i\,N_1(p,\ell_2,-p')^2,$$

$$M_2^{\rm tree}(p,\ell_2,\ell_3,-p') = i\,N_2(p,2,3,-p')A_2(p,2,3,-p') + \text{perm.}\{2,3\}$$
$$= \frac{iN_2(p,2,3,-p')^2}{(\ell_2+p)^2-m^2+i\varepsilon} + \frac{iN_2(p,3,2,-p')^2}{(\ell_3+p)^2-m^2+i\varepsilon} + \frac{i(N_2{}^{[2,3]})^2}{(\ell_2+\ell_3)^2+i\varepsilon}$$

$$N_1(p,\ell_2,-p') = i\sqrt{2}\,\zeta_2\cdot p, \qquad A_1(p,\ell_2,-p') = N_1(p,\ell_2,-p').$$

$$N_2(p,\ell_2,\ell_3,-p') = \frac{i}{2}\Big(s_{2p}(\zeta_2\cdot\zeta_3) - 4(\zeta_2\cdot p)\zeta_3\cdot(p+\ell_2)\Big)$$

Straightforward to compute any tree order needed with manifest color-kinematic numerators

- no double poles (from KLT)
- Spin-0, spin-1/2 .. easy to derive

(NEJB, Brown, Gomez)

Back: Next integral basis

New integrals

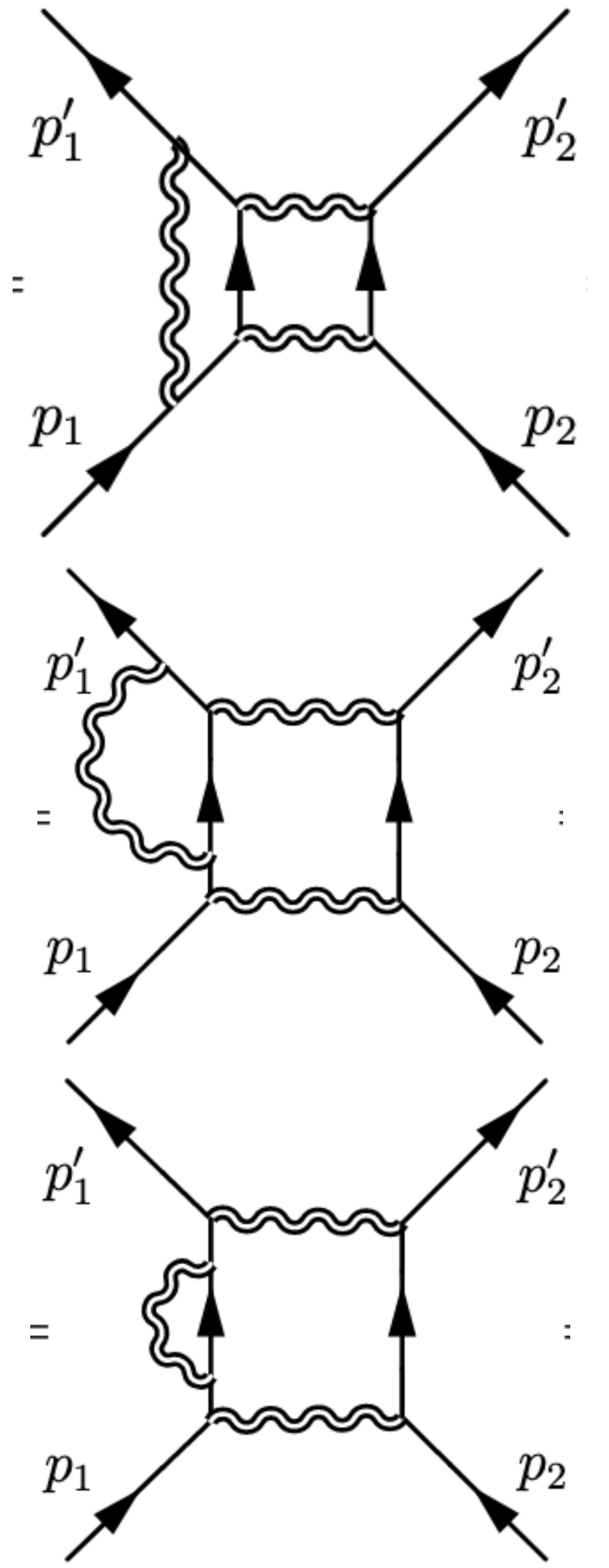
$$\mathcal{M}_2^{3-\mathrm{cut}}(\sigma, q^2) = \mathcal{M}_2^{\square\square} + \mathcal{M}_2^{\triangleleft\square} + \mathcal{M}_2^{\square\triangleright} + \mathcal{M}_2^{\triangleleft\triangleleft} + \mathcal{M}_2^{\triangleright\triangleright} + \mathcal{M}_2^{H} + \mathcal{M}_2^{\square\circ}$$

We use unitarity cut to fix coefficients in front of master-integrals. The full result can be written

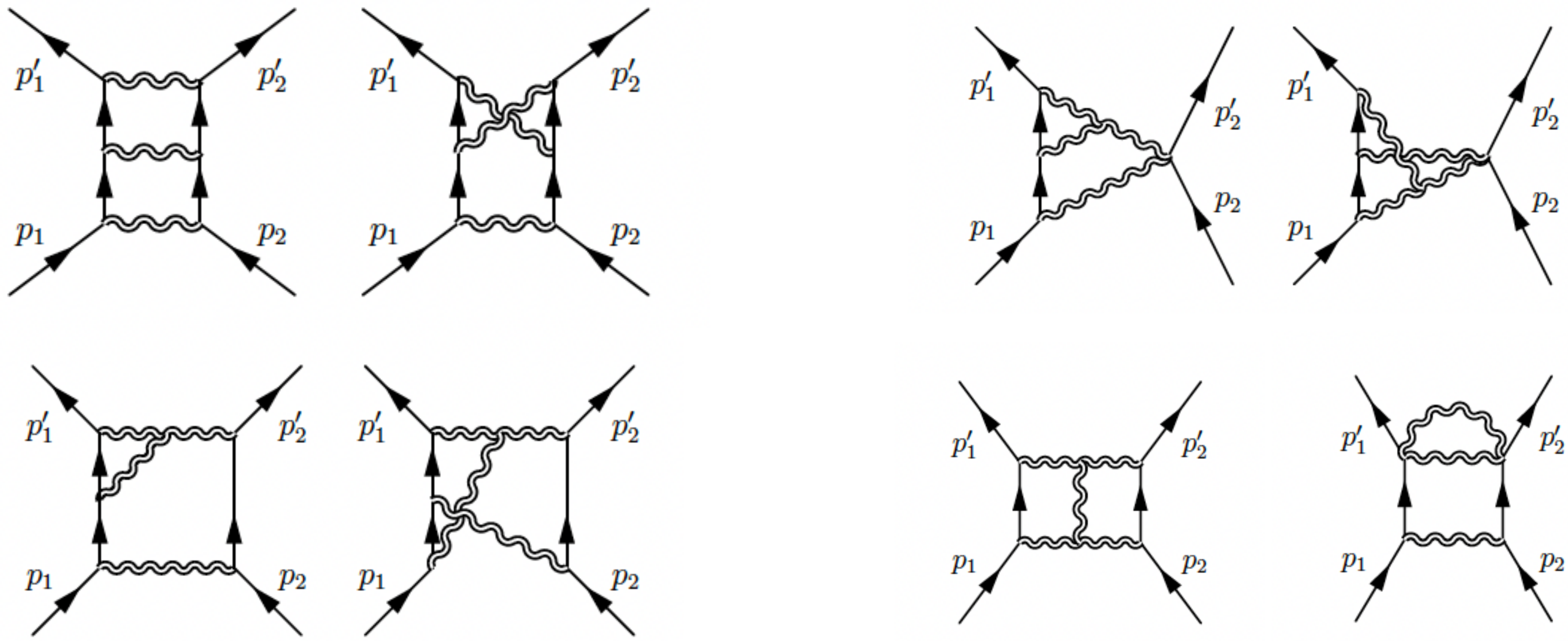
$$\mathcal{M}_2(\gamma, q^2) = \mathcal{M}_2^{3-\mathrm{cut}}(\gamma, q^2) + \mathcal{M}_2^{\mathrm{SE}}(\gamma, q^2)$$

Where the SE contribution is

$$\mathcal{M}_2^{\mathrm{self-energy}}(\gamma, \underline{q}^2) = -4(16\pi G_N)^3 \sum_{i=I}^{IV} (J_{SE}^{i,s} + J_{SE}^{i,u}) + (m_1 \leftrightarrow m_2)$$



Einstein gravity at two-loop order



Needed master integrals at two-loops for the conservative part of the amplitude - determined by LiteRed/FIRE6/KIRA etc.

Some examples of numerators

$$\mathcal{N}_{\square\square}^{(s)} = 512\pi^3 G_N^3 (m_1^4 + m_2^4 - 2(m_1^2 + m_2^2)s + s^2)^3 = 2^{12}\pi^3 G_N^3 m_1^6 m_2^6 (2\sigma^2 - 1)^3$$

$$\mathcal{N}_{\square\square}^{(cross,s)} = 2^{13}\pi^3 G_N^3 \left(96 m_1^6 m_2^6 (2\sigma^2 - 1)^3 + 8 m_1^5 m_2^5 \sigma (2\sigma^2 - 1)^2 (\hbar \underline{\vec{q}})^2 (l_2 \cdot l_3) + \mathcal{O}((\hbar \underline{\vec{q}})^4)\right)$$

$$\begin{aligned} \mathcal{N}_{\square\square}^{(u)} &= 512\pi^3 G_N^3 (m_1^4 + m_2^4 - 2(m_1^2 + m_2^2)u + u^2)^3 \\ &= 2^{12}\pi^3 G_N^3 \left(96 m_1^6 m_2^6 (2\sigma^2 - 1)^3 - 6 m_1^5 m_2^5 \sigma (2\sigma^2 - 1)^2 (\hbar \underline{\vec{q}})^2 + \mathcal{O}((\hbar \underline{\vec{q}})^4)\right) \end{aligned}$$

$$\begin{aligned} \mathcal{N}_H = \frac{128\pi^3 G_N^3}{3} \Big(&- 48(-4 m_1^2 m_2^4 ((l_2 + l_3)^2 - (l_1 + l_3)^2 + 4\sigma^2)(\bar{p}_1 \cdot l_2)^2 \\ &- 8 m_2^4 (\bar{p}_1 \cdot l_2)^4 + 16 m_1^3 m_2^3 \sigma (\bar{p}_1 \cdot l_2)(\bar{p}_2 \cdot l_1) \\ &+ m_1^4 \Big(m_2^4 (-1 - 2(l_2 + l_3)^2 (1 + (l_2 + l_3)^2) - 2(l_1 + l_3)^2 (1 + (l_1 + l_3)^2) \\ &+ 4\sigma^2 + 4((l_2 + l_3)^2 + (l_2 + l_3)^4 + (l_1 + l_3)^2 - 2(l_2 + l_3)^2 (l_1 + l_3)^2 + (l_1 + l_3)^4 \Big) \sigma^2 \\ &- 4\sigma^4) + 4 m_2^2 ((l_2 + l_3)^2 - (l_1 + l_3)^2 - 4\sigma^2)(\bar{p}_2 \cdot l_1)^2 - 8(\bar{p}_2 \cdot l_1)^4))(\hbar \underline{\vec{q}})^4 + \mathcal{O}((\hbar \underline{\vec{q}})^5) \Big). \end{aligned}$$

Einstein gravity at two-loop order

$$\mathcal{M}_2^{3-\mathrm{cut}(-1)}(\sigma, q^2) = \frac{2(4\pi e^{-\gamma_E})^{2\epsilon}\pi G_N^3 m_1^2 m_2^2}{3\epsilon|\underline{q}|^{4\epsilon}\hbar}\Bigg(\frac{3s(2\sigma^2-1)^3}{(\sigma^2-1)^2}$$
$$+\frac{im_1m_2(2\sigma^2-1)}{\pi\epsilon(\sigma^2-1)^{\frac{3}{2}}}\left(\frac{1-49\sigma^2+18\sigma^4}{5}-\frac{6\sigma(2\sigma^2-1)(6\sigma^2-7)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)$$
$$-\frac{9(2\sigma^2-1)(1-5\sigma^2)s}{2(\sigma^2-1)}+\frac{3}{2}(m_1^2+m_2^2)(-1+18\sigma^2)-m_1m_2\sigma(103+2\sigma^2)$$
$$+\frac{12m_1m_2(3+12\sigma^2-4\sigma^4)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$
$$-\frac{6im_1m_2(2\sigma^2-1)^2}{\pi\epsilon\sqrt{\sigma^2-1}}\left(\frac{-1}{4(\sigma^2-1)}\right)^{\epsilon}\frac{d}{d\sigma}\left(\frac{(2\sigma^2-1)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\Bigg).$$

Imaginary

Gravity amplitude in powers of hbar

$$\mathcal{M}_2(\sigma,|\underline{q}|) = \frac{1}{|\underline{q}|^{4\epsilon}}\left(\mathcal{M}_2^{(-3)}(\sigma,|\underline{q}|) + \mathcal{M}_2^{(-2)}(\sigma,|\underline{q}|) + \mathcal{M}_2^{(-1)}(\sigma,|\underline{q}|) + \mathcal{O}(\hbar^0)\right)$$

$$\mathcal{M}_2^{(-3)}(\sigma,|\underline{q}|) = -\frac{8\pi G_N^3 m_1^4 m_2^4 (2\sigma^2-1)^3\Gamma(-\epsilon)^3\Gamma(1+2\epsilon)}{3\hbar^3|\underline{q}|^2(\sigma^2-1)(4\pi)^{-2\epsilon}\Gamma(-3\epsilon)}$$

(Bern et al, Parra-Martinez et al)

$$\mathcal{M}_2^{(-2)}(\sigma,|\underline{q}|) = \frac{6i\pi^2 G_N^3(m_1+m_2)m_1^3m_2^3(2\sigma^2-1)(1-5\sigma^2)(4\pi e^{-\gamma_E})^{2\epsilon}}{\epsilon\sqrt{\sigma^2-1}\hbar^2|\underline{q}|} + \mathcal{O}(\epsilon^0)$$

$$\begin{aligned}\mathcal{M}_2^{(-1)}(\sigma,|\underline{q}|) = \frac{2\pi G_N^3(4\pi e^{-\gamma_E})^{2\epsilon} m_1^2 m_2^2}{\hbar\epsilon}\Bigg(&\frac{s(2\sigma^2-1)^3}{(\sigma^2-1)^2}\\
&+\frac{im_1m_2(2\sigma^2-1)}{\pi\epsilon(\sigma^2-1)^{\frac{3}{2}}}\left(\frac{1-49\sigma^2+18\sigma^4}{15}-\frac{2\sigma(7-20\sigma^2+12\sigma^4)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\\
&-\frac{3(2\sigma^2-1)(1-5\sigma^2)s}{2(\sigma^2-1)}+\frac{1}{2}(m_1^2+m_2^2)(18\sigma^2-1)-\frac{1}{3}m_1m_2\sigma(103+2\sigma^2)\\
&+\frac{4m_1m_2(3+12\sigma^2-4\sigma^4)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\\
&-\frac{2im_1m_2(2\sigma^2-1)^2}{\pi\epsilon\sqrt{\sigma^2-1}}\left(\frac{-1}{4(\sigma^2-1)}\right)^{\epsilon}\left(-\frac{11}{3}+\frac{d}{d\sigma}\left(\frac{(2\sigma^2-1)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\right)\Bigg).\end{aligned}$$

Laurant expansion in Planck's constant

- imaginary contribution cancelled by radiative contributions

(Di Vecchia, Heissenberg, Russo, Veneziano)

Gravity amplitude in b-space

$$\widetilde{\mathcal{M}}_2(\sigma, b) = \frac{1}{4E_{\text{c.m.}}P} \int_{\mathbb{R}^{D-2}} \frac{d^{D-2}\vec{\underline{q}}}{(2\pi)^{D-2}} \mathcal{M}_2(p_1, p_2, p_1', p_2') e^{i\vec{\underline{q}}\cdot\vec{b}}$$

$$\widetilde{\mathcal{M}}_2(\sigma, b) = -\frac{1}{6}\left(\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\right)^3 + i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\left(\widetilde{\mathcal{M}}_1^{\text{Cl.}}(\sigma, b) + \widetilde{\mathcal{M}}_1^{\text{Qt.}}(\sigma, b)\right) + \widetilde{\mathcal{M}}_2^{\text{Cl.}}(\sigma, b) + \mathcal{O}(\hbar^0).$$

$$\widetilde{\mathcal{M}}_2^{\square\square(-3)}(\sigma, b) = -\frac{1}{6}\left(\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\right)^3,$$

$$\widetilde{\mathcal{M}}_2^{\square\square(-2)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\widetilde{\mathcal{M}}_1^{\square(-1)}(\sigma, b),$$

$$\widetilde{\mathcal{M}}_2^{\triangleleft\square(-2)}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\square\triangleright(-2)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\left(\widetilde{\mathcal{M}}_1^{\triangleleft(-1)}(\sigma, b) + \widetilde{\mathcal{M}}_1^{\triangleright(-1)}(\sigma, b)\right)$$

$$\widetilde{\mathcal{M}}_2^{\square\square(-1)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\widetilde{\mathcal{M}}_1^{\square(0)}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\square\square\ \text{Cl.}}(\sigma, b),$$

$$\widetilde{\mathcal{M}}_2^{\triangleleft\square(-1)}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\square\triangleright(-1)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\left(\widetilde{\mathcal{M}}_1^{\triangleleft(0)}(\sigma, b) + \widetilde{\mathcal{M}}_1^{\triangleright(0)}(\sigma, b)\right) + \widetilde{\mathcal{M}}_2^{\triangleleft\square\ \text{Cl.}}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\square\triangleright\ \text{Cl.}}(\sigma, b),$$

$$\widetilde{\mathcal{M}}_2^{\square\circ(-1)}(\sigma, b) = i\widetilde{\mathcal{M}}_0^{(-1)}(\sigma, b)\widetilde{\mathcal{M}}_1^{\circ(0)}(\sigma, b) + \widetilde{\mathcal{M}}_2^{\square\circ\ \text{Cl.}}(\sigma, b),$$

Again iterative structure like one-loop, part of a bigger scheme..Seen after Fourier transform to b space

Scattering angle from amplitudes

$$1 + i \sum_{L \geq 0} \widetilde{\mathcal{M}}_L(\sigma, b) = (1 + 2i\Delta(\sigma, b)) \exp\left(\frac{2i}{\hbar} \sum_{L \geq 0} \delta_L(\sigma, b) \right)$$

Gravity eikonal

$$\delta_0(\sigma, b) = -\frac{G_N m_1 m_2 (2\sigma^2 - 1)}{2\epsilon\sqrt{\sigma^2 - 1}} (\pi b^2 e^{\gamma_E})^\epsilon + \mathcal{O}(\epsilon),$$

$$\delta_1(\sigma, b) = \frac{3\pi G_N^2 (m_1 + m_2) m_1 m_2 (5\sigma^2 - 1)}{8b\sqrt{\sigma^2 - 1}} (\pi b^2 e^{\gamma_E})^{2\epsilon}$$

Scattering angle from amplitudes

$$1 + i\sum_{L\geq 0}\widetilde{\mathcal{M}}_L(\sigma,b) = (1+2i\Delta(\sigma,b))\exp\left(\frac{2i}{\hbar}\sum_{L\geq 0}\delta_L(\sigma,b)\right)$$

Gravity eikonal

$$2\Delta_1 = \widetilde{\mathcal{M}}_1^{\text{Qt.}}(\sigma,b)$$

$$\delta_2(\sigma,b) = \frac{G_N^3 m_1 m_2 (\pi b^2 e^{\gamma_E})^{3\epsilon}}{2b^2\sqrt{\sigma^2-1}}\Bigg(\frac{2s(12\sigma^4-10\sigma^2+1)}{\sigma^2-1} - \frac{4m_1m_2\sigma}{3}(25+14\sigma^2) + \frac{4m_1m_2(3+12\sigma^2-4\sigma^4)\,\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + \frac{2m_1m_2(2\sigma^2-1)^2}{\sqrt{\sigma^2-1}}\frac{1}{(4(\sigma^2-1))^\epsilon}\left(-\frac{11}{3}+\frac{d}{d\sigma}\left(\frac{(2\sigma^2-1)\,\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\right)\Bigg).$$

Scattering angle from amplitudes

$$\sin\left(\frac{\chi}{2}\right)\Big|_{3PM} = -\frac{\sqrt{s}}{m_1 m_2\sqrt{\sigma^2-1}}\frac{\partial\delta_2(\sigma,b)}{\partial b}$$

$$J = \frac{m_1 m_2\sqrt{\sigma^2-1}}{\sqrt{s}} b\cos\left(\frac{\chi}{2}\right)$$

$$\chi_{1PM} = \frac{2G_N m_1 m_2(2\sigma^2-1)}{J\sqrt{\sigma^2-1}},$$

$$\chi_{2PM} = \frac{3\pi G_N^2 m_1^2 m_2^2(m_1+m_2)(5\sigma^2-1)}{4J^2\sqrt{s}};$$

Scattering angle from amplitudes

$$\widehat{\chi}_{3PM} = \frac{2G_N^3 m_1^3 m_2^3 \left(64\sigma^6 - 120\sigma^4 + 60\sigma^2 - 5\right)}{3J^3\left(\sigma^2-1\right)^{\frac{3}{2}}}$$
$$+ \frac{8G_N^3 m_1^4 m_2^4 \sqrt{\sigma^2-1}}{3J^3 s}\left(\sigma(-25-14\sigma^2) + \frac{3(3+12\sigma^2-4\sigma^4)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)$$

$$\chi_{3PM}^{\mathrm{Rad.}} = \frac{4G_N^3 m_1^4 m_2^4 (2\sigma^2-1)^2}{J^3 s}\frac{1}{(4(\sigma^2-1))^{\epsilon}}\left(-\frac{11}{3} + \frac{d}{d\sigma}\left(\frac{(2\sigma^2-1)\,\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\right)$$

Match with expectations

(Bern at al, Damour; Di Vecchia et al; Hermann et al)

(NEJB, Damgaard, Plante, Vanhove)

What is nice to see is the fact that everything matches up!

- the cancellation of terms that is demonstrated explicitly gives important consistency of computations.

Even simpler organisation of results — velocity cuts, exponentiation and soft expansion

An example of this is the 'velocity cuts' is a clever to organise the integrand for simpler computations. The basic observation is that the combination of linear propagators

$$\left(\frac{1}{(p_A\cdot\ell_A+i\varepsilon)(p_A\cdot\ell_B-i\varepsilon)}-\frac{1}{(p_A\cdot\ell_B+i\varepsilon)(p_A\cdot\ell_A-i\varepsilon)}\right)\times$$
$$\left(\frac{1}{(p_B\cdot\ell_A-i\varepsilon)(p_B\cdot\ell_C+i\varepsilon)}-\frac{1}{(p_B\cdot\ell_C-i\varepsilon)(p_B\cdot\ell_A+i\varepsilon)}\right)$$

can be expressed as

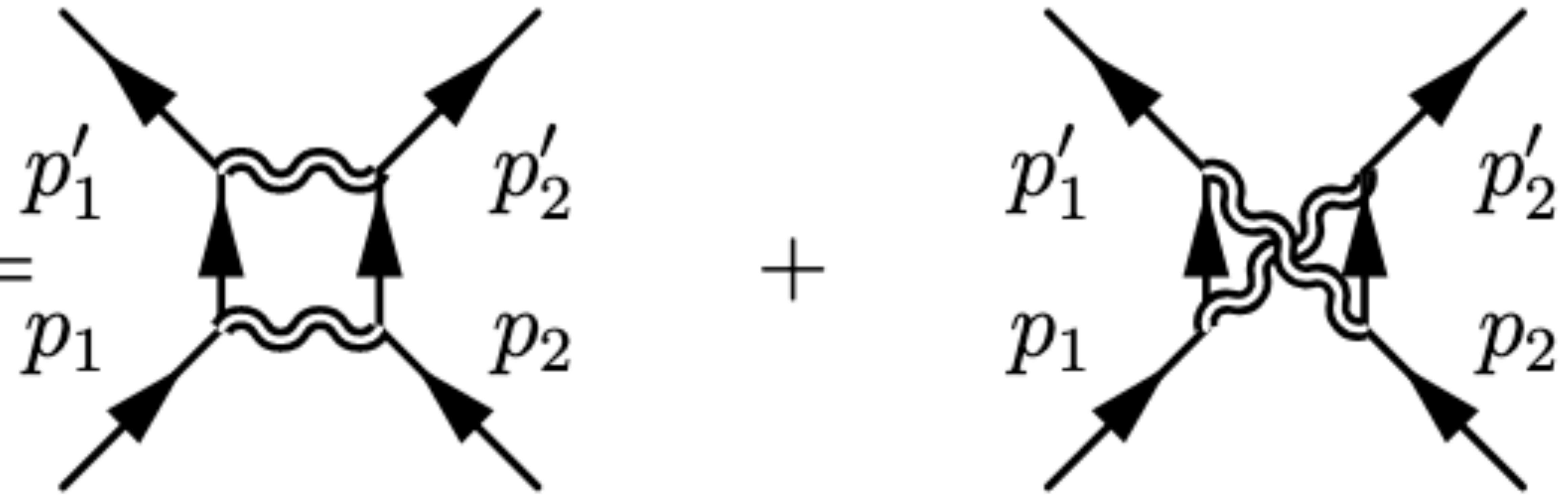
$$\left(\frac{\delta(p_A\cdot\ell_A)}{p_A\cdot\ell_B+i\varepsilon}-\frac{\delta(p_A\cdot\ell_B)}{p_B\cdot\ell_A+i\varepsilon}\right)\times\left(\frac{\delta(p_B\cdot\ell_C)}{p_B\cdot\ell_A+i\varepsilon}-\frac{\delta(p_B\cdot\ell_A)}{p_B\cdot\ell_C+i\varepsilon}\right)$$

using

$$\frac{1}{x+i\varepsilon}-\frac{1}{x-i\varepsilon}=-2i\pi\delta(x)$$

We can see this in the organisation of the one-loop

$I_\square =$



$$= \int \frac{d^D\ell}{(2\pi\hbar)^D} \frac{1}{\ell^2(\ell+q)^2} \left(\frac{1}{(-p_1+\ell)^2 - m_1^2 + i\varepsilon} + \frac{1}{(p_1'+\ell)^2 - m_1^2 + i\varepsilon} \right)$$

$$\times \left(\frac{1}{(-p_2+\ell)^2 - m_2^2 + i\varepsilon} + \frac{1}{(p_2'+\ell)^2 - m_2^2 + i\varepsilon} \right) .$$

We can see this in the organisation of the one-loop

$$I_\square = -\frac{|\vec{q}|^{D-6}}{8\hbar^2}\int \frac{d^D k}{(2\pi)^D}\frac{1}{k^2(k+u_q)^2}$$

$p_1 = \bar{p}_1 + \frac{\hbar}{2}\underline{q},\ p_1' = \bar{p}_1' - \frac{\hbar}{2}\underline{q},\ p_2 = \bar{p}_2 - \frac{\hbar}{2}\underline{q},\ p_2' = \bar{p}_2' + \frac{\hbar}{2}\underline{q}$

$\ell = \hbar|\underline{q}|l \qquad \underline{q} = |\underline{q}|u_q$

$$\times\left(\frac{1}{\bar{p}_1\cdot k + \frac{\hbar|\vec{q}|u_q\cdot k}{2} + i\varepsilon} - \frac{1}{\bar{p}_1\cdot k - \frac{\hbar|\vec{q}|u_q\cdot k}{2} - i\varepsilon}\right)$$

$$\times\left(\frac{1}{\bar{p}_2\cdot k - \frac{\hbar|\vec{q}|u_q\cdot k}{2} - i\varepsilon} - \frac{1}{\bar{p}_2\cdot k + \frac{\hbar|\vec{q}|u_q\cdot k}{2} + i\varepsilon}\right)$$

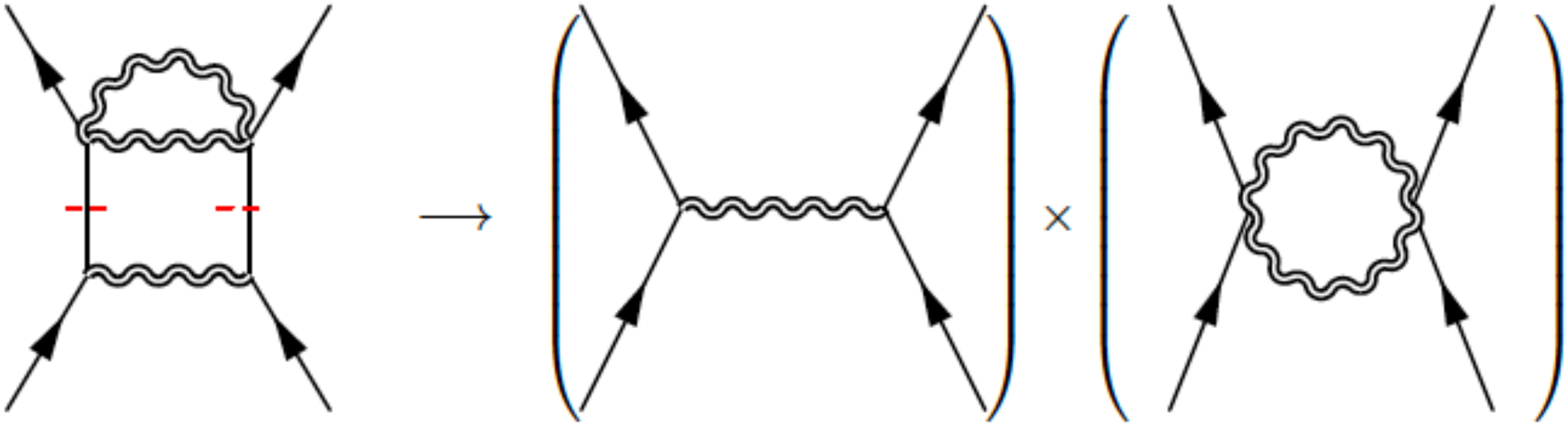
Can be seen to be cancelled in subtractions

$$I_\square = I_\square^{1-\mathrm{cut}} + \frac{|\vec{q}|^{D-5}}{16\hbar}\int \frac{d^D l}{(2\pi)^{D-1}}\frac{1}{\ell^2(\ell+u_q)^2}\left(\frac{\delta(\bar{p}_2\cdot l)}{(\bar{p}_1\cdot\ell)^2} + \frac{\delta(\bar{p}_1\cdot l)}{(\bar{p}_2\cdot\ell)^2}\right) + \mathcal{O}(|\underline{q}|^{D-4})$$

$$I_\square^{1-\mathrm{cut}} = \frac{|\vec{q}|^{D-6}}{4\hbar^2}\left(1 + \frac{\hbar^2|\vec{q}|^2\mathcal{E}^2_{\mathrm{C.M.}}}{4m_1^2m_2^2(\gamma^2 - 1 - \frac{\hbar^2|\vec{q}|^2\mathcal{E}^2_{\mathrm{C.M.}}}{4m_1^2m_2^2})}\right)^{\frac{D-5}{2}}\int \frac{d^D k}{(2\pi)^{D-2}}\frac{\delta(\bar{p}_1\cdot k)\delta(\bar{p}_2\cdot k)}{k^2(k+u_q)^2}$$

Velocity cuts and relation to world lines

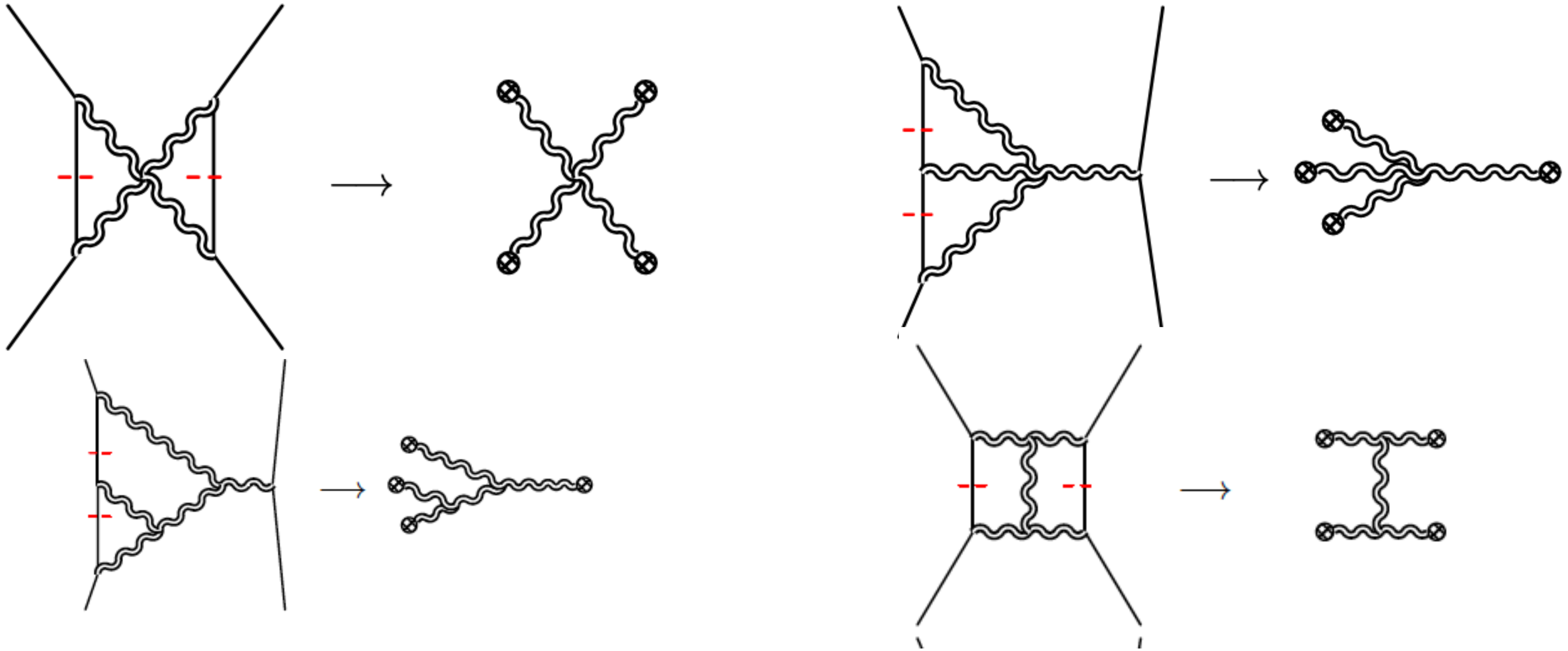
Can open up massive propagators - direct connection to world-line formulation - direct computation of probe amplitude to four-loop order, e.g. subtraction term



Terms like these that are disconnected using velocity cuts are only there to be subtracted

Velocity cuts and relation to world lines

Can open up massive propagators - direct connection to world-line formulation - direct computation of probe amplitude to four-loop order - classification of subtraction terms and classical contributions



Terms like these give non-cancelling contributions

(NEJBB, Damgaard, Plante, Vanhove; NEJBB, Plante, Vanhove)

(See Jan’s talk)

Lessons from exponentiation of the S-matrix

This can be further refined via the direct identification of the radial action. Considering the following representation of the exponentiated amplitude, one has

$$\widehat{S} = \mathbb{I} + \frac{i}{\hbar}\widehat{T} = \exp\left(\frac{i\widehat{N}}{\hbar}\right)$$

$$\hat{N}_0 = \hat{T}_0, \qquad \hat{N}_0^{\mathrm{rad}} = \hat{T}_0^{\mathrm{rad}},$$

$$\hat{N}_1 = \hat{T}_1 - \frac{i}{2\hbar}\hat{T}_0^2, \qquad \hat{N}_1^{\mathrm{rad}} = \hat{T}_1^{\mathrm{rad}} - \frac{i}{2\hbar}(\hat{T}_0\hat{T}_0^{\mathrm{rad}} + \hat{T}_0^{\mathrm{rad}}\hat{T}_0),$$

$$\hat{N}_2 = \hat{T}_2 - \frac{i}{2\hbar}(\hat{T}_0^{\mathrm{rad}})^2 - \frac{i}{2\hbar}(\hat{T}_0\hat{T}_1 + \hat{T}_1\hat{T}_0) - \frac{1}{3\hbar^2}\hat{T}_0^3\,,$$

Bern et al   Damgaard, Plante, Vanhove

- It is easy to see which terms needs to be computed and identify the classical contributions to the radial action
- new radiation terms allow ‘radiation reaction’ to be automatically correctly accounted for

Simplifications from the exponentiation of the S-matrix

Now it is clear how 'unitarity' removes certain terms when computing the radial action N

$$\mathcal{M}_1 \propto \frac{1}{\hbar}\langle p_1, p_2|\hat{T}_1|p_1', p_2'\rangle \propto \frac{1}{\hbar}\langle p_1, p_2|\hat{N}_1|p_1', p_2'\rangle + \frac{i}{2\hbar^2}\langle p_1, p_2|\hat{T}_0^2|p_1', p_2'\rangle$$

Cancelled in subtractions

$$\mathcal{M}_1(|\underline{\vec{q}}|, \gamma, \hbar) = \frac{i\hbar}{2}(16\pi G_N m_1^2 m_2^2(2\gamma^2-1))^2 I_{\Box}^{1-\text{cut}} + N_1(|\underline{\vec{q}}|, \gamma) + \mathcal{O}(\hbar)$$

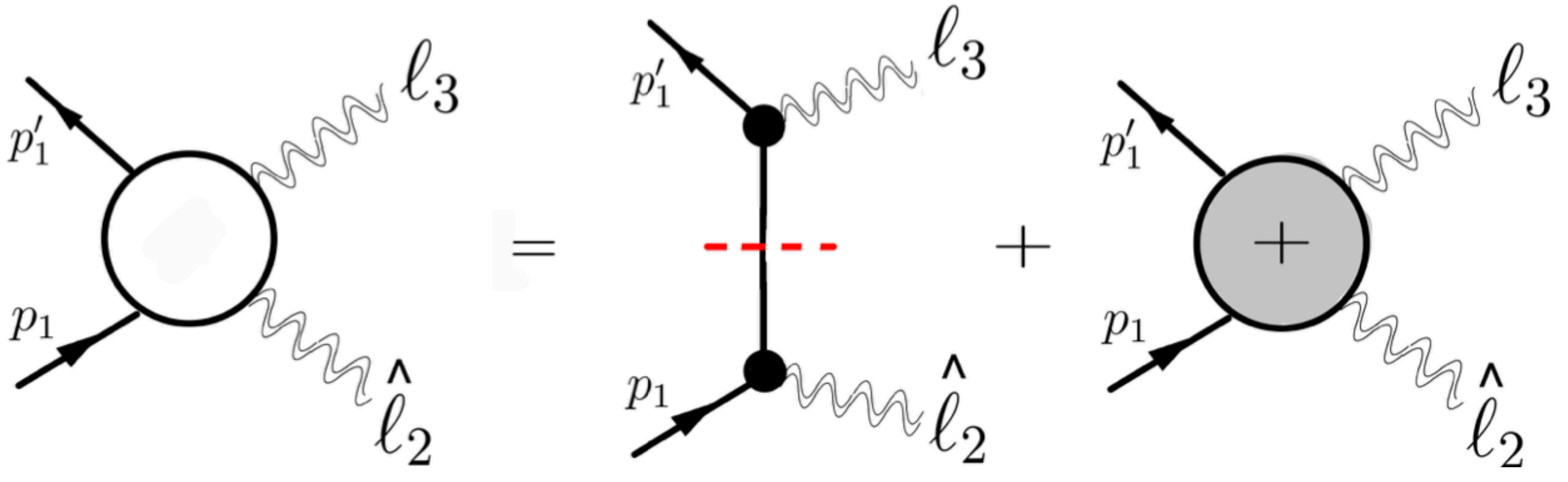
$$N_1(|\underline{\vec{q}}|, \gamma) = \frac{3\pi^2 G_N^2 m_1^2 m_2^2 (m_1+m_2)(5\gamma^2-1)(4\pi e^{-\gamma_E})^{\frac{4-D}{2}}}{|\underline{\vec{q}}|^{5-D}}$$
$$- \frac{8G_N^2 m_1^2 m_2^2 (4\pi e^{-\gamma_E})^{\frac{4-D}{2}}\hbar}{(4-D)|\underline{\vec{q}}|^{4-D}}\Big(\frac{2(2\gamma^2-1)(7-6\gamma^2)\,\text{arccosh}(\gamma)}{(\gamma^2-1)^{\frac{3}{2}}} + \frac{1-49\gamma^2+18\gamma^4}{15(\gamma^2-1)}\Big)$$
$$+ \mathcal{O}(|\underline{\vec{q}}|^{5-D}) .$$

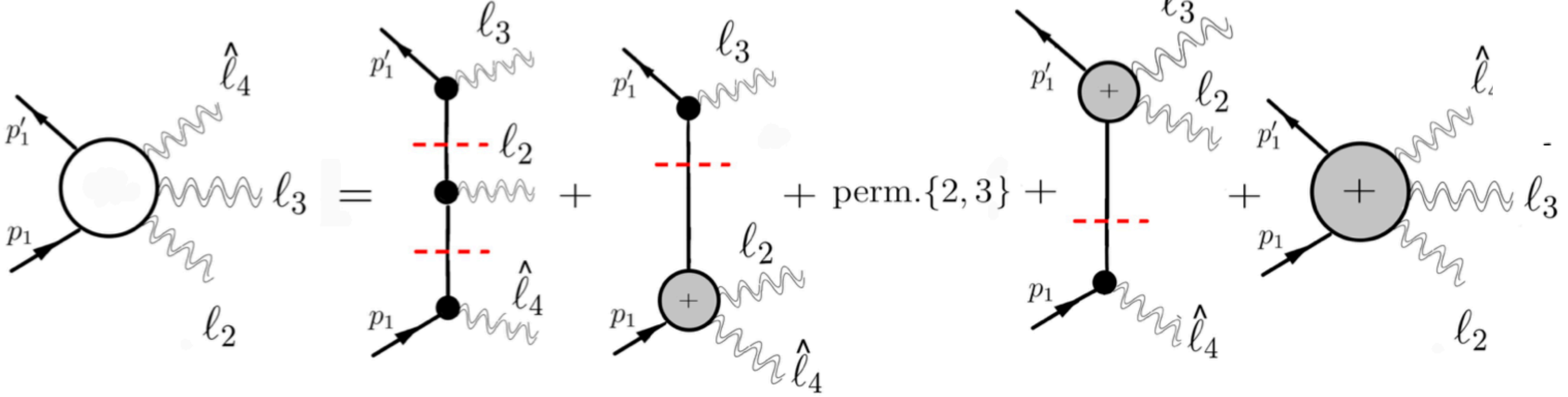
Simplifications from the exponentiation of the S-matrix

Two-loop radial action contribution

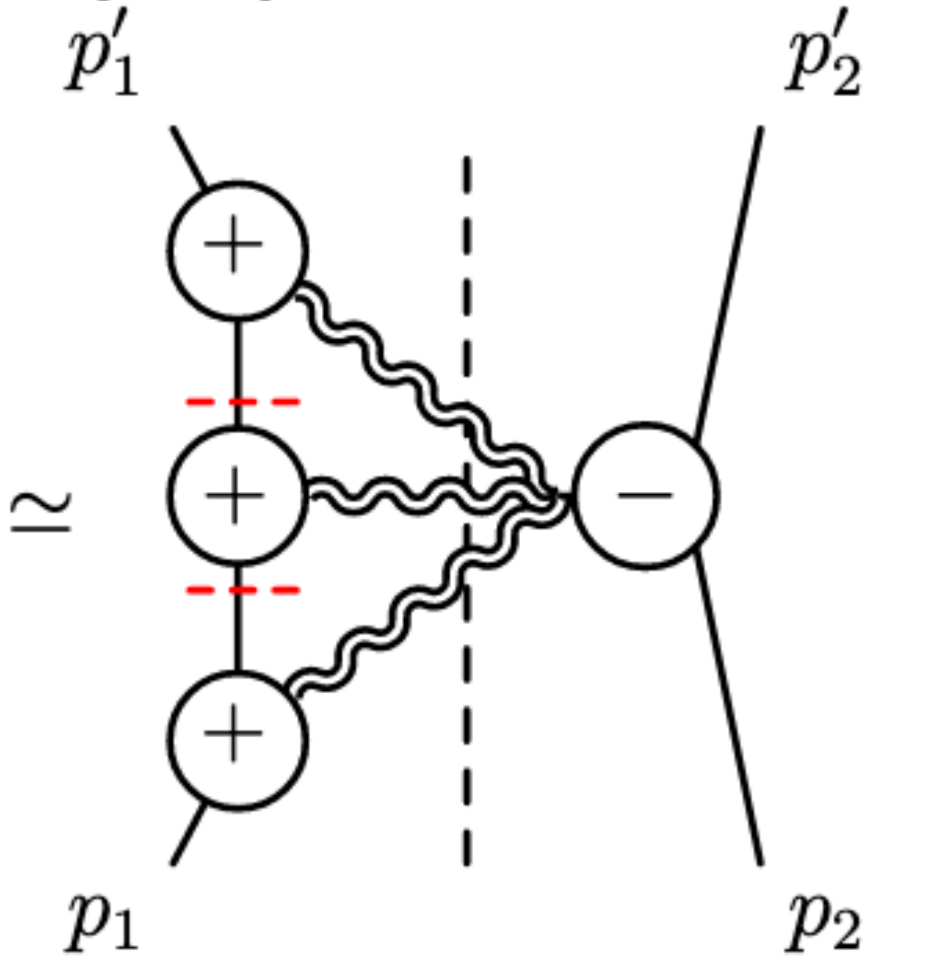
$$N_2(|\underline{\vec{q}}|,\gamma) = \frac{4\pi G_N^3 (4\pi e^{-\gamma_E})^{4-D} m_1^2 m_2^2}{(4-D)|\underline{\vec{q}}|^{8-2D}} \Bigg(\frac{\mathcal{E}_{\text{C.M.}}^2 (64\gamma^6 - 120\gamma^4 + 60\gamma^2 - 5)}{3\left(\gamma^2-1\right)^2}$$
$$- \frac{4}{3} m_1 m_2 \gamma \left(14\gamma^2 + 25\right) + \frac{4 m_1 m_2 (3 + 12\gamma^2 - 4\gamma^4) \operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}$$
$$+ \frac{2 m_1 m_2 (2\gamma^2-1)^2}{\sqrt{\gamma^2-1}} \Big(-\frac{11}{3} + \frac{d}{d\gamma} \Big(\frac{(2\gamma^2-1)\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} \Big) \Big) \Bigg) + \mathcal{O}(\hbar)$$

Velocity cuts tree diagrams / soft expansion





$m_1^4 m_2^2\, c_{3,1}(\gamma, D) \simeq$

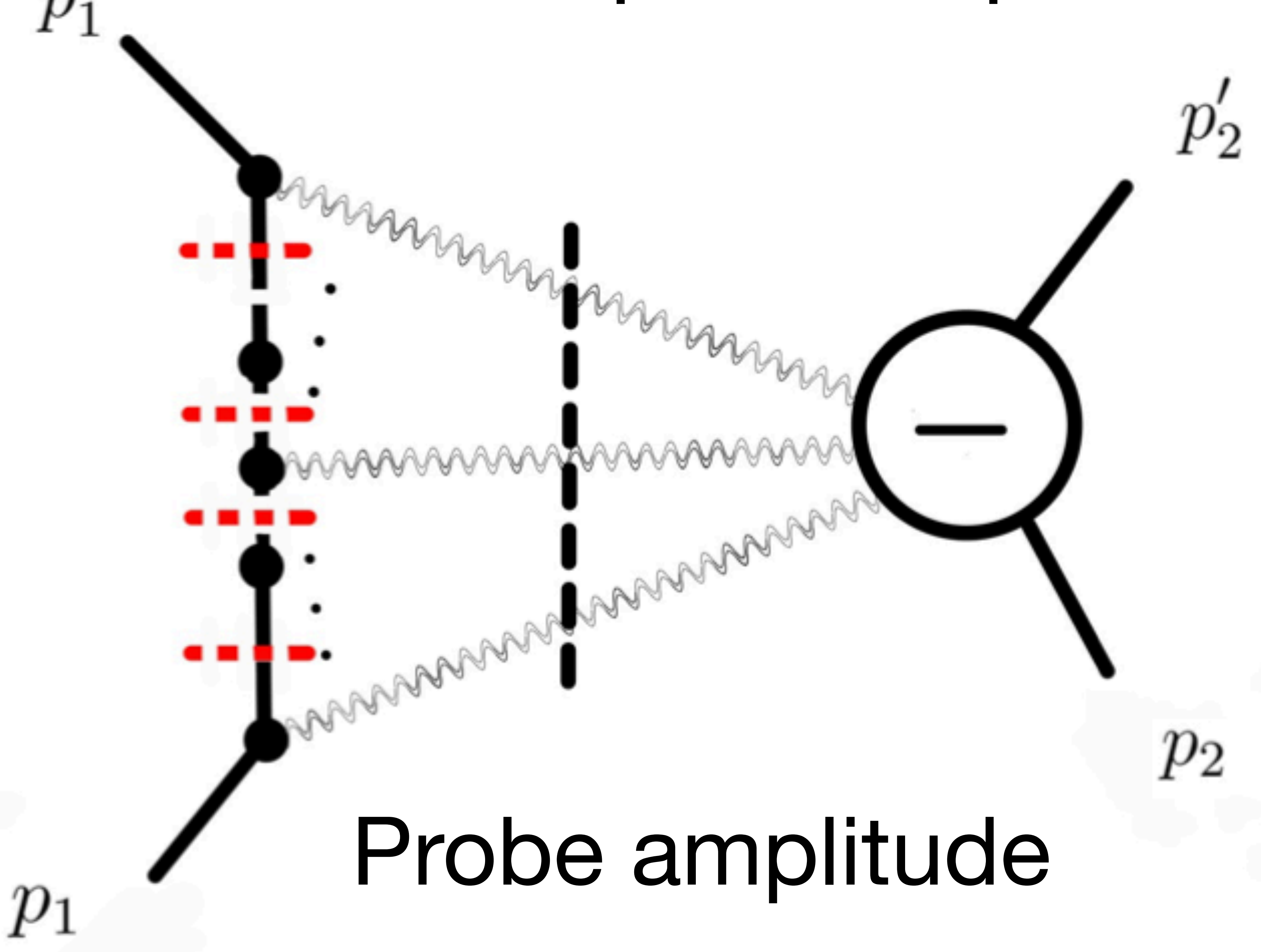


$$\mathcal{M}^{\text{tree}}_{L+1} \sim (\mathcal{M}_1^{\text{tree}(+)})^{L+1} \prod_i^{L} \delta_i(\ldots) + (\mathcal{M}_1^{\text{tree}(+)})^{L-1} (\mathcal{M}_2^{\text{tree}(+)}) \prod_i^{L-1} \delta_i(\ldots) + \cdots$$
$$+ \mathcal{M}_1^{\text{tree}(+)} \mathcal{M}_L^{\text{tree}(+)} \delta(\ldots) + \mathcal{M}_{L+1}^{\text{tree}(+)}$$

(NEJBB, Plante, Vanhove)

Simpler integrand - velocity cuts tree topologies!

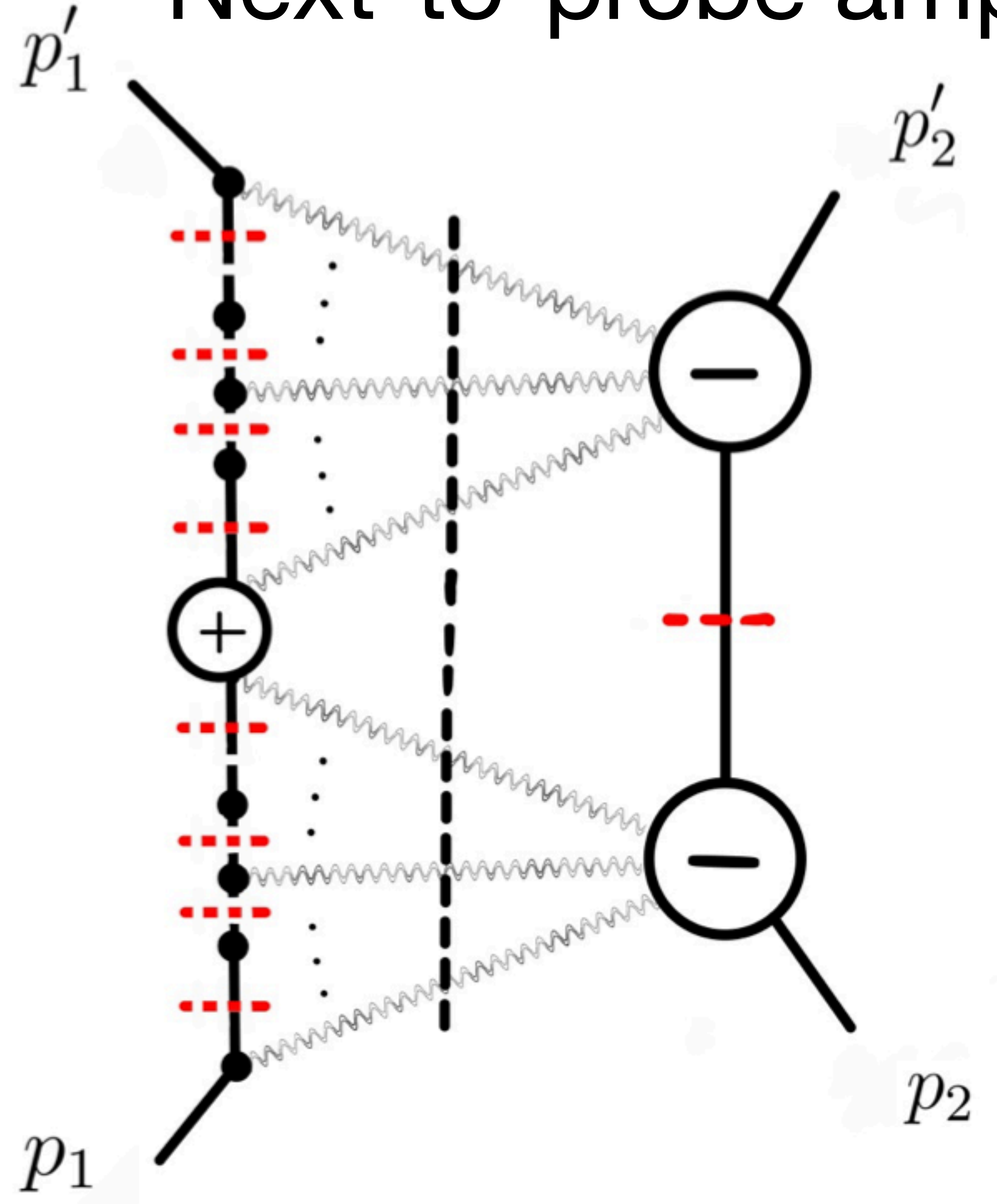
Simpler computation of integrands



Probe amplitude

(NEJBB, Plante, Vanhove)

Next-to-probe amplitude



Interesting stuff to investigate

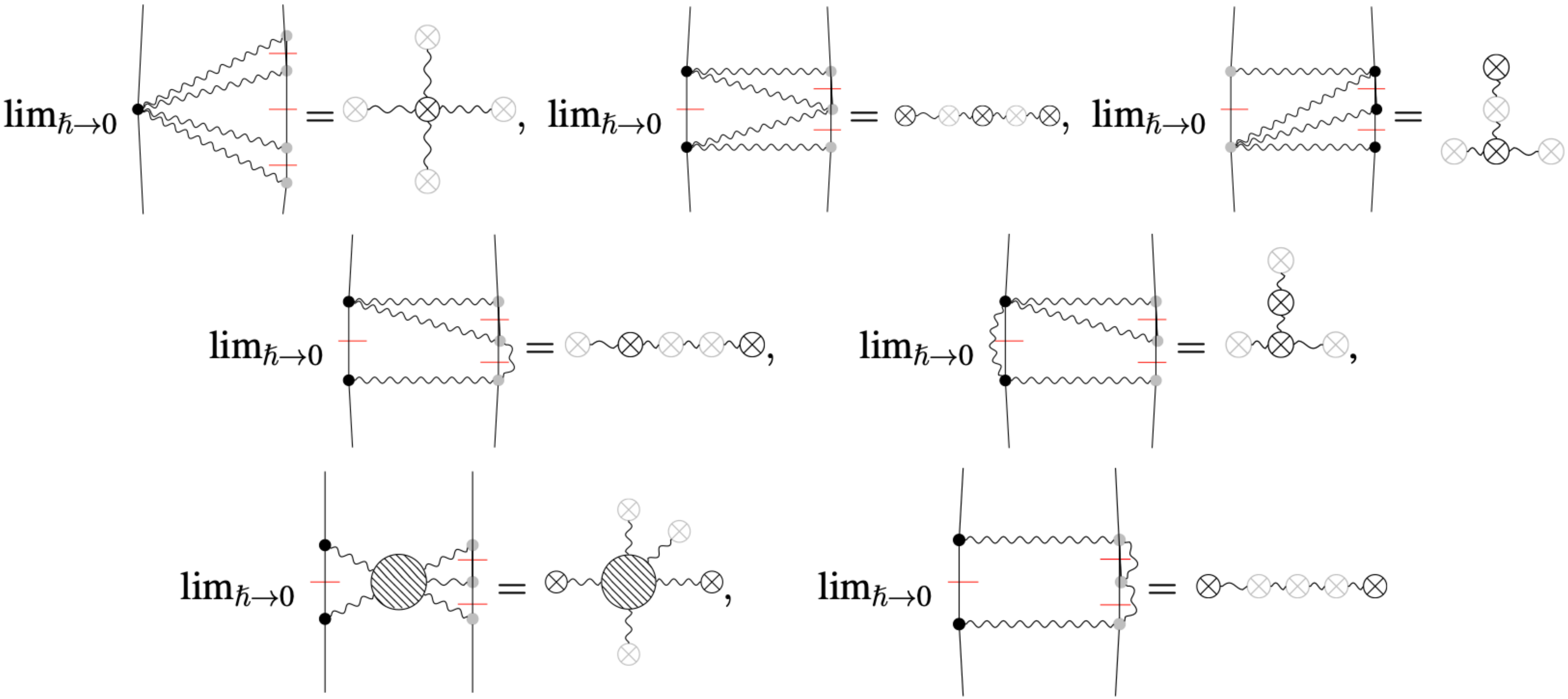
Heavy-quark—EFT inspiration:

(Damgaard, Haddad, Helset)

(Brandhuber, Chen, Travaglini, Wen)

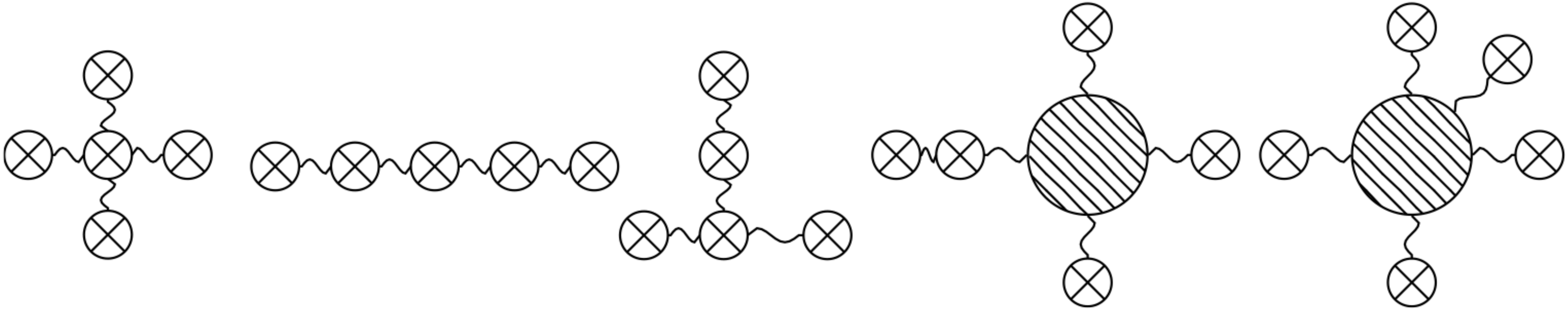
- heavy mass vs small |q| expansion?
- some similarities / some differences

Extension to fourth order in Newton's constant



Extension to fourth order in Newton's constant

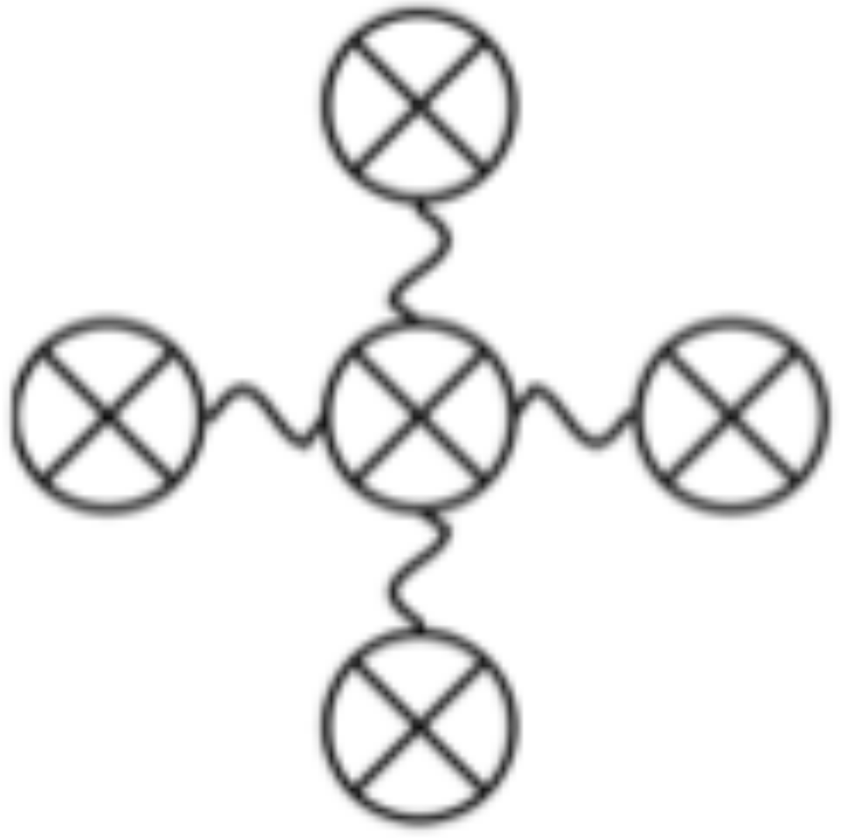
Only five integrand topologies have to be considered



Extension to fourth order in Newton's constant

$$\mathcal{M}_{4\mathrm{PM}}(\gamma, \underline{q}^2) = \lim_{\varepsilon \to 0} \sum_{i=1}^{40} c\left(\{n_j\}; \gamma, \underline{q}^2\right) \mathcal{I}\left(\{n_j\}; \gamma, \varepsilon\right)$$

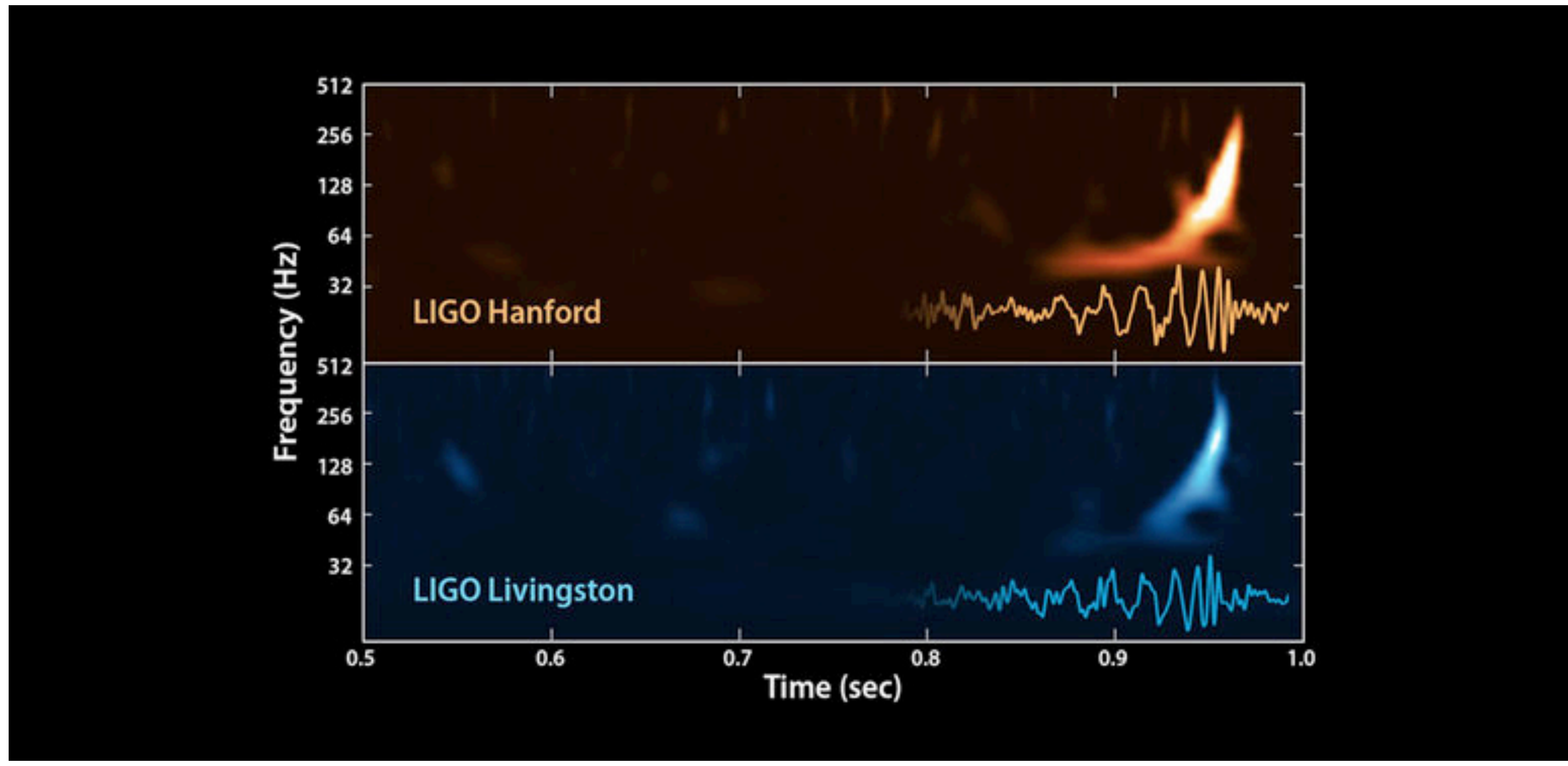
For instance the probe result is



$$\begin{aligned} \mathcal{M}^{\mathrm{probe}}_{4\mathrm{PM}}(\gamma, \underline{q}^2) &= \lim_{\varepsilon \to 0} \frac{(8\pi G_N)^4}{|\underline{q}|^{-1+3\varepsilon}} m_1^2 m_2^2 (m_1^3 + m_2^3) \frac{(1-2\varepsilon)^3}{(2-2\varepsilon)^4} \frac{c_3(\gamma, \varepsilon)}{(\gamma^2-1)^3} I^1_{\mathrm{PP}}(1, \varepsilon), \\ &= G_N^4 (m_1^3 + m_2^3) m_1^2 m_2^2 |\underline{q}| \pi^3 \frac{35i\left(33\gamma^4 - 18\gamma^2 + 1\right)}{8\left(\gamma^2 - 1\right)}, \end{aligned}$$

Conclusion

- So amplitude techniques are surprisingly efficient in Post-Minkowskian gravity computations and bridging the gap to current data.
  - **NB:** different setup from QCD
  - Gravity: New insights have been necessary to develop alongside brute-force computations
    - We have efficient frameworks for computation but still much more to learn
- For GW community: automatic programs could be useful
    - We are still far from that… each new loop order brings new problems…
- **Current bottlenecks:** Solving the integral-system: identifying IBP-relations, solving the DE equations/integrals, managing high multiplicities.
    - Better understanding of what the **minimal** computation is could lead to much simplified analysis.



Outlook

Amplitude toolbox for computations already provided many new efficient methods for computation

- Amplitude tools very useful for computations
  - Double-copy/KLT
  - Unitarity
  - Spinor-helicity
  - CHY formalism
  - Low energy limits of string theory

- Identifying IBP-relations solving DE equations/ integral
- Recycling tools from QCD computations
- Numerical programs for amplitude computation

Conclusion

Already a number of very impressive PM amplitude computations.
(Bern, Cheung, Roiban, Shen, Solon, Zeng, Bern, Ita, Parra-Martinez, Ruf; Abreu, Febres Cordero, Ita, Jaquier, Page, Cheung, Solon, Parra-Martinez, Ruf, Zeng, Bern, Luna, Roiban, Shen, Zeng; Di Vecchia, Heissenberg, Russo, Veneziano; Porto, Kalin, Liu et al; Helset, Haddad et al; Travaglini, Brandhuber, Chen at al) etc +

Clearly many more to come.. much more to learn

Conclusion

Endless tasks ahead

— spin effects (a current hot topic, very recent papers)

- higher (classical) spin from amplitudes

Spin: covariant formalism for spinning sources (NEJBB, Chen, Santos)

— radiation / validity of exponentiation / validity of perturbative amplitudes at high energy scattering (open questions…)

— quantum terms?? and inclusion of high order curvature terms / finite size effects

- String theory amplitudes useful? Gravity from amplitude geometry/twistors etc?

Clearly much more physics to learn….

THANKS!