Geometry from strong coupling

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Work with Hadleigh Frost & Omer Gurdogan

Based on Alday, Maldacena, Gaiotto, Sever, Vieira, 2007-10.

Determine new geometric structures on space of kinematic data for super Yang-Mills amplitudes at strong coupling.

Amplitudes, Wilson loops & Strings in AdS

Planar $N = 4$ dualities at strong coupling [Alday, Maldacena, Gaiotto, Sever & Vieira 2007-10]

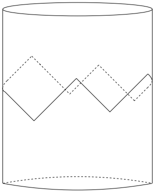
Conjecture [Alday-Maldacena]: 3-way correspondence

$$\mathcal{A}_\lambda = \langle \mathcal{W}_\gamma \rangle_\lambda = \int_{\partial\Sigma=\gamma} \mathcal{D}[\Sigma \subset AdS_5 \times S^5] \, \mathrm{e}^{-\frac{1}{\alpha'} S_{\mathrm{string}}} \, .$$

- $\mathcal{A} = \mathcal{A}(k_1, \ldots, k_n)$ amplitude,
- $\gamma =$ polygon of ordered null momenta,
- $\mathcal{W}_\gamma =$ Wilson loop around γ
- $\lambda =$ t'Hooft coupling, $\frac{R^2_{AdS}}{\alpha'} = \sqrt{\lambda}$.
- $\mathcal{A}_\lambda = \langle \mathcal{W}_\gamma \rangle_\lambda$ proved for loop-integrand [M., Skinner, Caron-Huot, 2010].
  - Strong coupling as $\alpha' \to 0, \lambda \to \infty$:

$$\langle \mathcal{W}_\gamma \rangle_\lambda \sim \mathrm{e}^{-\sqrt{\lambda}\,\mathrm{Area}_\Sigma / R^2} + \ldots$$

  - solved with *Y-system* $\rightsquigarrow$ good qualitative agreement!



$\gamma \subset \partial$AdS$= \mathbb{M}$.

Sketch of integrability approach [Alday, Maldacena, Gaiotto, Sever, Vieira]

- Exploit integrability of Minimal surfaces in AdS.
- Equation reduces to generalized Sinh-Gordon in AdS_3.
- Reformulate as $\mathbb{Z}_2$-invariant $SU(2)$ Hitchin system in AdS_3, or $\mathbb{Z}_4$-invariant $SU(4)$ in AdS_5.
- Wilson loop boundary conditions gives worldsheet $\mathbb{CP}^1$ with irregular singularity at ∞ depending on kinematic data.
- Stokes sectors at ∞ correspond to edges of polygon.
- Solution is encoded in $\mathcal{Y}_s(\zeta)$-functions, holomorphic in spectral parameter ζ using Lax pair.
- Area can be computed directly from $\mathcal{Y}_s$

Question: Moduli-spaces of regular Hitchin systems admit Hyperkahler structures (and hence twistor spaces).
Is there some version of this story here?

Space of kinematic data for minimal surfaces in AdS_3

Lemma

Null polygon $\gamma \subset \mathbb{M}^{1+1}$ has 2n sides with parameter space

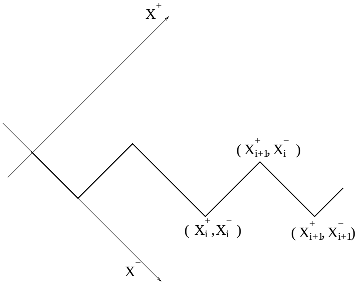
$$\mathcal{K} := \mathcal{M}^+_{0,n} \times \mathcal{M}^+_{0,n} = A_{n-3} \times A_{n-3}\,.$$

Here $\mathcal{M}_{0,n} = A_{n-3} = \{X_i \in \mathbb{RP}^1 | X_i \neq X_j\}/$Mobius, dim. $= n-3$.

Proof:

- Let (X^+, X^-) be coords on $\mathbb{M}^{1+1}$ s.t.

$$ds^2 = 2dX^+ \odot dX^-.$$



- Label edges of γ alternately by constant X^+ and X^-
- gives coords $\{X_i^+\}, \{X_i^-\}$, $i = 1, \ldots, n$.
- Conf. group=Mobius$^+\times$ Mobius$^-$,
- quotient gives result.□

These are *cluster varieties*.

Cluster coordinates on $\mathcal{K}$

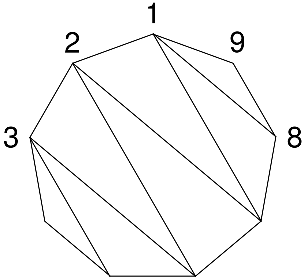
Define cluster coordinates on each $\mathcal{M}^{\pm}_{0,n}$ factor by:

- $\chi_{i,j}$ = cross ratio $\leftrightarrow$ diagonal (i,j) of n-gon

$$\chi_{i,j} := \frac{(X_i - X_j)(X_{i-1} - X_{j+1})}{(X_i - X_{i-1})(X_{j+1} - X_j)},$$

- Need *clusters* $\{\chi_s\}, s = 1, \ldots, n-3$.
- Clusters $\leftrightarrow$ triangulation of n-gon.
- E.g., zig-zag:

$$\{\chi_s\} := \{\chi_s | \chi_{2i-1} = \chi_{-i,i}\,, \chi_{2i} = \chi_{-i-1,i}\}\,.$$



- Flipping a diagonal $\leftrightarrow$ *mutations*.

Hereon, we use zig-zag and will see sets of mutations for rotation of n-gon.

Introduce *spectral parameter* $\zeta \in \mathbb{CP}^1$ and functions

$$\mathcal{Y}_s = \mathcal{Y}_s(\chi_r^+, \chi_r^-, \zeta) : \mathcal{K} \times \mathbb{CP}^1 \to \mathbb{C}\,, \qquad s = 1, \ldots, n-3\,,$$

subject to:

1. $$\mathcal{Y}_s(1) = \chi_s^+\,, \qquad \mathcal{Y}_s(i) = \chi_s^-\,.$$
2. $\mathcal{Y}_s$ analytic in ζ with branches at $\zeta = 0, \infty$ & cut: $\zeta \in \mathbb{R}^-$.
3. As $\zeta \to 0, \infty$, $\exists!(Z_s, \bar{Z}_s)$ functions of (χ_r^+, χ_r^-) s.t.
$$\log \mathcal{Y}_s(\zeta) \sim \frac{Z_s}{\zeta} + \bar{Z}_s \zeta + O(1) \quad \text{as} \quad \zeta \to 0, \infty.$$
4. For $\mathcal{Y}_s^{++}(\zeta) = \mathcal{Y}(\mathrm{e}^{i\pi}\zeta)$ analytic continuation as $\zeta \to \mathrm{e}^{i\pi}\zeta$:
$$\mathcal{Y}_{2k+1}^{++}\mathcal{Y}_{2k+1} = (1 + \mathcal{Y}_{2k+2})(1 + \mathcal{Y}_{2k})\,,$$
$$\mathcal{Y}_{2k}\mathcal{Y}_{2k}^{++} = (1 + \mathcal{Y}_{2k+1}^{++})(1 + \mathcal{Y}_{2k-1}^{++})\,.$$

Branching of $\mathcal{Y}_s \leftrightarrow$ set of cluster mutations given by rotating zig-zag triangulation by $2\pi/n$.

Output of Y-system

- $\exists!$ solution $\mathcal{Y}_s(\chi_r^+, \chi_r^-, \zeta)$ to Y-system (from TBA).
- Determines $Z_s(\chi_r^+, \chi_r^-)$ too with

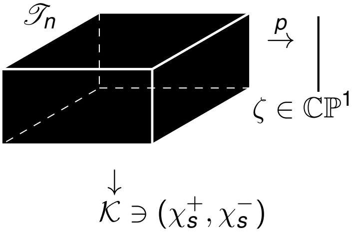
$$Z_s \sim \log \chi_s^+ - i \log \chi_s^- \,, \quad \text{as} \quad \chi_r^\pm \to 0, \infty$$

- Remainder function (regularized area) obtained as

$$R(\chi_r^+, \chi_r^-) = -i\varepsilon^{rs} Z_r \bar{Z}_s - \sum_s \int_{\zeta/Z_s \in \mathbb{R}^-} \frac{d\zeta}{\pi\zeta^2} Z_s \log(1 + \mathcal{Y}_s(\zeta))\,.$$

Gives procedure to construct amplitude via complex analysis, but what does it mean geometrically?

The Y-system defines a twistor space



Definition

Define the twistor space $\mathscr{T}_n$ for $\mathcal{K}_n$ to be

$$\begin{aligned}\mathscr{T}_n &= \mathcal{K}_n \times \mathbb{CP}^1 \\ &= A^+_{n-3} \times A^-_{n-3} \times \mathbb{CP}^1 \,.\end{aligned}$$

Smooth coords: $(\chi_s^+, \chi_s^-, \zeta)$ or $(Z_r, \bar{Z}_r, \zeta)$.

- $\mathscr{T}_n$ is a complex-manifold: $(\mathcal{Y}_s, \zeta)$ give $n-2$ local holomorphic coords.
- Holomorphic projection $p : \mathscr{T}_n \to \mathbb{CP}^1$, projecting to ζ.
- The fibres of p admit holomorphic symplectic structures

$$\Sigma(\zeta) = \epsilon^{rs} dy_r \wedge dy_s \,, \quad y_s = \log \mathcal{Y}_s \,, \quad \epsilon^{2k\,2k\pm1} = \pm 1 \,.$$

- Non-degenerate only for n odd.
- The y_r are invariant under S^1 symmetry generated by

$$V = i\zeta \frac{\partial}{\partial \zeta} + iZ_r \frac{\partial}{\partial Z_r} - i\bar{Z}_r \frac{\partial}{\partial \bar{Z}_r} \,.$$

From twistor space to pseudo-hyperkahler structure

Key device: $\Sigma(\zeta) = \epsilon^{rs} dy_r \wedge dy_s$ is global on $\mathbb{CP}^1$; *no branching*.

- Recall $(\mathcal{Y}_s(1), \mathcal{Y}_s(i)) = (\chi_s^+, \chi_s^-)$ so setting $x_s^\pm = \log \chi_s^\pm$:

$$\Sigma(1) = \sum \epsilon^{rs} dx_r^+ \wedge dx_s^+ , \qquad \Sigma(i) = \sum \epsilon^{rs} dx_r^- \wedge dx_s^- .$$

- $\Sigma(\zeta)$ has *double poles* at $\zeta = 0, \infty$ as y_s has single poles.
- But $\Sigma(-\zeta) = \Sigma(\zeta) \rightsquigarrow$ no terms in ζ, ζ^{-1}.
- So Laurent expansion is:

$$\Sigma(\zeta) = \frac{(\zeta^2+1)^2}{4\zeta^2}\Sigma(1) - \frac{(\zeta^2-1)^2}{4\zeta^2}\Sigma(i) + \frac{(\zeta^4-1)}{4\zeta^2}\Omega .$$

  for some ζ-independent closed 2-form Ω.
- Rank $n-3$ of $\Sigma(\zeta) \Rightarrow$ implies $\Omega = J^{rs} dx_r^+ \wedge dx_s^-$ with

$$J^{rs} = \frac{\partial^2 J(x_p^+, x_q^-)}{\partial x_r^+ \partial x_s^-} , \qquad J^{pq} J^{rs} \epsilon_{pr} = \epsilon^{qs} .$$

Plebanski: J = Kahler scalar for pseudo-hyperkahler structure.

Lemma (Alday, Maldacena)

$R(\chi_r^+, \chi_r^-)$ = Hamiltonian of the circle symmetry V.

Proposition

For n odd, the remainder function is $R(x_r^+, x_r^-) = J(x_r^+, x_r^-)$, and so satisfies

$$R^{rs} = \frac{\partial^2 R(x_p^+, x_q^-)}{\partial x_r^+ \partial x_s^-}\,, \qquad R^{pq} R^{rs} \epsilon_{pr} = \epsilon^{qs}\,.$$

Defines a pseudo-hyperkahler structure on $\mathcal{K}_n$:

$$ds^2 := R^{rs} dx_r^+ \odot dx_s^-\,, \quad \omega^{\pm} = \epsilon^{rs} dx_r^{\pm} \wedge dx_s^{\pm}\,, \quad \Omega = R^{rs} dx_r^+ \wedge dx_s^-$$

- Follows from $\mathcal{L}_V \Sigma(\zeta) = 0$.
- Completely integrable system of overdetermined PDE.
- Lax: $\mathcal{L}_r = (\zeta^2 - 1)\frac{\partial}{\partial x_r^+} + (\zeta^2 + 1) i J^{rs} \frac{\partial}{\partial x_s^-}\,, \;\; [\mathcal{L}_r, \mathcal{L}_s] = 0.$

Summary & directions

Summary of geometry:

- Remainder function generates pseudo-hyperkahler geometry on $\mathcal{K}_n$ for n-odd enhancing cluster geometry.
- Has circle symmetry, but not manifest in $x_r^\pm$ coords.
- Expect orbifold point for fixed point $\leftrightarrow$ regular polygon.
- Standard lore: R smooth in double soft limit $\chi_{n-3}^\pm \to 0$.
- For n even must understand geometry of such double soft 'corners': $\rightsquigarrow$ reduction to even n.
- $\rightsquigarrow$ hierarchy of nested boundaries determining full solution.

Outlook

- Expect pseudo-hyperkahler for full kinematics when $3(n-5)$ divisible by 4, soft limits give other n too.
- Can we see role for geometry and differential equations at weak/intermediate coupling? Hints from Origin story.
- Note good numerical agreement with weak coupling!
- How does differential geometry tie into positive geometry?

Thank You!