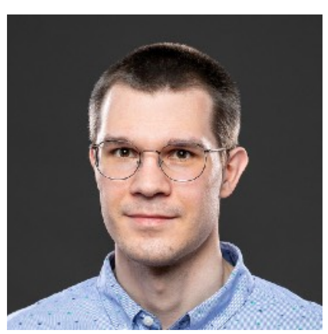
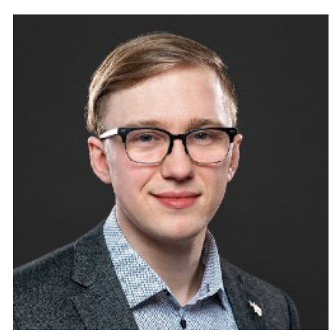
Finite four-particle amplitudes from a deformed Amplituhedron geometry

Johannes M. Henn

Based on work in progress with Nima Arkani-Hamed, Wojciech Flieger, Anders Schreiber, and Jaroslav Trnka.





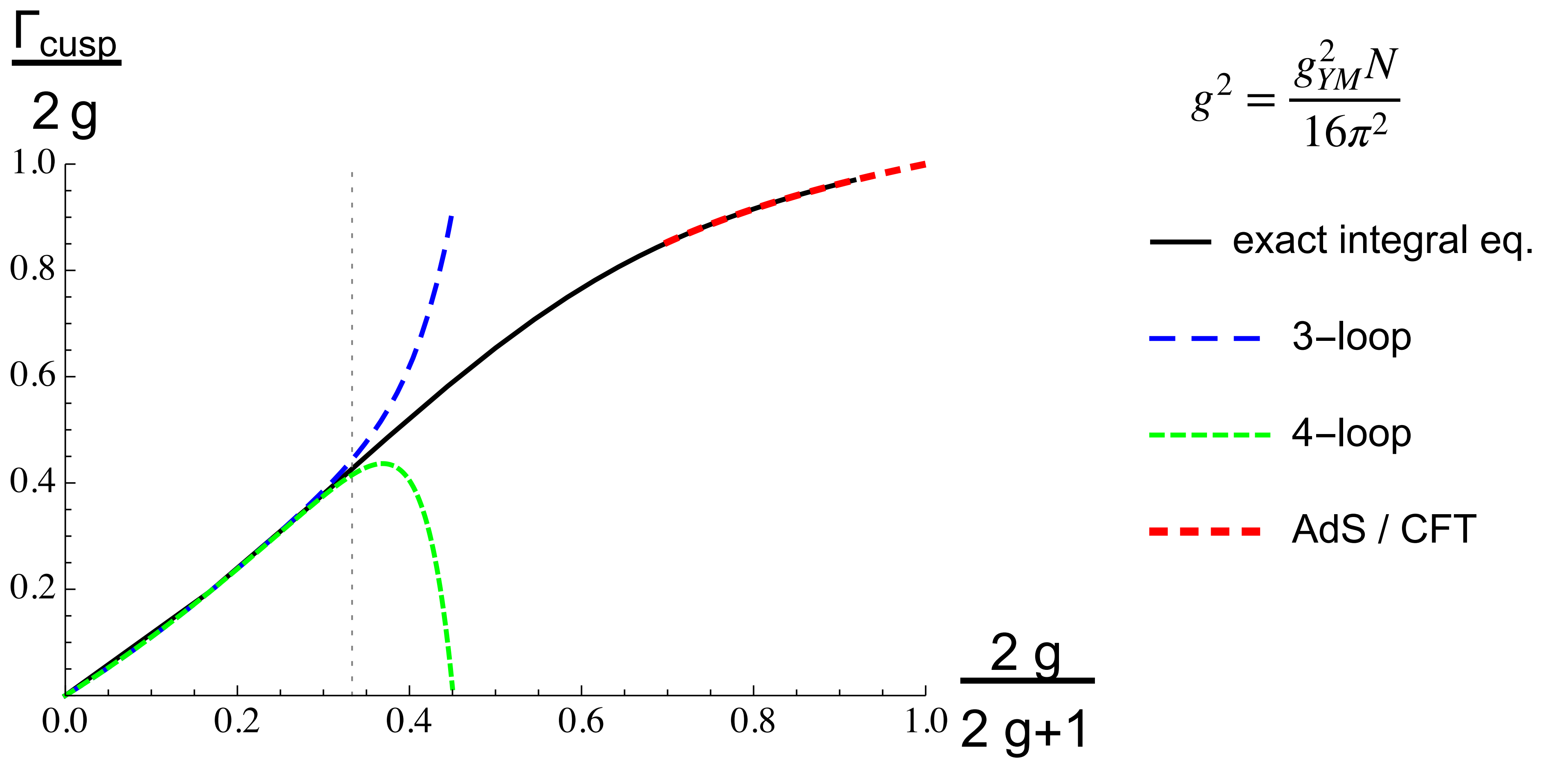
Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

Amplitudes in Prague, May 17, 2023

European Research Council
Established by the European Commission

Motivation

In N=4 super Yang-Mills, the form of four-point amplitudes is known to all orders in the coupling (thanks to dual conformal symmetry). It depends on the cusp anomalous dimension, which is known exactly from integrability.



What can we learn from this for QFT calculations?

What methods allow us to see the AdS picture emerge?

A geometric approach to amplitudes

The Amplituhedron gives a novel definition of the four-dimensional loop *integrand*, without reference to Feynman diagrams.

Unfortunately, dimensional regularization breaks the beautiful geometric picture. We propose a deformation of the Amplituhedron geometry that allows us to obtain *integrated amplitudes*, working in four dimensions.

Outline

1. Deformed Amplituhedron geometry

2. Result for one- and two-loop amplitudes

3. (Extra slides) Differential equations algorithm for finite loop integrals.

Part 1: Deformed Amplituhedron Geometry

Simple examples of canonical forms from geometry

Consider the interval $z \in [0,1]$

Canonical form with logarithmic divergence at boundaries:

$$dz\, r(z) = \frac{dz}{z} - \frac{dz}{z-1} = d\log\left(\frac{z}{z-1}\right)$$

Residues, e.g.: $\oint_{z=0} dz\, r(z) = 1$

This is the analog of the Feynman integrand, with integration variable z. More generally, the geometry depends additionally on external variables/parameters x.

Workflow: from geometry to functions

1. Geometry defined by set of inequalities
(in physical or auxiliary space x, z)

↓

2. Canonical differential form:
(rational) Feynman *integrand* $\omega(x, z)$

↓

3. Feynman *integral:*
special function $f(x) = \int \omega(x, z)$

Workflow: from geometry to functions

1. Geometry defined by set of inequalities
(in physical or auxiliary space x, z)

$\longrightarrow$ which geometries are relevant?

2. Canonical differential form:
(rational) Feynman *integrand* $\omega(x, z)$

$\longrightarrow$ how to construct the form in practice?

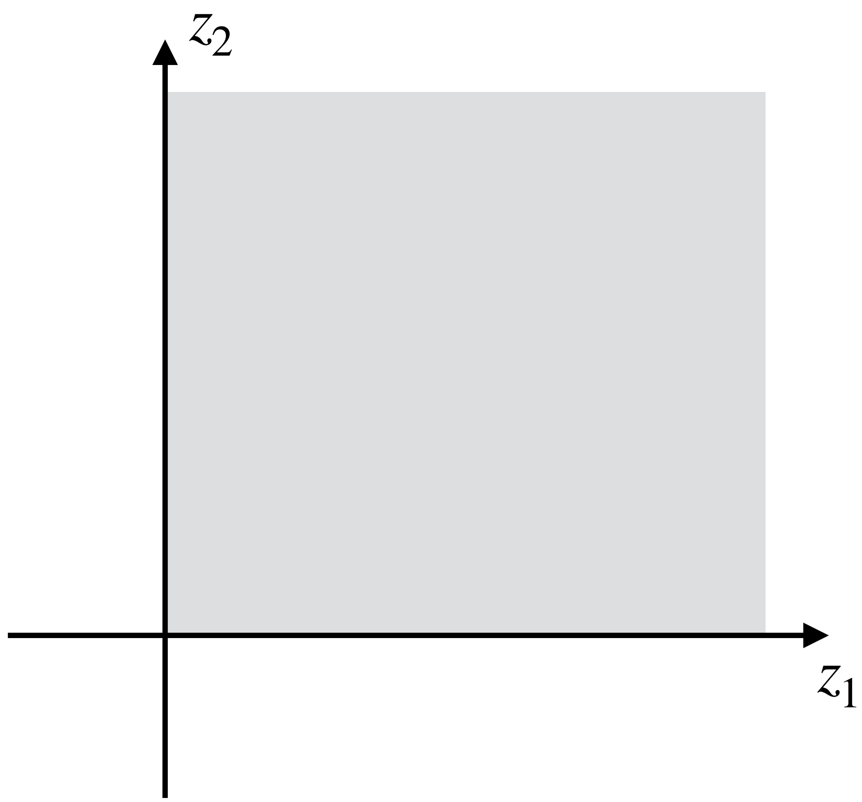
3. Feynman *integral:*
special function $f(x) = \int \omega(x, z)$

$\longrightarrow$ how to perform the integrations?

Sketch of Amplituhedron deformation

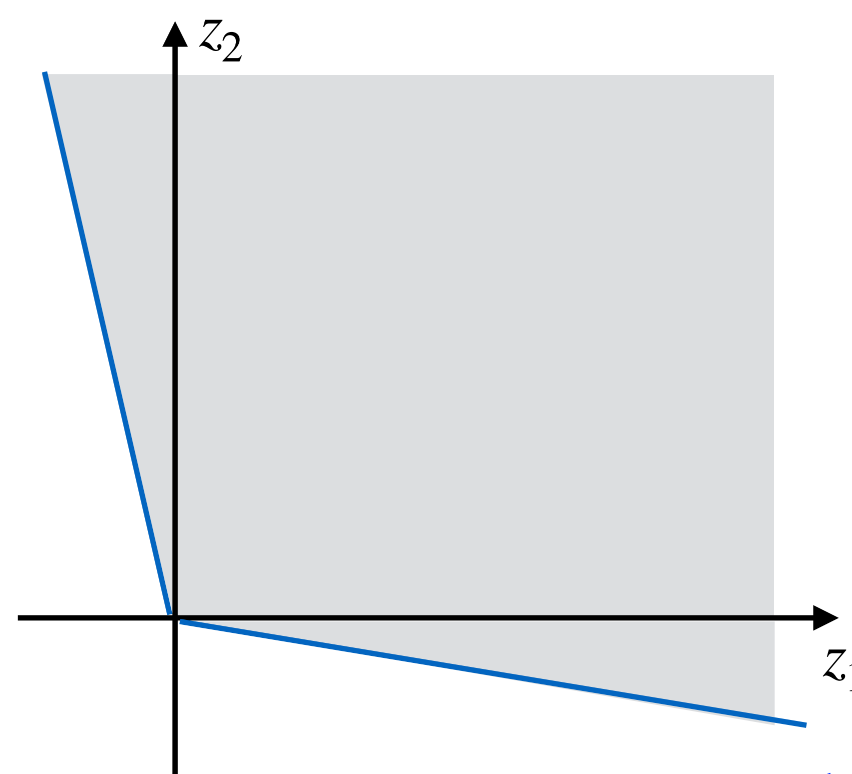
Amplituhedron:

$$z_1 > 0\,, z_2 > 0$$



Deformation:

$$\tilde{\xi} z_1 + z_2 > 0$$



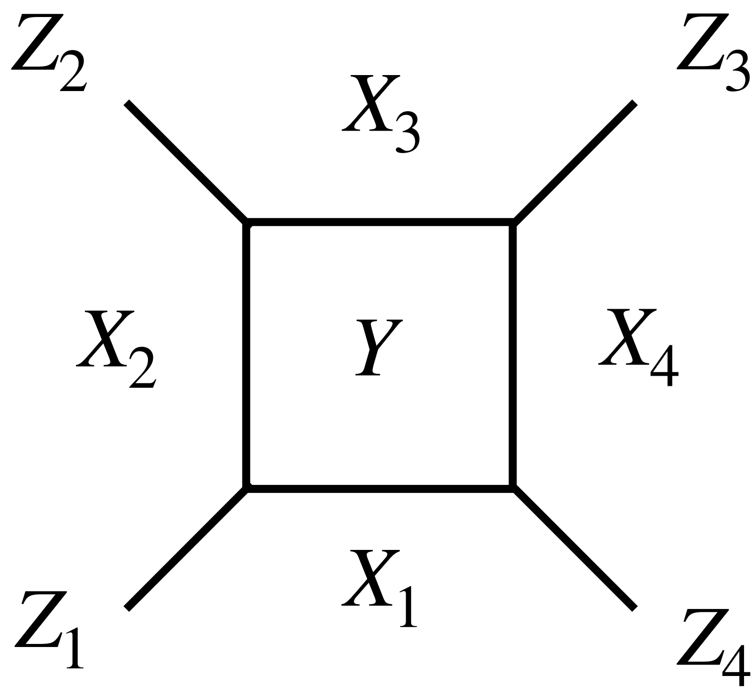
$$z_1 + \xi z_2 > 0$$

Integrand:

$$\omega = \frac{dz_1 dz_2}{z_1 z_2}$$

$$\omega = \frac{dz_1 dz_2 (1 - \xi\tilde{\xi})}{(z_1 + \xi z_2)(\tilde{\xi} z_1 + z_2)}$$

Deformed four-point Amplituhedron in terms of momentum twistors



$$X_1 = Z_1 Z_2 \,, X_2 = Z_2 Z_3 \,,$$
$$X_3 = Z_3 Z_4 \,, X_4 = Z_4 Z_1 \,,$$
$$Y = Z_A Z_B$$

Massless on-shell kinematics: $X_1^2 := (X_1 X_1) = 0$
$(X_1 X_2) = 0$ (+ cyclic)

Amplituhedron geometry defined from inequalities:

$$\langle X_1 Y \rangle > 0 \,, \langle X_2 Y \rangle > 0 \,, \langle X_3 Y \rangle > 0 \,, \langle X_4 Y \rangle > 0 \,.$$

$$\langle 13 Y \rangle < 0 \,, \langle 24 Y \rangle < 0 \,.$$

Deformed four-point Amplituhedron in terms of momentum twistors

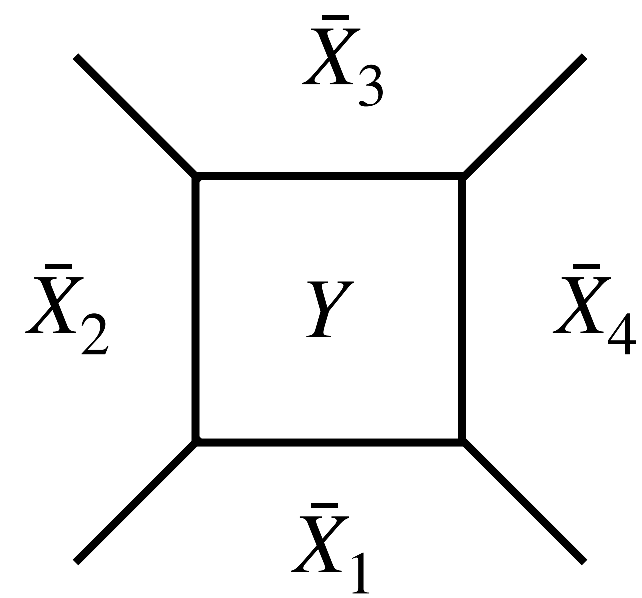
Deformation: $\bar{X}_1 = X_1 + \xi X_3$ $\quad \bar{X}_3 = X_3 + \tilde{\xi} X_1$

$\bar{X}_2 = X_2 + \eta X_4$ $\quad \bar{X}_4 = X_4 + \tilde{\eta} X_2$

One-loop geometry leads to 'massive' box integral

$$\langle \bar{X}_1 Y \rangle > 0\,, \langle \bar{X}_2 Y \rangle > 0\,, \langle \bar{X}_3 Y \rangle > 0\,, \langle \bar{X}_4 Y \rangle > 0\,.$$

$$\langle 13AB \rangle < 0\,, \langle 24AB \rangle < 0\,.$$

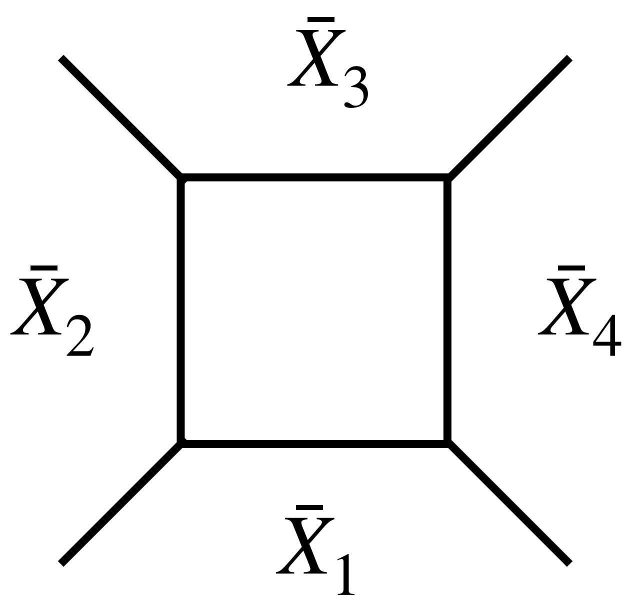


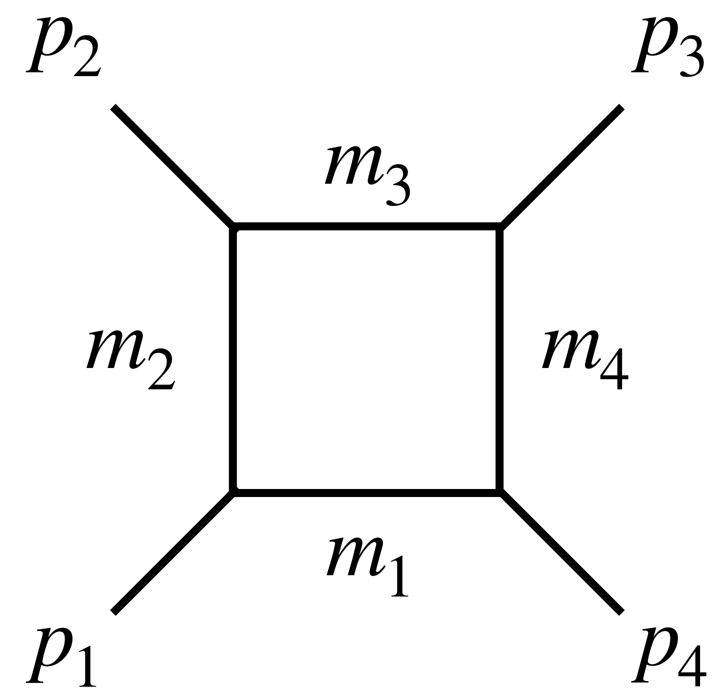
$$M^{(1)} = \oint_Y \frac{N}{\langle \bar{X}_1 Y \rangle \langle \bar{X}_2 Y \rangle \langle \bar{X}_3 Y \rangle \langle \bar{X}_4 Y \rangle}$$

Massive propagators due to $\bar{X}_i^2 \neq 0$

Deformation preserves condition $(\bar{X}_i \bar{X}_{i+1}) = 0$

Deformed kinematics in momentum space





$$(\bar{X}_i \bar{X}_{i+1}) = 0$$

$$(\bar{X}_i \bar{X}_i) \neq 0$$

$$p_i^2 = m_i^2 + m_{i+1}^2$$

$$s = (p_1 + p_2)^2 \quad t = (p_2 + p_3)^2$$

Two dual conformal invariants:

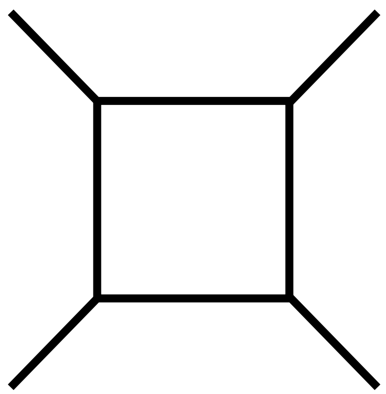
$$u = \frac{(\bar{X}_1 \bar{X}_3)^2}{\bar{X}_1^2 \bar{X}_3^2} = \frac{(1 + \xi\bar{\xi})^2}{4\xi\bar{\xi}} = \frac{(-s + m_1^2 + m_3)^2}{4m_1^2 m_3^2}$$

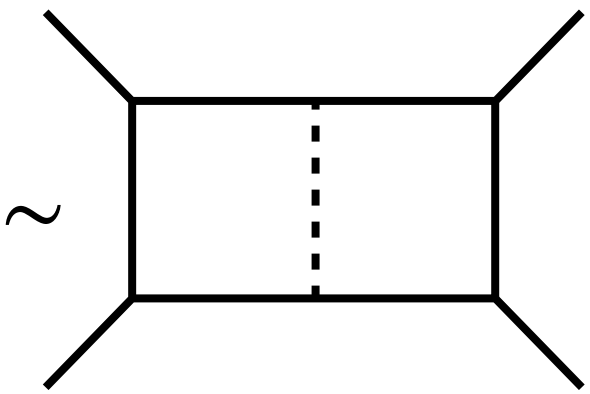
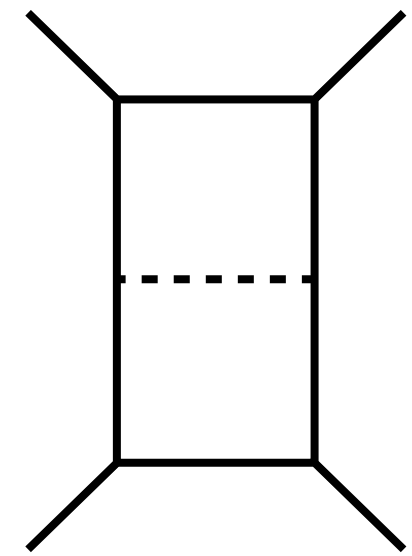
$$v = \frac{(\bar{X}_2 \bar{X}_4)^2}{\bar{X}_2^2 \bar{X}_4^2} = \frac{(1 + \eta\bar{\eta})^2}{4\eta\bar{\eta}} = \frac{(-t + m_2^2 + m_4)^2}{4m_2^2 m_4^2}$$

Part 2: Result for one- and two-loop amplitudes

Deformed integrands

The deformed canonical form is given by

$M^{(1)} \sim$ 

$M^{(2)} \sim$  $+$ 

They depend on two cross-ratios, which we parametrize by

$$u = \frac{1}{4}\left(x + \frac{1}{x}\right)^2, \qquad v = \frac{1}{4}\left(y + \frac{1}{y}\right)^2,$$

Undeformed case is recovered as $x, y \to 0\,, \quad x/y = t/s$

We chose the normalisation so that they have unit leading singularities, as in the undeformed case.*

Quick recap

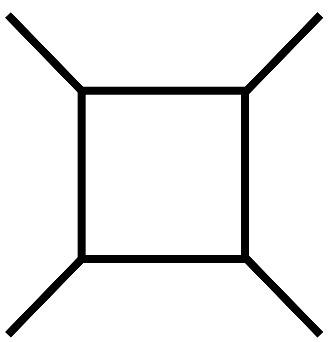
We defined deformed four-point amplitudes $M(x, y)$ that depend on two variables.

Up to two loops, the amplitude is given by infrared- and ultraviolet-finite box and double box integrals.

Our next goal is to compute these integrals. We can then take the massless limit to make contact with the undeformed case.

The deformed amplitudes are surprisingly simple!

E.g. simply by Feynman parametrization:



$$= \int_0^\infty \frac{d^4\alpha}{GL(1)} \frac{1}{(\alpha_1^2 x + \alpha_2^2 y + \alpha_3^2 x + \alpha^4 y + \alpha_1\alpha_3(1+x^2) + \alpha_2\alpha_4(1+y^2))^2}$$

$$= \frac{2\log x \log y}{(1-x^2)(1-y^2)}$$

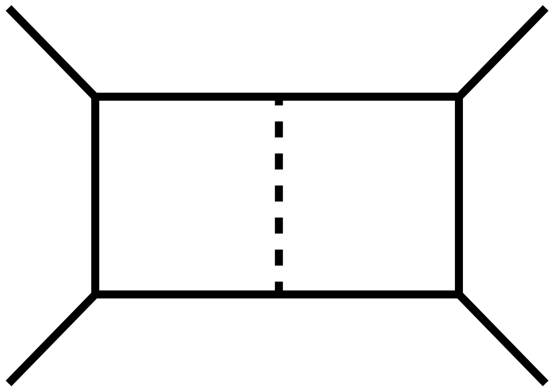
(There is a simple way to obtain this using four-dimensional differential equations, but that is another story…)

The (leading-singularity-)normalized amplitude is:

$$M(x,y;g) = 1 + g^2 M^{(1)}(x,y) + g^4 M^{(2)}(x,y) + \dots \qquad g^2 = \frac{g_{\mathrm{YM}}^2 N_c}{16\pi^2}$$

$$M^{(1)}(x,y) = -2\log x \log y$$

Result for two-loop box integral



$$= Q(y^2) - \frac{1}{2} Q\left(\frac{y^2}{x^2}\right) - \frac{1}{2} Q(x^2 y^2) - J_3(x^2)\log(y^2)$$

$$\begin{aligned} Q(z) =& 3J_4(z) + \frac{3\pi^4}{10} + \frac{\pi^2}{4}\log(z)^2 + \frac{1}{4}\log(z)^4 + \log(z)^2 \mathrm{Li}_2(1-z) \\ &+ 4\pi^2 \mathrm{Li}_2(-\sqrt{z}) - \log(z)\mathrm{Li}_3\left(1-\frac{1}{z}\right) - \log(z)\mathrm{Li}_3\left(1-z\right) , \end{aligned}$$

$$J_4(z) = \mathrm{Li}_4(z) - \log(z)\mathrm{Li}_3(z) + \frac{1}{2}\log^2(z)\mathrm{Li}_2(z) + \frac{1}{6}\log^3(z)\log(1-z) - \frac{1}{48}\log^4(z) \,.$$

$$J_3(z) = \frac{1}{4}\log^3 z + \log z \mathrm{Li}_2\left(1-z\right) - 2\mathrm{Li}_3\left(1-z\right) - 2\mathrm{Li}_3\left(1-\frac{1}{z}\right)$$

Only subset of two-loop functions needed, no $\mathrm{Li}_{2,2}, (\mathrm{Li}_2)^2$

[similar to Goncharov, Spradlin, Vergu, Volovich, 2010]

Symbol alphabet

$$\{x, x-1, x+1, y, y-1, y+1, x-y, x+y, 1-xy, 1+xy\}$$

Geometric amplitudes are simpler compared to Coulomb branch ones

- Coulomb branch: [Alday, JMH, Plefka, Schuster 2009; Caron-Huot, JMH 2014]

$$M^{(1)} = -\frac{2}{\beta_{uv}}\Big\{2\log^2\left(\frac{\beta_{uv}+\beta_u}{\beta_{uv}+\beta_v}\right) + \log\left(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}\right)\log\left(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}\right) - \frac{\pi^2}{2}$$
$$+\sum_{i=1,2}\left[2\,\mathrm{Li}_2\left(\frac{\beta_i-1}{\beta_{uv}+\beta_i}\right) - 2\,\mathrm{Li}_2\left(-\frac{\beta_{uv}-\beta_i}{\beta_i+1}\right) - \log^2\left(\frac{\beta_i+1}{\beta_{uv}+\beta_i}\right)\right]\Big\}.$$

$$\beta_u = \sqrt{1+u}\,,\qquad \beta_v = \sqrt{1+v}\,,\qquad \beta_{uv} = \sqrt{1+u+v}\,.$$

$$u = \frac{4m^2}{-s}\,,\qquad v = \frac{4m^2}{-t}\,,$$

At two loops, $\mathrm{Li}_{2,2}$ needed.

- Deformed Amplituhedron:

$$M^{(1)}(x,y) = -2\log x \log y$$

However, the Coulomb branch amplitudes have a Lagrangian formulation. We currently only have the geometric definition.

Taking the deformation to zero

We know the deformed amplitude $M(x, y)$ up to two loops.

Undeformed case is recovered as $x, y \to 0\,, \quad x/y = t/s$

We propose the following exact formula:

$$\lim_{x,y\to 0} \log M = -\frac{1}{2}\Gamma_{\rm cusp}(g)\log x \log y + \mathcal{G}_{\rm deformed}(g)\log(xy) + C_{\rm deformed}(g)\,,$$

[similar to Coulomb amplitudes: Alday, JMH, Plefka, Schuster 2009;
comparison to dimensional regularization: JMH, Moch, Naculich 2011]

We confirm this to two loops, with the following values:

$$\Gamma_{\rm cusp}(g) = 4g^2 - 8\zeta_2 g^4 + \ldots\,.$$

$$\mathcal{G}_{\rm deformed}(g) = -4\zeta_3 g^4 + \ldots\,, \qquad C_{\rm deformed}(g) = -\frac{3}{10}\pi^4 g^4 + \ldots$$

Other interesting limits

Recall the two-loop alphabet

$$\{x, x-1, x+1, y, y-1, y+1, x-y, x+y, 1-xy, 1+xy\}$$

Zeros correspond to (potentially singular physical limits).
It is interesting to study universal formulas in those.

$$x = \frac{\sqrt{4m^2 - s} - \sqrt{-s}}{\sqrt{4m^2 - s} + \sqrt{-s}} \qquad y = \frac{\sqrt{4m^2 - t} - \sqrt{-t}}{\sqrt{4m^2 - t} + \sqrt{-t}}$$

Examples:

$x \to 1$	low energy limit	$s \to 0$
$x \to -1$	threshold limit	$s \to 4m^2$
$x \to 0$	Regge limit	$s \to \infty$

Interesting to explore further the full kinematic space!

Recap

Our Amplituhedron deformation leads to finite four-point amplitudes that depend on two cross-ratios x,y.

We determined the integrals in the two-loop amplitude. They are simpler compared to Coulomb-branch amplitudes.

We proposed an exact form of the amplitude in the limit where the deformation parameters are taken to zero.

Extra slides: Differential equations algorithm for finite loop integrals

Key features of the method

[Caron-Huot and JMH, 2014]

We write all integrals in embedding space. This makes dual conformal symmetry and ultraviolet finiteness manifest.

The differential operators and integration-by-parts identities (IBP) stay in this space. Therefore only a subset of integrals needs to be considered (compared to dimensional regularization).

Four-dimensional Laplace-type equations relate integrals at different loop orders.

One-loop differential equations

$$G_{a_1,a_2,a_3,a_4} := \int_Y \frac{1}{(X_1Y)^{a_1}(X_2Y)^{a_2}(X_3Y)^{a_3}(X_4Y)^{a_4}}\,, \qquad \sum_{i=1}^4 a_i = 4$$

Differential operator:

$$\partial_x = \frac{1}{(-1+x)(1+x)}(xO_{1,1} - O_{1,3} - O_{3,1} + xO_{3,3}) \qquad O_{i,j} = (X_i \partial_{X_j})$$

Differential equation:

$$\partial_x G_{1,1,1,1} = \frac{2x}{1-x^2} G_{1,1,1,1} - \frac{2}{1-x^2} G_{0,1,2,1}$$

(similar for y derivative)

Solve for normalization (leading singularity):

$$g_4 = (1-x^2)(1-y^2)G_{1,1,1,1}$$

$$\partial_x g_4 = -\frac{2}{1-x^2} G_{0,1,2,1}$$

Repeat procedure for triangle integral $\longrightarrow$ iterative algorithm!

One-loop differential equations (2)

Outcome of iterative algorithm:

$$
\begin{aligned}
g_1 &= 2xy\, G_{2,2,0,0}\,,\\
g_2 &= x(1-y^2)\, G_{0,1,2,1}\,,\\
g_3 &= (1-x^2)y\, G_{1,2,1,0}\,,\\
g_4 &= (1-x^2)(1-y^2)\, G_{1,1,1,1}\,.
\end{aligned}
$$

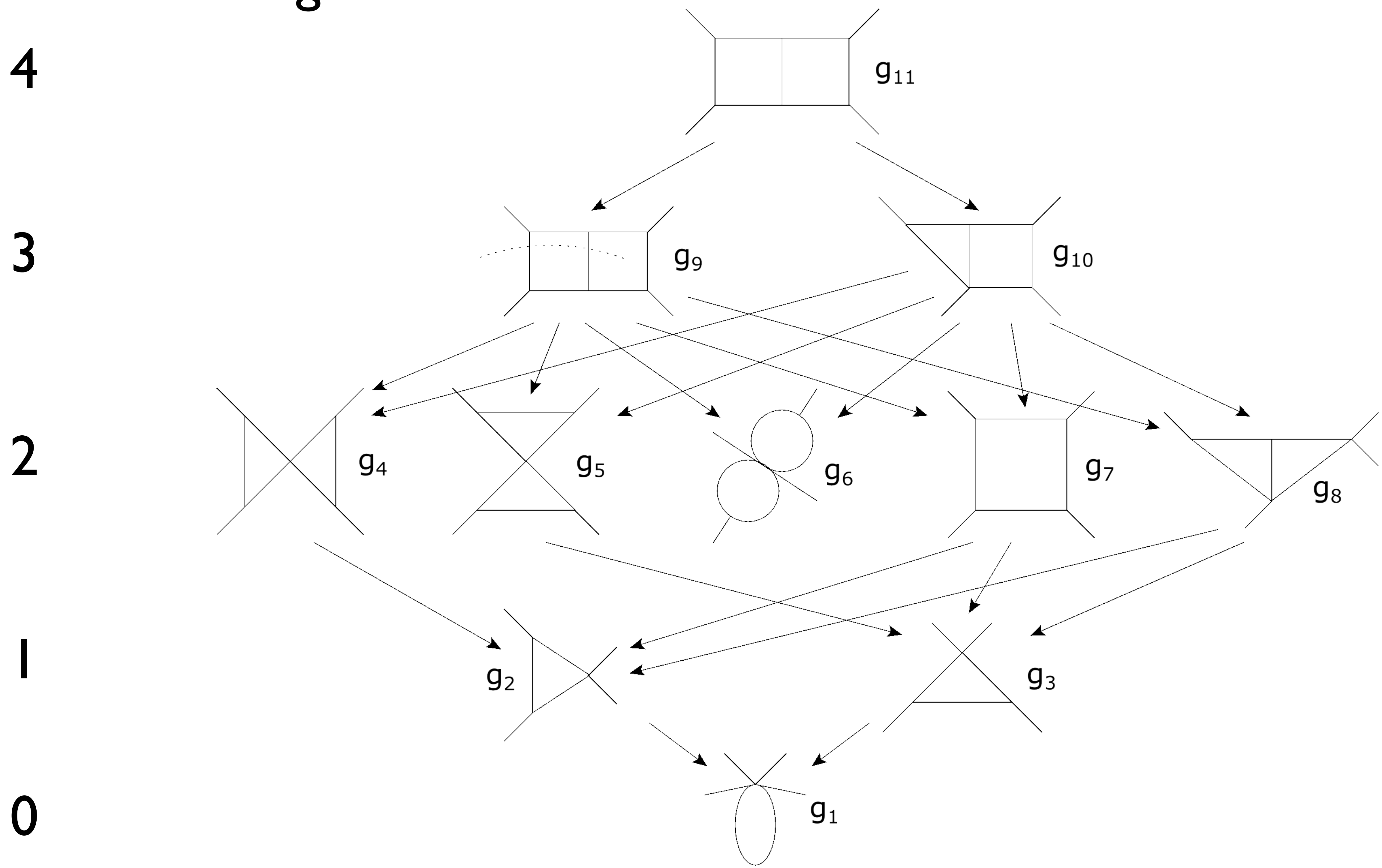
$$
\partial_x \vec{g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{x} & 0 & 0 & 0 \\ 0 & -\frac{2}{x} & 0 & 0 \end{pmatrix} \qquad \partial_y \vec{g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{y} & 0 \end{pmatrix} \vec{g}
$$

Integrate up solutions with boundary condition at x=y=1.

$$
\begin{aligned}
g_1 &= 1\,,\\
g_2 &= -\log(y)\,,\\
g_3 &= -\log(x)\,,\\
g_4 &= 2\log(x)\log(y)\,.
\end{aligned}
$$

Two-loop differential equations (DE)

Transcendental weight:



Non-zero entries of DE represented arrows.

Open questions

Is there a quantum field theory that corresponds to the deformation? What is the role of unitarity?

All integrals needed in the differential equations method are finite functions. Is there a streamlined geometry-based approach for obtaining the integral basis and the differential equations matrix?

The amplitude has many interesting limits. Can we determine them using the underlying geometry?

Thank you!