Pentagon functions for one-mass scattering amplitudes

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based on work in collaboration with

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Precision physics at hadron colliders

- Theoretical predictions (QCD corrections) at Next-to-Next-to-Leading-Order are required nowadays

$$\sigma = \sigma_{\rm LO} + \alpha_s \sigma_{\rm NLO} + \underbrace{\alpha_s^2 \sigma_{\rm NNLO}}_{\approx 1-10\%} + \mathcal{O}(\alpha_s^3)$$

- $2 \to 1$ and $2 \to 2$ extensively studied @NNLO

- Great interest in QCD corrections @NNLO for $2 \to 3$ production

$$pp \to V + 2j\,,\; pp \to VV' + j\,,\; pp \to H + 2j\,,\; pp \to V + b\bar{b}\,,\; pp \to t\bar{t} + j$$
$$pp \to t\bar{t} + \gamma\,,\; pp \to t\bar{t} + W\,,\; pp \to t\bar{t} + Z\,,\; pp \to t\bar{t} + H\,,\; pp \to \gamma\gamma\gamma$$

[from Les Houches 2021 wish list]

- Double-virtual corrections is an essential ingredient of NNLO calculations $\Rightarrow$ Two-loop five-particle scattering amplitudes

Analytic, algebraic, numeric complexity of scattering amplitudes and Feynman integrals

Rapid growth of complexity with

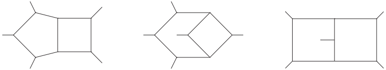
- # loops
- # legs
- # scales (also internal and external masses)

Efficient computational tools for complex multi-scale amplitudes are needed

Huge recent progress in calculation of five-particle two-loop amplitudes and four-particle three-loop amplitudes

Five-particle massless scattering @NNLO

- Feynman integrals



[Gehrmann, Henn, Lo Presti '15] [DCh, Gehrmann, Henn, Lo Presti, Mitev, Wasser '18]

[Abreu, Page, Zeng '18][Abreu, Dixon, Herrmann, Page, Zeng '18]

[DCh, Gehrmann, Henn, Wasser, Zhang, Zoia '18]

- **Basis of special functions**

[DCh, Sotnikov '20][Gehrmann, Henn, Lo Presti '18]

`gitlab.com/pentagon-functions/`

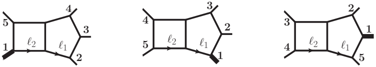
`pentagonfunctions.hepforge.org`

- QCD amplitudes (two-loop, planar and non-planar)
  - $q\bar{q} \to \gamma\gamma\gamma$ planar [Abreu, Page, Pascual, Sotnikov '20][Chawdhry, Czakon, Mitov, Poncelet '20]
  - $q\bar{q} \to g\gamma\gamma$, $qg \to q\gamma\gamma$ full color [Agarwal, Buccioni, von Manteuffel, Tancredi '21]
  - $gg \to g\gamma\gamma$ full color [Badger, Brönnum-Hansen, DCh, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]
  - $gg \to ggg$, $qg \to qgg, \ldots$ all planar five-parton [Abreu, Febres Cordero, Ita, Page, Sotnikov '21]

- QCD cross-sections @NNLO leading color
  - $pp \to \gamma\gamma\gamma$ [Chawdhry, Czakon, Mitov, Poncelet '19][Kallweit, Sotnikov, Wiesemann '20]
  - $pp \to j\gamma\gamma$ [Chawdhry, Czakon, Mitov, Poncelet '21][Badger, Gehrmann, Marcoli, Moodie '21]
  - $pp \to jjj$ [Czakon, Mitov, Poncelet '21][Chen, Gehrmann, Glover, Huss, Marcoli '22]

Planar five-particle one-mass scattering @NNLO

- Feynman integrals



[Abreu, Ita, Moriello, Page, Tschernow, Zeng '20][Canko, Papadopoulos, Syrrakos '20]

[Papadopoulos, Tommasini, Wever '15]

- **Basis of special functions** [DCh, Sotnikov, Zoia '21] gitlab.com/pentagon-functions/

- QCD amplitude for electroweak processes @leading color (two-loop, planar)

  $u\bar{d} \to W^+ b\bar{b}$, $pp \to b\bar{b}H$ [Badger, Hartanto, Zoia '21][Badger,Hartanto,Kryś,Zoia '21]

  $pp \to W(\to \ell\nu)\gamma + j$ [Badger,Hartanto,Kryś,Zoia '22]

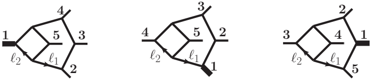
  $4p + W$ planar: $gg \to q\bar{q} + W(\to \ell\bar{\ell})$, $Q\bar{Q} \to q\bar{q} + W(\to \ell\bar{\ell})$
  and also planar Z/γ^* [Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov '21]

- QCD corrections @NNLO leading color

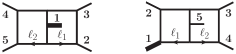
  $pp \to Wb\bar{b}$ [Hartanto, Poncelet, Popescu, Zoia '22]

Non-planar five-particle one-mass scattering @NNLO

- Feynman integrals



[Abreu, Ita, Page, Tschernow '21][Kardos, Papadopoulos, Smirnov, Syrrakos, Wever '22]



- **Basis of special functions**

? QCD amplitudes for electroweak processes

$4p + Z/\gamma^*$ @leading color: two-loop, planar and **nonplanar**

? QCD corrections @NNLO for electroweak production

Finite remainders of amplitudes in the pentagon function basis

$$\mathcal{R} = \sum_a \underbrace{r_a(X)}_{\substack{\text{rational}\\\text{coefficients}}} \underbrace{\text{mon}_a(\mathcal{F})}_{\substack{\text{special}\\\text{functions}}}$$

[depend on QFT,
type of scattered
particles and helicities]

[universal]

Basis of special functions $\mathcal{F} := \{f_i^{(w)}(X)\}$ for five-particle scattering

Analytics

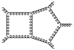
- compact expressions
- manifest analytic properties
- analytic cancellation of IR/UV poles
- avoid/control spurious cancellations
- embedded in the most efficient calculation strategies of QCD amplitudes

Numerics

- `C++` implementation
- fast
- high precision
- stable across phase space

Pentagon function basis is a prerequisite for efficient rational reconstruction of amplitudes

Calculation workflow

$$\mathcal{A}^{(2)}(X,\epsilon) = \sum$$ 

$$\xrightarrow{\text{reduction}} \quad \sum_i \int d^D\ell \, \frac{\mathcal{N}_i(\ell)}{\prod_j \rho_j(\ell)}$$

$$\Big\downarrow \ \text{IBP}$$

$$\sum_i \underbrace{c_i(X,\epsilon)}_{\text{rational}} \underbrace{\mathcal{I}_i(X,\epsilon)}_{\text{Master Integrals}}$$

$$\xleftarrow{\text{to funct basis}}$$

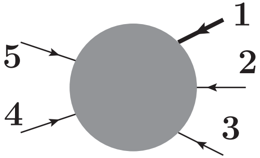
$$\sum_{w\geq -4} \epsilon^w \sum_a \underbrace{d_{a,w}(X)}_{\text{rational}} \mathrm{mon}_a(\mathcal{F})$$

$$\Big\downarrow \ \text{IR/UV subtraction}$$

$$\mathcal{R}^{(2)}(X) = \sum_a \underbrace{r_a(X)}_{\text{rational}} \mathrm{mon}_a(\mathcal{F})$$

Modular arithmetics $\mathbb{F}_p$ and rational reconstructions [von Manteuffel,Schabinger '14] [Peraro '16 '19] helps to bypass complexity of intermediate steps

Kinematics of the five-particle one-mass scattering



Four light-like and one massive momenta

$$p_1^2 > 0\,, \quad p_2^2 = p_3^2 = p_4^2 = p_5^2 = 0\,, \quad \sum_i p_i = 0$$

Six independent Mandelstam variables $s_{ij} := (p_i + p_j)^2$,

$$X := \left(p_1^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\right)$$

and a parity-odd invariant

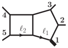
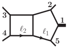
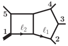
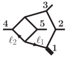
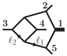
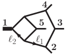
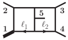
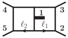
$$4i\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta = \pm\sqrt{\Delta_5}$$

Eight families of one-mass five-particle two-loop Feynman integrals

Feynman integrals : eight propagators and three ISP

$$I_{\vec{\nu}}(X,\epsilon) = \int \frac{d^D \ell_1}{i\pi^{\frac{D}{2}}} \int \frac{d^D \ell_2}{i\pi^{\frac{D}{2}}} \frac{\rho_9^{-\nu_9} \rho_{10}^{-\nu_{10}} \rho_{11}^{-\nu_{11}}}{\rho_1^{\nu_1} \rho_2^{\nu_2} \dots \rho_8^{\nu_8}}\,, \qquad \begin{array}{l} \nu_1, \nu_2, \dots, \nu_8 \in \mathbb{Z}\,, \\ \nu_9, \nu_{10}, \nu_{11} \in \mathbb{Z}_{\leq 0} \end{array}$$

`Laporta algorithm` ['00] solves the Integration-by-Part identities (IBP) and finds a finite basis of **Master Integrals**

								
# MI	74	75	86	86	86	135	142	179
top sector	3	3	3	3	3	3	8	9

Efficient implementation in the finite-field framework `FiniteFlow`, `FIRE6`, `Kira 2.0` [von Manteuffel,Schabinger '14; Peraro '16 '19; Smirnov,Chukharev '19; Klappert,Lange,Maierhöfer,Usovitsch '20]

Polylogarithmic alphabet for all two-loop families of five-particle one-mass integrals

204-letter **alphabet**

$$\{W_i = W_i(X)\}_{i=1}^{204}$$

is closed under 4! permutations $\sigma \in \mathcal{S}_4$ of the massless momenta

$$\sigma(d\log(W_i)) \in \langle d\log(W_1), \ldots, d\log(W_{204})\rangle_{\mathbb{Q}}$$

93 letters of the alphabet are linear or quadratic in Mandelstam variables, e.g.

$$W_1 = p_1^2\,,\;\; W_2 = s_{12}\,, \ldots,\; W_{16} = s_{15} - s_{34}\,, \ldots$$
$$W_{28} = s_{12}s_{15} - p_1^2 s_{34}\,, \ldots,\; W_{70} = s_{12}s_{15} - s_{12}s_{23} - p_1^2 s_{34}\,, \ldots$$

and the remaining letters are more complicated and involve square roots

$1 + 3 + 6 = 10$ square roots in the alphabet

Analytic solution of the Master Integrals in terms of the iterated integrals

- DE is a powerful method to solve **Master Integrals** analytically

- DE takes the *canonical form* for a natural choice of **Master Integrals**

[Henn '13]

- **Easy problem:** formally solve DE in terms of the **iterated integrals**, i.e. iterated integrations of the $d\log$ forms

$$[W_{i_1}, \ldots, W_{i_w}](X) := \int_\gamma [W_{i_1}, \ldots, W_{i_{w-1}}]\, d\log(W_{i_w})$$

along a path γ linking point X and a reference point X_0

- **Difficult problem:** finding the **canonical basis** of **Master Integrals**

Algorithmic construction of the pentagon function basis

- Iterated integrals form Shuffle algebra, and they are graded by the transcendental weight

- Pentagon functions : algebraically independent set of iterated integrals $\mathcal{F} := \{f_i^{(w)}\}$

weight	P ∪ PB	HB	DP	Total
1	11	0	0	11
2	25	10	0	35
3	145	72	0	217
4	675	305	48	1028

- Series expansion coefficients of the **canonical Master Integrals** in dim reg

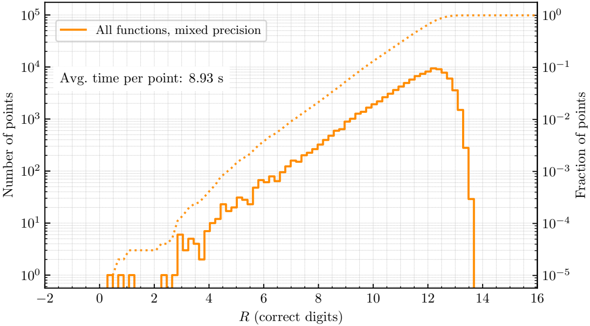
$$\mathbf{g}_\sigma(X,\epsilon) = \sum_{w\geq 0} \epsilon^w \mathbf{g}_\sigma^{(w)}(X)\;, \qquad \sigma \in \mathcal{S}_4$$

  are pure weight-w functions (linear combinations of iterated integrals)

$$\mathbf{g}_\sigma^{(w)}(X) = \text{weight-}w \text{ polynomial in } \mathcal{F} := \{f_i^{(w)}\},\, \zeta_2,\, \zeta_3$$

Efficient numerical evaluation of the one-mass pentagon functions

Implemented in the public `C++` library `PentagonFunctions++` and ready for phenomenological applications



✓ Stable

✓ Fast

✓ Precise

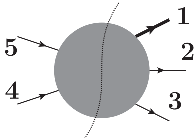
Physical simulation of 10^5 phase-space points evaluated in double precision

$$R(X) := \min_{f \in \mathcal{F}_{\text{planar}}} \left(-\log_{10} \left| \frac{f(X)_{\text{double}}}{f(X)_{\text{exact}}} - 1 \right| \right)$$

Evaluations of planar pentagon functions are much faster, $\approx 0.22s$ per point

Permutations of the Master Integrals vs Analytic continuation

Define pentagon functions $\mathcal{F} := \{f_i^{(w)}\}$ in the region $45 \to \mathbf{1}23$



Master integrals/amplitudes/finite remainders in any scattering region with massive production are expressible in the same basis $\mathcal{F}$,

$$\mathbf{g}(\underbrace{X}_{\text{region } \sigma_4\sigma_5\to\mathbf{1}\sigma_2\sigma_3}) \;=\; \mathbf{g}_\sigma\Big(X' := \underbrace{\sigma^{-1}\circ X}_{\text{region } 45\to\mathbf{1}23}\Big)$$

Permutations of the Master Integrals vs Analytic continuation

Difficult!		**Easy and Automatized!**
Analytically continue from $45 \to \mathbf{123}$ to all scattering regions	$\Leftrightarrow$	Solve DE in 4! permutations

Moreover, the pentagon functions are closed upon permutations

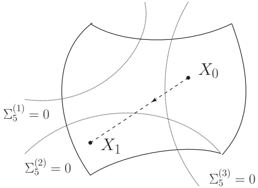
$$\sigma\left(f_i^{(w)}\right) = \text{weight-}w \text{ polynomial in } \mathcal{F} := \{f_i^{(w)}\},\, \zeta_2,\, \zeta_3$$

Finite remainders of amplitudes calculated in the region $45 \to \mathbf{123}$

$$\mathcal{R}(X) = \sum_a r_a(X)\, \mathrm{mon}_a(\mathcal{F})(X)$$

are automatically transferred to all regions

Singularities of the non-planar pentagon functions inside the scattering region



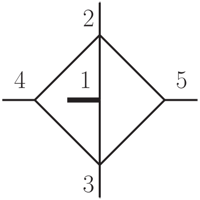
Six square-roots (related by $\mathcal{S}_4$ permutations)

$$\Sigma_5^{(1)}, \Sigma_5^{(2)}, \ldots, \Sigma_5^{(6)}$$

are singular surfaces

$$\Sigma_5^{(1)} := (s_{12}s_{15} - s_{12}s_{23} - s_{15}s_{45} + s_{34}s_{45} + s_{23}s_{34})^2 - 4 s_{23}s_{34}s_{45}(s_{34} - s_{12} - s_{15})$$

Example:



$$\sim \frac{1}{\sqrt{\Sigma_5^{(3)}}}\left[\frac{1}{\epsilon^2} f^{(2)} + \frac{1}{\epsilon} f^{(3)} + \mathcal{O}(\epsilon^0)\right]$$

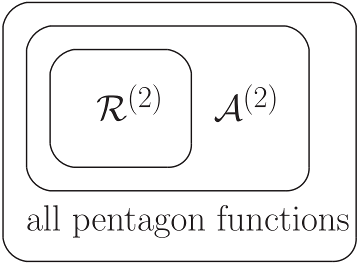
pentagon functions $f^{(w)}$ are singular at $\Sigma_5^{(3)} \sim 0$,

$$f^{(2)} \sim -4\pi^2 \,, \quad f^{(3)} \sim 256\pi^2 \log\left(\Sigma_5^{(3)}\right)$$

A subset of the pentagon functions is required for phenomenological applications

- Some alphabet letters drop out from the **amplitudes**

$$\sum_{w=0}^{4} \epsilon^{-4+w} \mathcal{A}^{(2)}_{[w]}(X) + \mathcal{O}(\epsilon)$$



- and further on some more letters drop out from the **finite remainders**

$$\mathcal{R}^{(2)} = \mathcal{A}^{(2)} - \mathbf{I}^{(1)}\mathcal{A}^{(1)} - \mathbf{I}^{(2)}\mathcal{A}^{(0)} + \mathcal{O}(\epsilon)$$

- We expect that seven square-root letters

$$\Delta_5,\, \Sigma_5^{(1)}, \ldots, \Sigma_5^{(6)}$$

are artifacts of dim-reg

Conclusions

Pentagon Functions

- Basis of transcendental functions to describe five-particle scattering
- Transparent analytic properties and efficient numerical implementation

  `gitlab.com/pentagon-functions/`
- A tool to facilitate QCD calculations @NNLO
- Successfully applied in numerous calculations

…and more generally, the approach is very efficient for multi-scale processes with many square roots

Backup slides

Differential equations for Master Integrals

- IBP identities provide differential equations for Master Integrals

$$\partial_{s_{12}} I_{\vec{\nu}} = \sum_{\vec{\mu}} \underbrace{d_{\vec{\nu},\vec{\mu}}(X,\epsilon)}_{\text{rational}} I_{\vec{\mu}} \underset{\text{reduction}}{\overset{\text{IBP}}{\Rightarrow}} \partial_{s_{12}} \mathcal{I}_i = \sum_j M_{ij}(X,\epsilon)\mathcal{I}_j$$

[Kotikov '91] [Bern,Dixon,Kosower '94][Remiddi '97][Gehrmann,Remiddi '00]

- DE is a powerful method to obtain analytic expressions for Master Integrals and to set up their numerical evaluations

- Huge arbitrariness in the choice of Master Integrals

$$\{\mathcal{I}_i\} \sim \{\mathcal{I}'_i \equiv \sum_j T_{ij}(X,\epsilon)\mathcal{I}_j\}$$

Canonical DE and pure basis of Master Integrals

- DE takes the *canonical form* for a natural choice of Master Integrals

[Henn '13]

$$\begin{cases} d\vec{\mathbf{g}} = \epsilon M \vec{\mathbf{g}} \\ M := \sum_i M_i \, d\log(W_i) \end{cases}$$

constant matrices (M_i) — alphabet letters (W_i)

$W_i = W_i(X)$

- Canonical DE is easy to solve

$$\vec{\mathbf{g}}(X,\epsilon) = \text{Pexp}\left(\epsilon \int_\gamma M\right) \underbrace{\vec{\mathbf{g}}(X_0,\epsilon)}_{\text{initial values}} \, ,$$

path γ links X_0 and X

- The difficult task is to find the basis $\vec{\mathbf{g}}$
  - Leading singularity analysis of the integrands [Arkani-Hamed,Bourjaily,Cachazo,Trnka '10]
  - Educated guess and semi-numerics

$$(\vec{c}_{\text{rnd}} \cdot \nabla_X)\, \vec{\mathbf{g}} \,\Big|_{X=X_{\text{rnd}}} \overset{\text{IBP}}{=} \epsilon \cdot M(X_{\text{rnd}}) \vec{\mathbf{g}}$$

Iterated integrals

Iterated integration of the $d\log$ forms along path γ linking points X_0 and X

$$[W_{i_1}, \ldots, W_{i_w}](X) := \int_\gamma [W_{i_1}, \ldots, W_{i_{w-1}}]\, d\log(W_{i_w})$$

Iterated integrals

- are linear independent

$$[W_{\alpha_1}, \ldots, W_{\alpha_w}] = [W_{\beta_1}, \ldots, W_{\beta_{w'}}] \;\Rightarrow\; w = w', \vec{\alpha} = \vec{\beta}$$

- satisfy the shuffle algebra

$$[W_{\alpha_1}, \ldots, W_{\alpha_w}][W_{\beta_1}, \ldots, W_{\beta_{w'}}] = \sum_{\vec{\gamma} \in \vec{\alpha} \sqcup\!\sqcup \vec{\beta}} [W_{\gamma_1}, \ldots, W_{\gamma_{w+w'}}]$$

- graded by the transcendental weight-w

Iterated integral solution of the canonical DE

Expansion coefficients of the Master Integrals in dim reg ϵ parameter

$$\vec{\mathbf{g}}(X,\epsilon) = \sum_{w \geq 0} \epsilon^w \vec{\mathbf{g}}^{(w)}(X)$$

are pure weight-w function (linear combinations of iterated integrals)

$$\vec{\mathbf{g}}^{(w)}(X) = \sum_{0 \leq w' \leq w} \underbrace{[W_{i_1}, W_{i_2}, \ldots, W_{i_{w'}}](X)}_{\text{iterated integral}} \underbrace{M_{i_1} M_{i_2} \ldots M_{i_{w'}}}_{\substack{\text{constant} \\ \text{rational} \\ \text{matrix}}} \underbrace{\vec{\mathbf{g}}^{(w-w')}(X_0)}_{\substack{\text{transcendental} \\ \text{numbers}}}$$

For example,

$$\mathbf{g}^{(0)} \in \mathbb{Q}\,,$$

$$\mathbf{g}^{(1)} \sim [W_i] + v^{(1)}\,,$$

$$\mathbf{g}^{(2)} \sim [W_i, W_j] + v^{(1)}[W_k] + v^{(2)}\,, \ldots$$

with transcendental numbers $v^{(1)}, v^{(2)}, \ldots$

Algorithmic construction of the pentagon function basis

Solve 4! permutations $\sigma \in \mathcal{S}_4$ of the Master Integrals, $\mathbf{g}_\sigma(X) := \mathbf{g}(\sigma \circ X)$

$$\begin{cases} d\vec{\mathbf{g}}_\sigma = \epsilon M_\sigma \vec{\mathbf{g}}_\sigma \\ \vec{\mathbf{g}}_\sigma = \sum_{w \geq 0} \epsilon^w \vec{\mathbf{g}}_\sigma^{(w)} \end{cases} \underset{\text{solutions}}{\overset{\text{iterated integral}}{\Rightarrow}} \{\vec{\mathbf{g}}_\sigma^{(w)}\} \underset{\text{iterated integrals}}{\overset{\text{basis of irreducible}}{\Rightarrow}} \mathcal{F} := \{f_i^{(w)}\}$$

$\{f_1^{(w)}, f_2^{(w)}, \ldots\} =$ Linear independent $\{\vec{\mathbf{g}}_\sigma^{(w)}\}$ modulo products of lower weigths $\{f_i^{(w')}\}_{w'<w}$

weight	P ∪ PB	HB	DP	Total
1	11	0	0	11
2	25	10	0	35
3	145	72	0	217
4	675	305	48	1028

All Master Integrals are decomposible in the pentagon function basis

Initial values of the canonical DE and pentagon function basis

- Initial values of the DE have to taken into account
- We choose $X_0 = (1, 3, 2, -2, 7, -2)$ and evaluate $\vec{\mathbf{g}}_\sigma^{(w)}(X_0)$ **numerically** with `AMFlow` [Liu, Ma '22] with 60-digit precision
- Is it enough for identifying all linear relations among iterated integrals?

$$\mathbf{g}^{(2)} \sim [W_i, W_j] + v^{(1)}[W_k] + v^{(2)}$$

and numbers $v^{(1)}$ and $v^{(1)}$ are evaluated numerically

- Yes! We find **exact** relations among initial values and perform decomposition in the pentagon function basis simultaneously. Recursively in weight w,

$$\vec{\mathbf{g}}_\sigma^{(w)}(X) = \text{weight-}w\text{ polynomial in } \{f_i^{(w')}\}_{w' \le w},\, \zeta_2,\, \zeta_3$$

For example,

$$\mathbf{g}^{(2)} = \sum_i a_i f_i^{(2)} + \sum_{i,j} a_{ij} f_i^{(1)} f_j^{(1)} + \tilde{a}\,\zeta_2 \quad \Longrightarrow \quad \text{exact} \quad a_i, a_{ij}, \tilde{a} \in \mathbb{Q}$$

From iterated integrals to explicit representation of the pentagon functions

Only pentagon functions $\{f_i^{(w)}\}$ up to weight $w \leq 4$ are required for NNLO

$$\{W_i = W_i(X)\}_{i=1}^{204}$$

No branch cuts in the physical scattering channel $45 \to 123$

- weight-1 and weight-2: logarithmic and dilogarithmic functions, e.g.

$$f^{(1)}(X) \sim \log\left(p_1^2\right),\ \log\left(-s_{34}\right), \ldots$$
$$f^{(2)}(X) \sim \text{Li}_2\left(1 - \frac{s_{12}s_{15}}{p_1^2 s_{34}}\right), \ldots$$

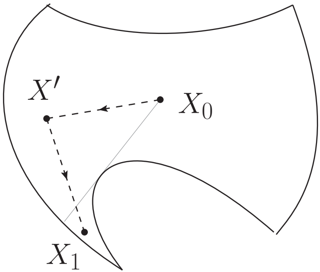
- weight-3 and weight-4: one-fold integration along a path γ linking X_0 and X

$$f^{(3)}(X) \sim \sum_i \int_\gamma h_i^{(2)} d\log(W_i)$$
$$f^{(4)}(X) \sim \sum_i \int_\gamma h_{ij}^{(2)} \log(W_j) d\log(W_i)$$

where $h_i^{(2)}$ and $h_{ij}^{(2)}$ are weight-2 polynomial in pentagon functions

Nontrivial geometry of the physical scattering region

The physical scattering region is "curvy" owing to $\Delta_5 < 0$



Integration along polygonal chain (two segments are sufficient)

$$\gamma = [X_0, X', X_1]$$

The region is **not star-shaped** owing to massive $p_1^2 > 0$: there is no X_0 which "sees" all other phase-space points

Only $\sim 3\%$ of the region is not visible from $X_0 = (1, 3, 2, -2, 7, 2)$

Refinement of the pentagon function basis

Huge arbitrariness in the choice of the pentagon function basis.

We order pentagon functions by their "simplicity":

- planar vs non-planar corrections

$$\text{planar} \prec \text{non-planar}$$

- ordering of the Feynman integral families

$$\text{PB} \prec \text{HB} \prec \text{DP}$$

- Some letters of the alphabet could appear only in Feynman integrals but they are irrelevant for physics!

$$\text{physical letters} \prec \text{non-physical letters}$$

- Numerical integration of some one-fold integrals is easier than others

$$\text{no spurious singularities} \prec \text{spurious singularities}$$

Spurious singularities of the planar pentagon functions

- We want to define the pentagon functions unambiguously in the physical region $45 \to 123$, i.e. no branch cuts
- Then we need to inspect **positivity** of the alphabet letters,

$$f^{(3)}(X) \sim \sum_i \int_\gamma h_i^{(2)} d\log(\underbrace{W_i}_{\text{can it vanish?}})$$

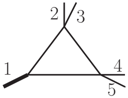
- Twenty letters of the planar alphabet can vanish in the physical region, e.g.

$$W_{72} := s_{12}s_{23} - p_1^2 s_{34} + s_{12}s_{34} \gtrless 0$$

but the pole in $f^{(3)}$ is always compensated, e.g.

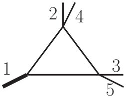
$$h_{72}^{(2)}\Big|_{W_{72}=0} = 0$$

Square roots of the alphabet (Massive Triangles)



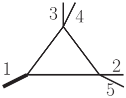
$$\frac{1}{\sqrt{\Delta_3^{(1)}}} \times \text{pure}$$

$$\Delta_3^{(1)} := \lambda(p_1^2, s_{23}, s_{45})$$



$$\frac{1}{\sqrt{\Delta_3^{(2)}}} \times \text{pure}$$

$$\Delta_3^{(2)} := \lambda(p_1^2, s_{24}, s_{35})$$



$$\frac{1}{\sqrt{\Delta_3^{(3)}}} \times \text{pure}$$

$$\Delta_3^{(3)} := \lambda(p_1^2, s_{34}, s_{25})$$

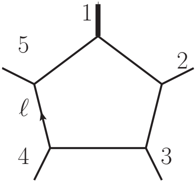
with Källen function $\lambda(a, b, c) := a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$

Three square roots upon $\mathcal{S}_4$ permutations

$$\sqrt{\Delta_3^{(1)}}, \sqrt{\Delta_3^{(2)}}, \sqrt{\Delta_3^{(3)}}$$

Square roots of the alphabet (Parity-Odd)

Dimensionally reduced $D = 6 - 2\epsilon \to D = 4 - 2\epsilon$ one-loop pentagon integral

$\mathcal{N}(\ell) \otimes$ 

$= \epsilon \frac{1}{\sqrt{\Delta_5}} \times \text{pure} + \mathcal{O}(\epsilon^2)$

with numerator $\mathcal{N}(\ell) = \ell^2 - (\ell_{D=4})^2$

Quartic polynomial in the Mandelstam variables

$$\Delta_5 := \det ||2p_i \cdot p_j||_{i,j=1}^4$$

is related to the parity of the kinematic configuration

$$4i\epsilon_{\alpha\beta\gamma\delta} p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta = \pm\sqrt{\Delta_5}$$

and invariant upon $\mathcal{S}_4$ permutations

$$\sigma(\Delta_5) = \Delta_5$$

Odd letters of the five-particle one-mass alphabet

Assign $\mathbb{Z}_2$-charge $\{+,-\}$ to each of the ten square roots

$$\sqrt{\Delta_5}, \sqrt{\Delta_3^{(1)}}, \sqrt{\Delta_3^{(2)}}, \sqrt{\Delta_3^{(3)}}, \sqrt{\Sigma_5^{(1)}}, \sqrt{\Sigma_5^{(2)}}, \ldots, \sqrt{\Sigma_5^{(6)}}$$

and classify letters according to their $(\mathbb{Z}_2)^{10}$-charge

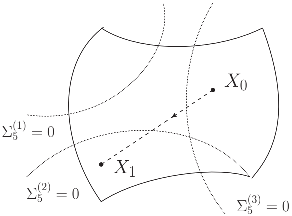
[Abreu, Ita, Moriello, Page, Tschernow, Zeng '20][Abreu, Ita, Page, Tschernow '21]

Alphabet letters with nontrivial charge

	$\frac{a_i-\sqrt{\Delta_5}}{a_i+\sqrt{\Delta_5}}$	$\frac{b_{ik}-\sqrt{\Delta_3^{(i)}}}{b_{ik}+\sqrt{\Delta_3^{(i)}}}$	$\frac{c_{ik}-\sqrt{\Sigma_5^{(i)}}}{c_{ik}+\sqrt{\Sigma_5^{(i)}}}$	$\frac{d_{ik}-\sqrt{\Delta_5}\sqrt{\Delta_3^{(i)}}}{d_{ik}+\sqrt{\Delta_5}\sqrt{\Delta_3^{(i)}}}$	$\frac{e_{ik}-\sqrt{\Delta_5}\sqrt{\Sigma_6^{(i)}}}{e_{ik}+\sqrt{\Delta_5}\sqrt{\Sigma_6^{(i)}}}$
# letter	32	12×3	24×6	1×3	1×6

with polynomial a, b, c, d, e in X

Singularities of the non-planar pentagon functions inside the scattering region



Additional spurious surfaces of the nonplanar families: $\Sigma_5^{(i)} \gtrless 0$ in the scattering region

When the integration path γ crosses $\Sigma_5^{(i)} = 0$ surface

$$\gamma^* \left(\Sigma_5^{(i)} \right)(t) = \mathcal{O}(t - t_0)$$

some of the weight-3 pentagon functions are singular or not smooth,

$$\int_\gamma h_i^{(2)} \, d \log \left(\Sigma_5^{(i)} \right) \sim \int \frac{dt}{t - t_0} \gamma^* \left(h_i^{(2)} \right)(t)$$

depending on the behavior of the weight-2 pentagon functions $h_i^{(2)}$,

$\Sigma_5^{(3)}$: logarithmic divergence $\int \frac{dt}{t-t_0}$

other five $\Sigma_5^{(i)}$: integrable singularity $\int \frac{dt}{t-t_0} \sqrt{t - t_0}$

A subset of the pentagon functions is required for phenomenological applications

Observed for two-loop leading color **planar** amplitudes: $W + 4\,\text{partons}$ and planar contributions $Z/\gamma^* + 4\,\text{partons}$

[Badger, Hartanto, Zoia '21] [Badger, Hartanto, Kryś, Zoia '21] [Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov '21]

- Master integrals require 49 letters – penta-box family
- Six letters drop out from the amplitudes $\mathcal{A}^{(2)}_{[w]}$ with $w = 0, 1, \ldots, 4$
- Letter $\sqrt{\Delta_5}$ drops out from the finite remainders $\mathcal{R}^{(2)}$

Extremely interesting to inspect nonplanar corrections!

We expect that seven square-root letters

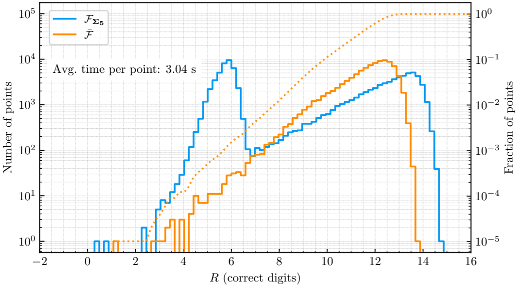
$$\sqrt{\Delta_5},\ \Sigma_5^{(1)}, \ldots, \Sigma_5^{(6)}$$

are NOT required for phenomenological applications (artifacts of dim-reg)

Efficient numerics for the complete set of one-mass pentagon functions

Added in the new version of `C++` library `PentagonFunctions++`

[Abreu, DCh, Ita, Page, Sotnikov, Tschernow, Zoia; to appear]



$\mathcal{F}$ – all pentagon functions

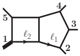
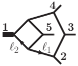
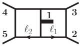
$$\mathcal{F} = \mathcal{F}_{\mathbf{\Sigma}_5} \bigcup \overline{\mathcal{F}}$$

$\mathcal{F}_{\mathbf{\Sigma}_5}$ – pentagon functions with letters $\Sigma_5^{(i)}$ (presumably irrelevant for phenomenology)

Also implemented a recovery system with quadruple precision for $\mathcal{F}_{\mathbf{\Sigma}_5}$ with paths crossing $\Sigma_5^{(i)} = 0$

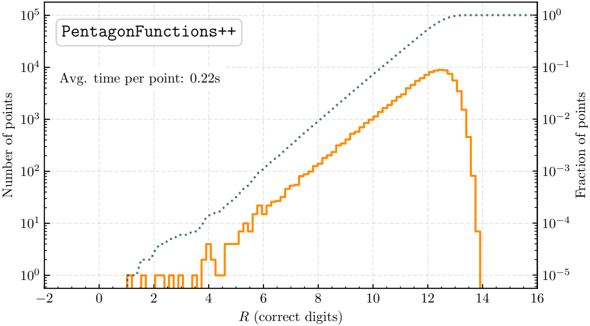
Performance is comparable with that for the planar subset and suits for phenomenology applications!

Available analytic results on one-mass five-particle two-loop Feynman integrals

	Penta-boxes	Hexa-boxes	Double-Pentagons
			
Canonical DE $d\vec{\mathbf{g}} = \epsilon M \vec{\mathbf{g}}$	[Abreu, Ita, Moriello, Page, Tschernow, Zeng ’20]	[Abreu, Ita, Page, Tschernow ’21]	*
Pentagon Functions $\{f_i^{(w)}\}$	[DCh, Sotnikov, Zoia ’21]	*	*
Multiple Polylogs	[Canko, Papadopoulos, Syrrakos ’20]	[Kardos, Papadopoulos, Smirnov, Syrrakos, Wever ’22] MPL and one-fold integrals	

Efficient numerical evaluation of the planar one-mass pentagon functions

Implemented in the public C++ library `PentagonFunctions++` and ready for phenomenological applications [DCh, Sotnikov, Zoia '21]



- ✓ Stable
- ✓ Fast
- ✓ Precise

Type	Correct digits	Timing (s)
dbl	12	0.22
quadr	28	159
octpl	60	1695

Calculated in double precision for 10^5 phase-space points

$$R(X) := \min_{f \in \mathcal{F}_{\text{planar}}} \left(-\log_{10} \left| \frac{f(X)_{\text{double}}}{f(X)_{\text{exact}}} - 1 \right| \right)$$