The amplituhedron, correlahedron and internal boundaries

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Based on
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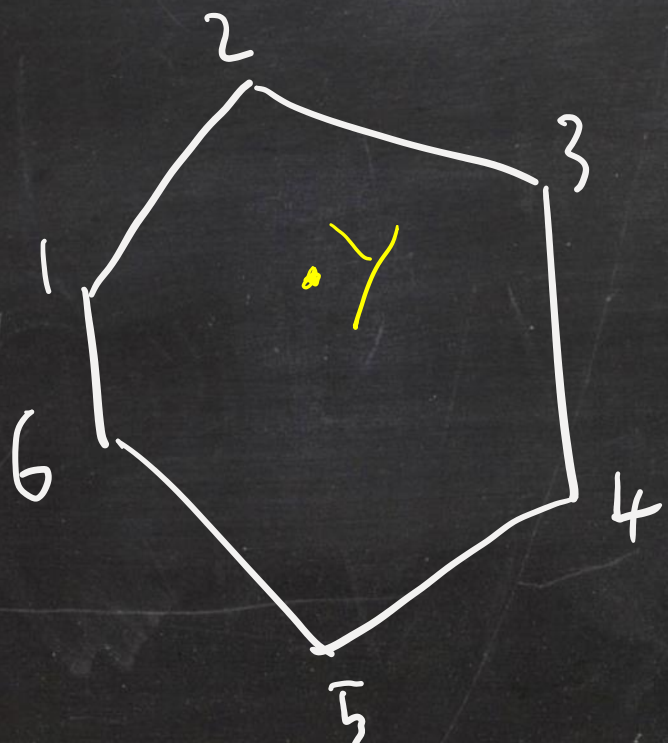
Outline

- Intro to amplituhedron (Beautiful mathematical geometry equivalent to planar loop amplitude integrands in N=4 SYM)
- Loop amplituhedron needs generalisation of previously understood concepts: Internal boundaries
- Weighted positive geometries (WPGs)
- Correlahedron / squared amplituhedron

The amplituhedron

- Toy model: polygons in $\mathbb{P}^2$, $Y, Z_i \in \mathbb{P}^2$



Y in polygon

Y to the right of line (i i+1)

$$\Downarrow$$

$$\langle Y\, i\; i{+}1 \rangle > 0$$

$\langle Y Z_i Z_{i+1} \rangle$

note

$$\langle Y\, 1\, i \rangle = (+,\ldots,+,-,\ldots,-)$$

$i=2,3,4,5$

Flips once from + to -   Flipping Number 1

Natural Generalization

[Arkani-Hamed, Trnka + Thomas]

Toy model
Polygons in P^2

"Tree Amplituhedron"

$A_{n,k,m}$

(physics $m=4$)

$Y \in P^2 = Gr(1,3) \longrightarrow Gr(k, k+m)$ k-planes in $\mathbb{R}^{k+m}$

$Z_i \in P^2 \longrightarrow P^{m+k-1}$ lines in $\mathbb{R}^{k+m}$

$\langle Y\, i\, i{+}1 \rangle > 0 \longrightarrow \langle Y\, i\, i{+}1\, j\, j{+}1 \rangle > 0$ $(m=4)$

$i = 1..n$

$\langle Y\, 1\, i \rangle$ has One sign flip $\longrightarrow$ $\langle Y\, 1\, 2\, 3\, i \rangle$ has k sign flips (max possible)

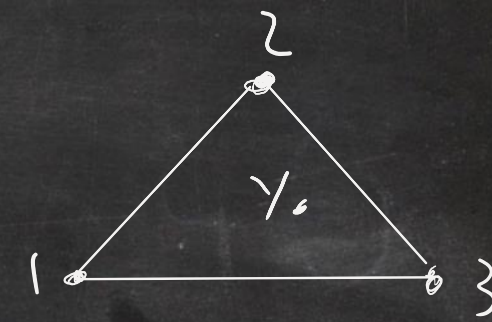
(Polygons in P^2 = $A_{n,1,2}$ amplituhedron)

Canonical Form

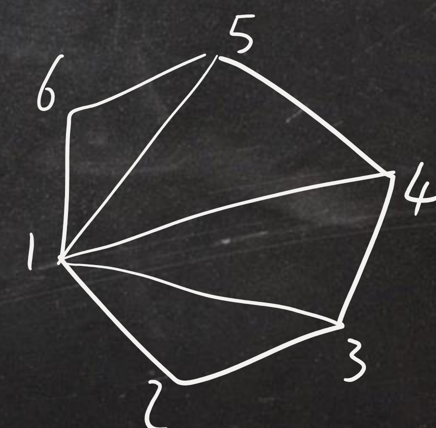
- Natural map from geometry to differential form: "canonical form"

- Eg Triangle: $A_{3,1,2}$

$$\Omega[\triangle] = [1,2,3] := \frac{\langle Y\, d^2Y\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$



- Polygon: $A_{n,1,2}$ $\Omega[\bigcirc]$ = sum of triangles $= \sum_{i=2}^{n-1} [1, i, i+1]$



Positive geometry = region possessing a canonical form

Amplitude = Canonical Form of (m=4) Amplituhedron

- Claim: The canonical form of the (n,k,4) amplituhedron is the n-point, $N^k MHV$, tree-level, planar scattering amplitude in N=4 SYM [Arkani-Hamed, Trnka]
  - Eg 2: $A_{5,1,4}$

$$[12345] = \frac{\langle Y d^4 Y\rangle \langle 12345\rangle^2}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}$$

  - $A_{n,1,4} = \sum_{i,j} [1, i, i+1, j, j+1]$

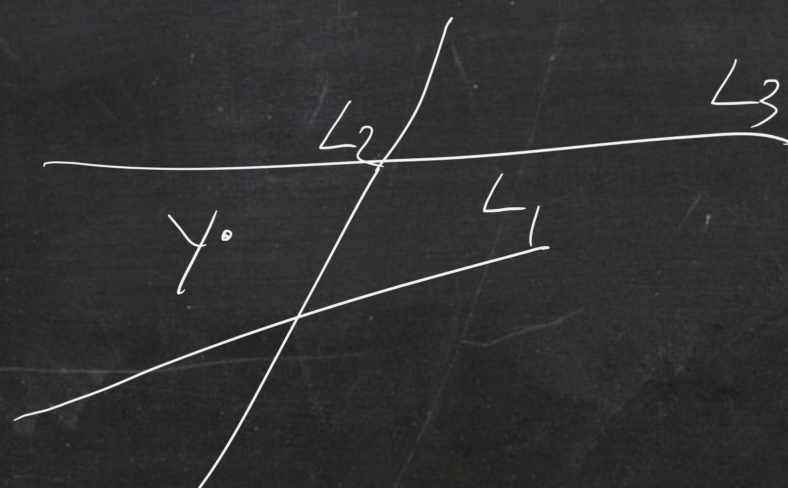
    n-point tree-level NMHV amplitude!

- Geometry with fewer sign flips? No sign flips?

Loops too!! (Integrand)

L-Loop amplituhedron $\mathcal{A}_{n,k,l,m}$ = amplituhedron Y and L "lines" (2-planes in (m+k)d) L_i, $i=1,\dots,L$ $\in Gr(2, m+k)$

in $\mathbb{P}^{m+k-1}$



$\langle Y L_i\, j\, j{+}1 \rangle > 0$

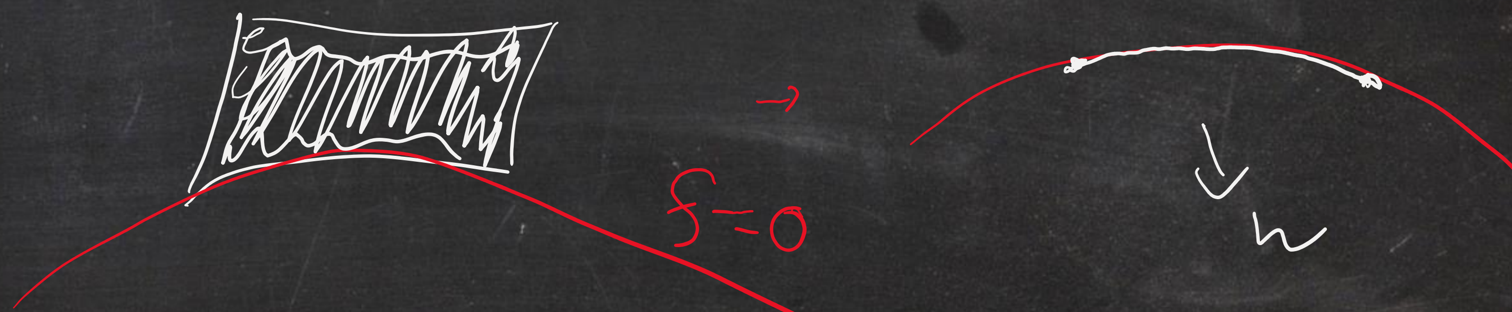
$\langle Y L_i L_j \rangle > 0$

loop flipping number.

Claim: canonical form of the (n.k.l,m=4) amplituhedron = planar, l-loop, N^kMHV, n-point amplitude (integrand) [Arkani-Hamed,Trnka]

Canonical form

- Canonical Form(and hence positive geometry) defined recursively via it's residues [Arkani-Hamed, Bai, Lam]
- Canonical form = volume form;
- Has simple poles only
- Location of simple poles = location of boundary components
- residue on these poles = canonical form of corresponding boundary cmpnts



boundary on f=0

⇓

Pole at f=0

$$\underset{f=0}{\mathrm{Res}}\,\Omega = \lim_{f\to 0} f\Omega = df \wedge \omega$$

canonical form of geometry on f=0 (pointing to ω)

(True for all boundary components)

- Eventually reach dimension 0 boundaries

- $\Omega(\cdot) = \pm 1$ (depending on orientation inherited from bulk)

$\Rightarrow$ **Maximal residues of positive geometries = +/-1**

Note:

Multiple residue = Residues of residue of residue ...

$\downarrow$

Boundary components of boundary components of boundary components....

Related comment: multiple residue not unique -> higher codimension boundaries not unique

- In general multiple residues depend on the order you take single residues
- Analogous statement true for boundaries
- Everything perfectly consistent!

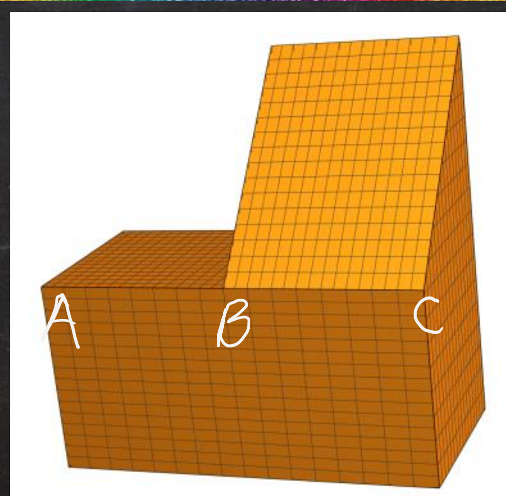
(Boundary component of)^k geometry

$\neq$

Codimension k boundary component of geometry

(not clear what interpretation of this is for k > 1)

Simple example



Codimension 2 boundary = [A,C]
Boundary of sloping roof boundary = [B,C]
Boundary of flat roof boundary = [A,B]

But:

Loop amplitude has max residues different from +/- 1 !!!

2 loop MHV:  (4 point)

eg.

$$\mathrm{MHV}(2) = \frac{\langle A_1B_1\mathrm{d}^2A_1\rangle\langle A_1B_1\mathrm{d}^2B_1\rangle\langle A_2B_2\mathrm{d}^2A_2\rangle\langle A_2B_2\mathrm{d}^2B_2\rangle\langle 1234\rangle^3}{\langle A_1B_1A_2B_2\rangle\langle A_1B_1 14\rangle\langle A_1B_1 12\rangle\langle A_2B_2 23\rangle\langle A_2B_2 34\rangle}\times$$

$$\times\left[\frac{1}{\langle A_1B_1 34\rangle\langle A_2B_2 12\rangle}+\frac{1}{\langle A_1B_1 23\rangle\langle A_2B_2 14\rangle}\right] \quad + \quad A_1B_1 \leftrightarrow A_2B_2\,.$$

Loop variables $L_i = A_i B_i$

- Parametrise 4×4 $Z=(Z_1Z_2Z_3Z_4)$ as identity and the loops as

$$\begin{pmatrix}A_i\\B_i\end{pmatrix}=\begin{pmatrix}1 & a_i & 0 & -b_i\\ 0 & c_i & 1 & d_i\end{pmatrix}.$$

Then (omitting the differential)

$$\mathrm{MHV}(2) = -\frac{a_2d_1+a_1d_2+b_2c_1+b_1c_2}{a_1a_2b_1b_2c_1c_2d_1d_2\left((a_1-a_2)(d_1-d_2)+(b_1-b_2)(c_1-c_2)\right)}$$

- Now we take the residues in $b_1 = 0, c_1 = 0, b_2 = 0, c_2 = 0$
- complicated pole factorises revealing new pole

$$-\frac{a_2 d_1 + a_1 d_2}{a_1 a_2 d_1 d_2 (a_1 - a_2)(d_1 - d_2)} .$$

- Now take the residue in a_1 at $a_1 = a_2$

$$-\frac{(d_1 + d_2)}{a_2 d_1 d_2 (d_1 - d_2)} .$$

- Now take residue in d_1 at $d_1 = d_2$,

max res=2

$$-\frac{2}{a_2 d_2} ,$$

Loop amplituhedron $\neq$ positive geometry!!??

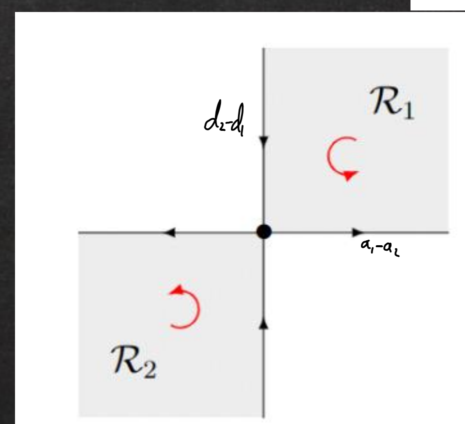
- Examine the above residues geometrically
- Start with amplituhedron. Carefully take boundaries corresponding to each of the above residues:

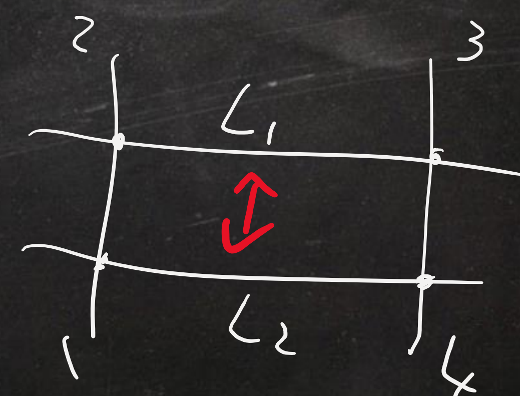
$$a_i > 0, \quad d_i > 0, \quad -(a_1 - a_2)(d_1 - d_2) > 0 \Rightarrow \Omega = -\frac{a_2 d_1 + a_1 d_2}{a_1 a_2 d_1 d_2 (a_1 - a_2)(d_1 - d_2)}$$

$$\parallel$$

$$\mathcal{R}_1 := \{a_1, a_2, d_1, d_2 \mid a_1 > a_2 > 0 \wedge d_2 > d_1 > 0\}$$
$$\mathcal{R}_2 := \{a_1, a_2, d_1, d_2 \mid a_2 > a_1 > 0 \wedge d_1 > d_2 > 0\}$$

R_1, R_2 same orientation





$d_2 - d_1 = 0$ boundary.

$a_1 - a_2$

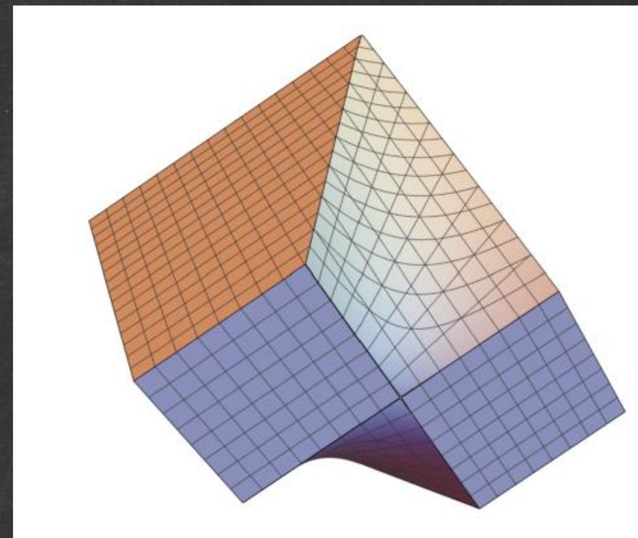
"internal boundary" separating two regions of opposite orientation (so not oriented which was an assumption of positive geometry!)

Previously unnoticed feature:

The (loop) amplituhedron contains internal boundaries!

Simple toy example:

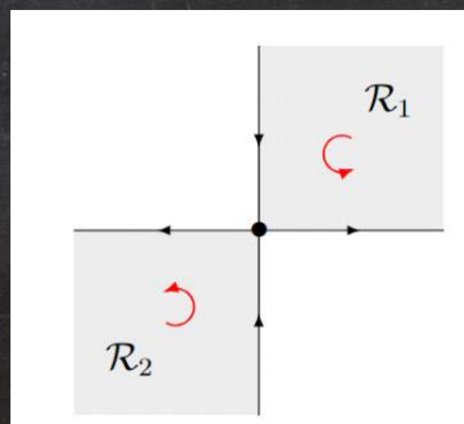
$xy + z > 0, \quad z > 0$



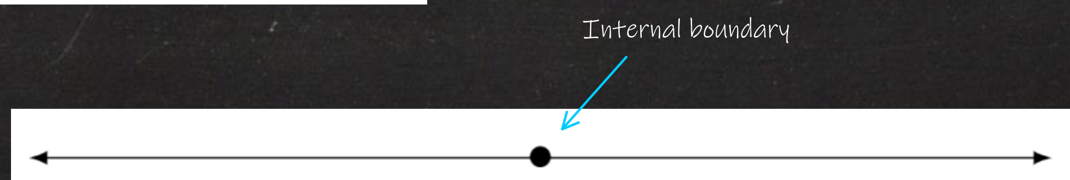
$\Omega = 2/(z(z+xy))$

Composite singularity [Arkani-Hamed, Cachazo, Cheung, and Kaplan]

$z=0$ boundary:



$y=0$ boundary:



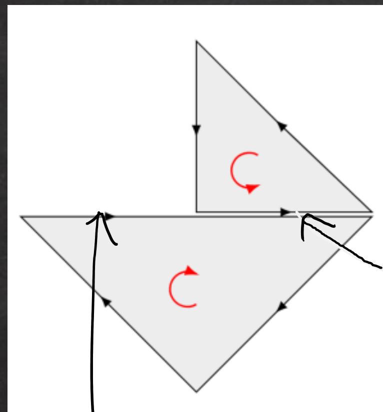
Need generalised positive geometry

Generalized canonical form recursive def:

$$\mathrm{Res}_f \Omega = \lim_{f \to 0} f\Omega = df \wedge (\omega_{\text{ext}} + 2\omega_{\text{int}})$$

canonical form of standard (external) boundary region

canonical form of internal boundary region eg.



$$\Omega(R_1) = \frac{dx\,dy}{xy(x+y-1)} + \frac{2dx\,dy}{y(x+y+1)(x-y-1)}$$

(just subtract the two triangles)

Internal boundary

External boundary

$$\underset{y=0}{\mathrm{Res}}\,\Omega = \lim_{y\to 0} y\,\Omega = dx\left(\frac{1}{x}-\frac{1}{x+1}\right) + 2dx\left(\frac{1}{x-1}-\frac{1}{x}\right) = \Omega([-1,0]) + 2\Omega([0,1])$$

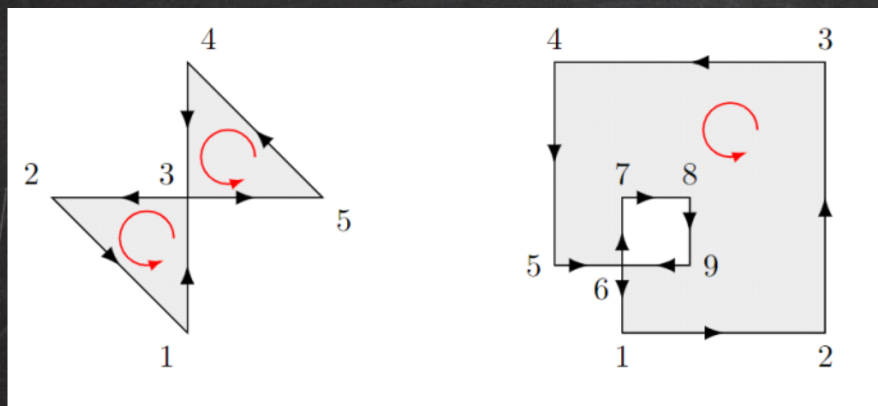
- Agrees with the formula

External boundary

Internal boundary

GPGs more complete than PGs: Anything that triangulates in GPGs is a GPG

- not true for positive geometries eg.



But suggests further generalisation:

Weighted Positive Geometry (WPG)

- Define geometry by a piecewise constant $\mathbb{Z}$-valued weight function w (and orientation form O)

Orientation = volume form $O \sim \lambda O$, $\lambda > 0$

Generalise: "weighted orientation"

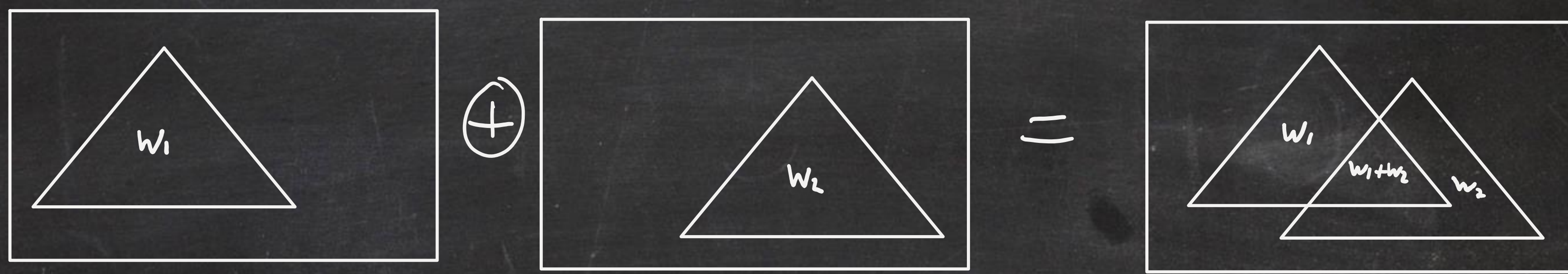
$$(w, O) \sim (\text{sign}(\lambda) w, \lambda O') \qquad \lambda \neq 0$$

Integer valued (w)

Volume form (O)

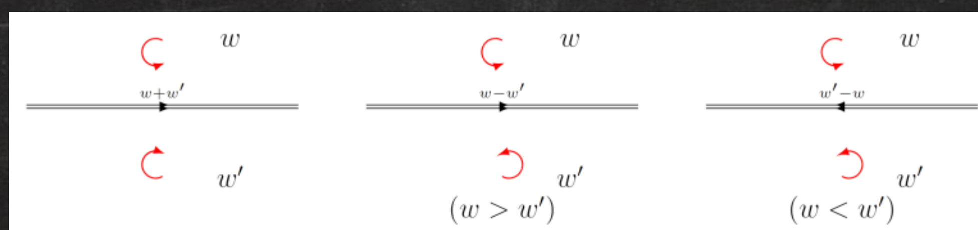
Weighted Positive Geometry (WPG):

- Natural additive structure (geometries can overlap - makes proofs easier!)
- $(w_1, O_1) \oplus (w_2, O_2) = (w_1 + \text{sign}(\lambda) w_2, O_1)$
- where $O_1 = \lambda O_2$



- Natural Projection operator onto boundaries (discontinuities):

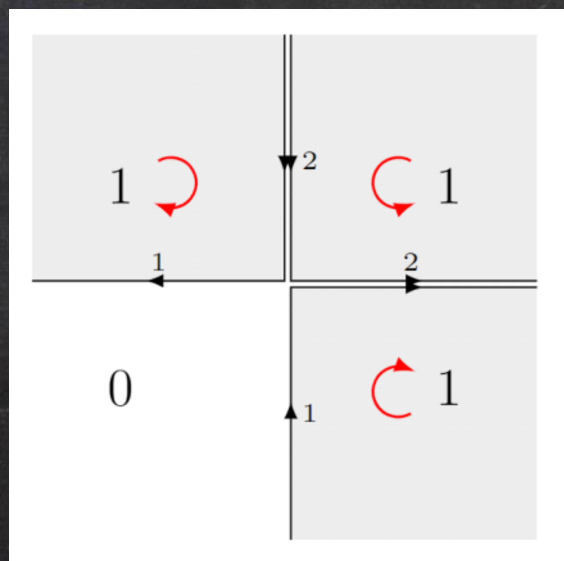
$$\Pi_C(w, O) = (w^+|_C, O^+|_C) \oplus (w^-|_C, O^-|_C)$$



Residue of canonical form is canonical form of the projection:

$$\mathrm{Res}_{\mathcal{C}}\Omega(w,O) = \Omega(\Pi_{\mathcal{C}}(w,O))$$

eg



$$\mathrm{Res}_{y=0,x=0}\Omega = -\mathrm{Res}_{x=0,y=0}\Omega = 3$$

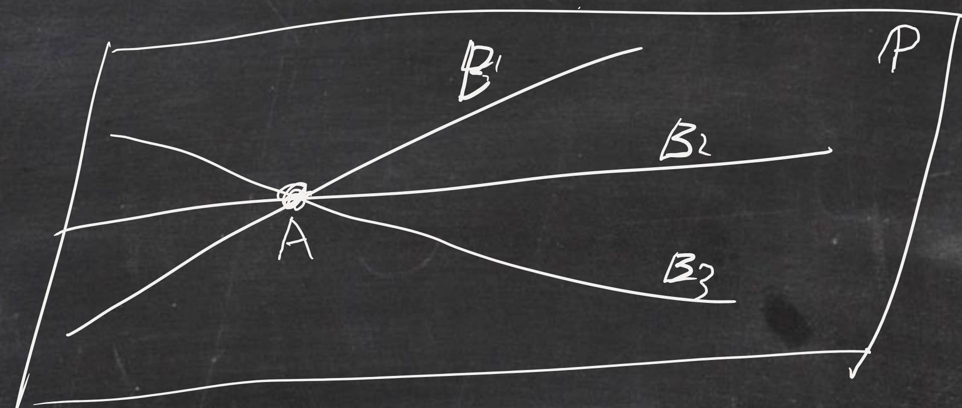
WPGs -> GPGs and PGs

- GPGs are WPGs with $w = 1$ (in the GPG), 0 everywhere else
- Positive Geometries (PGs) are WPGs with $w = \pm 1, 0$ everywhere (so a GPG) AND induced weight on all nested boundary components is $\pm 1, 0$.

Maximal loop-loop residue

[see Gabriele Dian talk later]

All in one point AND all in one plane AND only three loop lines remaining



CLAIM: ANY way you reach this configuration gives the same answer (up to a numerical factor = number of internal boundaries crossed)

The correlahedron

- Internal boundaries first found in squared amplituhedron [Dian,PH]
- Amplituhedron-like (non max winding number) = amplitude x amplitude
- Sum of amplituhedron-like = squared amplituhedron (no winding number) = limit of correlahedron (gives HBPS correlators) [Eden, Mason, PH]
- Squared amplitude contains non-unit max residues (but less subtley - almost disconnected sum of positive geometries)

Amplituhedron, Amplituhedron-like geometries

Amplituhedron: [Arkani-Hamed, Thomas, Trnka]

$$\mathcal{A}_{n,k} := \left\{ Y \in Gr(k,k+4) \middle| \begin{array}{ll} \langle Yii+1jj+1\rangle > 0 & 1 \le i < j-1 \le n-2 \\ \langle Yii{+}11n\rangle\,(-1)^k > 0 & 1 \le i < n-1 \\ \{\langle Y123i\rangle\} & \text{has } k \text{ sign flips as } i = 4,..,n \end{array} \right\}$$

for $Z \in Gr_>(k+4,n)$,

(tree level)

Natural generalization

Amplituhedron-like: [Arkani-Hamed, Thomas, Trnka; Dian, PH]

$$\mathcal{H}^{(f)}_{n,k} := \left\{ Y \in Gr(k,k+4) \middle| \begin{array}{ll} \langle Yii+1jj+1\rangle > 0 & 1 \le i < j-1 \le n-2 \\ \langle Yii{+}11n\rangle\,(-1)^f > 0 & 1 \le i < n-1 \\ \{\langle Y123i\rangle\} & \text{has } f \text{ sign flips as } i = 4,..,n \end{array} \right\}$$

for $Z \in Gr_+(k+4,n)$

$0 \le f \le k$

- We only consider $k = n-4$

$$\mathcal{A}_{n,k} = \mathcal{H}^{(k)}_{n,k}$$

What do these give?

Loop versions also ($k=n-4$)

Amplituhedron -> amplitudes (N=4 SYM planar, perturbative integrands)

Amplituhedron-like -> products of amplitudes

Main claim:

$$\Omega\left(\mathcal{H}^{(f)}_{n,n-4}\right) = \boxed{H^{(f)}_{n,n-4} = A_{n,f} * A_{n,n-f-4} \,.}$$

canonical form

products of superamplitudes

M=4

- Loop version too!

Problem of non unit residues

observed here first

- Canonical form (amplitude from amplituhedron) means max residues = 0, +/-1

- the maximal residues of the squared amplituhedron are not only +/- 1

e.g. $(A^2)_{6,2} = 2A_{6,2} + A_{6,1} * A_{6,1}$

max residues = 0, +/-2, +/-4

- Therefore GPGs or WPGs

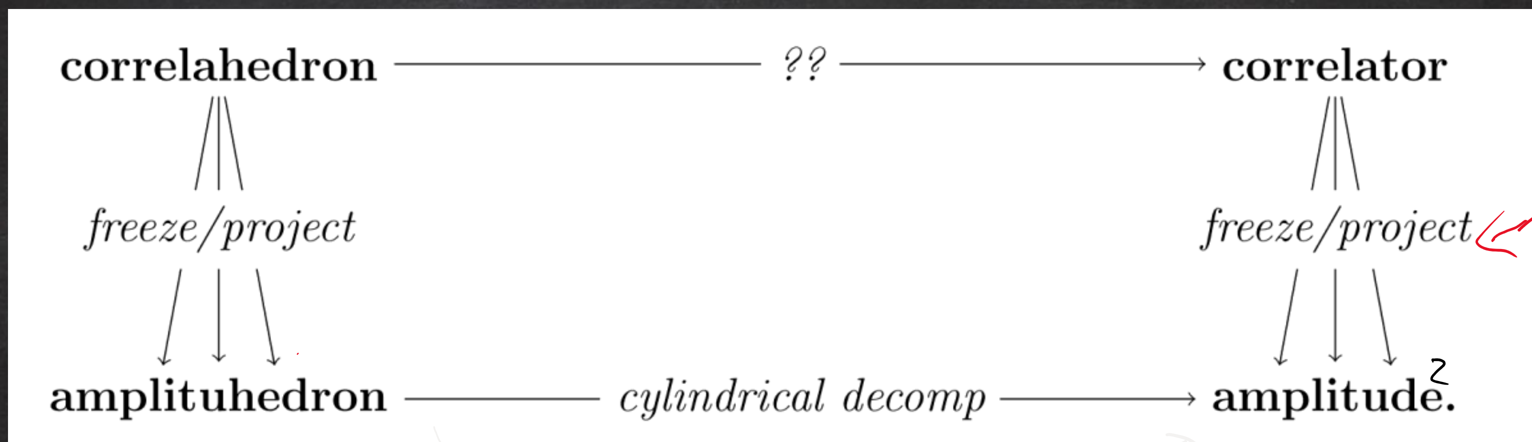
Correlahedron [Eden, Mason, PH]

$Y, L_{i=1..\ell}, z_{i=1..n} \to Y, X_{i=1..n}$

- Geometry: $Y \in Gr(n+k; n+k+4)$ $\langle Y X_i X_j \rangle > 0 \equiv G_{n;k}$

n+k-plane, 2-planes

Large dimension. Simple definition (no winding number)



squared

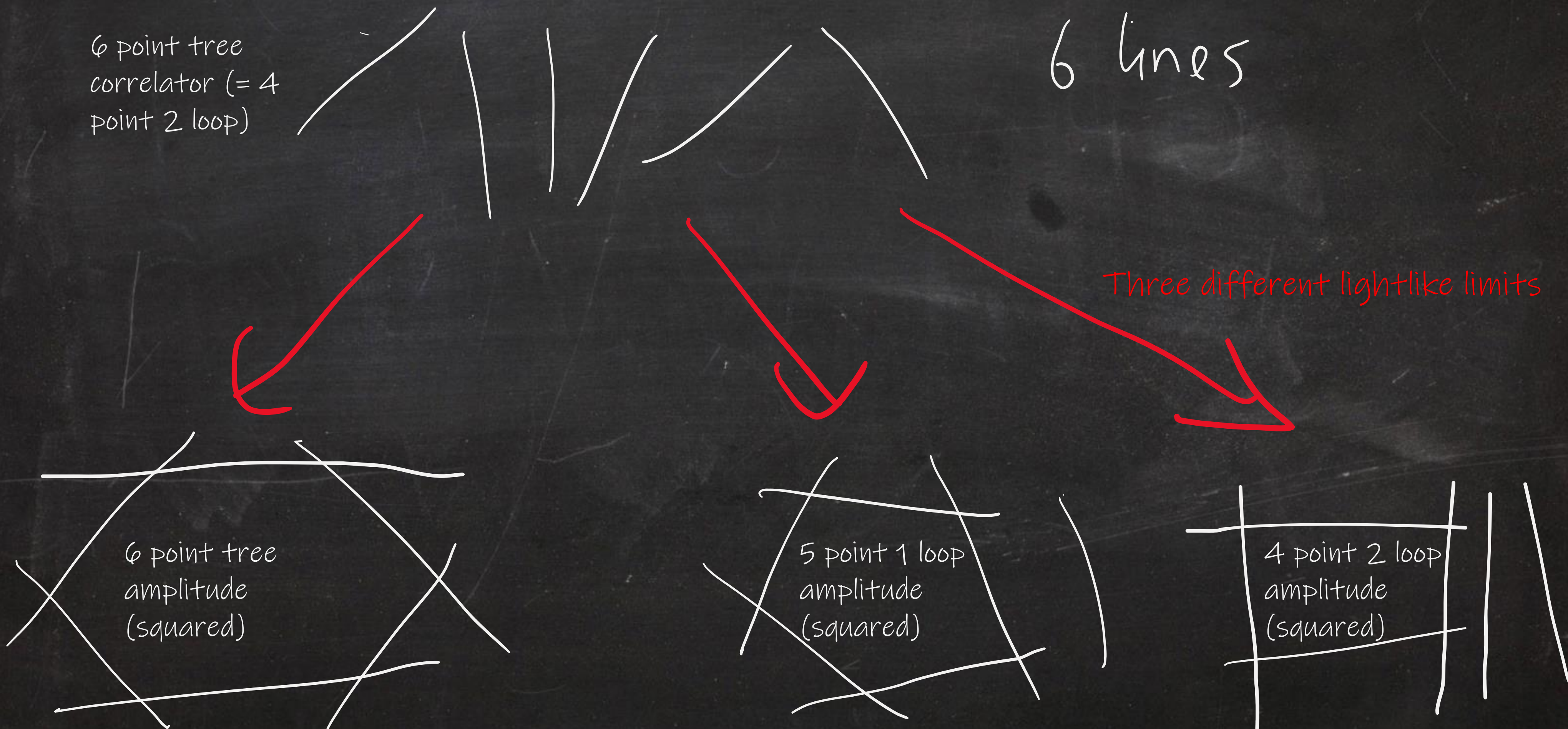
light-like limit in amplituhedron space

Prediction

- Correlahedron gives all half BPS single trace correlators
- All correlators = new observation!! Consequence from [Caron-Huot, Coronado]
- Equivalent to all IIB gravity amplitudes in AdS

Correlators projected to twistor space

- Functions of lines in 3d projective twistor space X
- Lightlike limit = intersecting lines (polygon)



Future:

- amplituhedron boundary structure, genus (+ relation to integral / symbol etc.): Eg [Dennen, Prlina, Spradlin, Stankowicz, Stanojevic, Volovich]
- Use cuts via amplituhedron to determine amplitude / correlator at higher loops (constructive approach?)
- Applications of weighted positive geometry? - [Cosmological polytope [Arkani-Hamed, Benincasa,Postnikov], negative geometries [Arkani-Hamed, Henn, Trnka] , non-planar amplitude [Arkani-Hamed, Bourjaily, Cachazo, Postnikov, and Trnka], momentum amplituhedron [Damgaard, L. Ferro, T. Lukowski, and R. Moerman] etc.?]
- Correlahedron projected to twistor space (more natural, sets of lines moving around in 3d, GL(2))
- More checks of correlahedron / connect with recent higher point/charge correlator activity [Bargheer, Fleury, Gonçalves; Caron-Huot, Frank Coronado and Beatrix Muehlmann]