Soliton moduli spaces

Chris Halcrow - KTH

Prague Spring Amplitudes, May 2023

Topological solitons

- A topological soliton is a localised solution of a PDE whose existence and stability relies on the topology of the system
- Examples:
  1D: domain walls
  2D: vortices, lumps
  3D: monopoles, skyrmions
  4D: instantons.
- They often have particle-like features, so are used as smooth models of particles. And they appear in condensed matter systems
- Space of static energy minimising solutions is called the ***moduli space***

Example: nonlinear sigma model

The nonlinear sigma model in (2+1)D is given by

$$\mathscr{L} = \partial_\mu \sigma_a \partial^\mu \sigma_a \,, \qquad \sigma \cdot \sigma = 1$$

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3), \qquad \sigma : \mathbb{R}^2 \to S^2$$

If we fix a boundary condition, then $\mathbb{R}^2$ compactifies to S^2. Then σ is a map between two-spheres. This has non-trivial topology as $\Pi_2(S^2) = \mathbb{Z}$.

=> Each configuration has a topological charge N, which cannot change under smooth deformations.

Example: nonlinear sigma model

Change coordinates to $z = x + iy$ and $R = \dfrac{\sigma_1 + i\sigma_2}{1 + \sigma_3}$. Lagrangian becomes

$$\mathscr{L} \propto (|\partial_z R|^2 + |\partial_{\bar{z}} R|^2)/(1 + |R|^2)$$

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$$\implies L = 2\pi N + 2\int |\partial_{\bar{z}} R|^2/(1 + |R|^2) \longleftarrow$$

Bogomolny argument

Lagrangian is bounded below by topological charge N

Solutions satisfy $\partial_{\bar{z}} R = 0$ => $R(z, \bar{z}) = R(z)$

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Overall: the moduli space of the N-lump is given by the order N rational maps

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Example: nonlinear sigma model

$N = 1$

Solutions are: $$R(z) = \frac{a}{z - c}$$

Symmetries help up understand the *moduli* a, c physically.

Translation symmetry => c ~ position

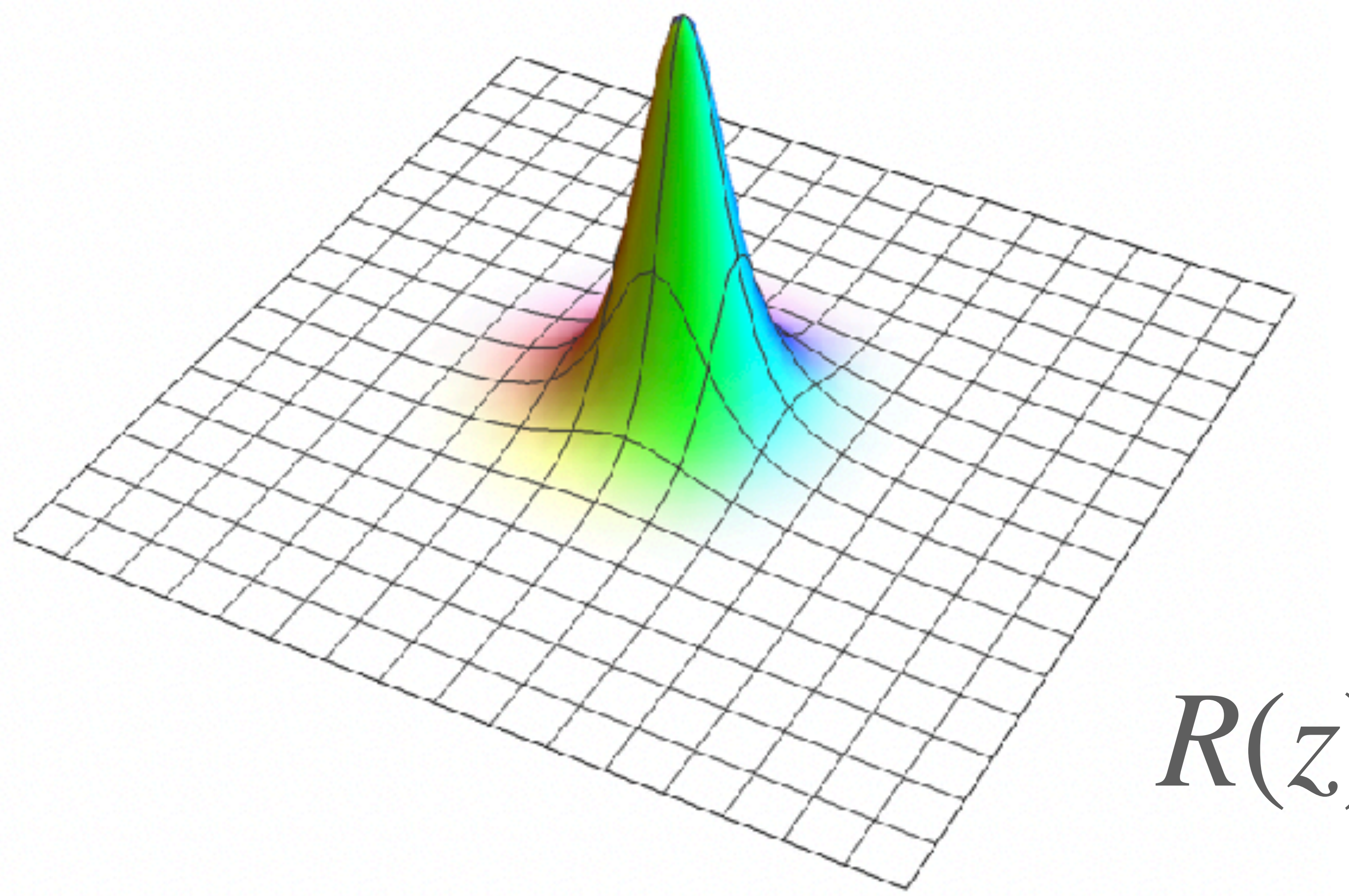
Scaling symmetry => $|a|$ ~ size

Internal symmetry $\sigma_1 + i\sigma_2 \rightarrow \exp(i\alpha)(\sigma_1 + i\sigma_2)$ => arg(a) ~ internal orientation

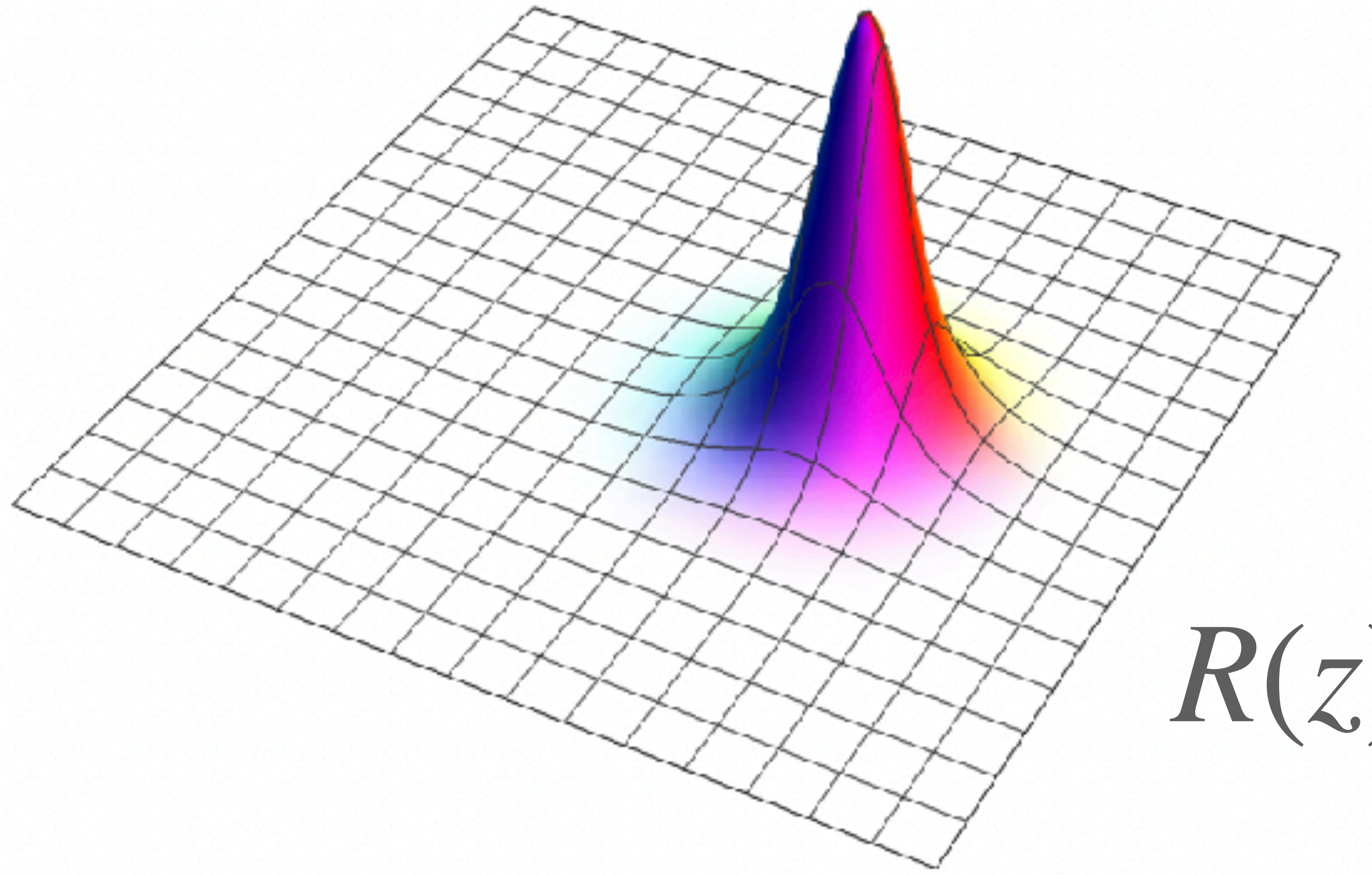
Example: nonlinear sigma model

$$R(z) = \frac{a}{z - c}$$

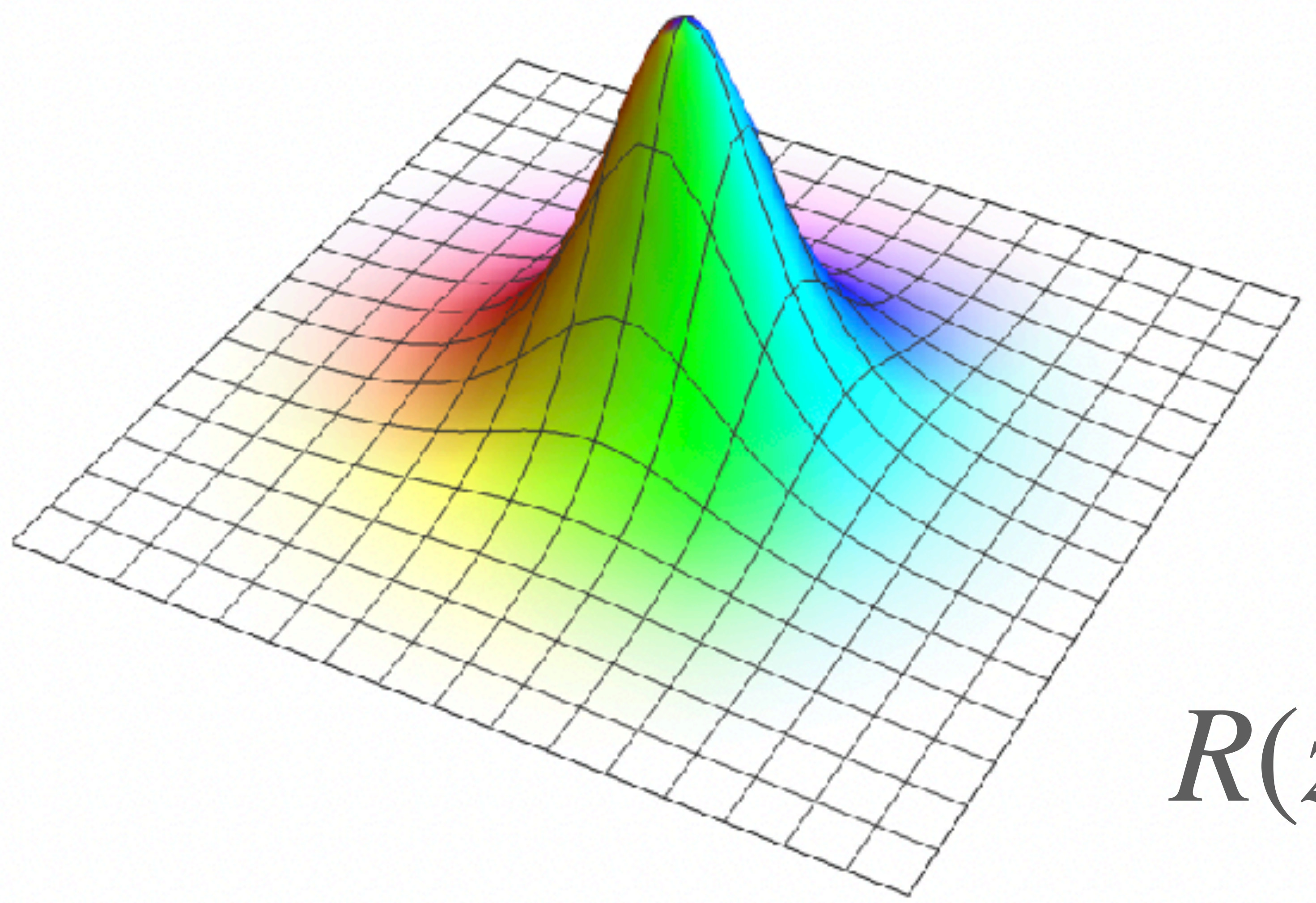
We plot the energy density.
The colour represents the
phase of R



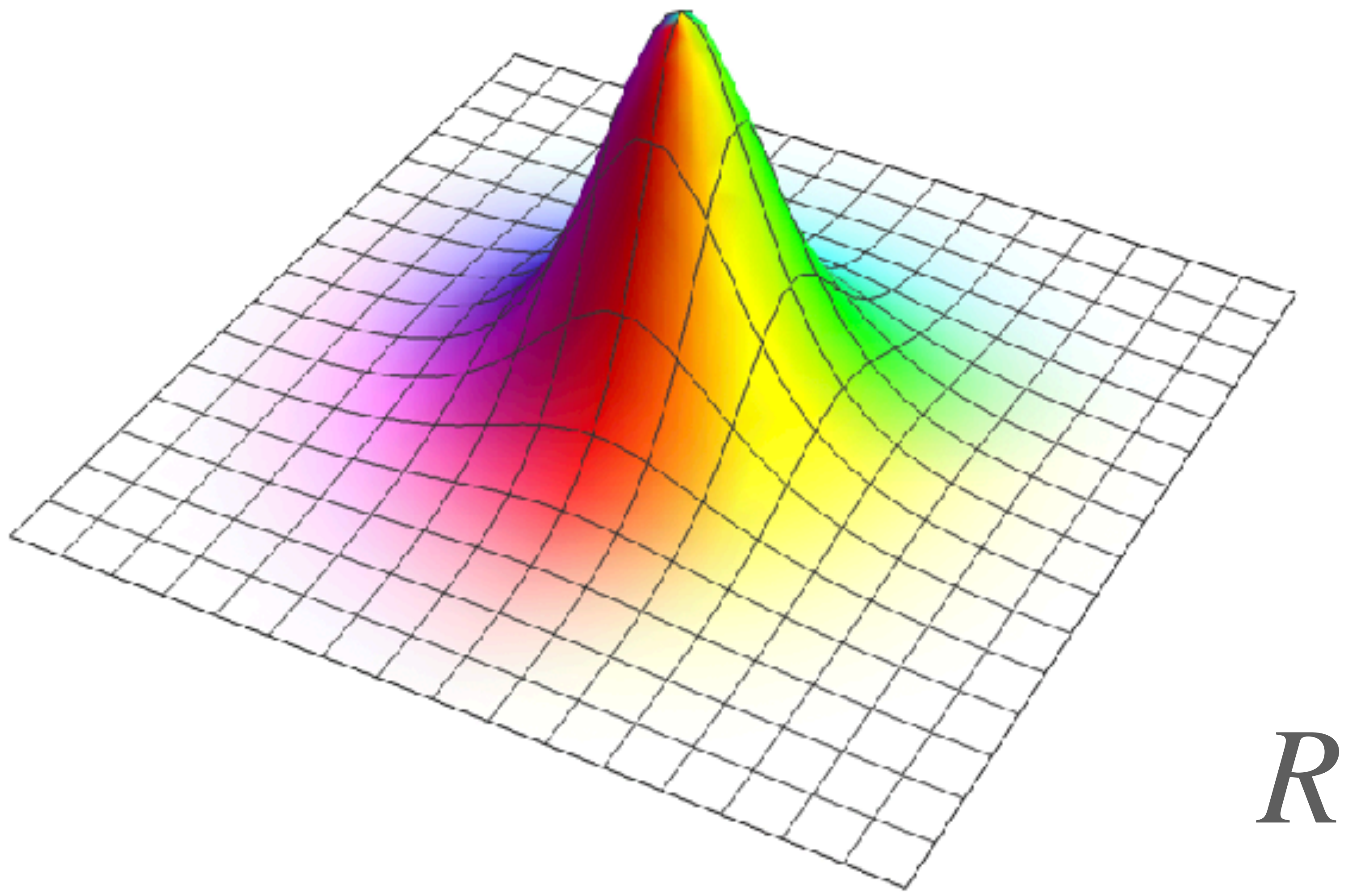
$R(z) = 1/z$



$R(z) = -1/(z-2)$



$R(z) = 2/z$



$R(z) = 2i/z$

Example: nonlinear sigma model

Why is this useful? Can describe dynamics by promoting parameters (also known as “moduli” or “collective coordinates”) to time dependent functions:

$$R(z,t) = \frac{\lambda(t)e^{i\theta(t)}}{z - P(t)}$$

We can then substitute these solutions into the original Lagrangian:

$$\mathscr{L} = \frac{1}{2}\left(\dot{\lambda}, \dot{\theta}, \dot{P}\right) g \left(\dot{\lambda}, \dot{\theta}, \dot{P}\right)^T - V(\lambda, \cancel{\theta}, P)$$

CONSTANT

This is just a free particle on a manifold (the moduli space) with metric g. The metric is induced by the field theory.

$$\mathscr{L} = \frac{1}{2}\dot{X}_a g_{ab} \dot{X}_b \implies \ddot{X}^a + \Gamma^a_{bc}(X)\dot{X}_b\dot{X}_c = 0$$ (geodesic equation)

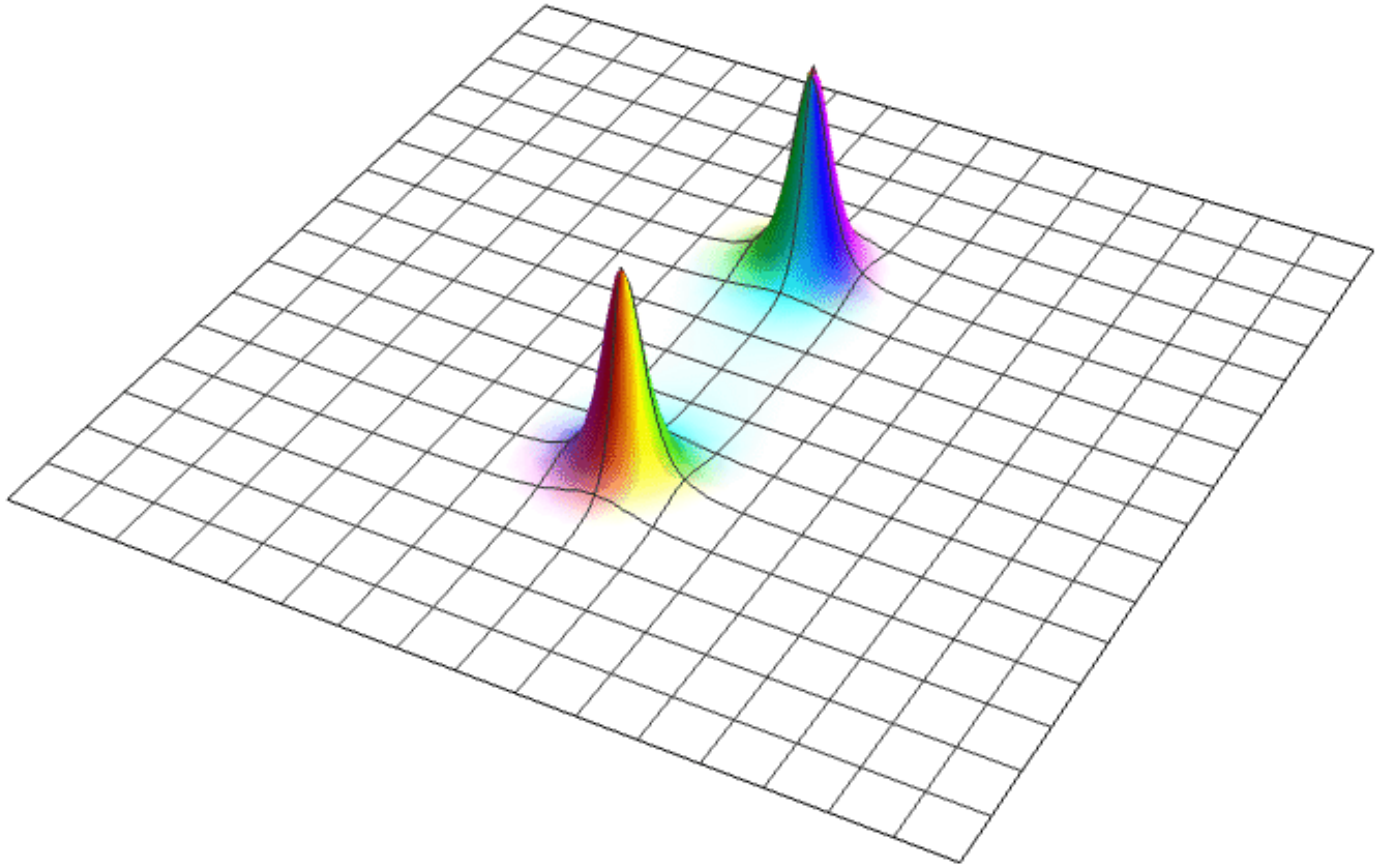
Example: nonlinear sigma model

$N = 2$, centered solutions are: $$R(z) = \frac{az + b}{z^2 - c}$$

Now: more fun! Can consider lump scattering on the 2-lump moduli space.

E.g.

$$R(z) = \frac{|b(t)|}{z^2 - c(t)}$$



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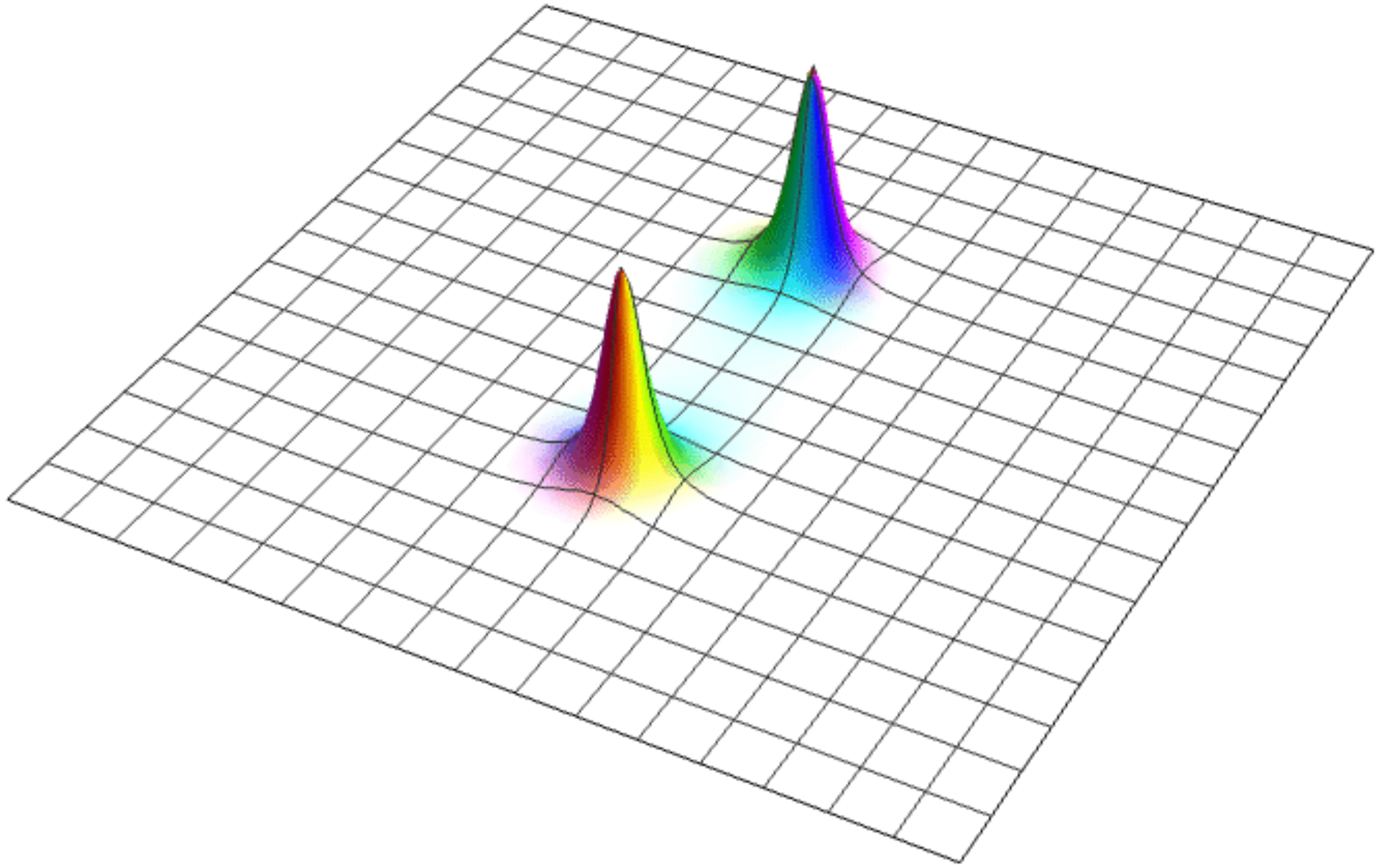
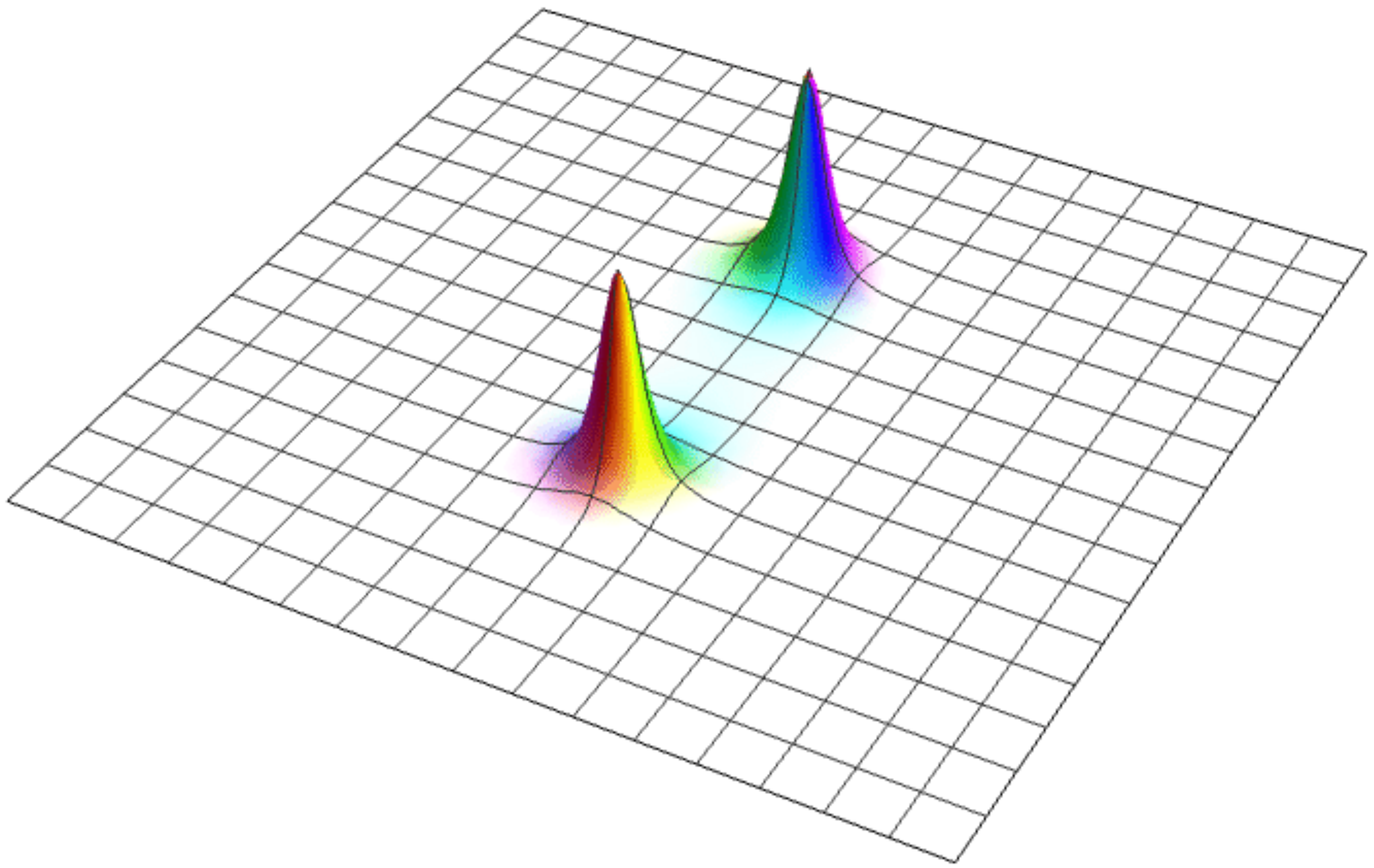
$$R(z) = \frac{|b(t)|}{z^2 - c(t)}$$

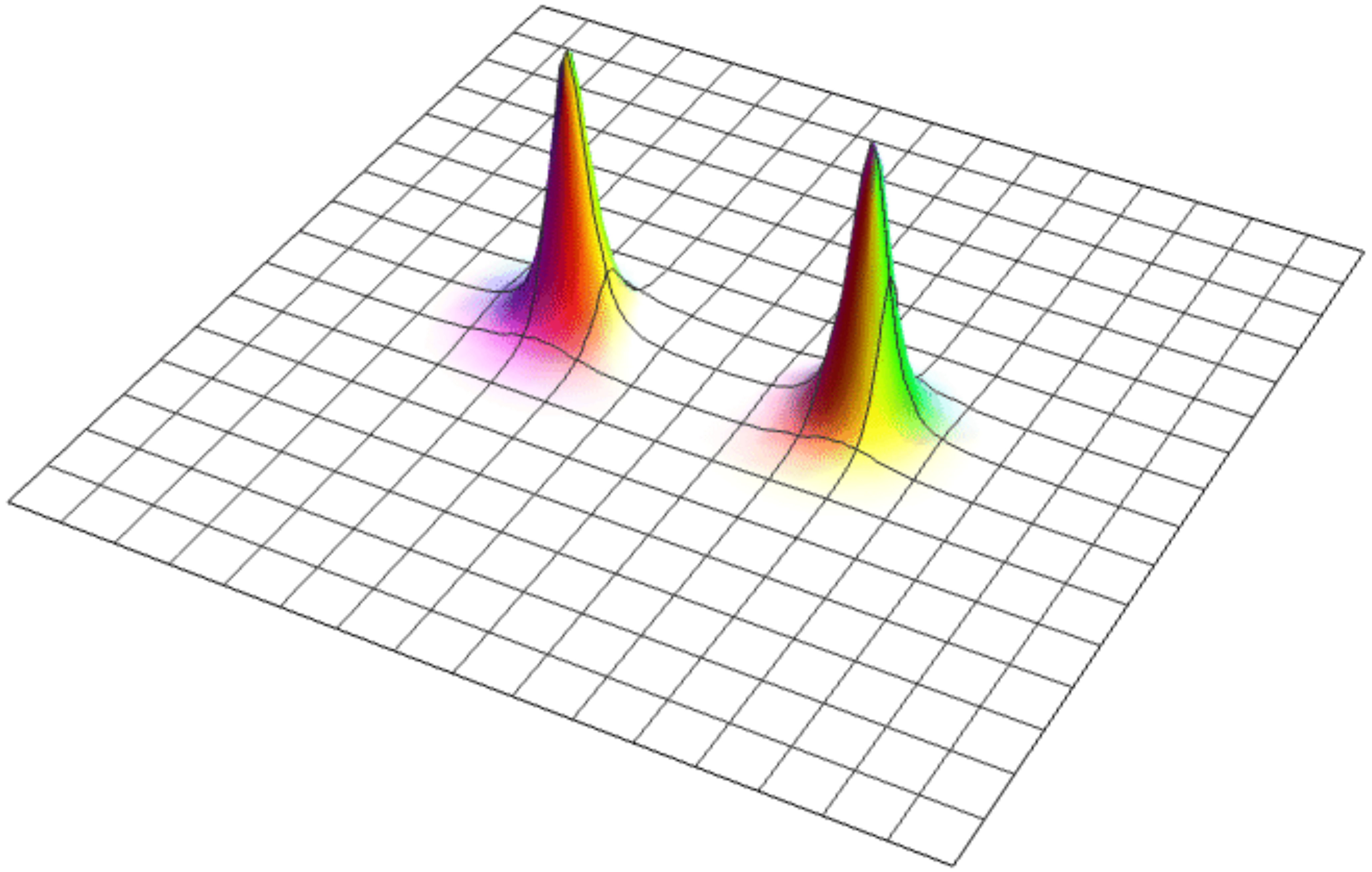
$$R_{\text{ring}}(z) = 1/z^2$$



Example: nonlinear sigma model

And we can consider other dynamics.



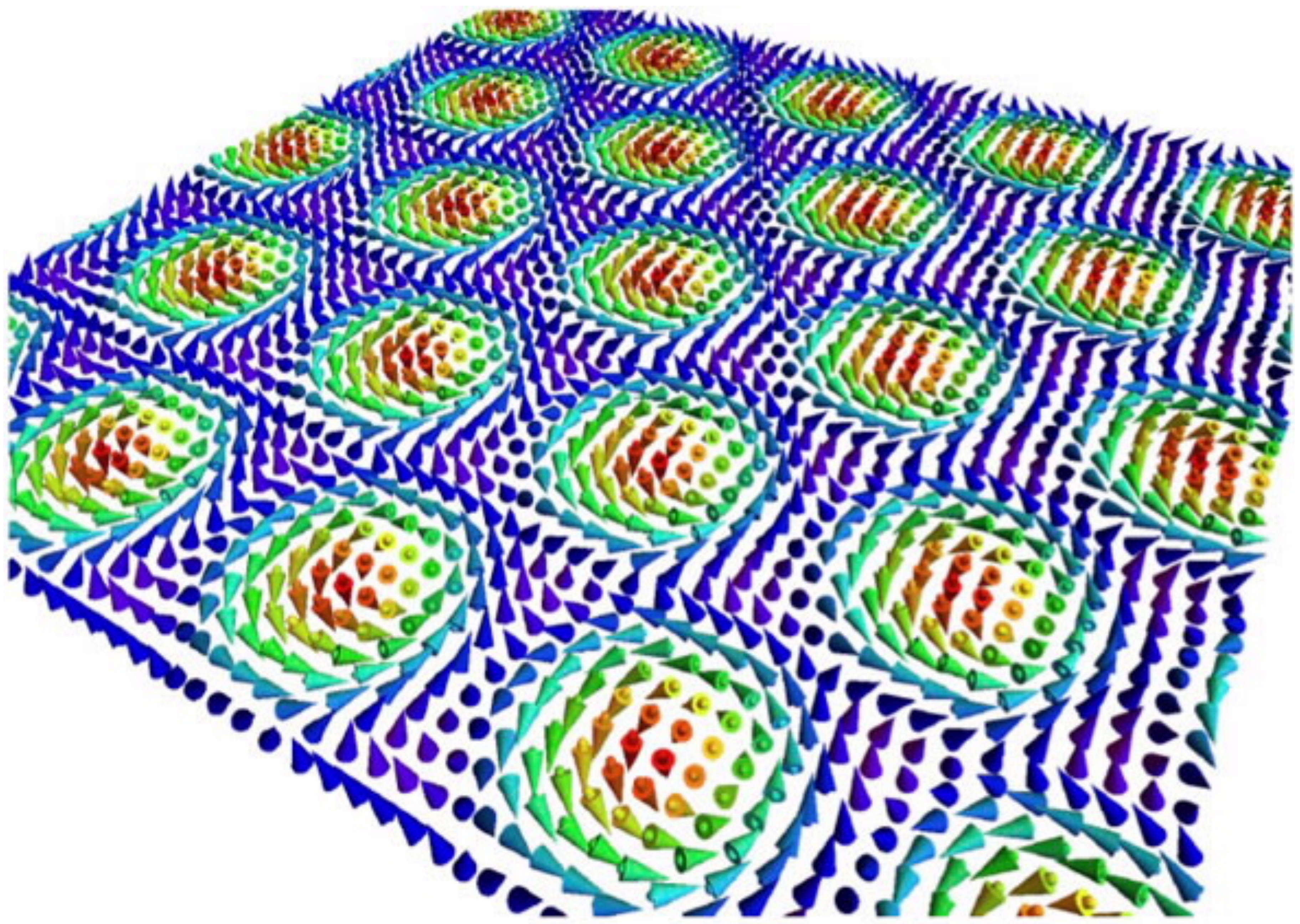


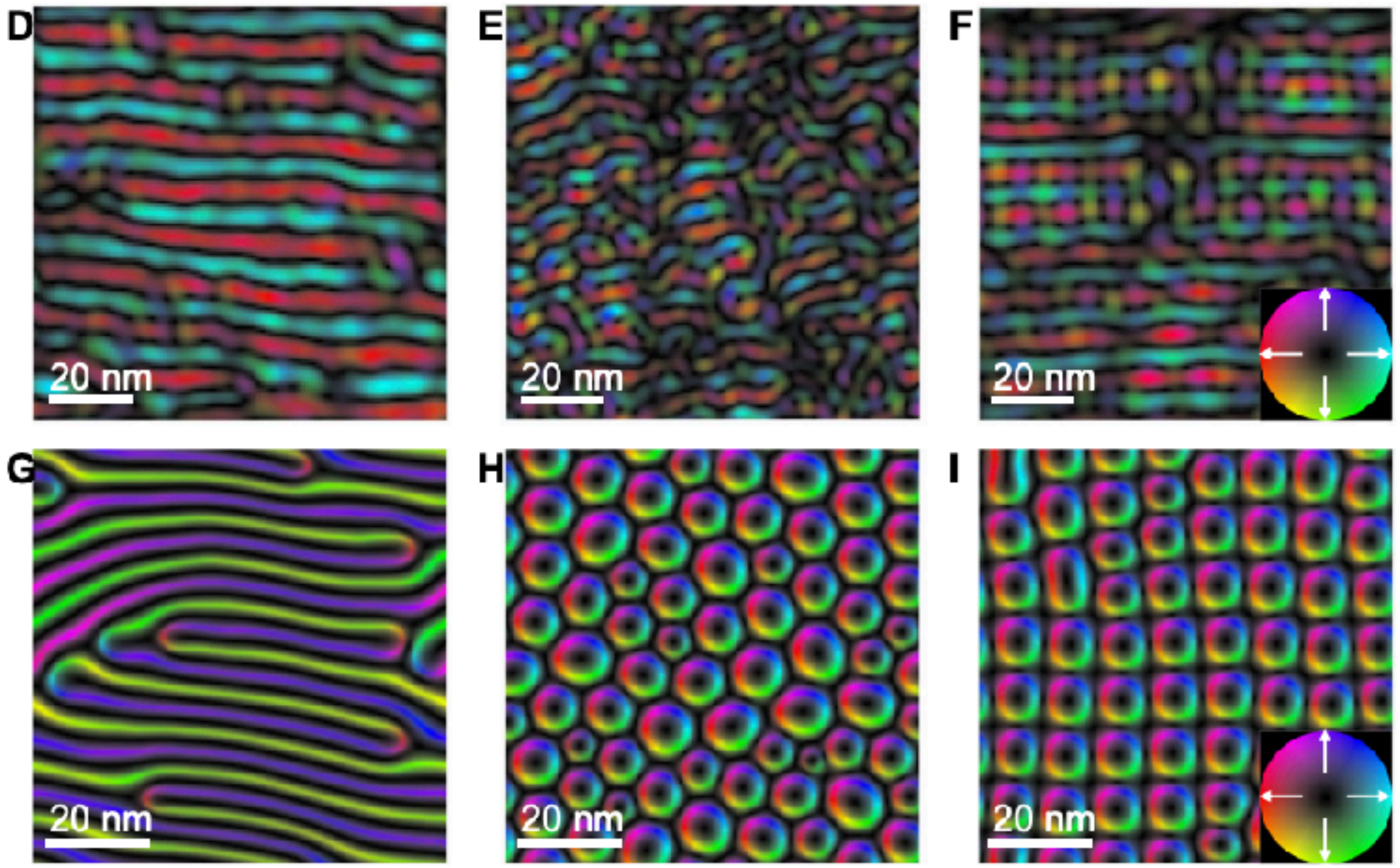
What we've discovered: a lump can exchange "phase energy" for "size energy".

So?

The nonlinear sigma model is closely related to models of magnetism and ferroelectricity. The vector $(\sigma_1, \sigma_2, \sigma_3)$ might model the magnetisation vector $\hat{m}$, or Polarisation vector P.

Lumps become ***skyrmions***.





Emergent chirality in a polar meron to skyrmion phase transition

Lumps to skyrmions

So the basic facts we've learned about lumps should apply to skyrmions too.

=> Should be able to exchange "phase energy" for "size energy".

Does this matter…?

Maybe not.

Skyrmions in magnetic systems have a fixed phase. And their dynamics are driven by external currents/forces => hard to see this in action.

Halfway summary

The simplest nonlinear sigma model contains topological solitons called lumps, which have a conserved integer N

The N-lump moduli space is isomorphic to the based order N rational maps

We can use trajectories on the space of rational maps to approximate lump dynamics

Information about lumps might help understand defects in condensed matter

Experiments on the moduli space of lumps gives us information about the dynamics/interactions of skyrmions in magnetic and ferroelectric systems

The next half

The same thing

Instantons

Consider SU(2) Yang-Mills theory in R^4

$$\mathscr{L} = \text{Tr}\, F_{\mu\nu} F_{\mu\nu}$$

Contains solitons called instantons, labelled by integer N. These satisfy

$$F_{\mu\nu} = \star\, F_{\mu\nu}$$

Instanton solutions are known, and given by ADHM data. This is an $N \times (N+1)$ matrix of quaternions, which satisfy a nonlinear constraint called the reality condition.

Here's some ADHM data:

$$L = \lambda \begin{pmatrix} 1 & \boldsymbol{k} \end{pmatrix}$$

$$M = \frac{\lambda}{\sqrt{2}} \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} \\ \boldsymbol{j} & -\boldsymbol{i} \end{pmatrix}$$

Instantons

Here's some more with N=8:

$$L = \begin{pmatrix} \lambda_1 \boldsymbol{k} & -\lambda_1 & \lambda_2 \boldsymbol{j} & \lambda_2 \boldsymbol{i} & 0 & 0 & \lambda_1 & \lambda_1 \boldsymbol{k} \end{pmatrix},$$

$$M = \begin{pmatrix} \mu_1 \boldsymbol{i} + R\boldsymbol{k} & \mu_1 \boldsymbol{j} & \nu \boldsymbol{j} & \nu \boldsymbol{i} & 0 & 0 & \eta & 0 \\ \mu_1 \boldsymbol{j} & -\mu_1 \boldsymbol{i} + R\boldsymbol{k} & \nu \boldsymbol{i} & -\nu \boldsymbol{j} & 0 & 0 & 0 & \eta \\ \nu \boldsymbol{j} & \nu \boldsymbol{i} & 0 & 0 & \mu_2 \boldsymbol{i} & \mu_2 \boldsymbol{j} & \nu \boldsymbol{i} & -\nu \boldsymbol{j} \\ \nu \boldsymbol{i} & -\nu \boldsymbol{j} & 0 & 0 & \mu_2 \boldsymbol{j} & -\mu_2 \boldsymbol{i} & -\nu \boldsymbol{j} & -\nu \boldsymbol{i} \\ 0 & 0 & \mu_2 \boldsymbol{i} & \mu_2 \boldsymbol{j} & \chi \boldsymbol{i} & \chi \boldsymbol{j} & 0 & 0 \\ 0 & 0 & \mu_2 \boldsymbol{j} & -\mu_2 \boldsymbol{i} & \chi \boldsymbol{j} & -\chi \boldsymbol{i} & 0 & 0 \\ \eta & 0 & \nu \boldsymbol{i} & -\nu \boldsymbol{j} & 0 & 0 & \mu_1 \boldsymbol{i} - R\boldsymbol{k} & \mu_1 \boldsymbol{j} \\ 0 & \eta & -\nu \boldsymbol{j} & -\nu \boldsymbol{i} & 0 & 0 & \mu_1 \boldsymbol{j} & -\mu_1 \boldsymbol{i} - R\boldsymbol{k} \end{pmatrix}.$$

“Nuclear” skyrmions

Nonlinear sigma model in 3D. Fundamental field,
$U = \sigma(x) + i\pi(x) \cdot \tau \in SU(2)$, with $\pi \cdot \pi + \sigma^2 = 1$, identified with pions.
Similar structure to chiral effective field theory:

$$\mathscr{L} = \mathscr{L}_2 + \mathscr{L}_4 + \mathscr{L}_6 + \ldots - V$$

Contains topological solitons called skyrmions with charge N.

Skyrme (’60):

Skyrmions = nuclei
N = baryon number

Witten (’79): this is a good idea, at least at large N_C.

Sakai-Suigimoto (’04): this is a low-energy limit of holographic QCD.

Instantons and Skyrmions

The Sakai-Sugimoto models tells us that skyrmions are related to instantons.
Originally an idea (from pure intuition) of Atiyah + Manton.

Instantons ⟶ Skyrmions

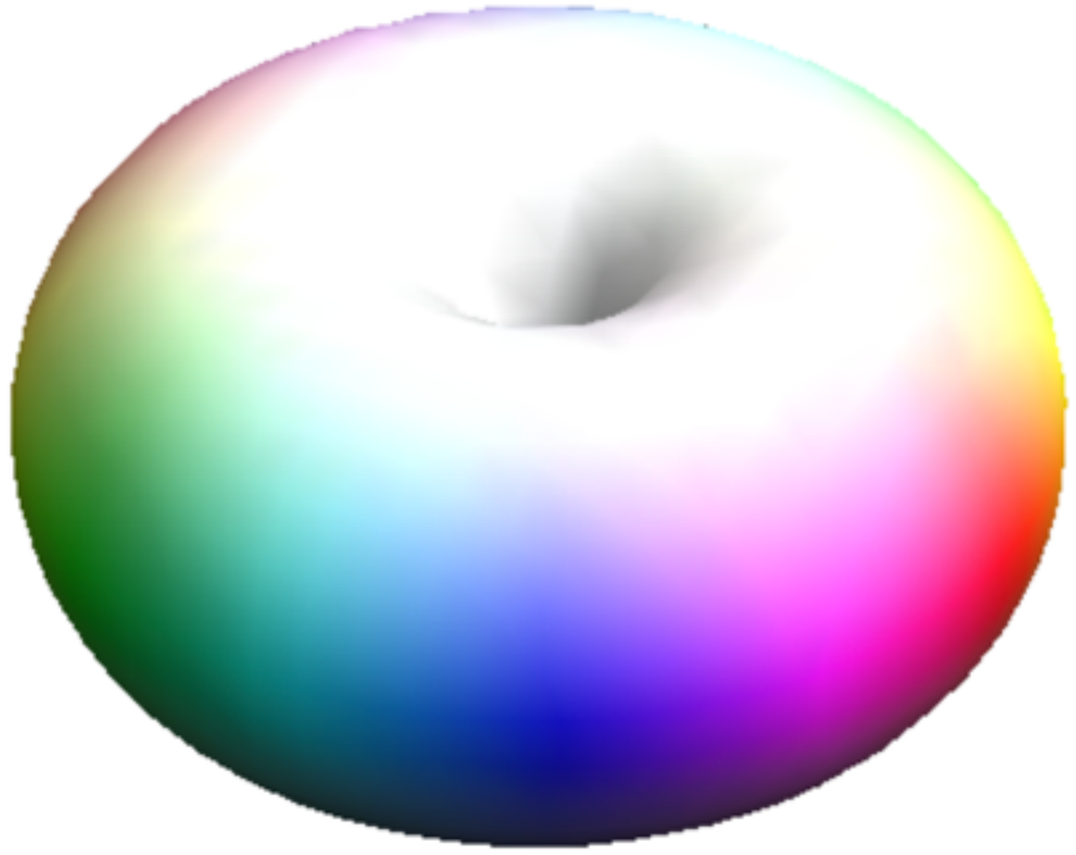
$$A_\mu(x) \qquad \exp\left(\int A_4(\boldsymbol{x}, x_4)\, dx_4\right) = U(\boldsymbol{x})$$

Instantons and Skyrmions

So, you give me ADHM data. I'll give you a skyrmion.

$$L = \lambda \begin{pmatrix} 1 & \boldsymbol{k} \end{pmatrix}$$

$$M = \frac{\lambda}{\sqrt{2}} \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} \\ \boldsymbol{j} & -\boldsymbol{i} \end{pmatrix}$$



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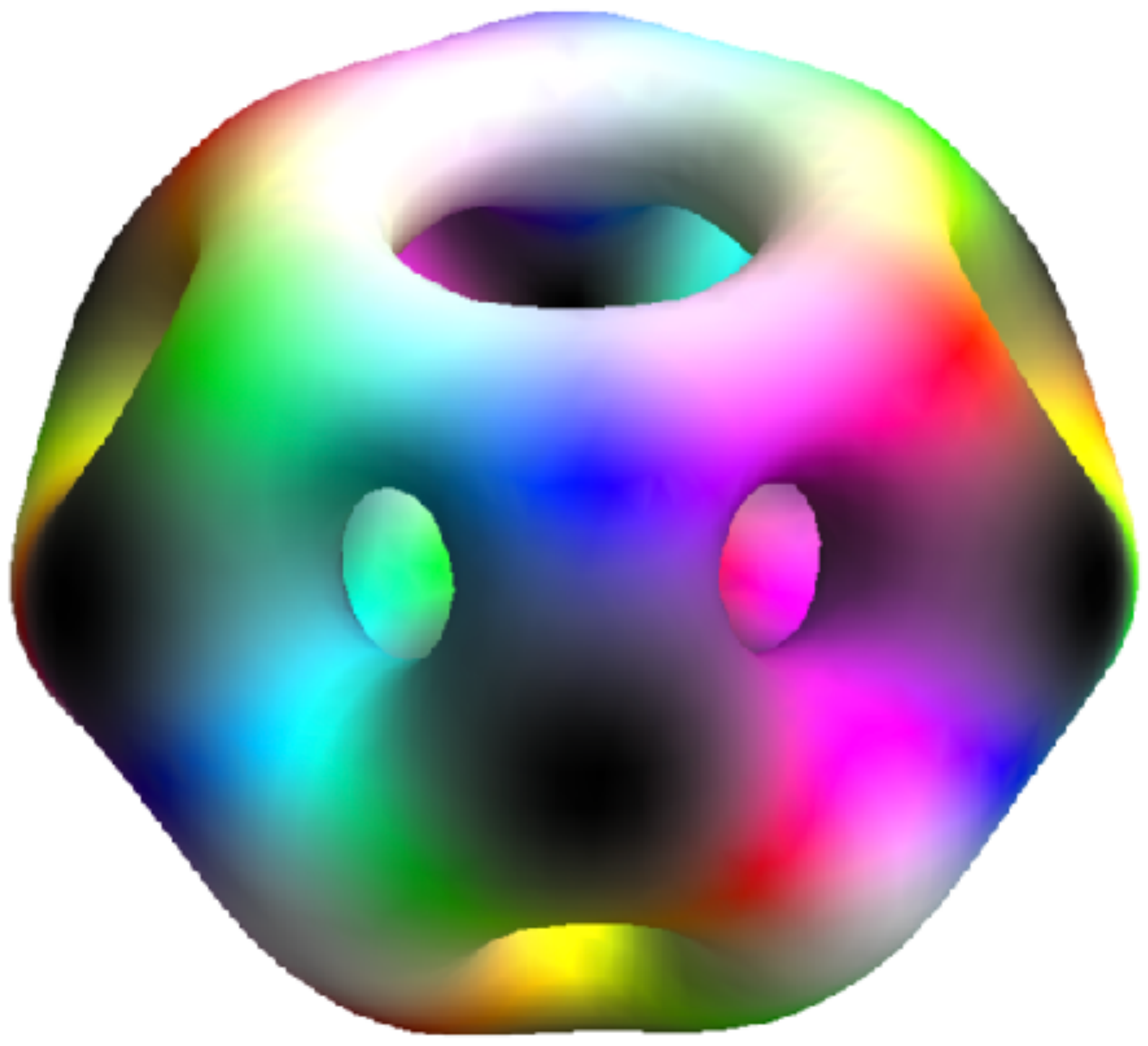

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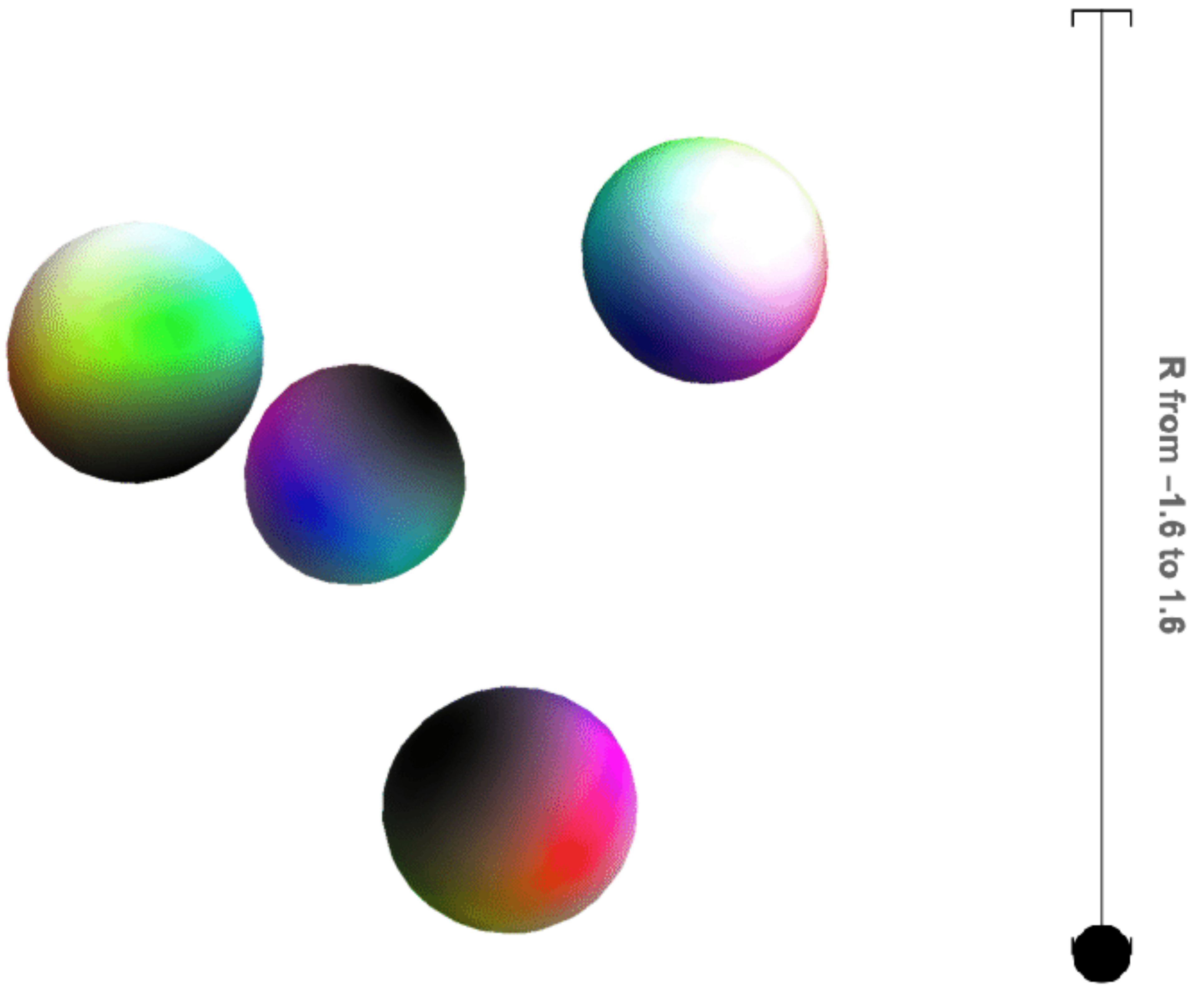
Instantons and skyrmions

Can also make families of configurations:

$$L = \kappa \begin{pmatrix} 1 & \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \end{pmatrix}$$

$$M = \begin{pmatrix} R(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) & \alpha\boldsymbol{j}+\beta\boldsymbol{k} & \beta\boldsymbol{i}+\alpha\boldsymbol{k} & \alpha\boldsymbol{i}+\beta\boldsymbol{j} \\ \alpha\boldsymbol{j}+\beta\boldsymbol{k} & R(\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}) & \alpha\boldsymbol{i}-\beta\boldsymbol{j} & \alpha\boldsymbol{k}-\beta\boldsymbol{i} \\ \beta\boldsymbol{i}+\alpha\boldsymbol{k} & \alpha\boldsymbol{i}-\beta\boldsymbol{j} & R(-\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}) & \alpha\boldsymbol{j}-\beta\boldsymbol{k} \\ \alpha\boldsymbol{i}+\beta\boldsymbol{j} & \alpha\boldsymbol{k}-\beta\boldsymbol{i} & \alpha\boldsymbol{j}-\beta\boldsymbol{k} & R(-\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}) \end{pmatrix}.$$



Instantons and skyrmions

Test data:

$$L_0 = (p_1 L_{T^2} q_1^{-1}, p_2 \boldsymbol{i} L_{T^2} q_2^{-1})$$

$$M_0 = \begin{pmatrix} q_1 M_{T^2} q_1^{-1} + d & 0 \\ 0 & q_2 M_{T^2} q_2^{-1} - d \end{pmatrix}$$

(L_{T^2}, M_{T^2}): toroidal ADHM data

p_1, p_2: *fixed* isorotations

$q_1 = q(\vec{e}_1, \theta)$, $q_2 = q(\vec{r}, \theta)$: rotations

$d = R\boldsymbol{k}$: positions

Instantons and skyrmions

This is…

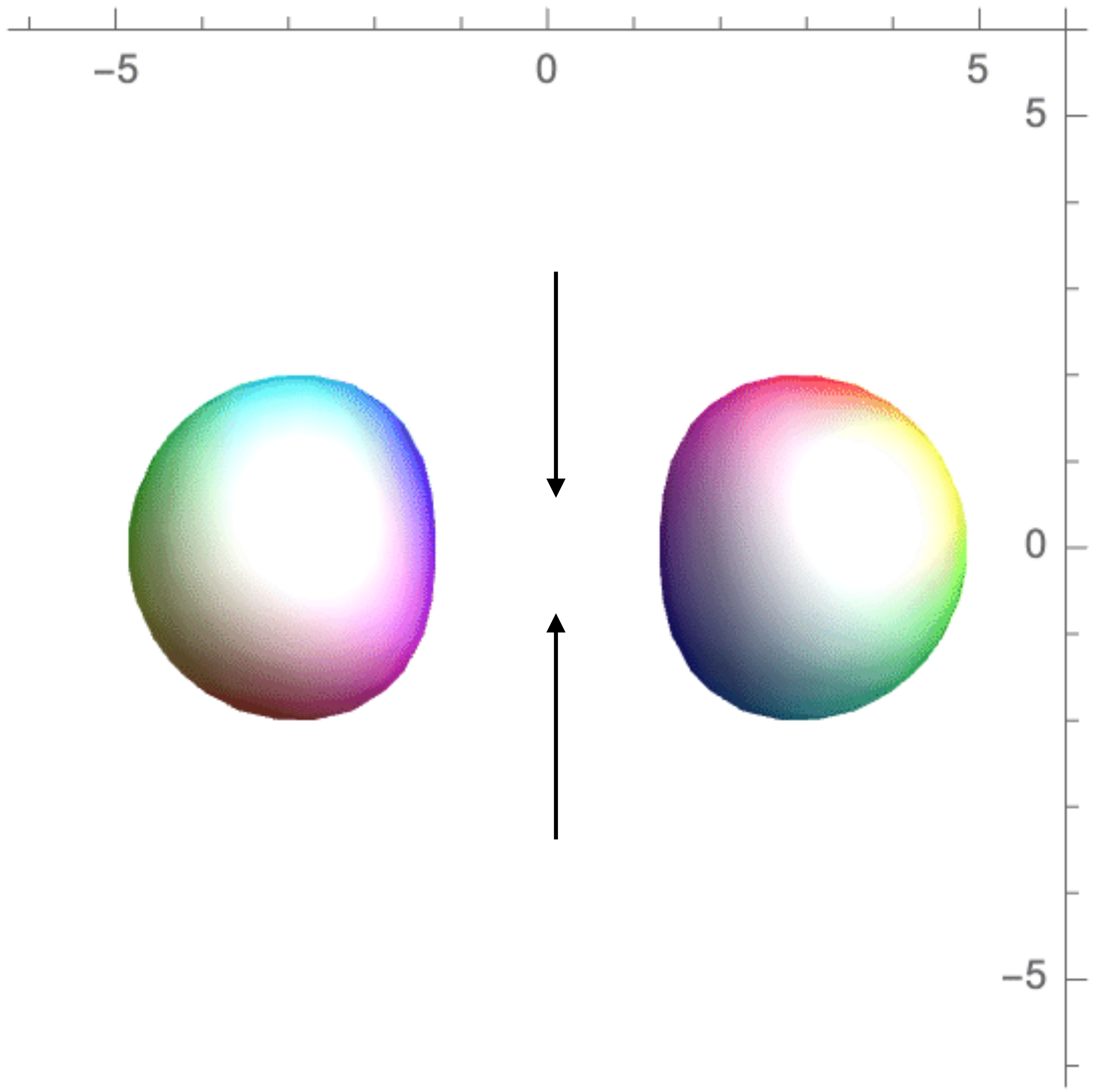
…very fun

Instantons and Skyrmions

Not just fun. Can repeat the questions from the previous section:

What are the dynamics?

What are the low-energy modes of a skyrmions? We know: 3 translations, 3 orientations.



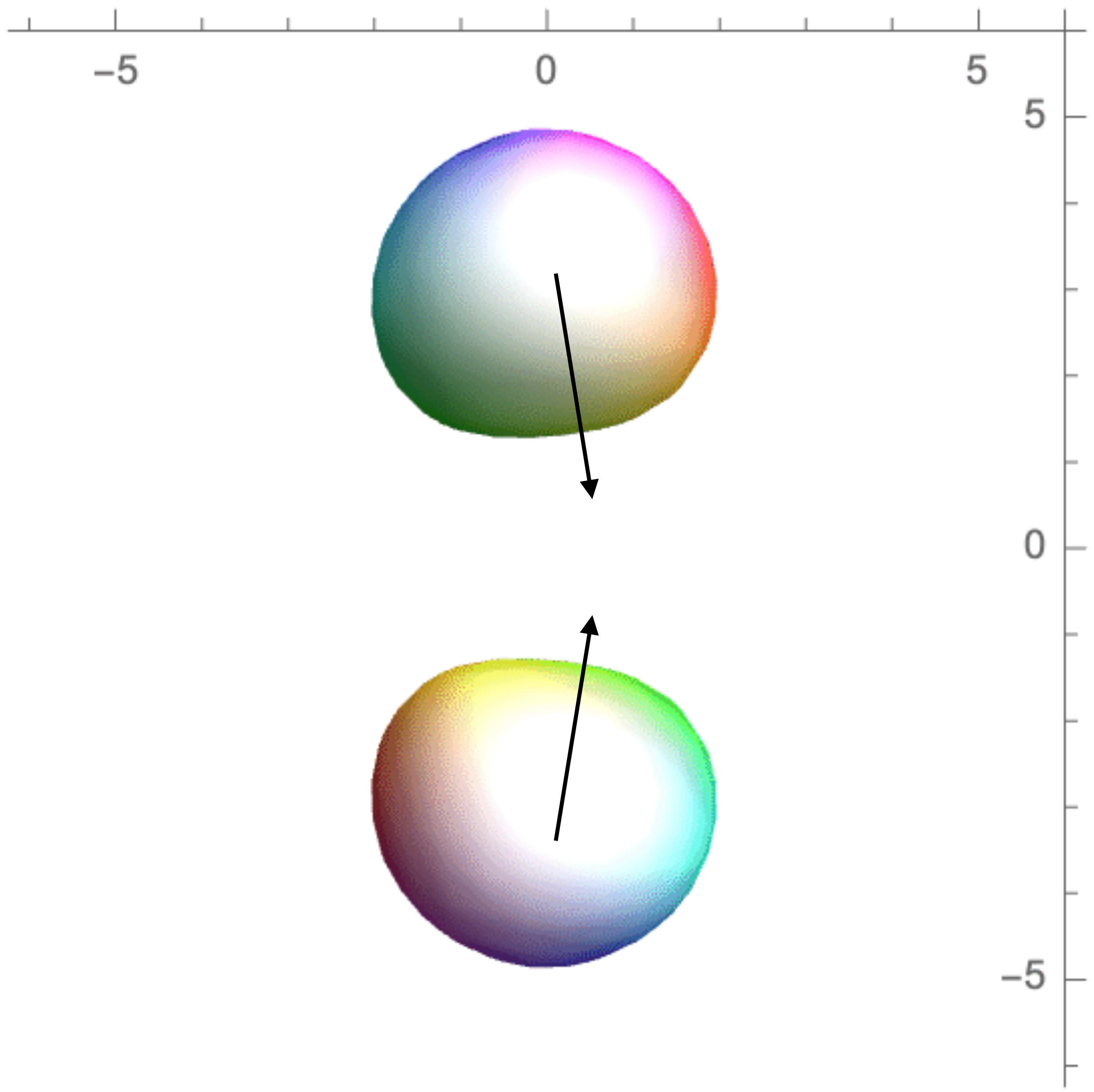
Instantons and Skyrmions

Not just fun. Can repeat the questions from the previous section:

What are the dynamics?

What are the low-energy modes of a skyrmions? We know: 3 translations, 3 orientations.

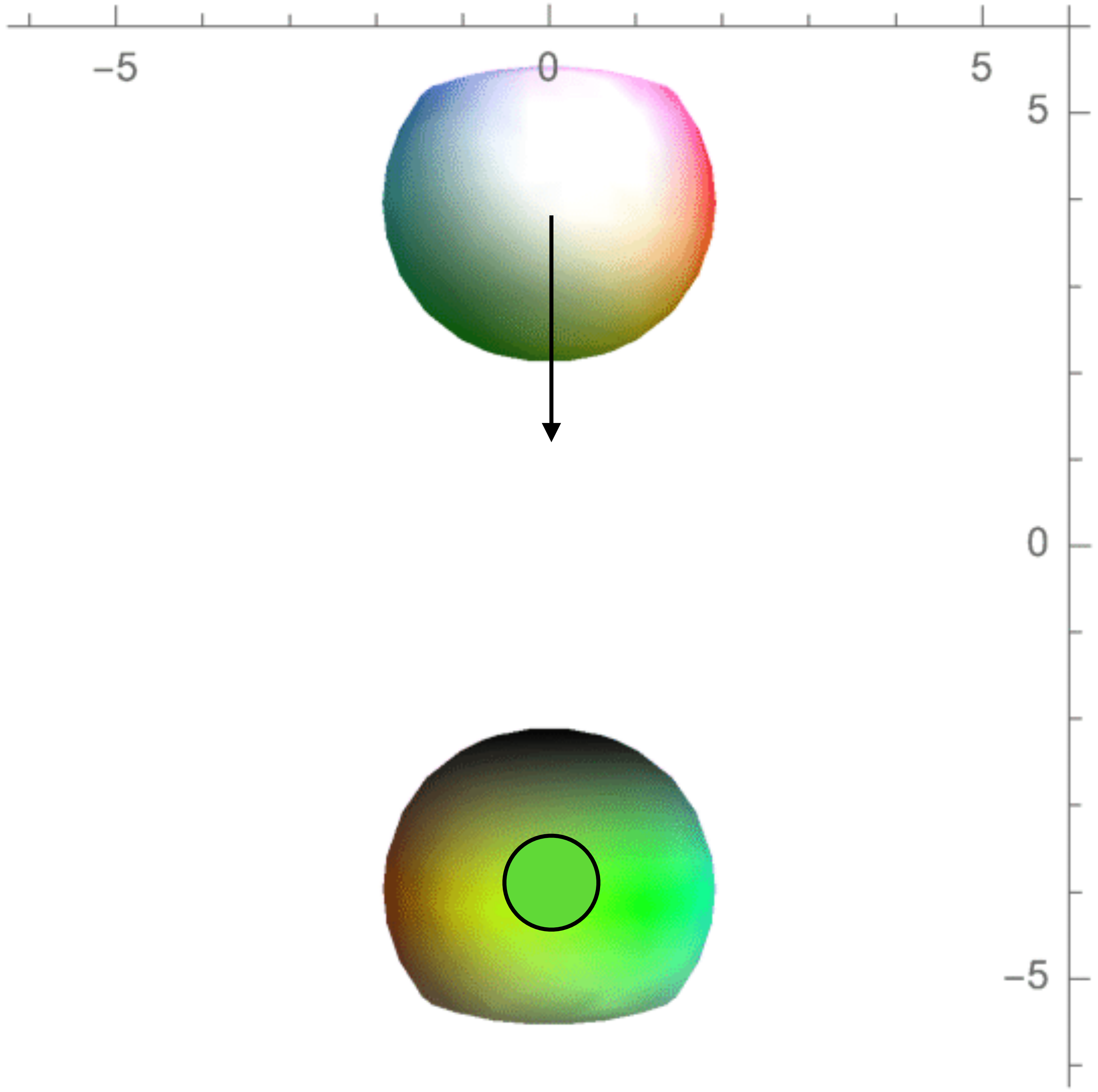
And one size mode.



Instantons and Skyrmions

But what happens when we rotate out of the plane?

What else is left to excite??



Instantons and Skyrmions

The Sakai-Sugimoto models tells us that skyrmions are projected instantons.
Originally an idea (from pure intuition) of Atiyah + Manton.

Instantons $\longrightarrow$ Skyrmions

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$$A_\mu(x) \qquad \exp\left(\int A_4(\boldsymbol{x}, x_4)\, dx_4\right) = U(\boldsymbol{x})$$

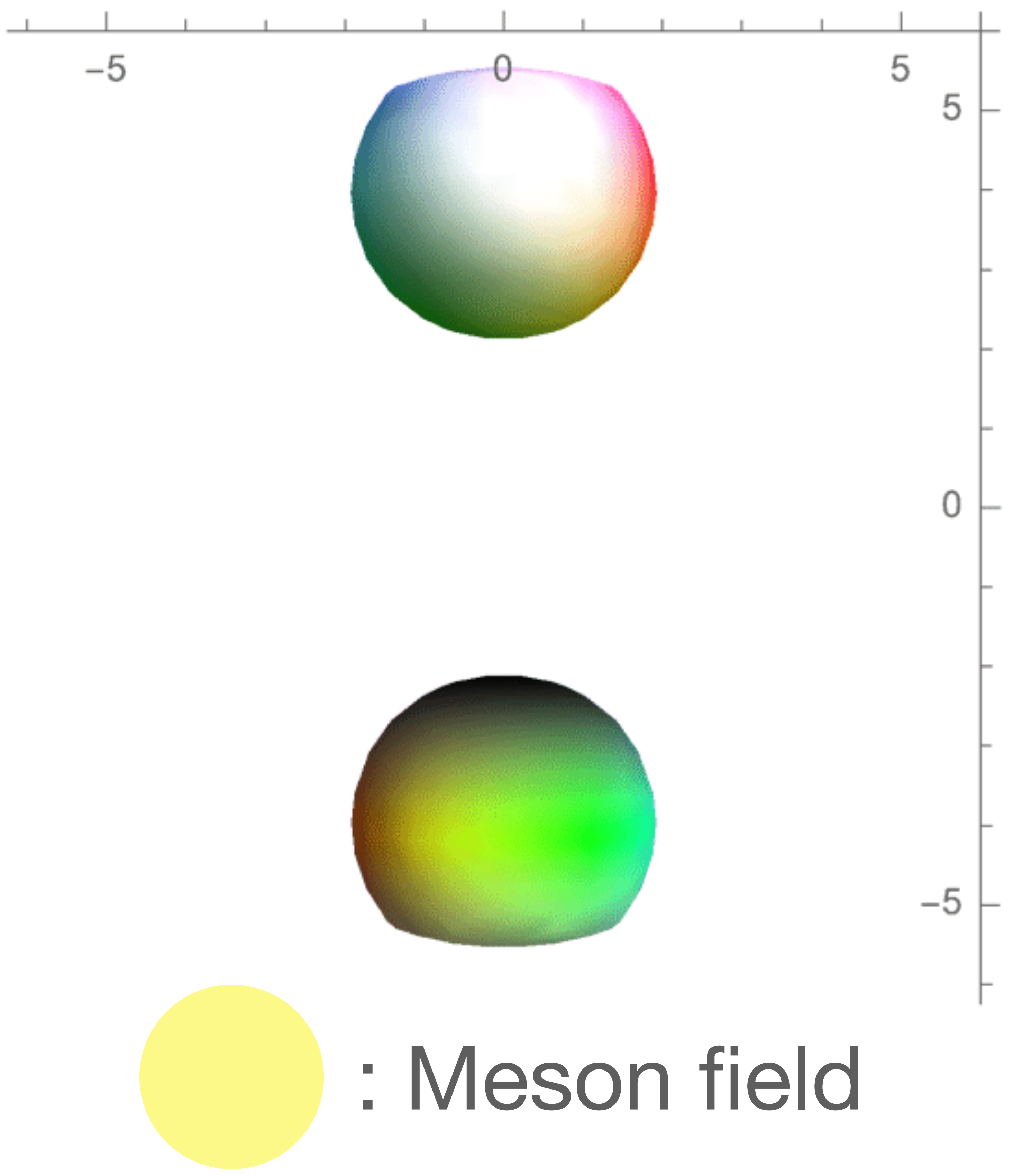
and vector mesons

Instantons and Skyrmions

But what happens when we rotate out of the plane?

What else is left to excite??

The final skyrmions are “facing each other” again, but now with an excited meson field.



Instantons and Skyrmions

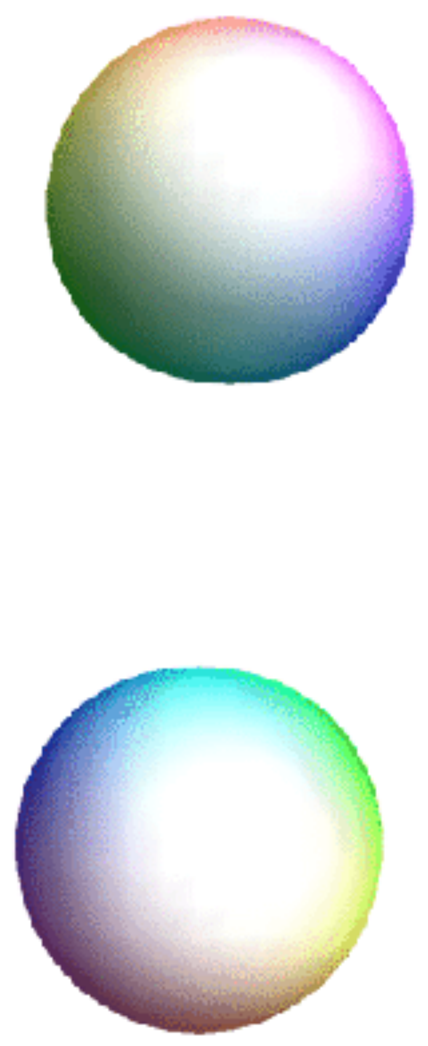
Overall, the low energy modes of a skyrmion are : 3 translations, 3 orientations, 1 size mode and a “vector meson size mode”.

=> To describe 2-skyrmion dynamics need a 16-dimensional space (11 can be dealt with pretty easily).

Discovered this using instantons and ADHM data!

Instantons and Skyrmions

Something more particular. There exists a special path in the 2-skyrmion space:



Spin: anti-clockwise
Ang mom: clockwise

Instantons and Skyrmions

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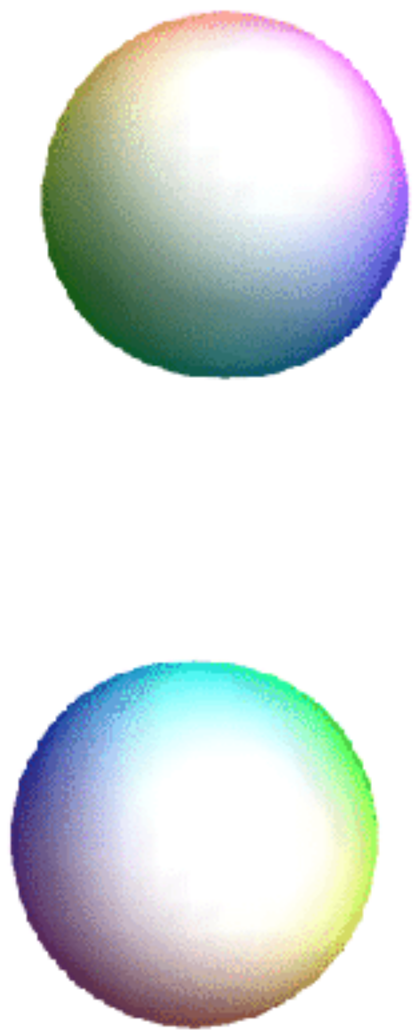

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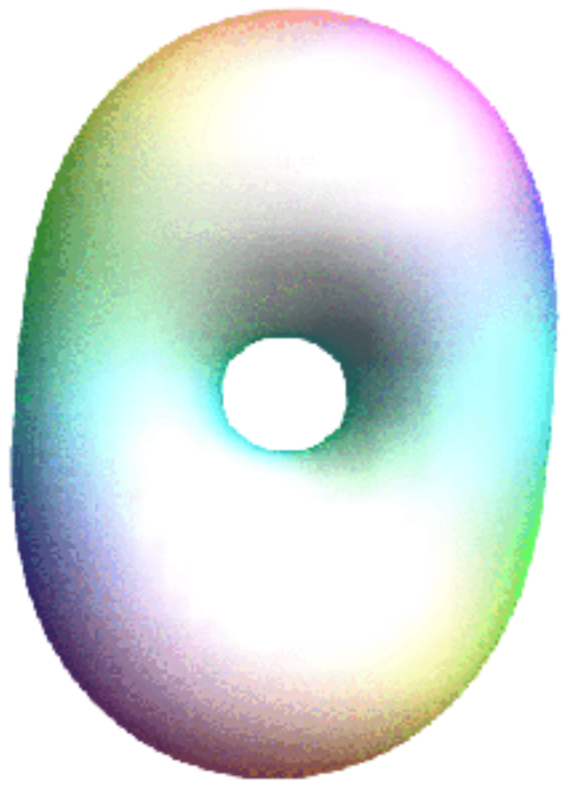

Barely deforms when close

Instantons and Skyrmions

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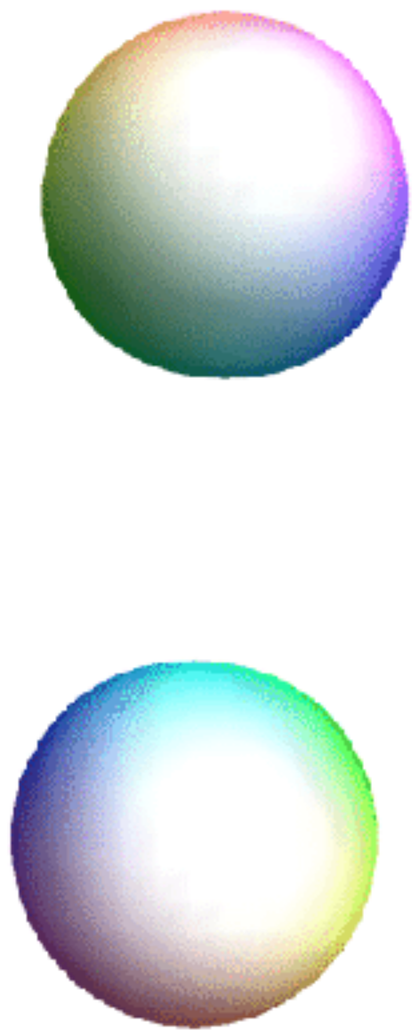

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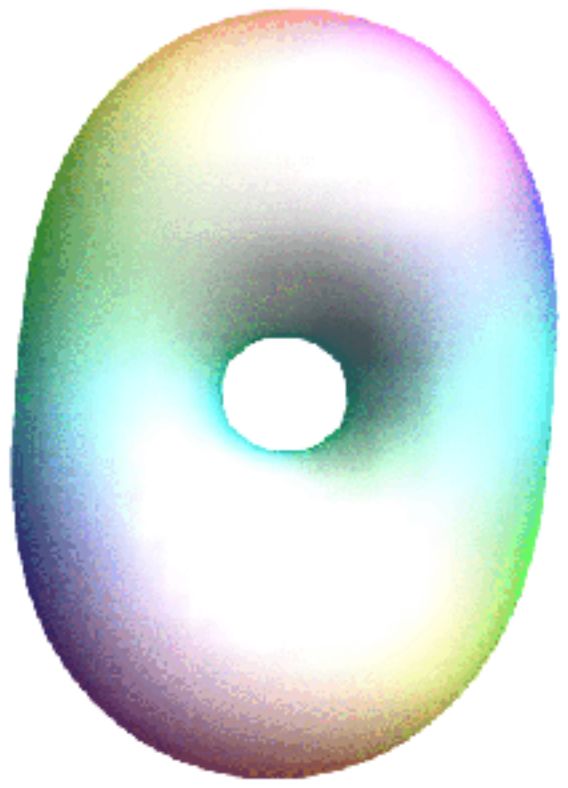

Barely deforms when close

Does nothing at the torus
=> path has zero length

Instantons and Skyrmions

Physically: a path where spin and angular momentum are correlated is unexpectedly short.

Short paths = high momentum = high energy

=> states with angular momentum and spin aligned have unexpectedly high energy

Short path = metric information

Can then do a semi-classical quantisation on the “moduli space” and follow this information through the calculation.

Instantons and Skyrmions

In a semi-classical quantisation, you are trying to derive the phenomenological nucleon nucleon interaction potentials.

This is hard.

The punchline:

The isoscalar spin-orbit potential from the calculation is negative, due to the short path:

The metric information

$$\frac{\hbar^2 A^1_{0;12}}{r\Lambda M} \in V^{IS}_{LS}(r),$$

which matches phenomenological models.

Instantons and Skyrmions

The key point: detailed geometric information about the moduli space can have genuine physical consequences.

In this case: a short path in the instanton moduli space affects the spin-orbit potential in the nucleon-nucleon interaction.

I think that studying the detailed structure of moduli spaces is very fun.

And I hope this talk has convinced you that it can be useful, too!

Soliton moduli spaces

Chris Halcrow - KTH

Counting skyrmion moduli: 2103.15669 (with Thomas Winyard)
Skyrmions from ADHM data: 2110.15190 (with Josh Cork)
The nucleon-nucleon force from instantons: 2208.04863 (with Derek Harland)