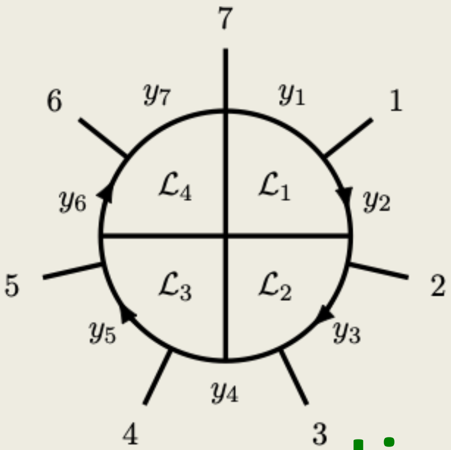
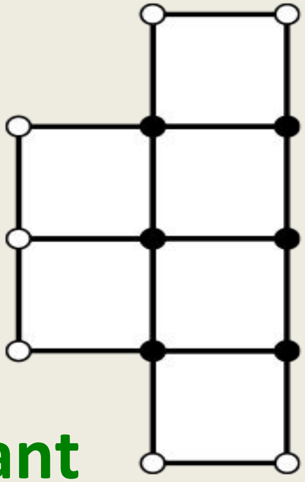
Landau Singularities of Ziggurat Graphs



Anastasia Volovich

Brown University



Lippstreu, Spradlin, Yelleshpur Srikant

2211.16425 + to appear

Prague Spring Amplitudes Workshop, May 2023

Outline

1. Some results that I will not talk about:
   - EFTs with Celestial duals
   - Coon amplitude
   - One-loop n-gon in n-dim

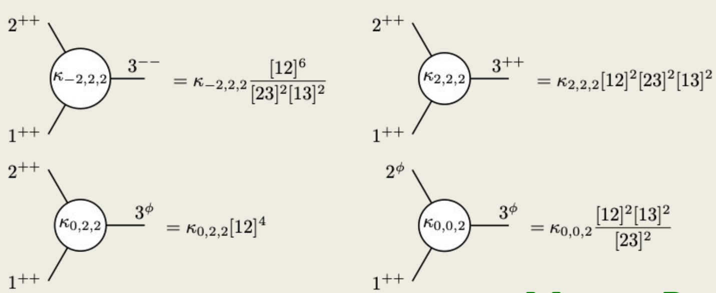
2. N=4 Yang-Mills symbol alphabet (review)

3. Symbol alphabet from Landau equations

1.1. EFTs with Celestial Duals

- **Strominger showed that the algebra of conformally soft graviton operators in CCFT is w-algebra.**
- **We computed the modification of the w-algebra due to non-minimal couplings and found that the Jacobi identity is satisfied only when the coupling constants obey the constraints:**

$$(\kappa_{-2,2,2} - \kappa_{0,0,2})\,\kappa_{0,2,2} = 0\,, \qquad (\kappa_{-2,2,2} - \kappa_{0,0,2})\,\kappa_{0,0,2} = 0$$
$$3\kappa_{0,2,2}^2 = 10\,\kappa_{-2,2,2}\,\kappa_{2,2,2}\,.$$



Mago, Ren, Yelleshpur Srikant, AV

1.1. EFTs with Celestial Duals

- **These coupling constants constraints can be derived directly from the associativity of the celestial OPEs.** Ren, Spradlin, Yelleshpur Srikant, AV
- **All four-point amplitudes constructible solely from holomorphic or anti-holomorphic three-point amplitudes vanish on the support of these constraints:** $\mathcal{A}_4\left(1^{++}, 2^{++}, 3^{++}, 4^{++}\right)$ $\mathcal{A}_4\left(1^{++}, 2^{++}, 3^{++}, 4^{\phi}\right)$
- **All theories with N≥1 SUSY and stable vacua satisfy these constraints.**

Ball, Yelleshpur Srikant, AV, in progress

1.2. On Unitarity of the Coon Amplitude

- **Coon amplitude**

Bhardwaj, De, Spradlin, AV 2212.00764

$$\mathcal{A}^C(s,t) = (1-q)\exp\left(\frac{\log\sigma\log\tau}{\log q}\right)\prod_{n=0}^{\infty}\frac{\left(1-\frac{q^n}{\sigma\tau}\right)\left(1-q^{n+1}\right)}{\left(1-\frac{q^n}{\sigma}\right)\left(1-\frac{q^n}{\tau}\right)} ,$$

$$\sigma = 1-(1-q)(\alpha_0+s) , \qquad \tau = 1-(1-q)(\alpha_0+t) ,$$

Remmen, Maldacena, Cheung, Geiser, Lindwasser, Figueroa, Tourkine Chakravarty, Maity, Mishra, Jepsen

- **Residue polynomials on each massive pole admits a partial wave expansion in terms of D-dimensional Gegenbauer polynomials**

$$\mathcal{A}^C(s,t) \to \frac{1}{s-[n]_q} \times R^C_{[n]_q}(t)$$

$$R^C_{[n]_q}(t) = \sum_j B^{(D)}_{n,j}(q)\, G^{(D)}_j(t)$$

1.2. On Unitarity of the Coon Amplitude

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Bhardwaj, De, Spradlin, AV 2212.00764

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$$\sigma = 1-(1-q)(\alpha_0+s)\,, \qquad \tau = 1-(1-q)(\alpha_0+t)\,,$$

- **We explored unitarity through partial wave expansion using q-calculus.**
- **Manifest positivity on leading (j=n-1) and sub-leading Regge (j=n-2) trajectories.**
- **Violation of unitarity for some (q, D) at sub-sub-leading Regge trajectories.**

1.3. One-loop n-gon in n-dimensions

- **The one loop hexagon integral in n-dim can be interpreted as the volume of a hyperbolic (n-1)-simplex.**
- **Such simplex can be dissected into *orthoschemes*, a special type of hyperbolic symplex for which the bounding hyperplanes can be ordered in such a way that they are mutually orthogonal.**
- **Rudenko 2012.05599, showed that the volume of the orthoscheme can be written as the polylog function of cross-ratios.**

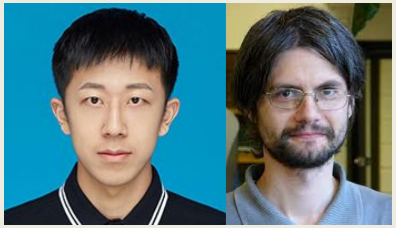
1.3. One-loop n-gon in n-dimensions

- **An explicit formula for the volume of a hyperbolic orthoscheme is given in terms of alternating polylogs**

$$\mathrm{ALi}_{m_1,\ldots,m_k}(\varphi_1,\ldots,\varphi_k) := \sum_{\epsilon_1,\ldots,\epsilon_k \in \{-1,1\}} \left(\prod_{i=1}^{k} \frac{\epsilon_i}{2} \right) \mathrm{Li}_{m_1,\ldots,m_k}(\epsilon_1 \sqrt{\varphi_1}, \ldots, \epsilon_k \sqrt{\varphi_k})$$

- **We can compute the volume of our full simplex by slicing it into orthoschemes recursively.**

Ren, Spradlin, Vergu AV to appear



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2. N=4 Yang-Mills symbol alphabet (review)

3. Symbol alphabet from Landau equations

Planar N=4 Yang-Mills Amplitudes

- Planar N=4 Yang-Mills scattering amplitudes have been computed to very high loop order.

- They have many remarkable properties, that have sparked interest from mathematicians working on combinatorics, algebraic geometry and number theory.

- At the same time, several methods that have been developed for N=4 Yang-Mills are directly applicable to, and have greatly aided, computations relevant to actual experiments.

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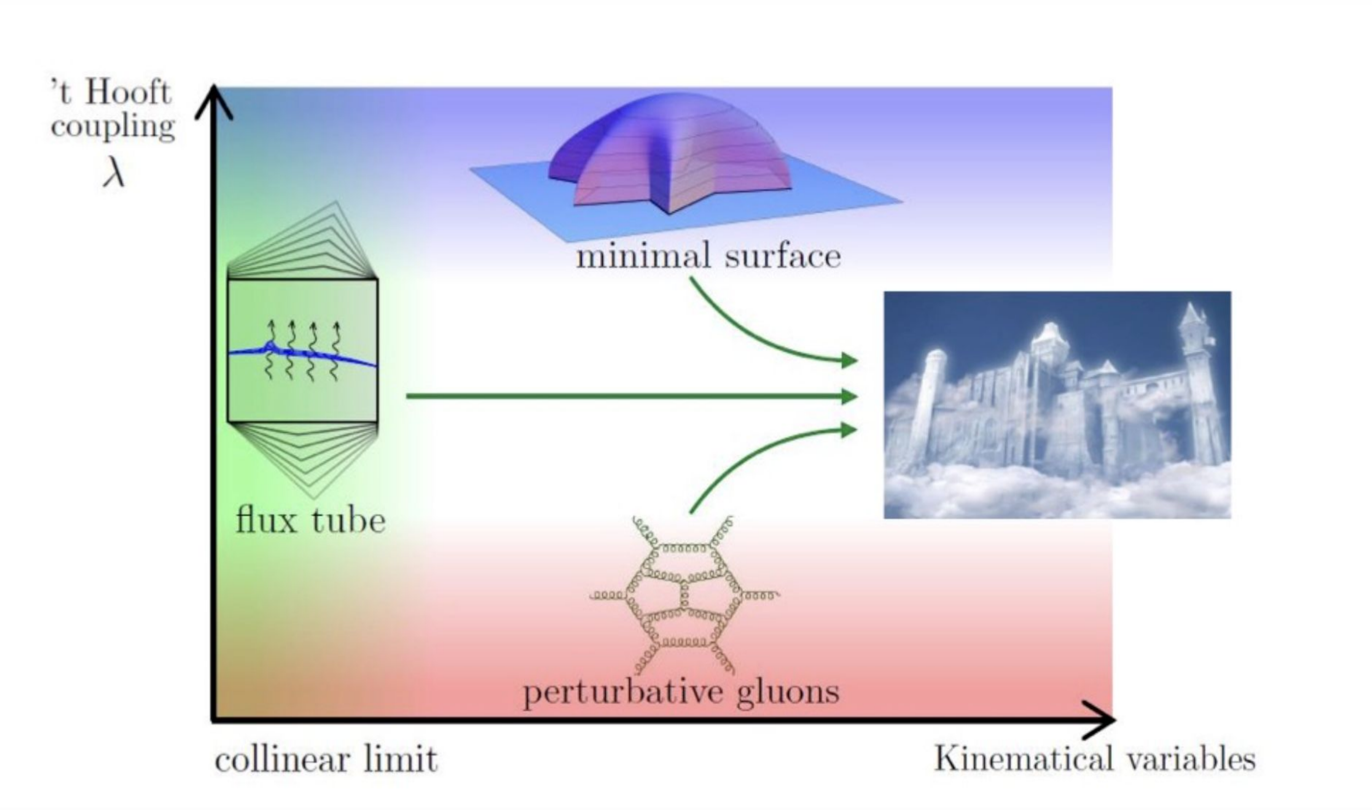
Solving N=4 super-Yang-Mills theory via Scattering Amplitudes: January 8, 2024 – March 8, 2024

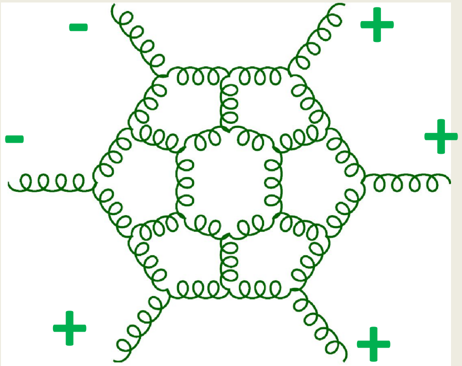
- Benjamin Basso (Ecole Normale Superieure),
- Lance Dixon (SLAC/Stanford U.)
- Jaroslav Trnka (UC Davis)
- Anastasia Volovich (Brown)

This program will bring together physicists and mathematicians with expertise in different facets of particle scattering in planar N=4 super-Yang-Mills theory, in order to try to solve the theory for generic values of the coupling and kinematical variables.



Status: N=4 Yang-Mills amplitudes

n-point L-loop MHV/NMHV



- n<6 all loops — Bern, Dixon, Smirnov 0505205
- n=6 through 7-loops
- n=7 through 4-loops — Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, McLeod, Papathanasiou, review: 2005.06735
- All n MHV through 2-loops — Caron-Huot 1105.5606
- n=8 MHV through 3-loop — Li, Zhang 2110.00350
- n=8, 9 NMHV through 2-loops — He, Li, Zhang 1911.01290 2009.11471

6 and 7-points: Amplitudes Bootstrap

Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, Papathanasiou, 2005.06735

Write down the answer as linear combo of functions and then determine the coefficients by solving a system of linear constraints.

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. $\mathscr{H}_6$	6	27	105	372	1214	3692?
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final-entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear	(0,0)	(0,0)	$(0^*,0^*)$	$(0^*,2^*)$	$(1^{*3},5^{*3})$	$(6^{*2},17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^{*2},2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^*,0^{*2})$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	$(1,0^*)$
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. Full MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. T^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Symbol Alphabet

- **One input into the 6 and 7-points amplitudes bootstrap is the symbol alphabet**

- **6-points: 15 letters <a a+1 b c>**
- **7-points: 49 letters <a a+1 b c>**

<1(23)(45)(67)>

<1(27)(34)(56)> +cyclic

Goncharov Spradlin Vergu AV
Golden Goncharov Spradlin Vergu AV
Caron-Huot

8, 9, ..n-points: $\bar{Q}$ equation

$$\bar{Q}_a^A = \sum_{i=1}^{n} \chi_i^A \frac{\partial}{\partial Z_i^a}$$

$$\bar{Q}_a^A R_{n,k} = \frac{\Gamma_{\text{cusp}}}{4} \ \text{Res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left(\mathrm{d}^{2|3} \mathcal{Z}_{n+1}\right)_a^A [R_{n+1,k+1} - R_{n,k} R_{n+1,1}^{\text{tree}}] + \text{cyclic}$$

Caron-Huot 1105.5606

He, Li, Zhang 1911.01290
2009.11471

Li, Zhang 2110.00350

8-point symbol alphabet

180+24 RATIONAL LETTERS Li, Zhang 2110.00350

- $\binom{8}{4} - 2 = 68$: all $\langle abcd \rangle$ except $\langle 1357 \rangle$ and $\langle 2468 \rangle$;
- 1 cyclic class of $\langle 12(345) \cap (678) \rangle$;
- 7 cyclic classes of $\langle 1(ij)(kl)(mn) \rangle$ with $2 \leq i < j < k < l < m < n \leq 8$;
  5 cyclic classes of $\langle 1(28)(kl)(mn) \rangle$ with $2 < k < l < m < n < 8$;
- 5 cyclic classes of $\langle \bar{2} \cap \bar{4} \cap (568) \cap \bar{8} \rangle$, $\langle \bar{2} \cap \bar{4} \cap \bar{6} \cap (681) \rangle$, $\langle (127) \cap (235) \cap \bar{5} \cap \bar{7} \rangle$, $\langle (127) \cap \bar{3} \cap (356) \cap \bar{7} \rangle$, $\langle \bar{2} \cap (278) \cap (346) \cap \bar{6} \rangle$.

$$\bar{a} = (a{-}1, a, a{+}1)$$
$$\langle a(bc)(de)(fg) \rangle := \langle abde \rangle \langle acfg \rangle - \langle acde \rangle \langle abfg \rangle,$$
$$\langle ab(cde) \cap (fgh) \rangle := \langle abde \rangle \langle cfgh \rangle + \langle abec \rangle \langle dfgh \rangle + \langle abcd \rangle \langle efgh \rangle,$$
$$\langle (a_1b_1c_1) \cap (a_2b_2c_2) \cap (a_3b_3c_3) \cap (a_4b_4c_4) \rangle := \langle (a_1b_1c_1) \cap (a_2b_2c_2), (a_3b_3c_3) \cap (a_4b_4c_4) \rangle$$

2 x 9 ALGEBRAIC LETTERS (SQUARE ROOTS)

$$\Delta_{1357} = (\langle 1256 \rangle \langle 3478 \rangle - \langle 1278 \rangle \langle 3456 \rangle - \langle 1234 \rangle \langle 5678 \rangle)^2 - 4 \langle 1234 \rangle \langle 3456 \rangle \langle 5678 \rangle \langle 1278 \rangle$$

\+ 1 cyclic

9-point symbol alphabet

He, Li, Zhang 2009.11471

59 x 9 RATIONAL LETTERS

- 13 cyclic classes of $\langle 12kl\rangle$ for $3 \leq k < l \leq 8$ but $(k,l) \neq (6,7),(7,8)$;
- 7 cyclic classes of $\langle 12(ijk)\cap(lmn)\rangle$ for $3 \leq i < j < k < l < m < n \leq 9$;
- 8 cyclic classes of $\langle \bar{2}\cap(245)\cap\bar{6}\cap(691)\rangle$, $\langle \bar{2}\cap(346)\cap\bar{6}\cap(892)\rangle$, $\langle \bar{2}\cap(346)\cap\bar{2}\cap(782)\rangle$, $\langle \bar{2}\cap(245)\cap\bar{7}\cap(791)\rangle$, $\langle \bar{2}\cap(245)\cap(568)\cap\bar{8}\rangle$, $\langle \bar{2}\cap(245)\cap(569)\cap\bar{9}\rangle$, $\langle \bar{2}\cap(245)\cap(679)\cap\bar{9}\rangle$, $\langle \bar{2}\cap(256)\cap(679)\cap\bar{9}\rangle$;
- 10 cyclic classes of $\langle 1(i\,i{+}1)(j\,j{+}1)(k\,k{+}1)\rangle$ for $2 \leq i,\ i+1 < j,\ j+1 < k \leq 8$;
- 6 cyclic classes $\langle 1(2i)(j\,j{+}1)(k9)\rangle$ for $3 \leq i < j,\ j+1 < k \leq 8$, but $(i,k) \neq (3,8),(4,7)$;
- 14 cyclic classes of $\langle 1(29)(ij)(k\,k{+}1)\rangle$ for $3 < i < j \leq 8,\ 3 \leq k \leq i-2$ or $j+1 \leq k \leq 7$;
- 1 cyclic class of $\langle 1,(56)\cap\bar{3},(78)\cap\bar{3},9\rangle$.

11 x 9 ALGEBRAIC LETTERS (SQUARE ROOTS)

$$\Delta_{1357} = (\langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle)^2 - 4\langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle \quad + \quad 8 \text{ cyclic}$$

Symbol Letters from Mathematics

- Cluster Algebras — Golden, Goncharov, Spradlin, Vergu, AV
- Tropical Geometry — Drummond, Foster, Gurdogan, Kalousios; Henke, Papathanasiou
- Dual Polytops — Arkani-Hamed, Lam, Spradlin
- Plabic Graphs — Mago, Schreiber, Spradlin, Srikant, AV; He, Li
- Tensor Diagrams — Ren, Spradlin, AV
- Scattering Diagrams — Herderschee
- Schubert Problem — Yang
- A-determinant — Dlapa, Helmer, Papathanasiou, Tellander

Symbol Letters from Physics

Can we have a general derivation of these letters?

Symbol letters encode information about the location of singularities of an amplitude.

Landau formulated the sufficient conditions for a Feynman integral to have singularities decades ago…

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Landau Singularities

Landau 1959, ELOP

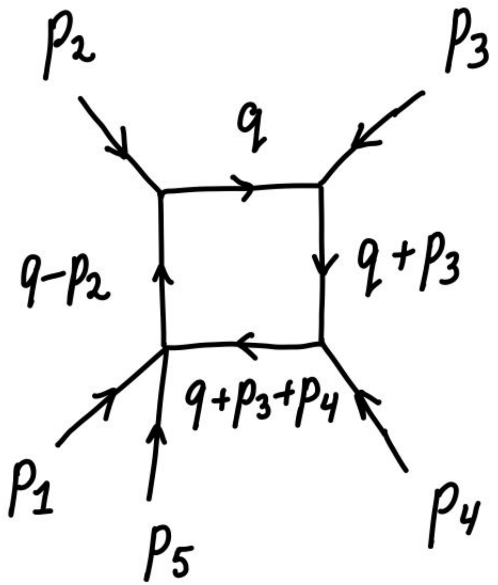
Landau Equations for a given Feynman integral – set of kinematic constraints that are necessary for the appearance of a pole or a branch point in the integrated function

$$\sum_{i \in \text{loop}} \alpha_i q_i^\mu = 0 \quad \forall \text{ loops},$$

$$\alpha_i (q_i^2 - m_i^2) = 0 \quad \forall i.$$

Landau Singularities occur for external momenta such that the Landau equations have solutions.

Example



$$\int \frac{d^4 q}{q^2 (q-p_2)^2 (q+p_3)^2 (q+p_3+p_4)^2}$$

$$\begin{cases} (q-p_2)^2 = q^2 = (q+p_3)^2 = (q+p_3+p_4)^2 \\ \alpha_1 (q-p_2) + \alpha_2 q + \alpha_3 (q+p_3) + \alpha_4 (q+p_3+p_4) = 0 \end{cases}$$

$$\to \quad (p_2+p_3)^2 (p_3+p_4)^2 = 0$$

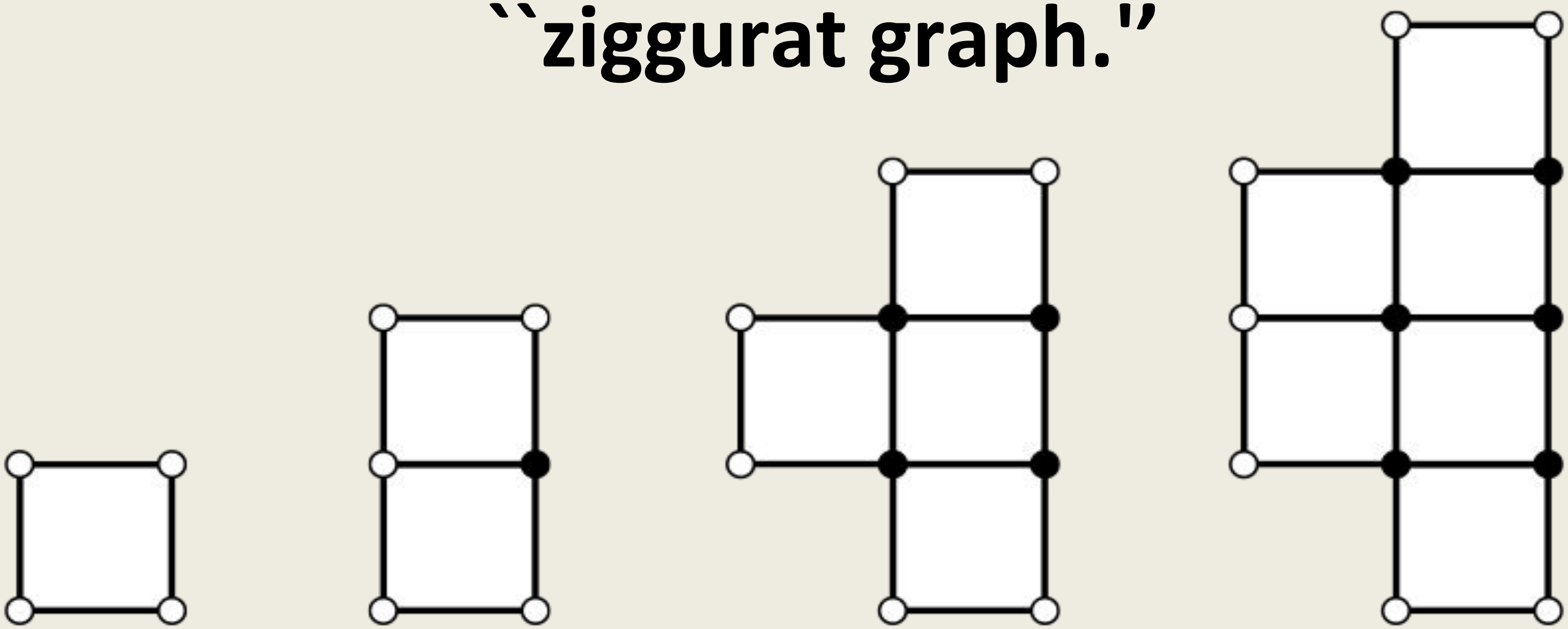
OR

$$\langle 1234 \rangle \langle 2345 \rangle = 0$$

Ziggurat Graphs

Prlina, Spradlin, Stanojevic

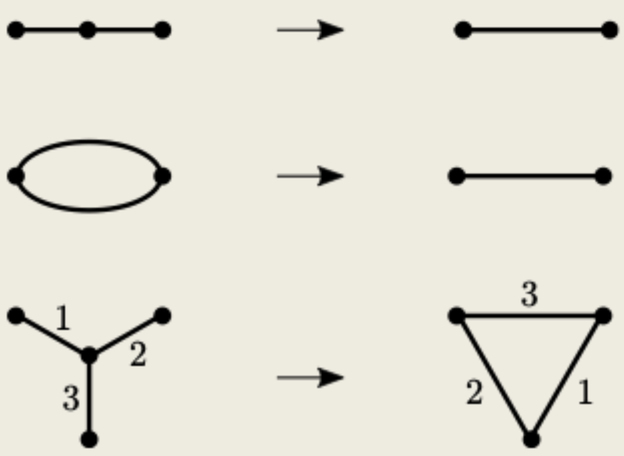
Landau singularities of any n-particle amplitude in any massless, planar theory are a subset of those of a special type of ``ziggurat graph.''



Landau singularities from Ziggurats

- Landau equations are invariant under circuit moves

Prlina, Spradlin, Stanojevic



- Any graph can be reduced to ziggurat (or a relaxation) by these moves.

Gitler

Six-point Ziggurat

3

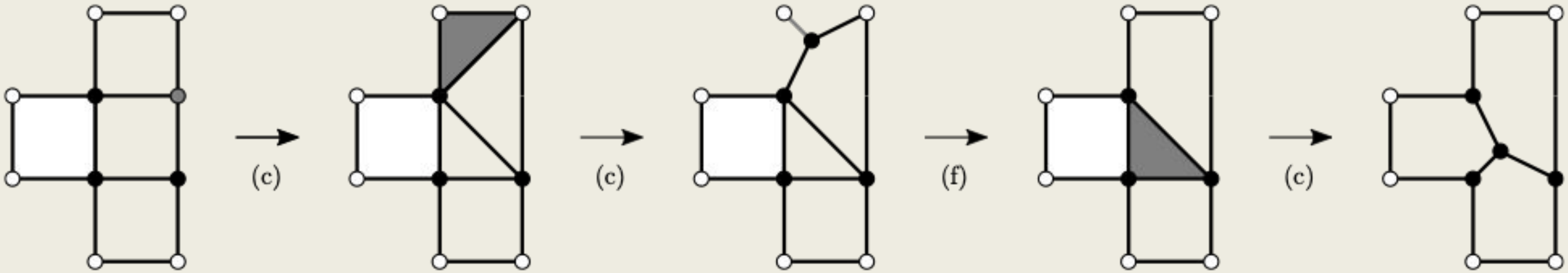
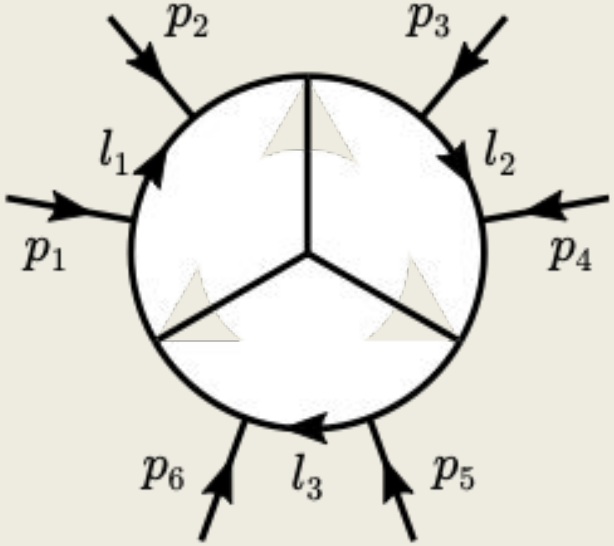


FIG. 3. The six-terminal ziggurat graph can be reduced to a three loop graph by a sequence of three Y-Δ reductions and one FP assignment. In each case the vertex, edge, or face to be transformed is highlighted in gray.



Matches six-point symbol letters!

Seven-point Ziggurat

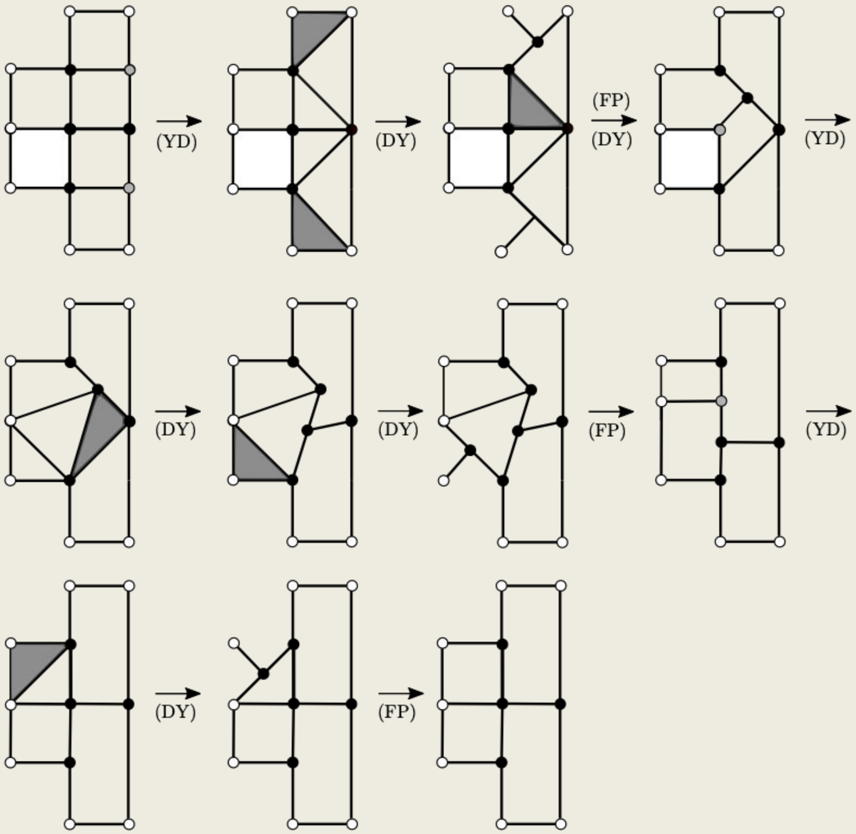
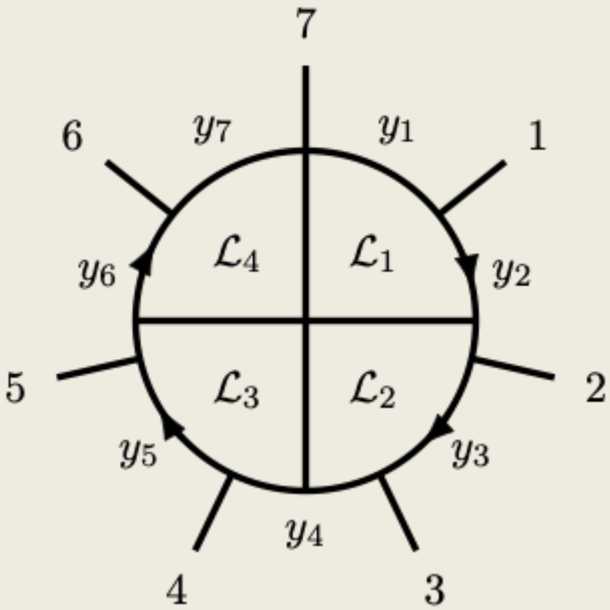


Figure 3: A sequence of graphical moves (see [15]) that transforms the 7-point ziggurat graph Fig. 1(d) into the wheel graph Fig. 2(b). YD indicates wye-delta transformation(s) on the node(s) shaded in grey; DY indicates delta-wye transformations(s) on the triangle(s) shaded in grey, and FP indicates a trivial contraction of external edges.



Seven-point Ziggurat

- **We computed leading Landau singularities and found agreement with all but 7 letters.**

Lippstreu, Spradlin, AV

- **We are now analysing subleading Landau singularities.**

Lippstreu, Spradlin, Yelleshpur Srikant, AV, to appear

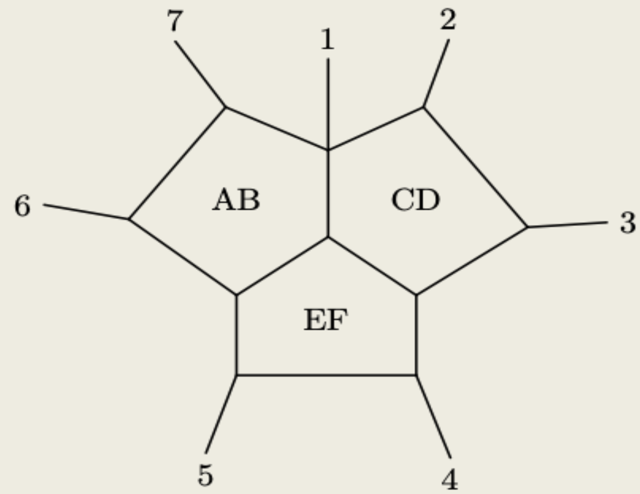


Figure 2: Graph 1. The Landau equations contain the letter $\langle 6(17)(25)(34)\rangle$, which is outside the heptagon symbol alphabet.

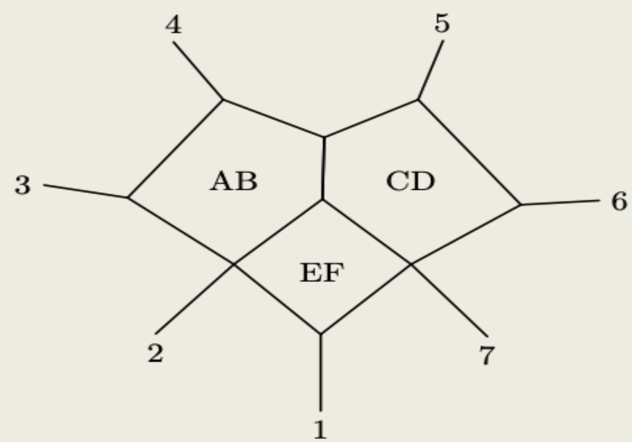


Figure 3: Graph 2. The Landau equations contain the letter $\langle 1(27)(36)(45)\rangle$, which is outside the heptagon symbol alphabet.

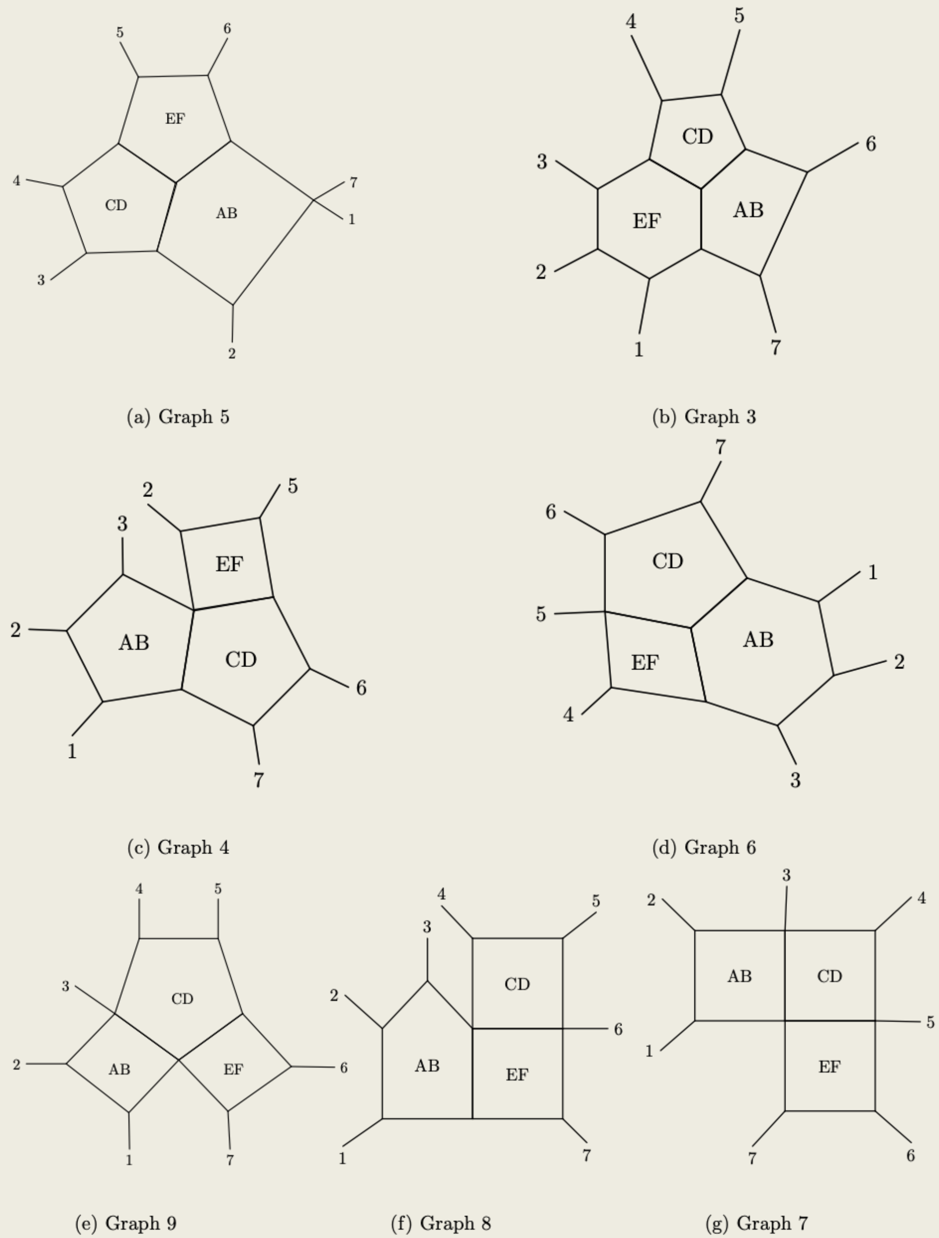


Figure 6: Seven other three-loop graphs we analyzed and did not find any letters beyond the heptagon symbol alphabet.

Seven-point Ziggurat

- **We computed leading Landau singularities and find agreement with all but 7 letters.**
- **We are now analysing subleading Landau singularities.**
- **We found the missing 7, and extra letters**

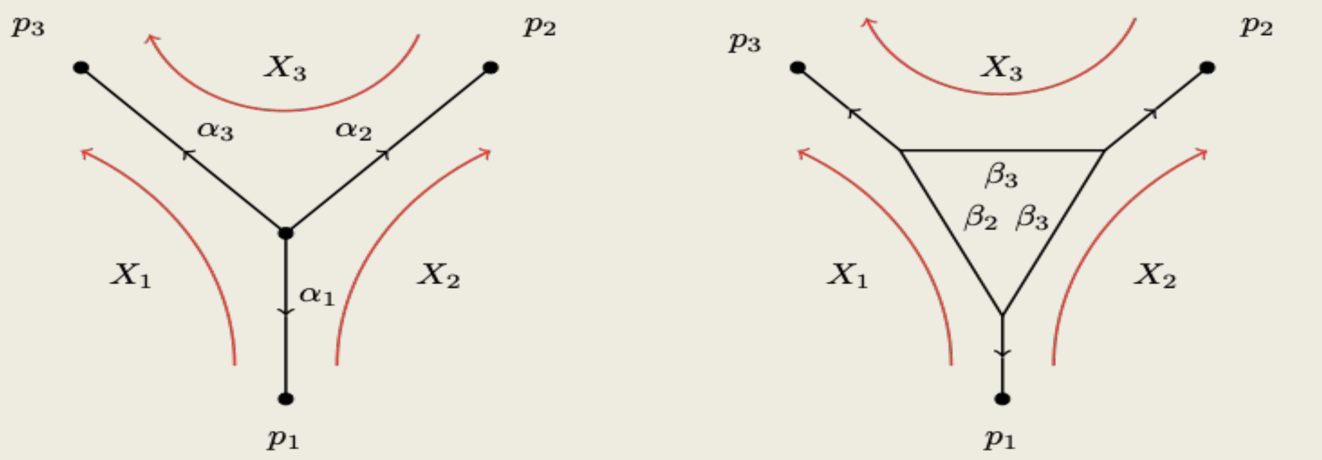
  <6(17)(25)(34)>and<1(27)(36)(45)>
- **They appear in certain 3-loop Feynman integrals and it will be interesting to see if they appear in higher loop amplitudes.**

Lippstreu, Spradlin, Yelleshpur Srikant, AV, to appear

Violation of Y->Delta

Lippstreu, Spradlin, Yelleshpur Srikant, AV, to appear

- Along the way, we found that there are some solutions that violate Y-Delta equivalence.



- When $X_1 = X_2 = X_3 = 0$, the Y has extra solutions that are not present in the Delta.

Conclusions

- **Planar N=4 Yang-Mills amplitudes have been computed to a very high order via bootstrap.**

- **Their singularity structure is an important input to the bootstrap.**

- **We were hoping to derive all n singularities from Landau analysis.**

- **Future: all n, all L, beyond polylogs, beyond planar, beyond N=4….**