TOWARDS THE FUNCTION SPACE FOR TWO-LOOP SIX-PARTICLE FEYNMAN INTEGRALS

BASED ON ARXIV:2210.13505 WITH J. HENN, A. MATIJAŠIĆ

AND WORK IN PROGRESS WITH J. HENN, A. MATIJAŠIĆ, T. PERARO, Y. ZHANG, Y. XU

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PRAGUE SPRING
AMPLITUDE WORKSHOP

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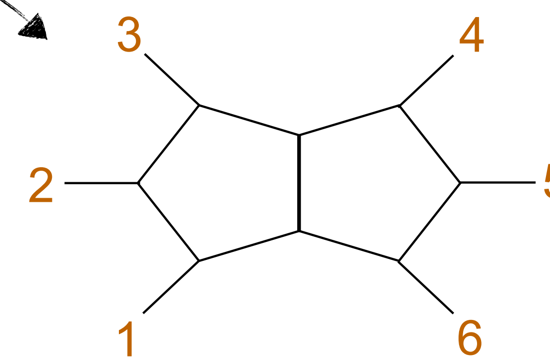
MOTIVATION

- Precision measurements have reached percent level: need two-loop integrals
- State-of-the-art: two-loop five-point Feynman integrals (five on-shell legs, four on-shell legs and one off-shell leg) [Gehrmann, Henn, Lo Presti '18; Abreu, Dixon, Herrmann, Page, Zeng '18; Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18; Abreu, Ita, Moriello, Page, Tschernow, Zeng '20; Abreu, Ita, Page, Tschernow '21]
- Very little is known about two-loop six-particle processes in general theories
- To understand their function space: find **planar two-loop hexagon alphabet**
- First time at six points:
  external kinematics satisfy **Gram determinant constraint**

NOTATION AND KINEMATICS

$$I^{d_0}(a_1, \ldots, a_{13}) = e^{2\epsilon\gamma_E} \int \frac{\mathrm{d}^{d_0-2\epsilon} l_1 \mathrm{d}^{d_0-2\epsilon} l_2}{i\pi^{(d_0-2\epsilon)}} \frac{1}{D_1^{a_1} \ldots D_{13}^{a_{13}}}$$



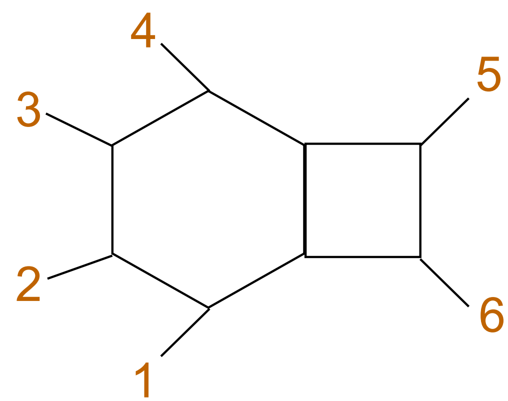
External momenta:

$$p_i^2 = 0, \quad i = 1, \ldots, 6 \qquad \sum_{i=1}^{6} p_i = 0 \qquad p_i \in \mathbb{R}^{D_{\text{ext}}}$$

- For $D_{ext} > 4$, nine independent Mandelstam invariants

$$\vec{v} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{123}, s_{234}, s_{345}\}$$

$$s_{ij} = (p_i + p_j)^2, \qquad s_{ijk} = (p_i + p_j + p_k)^2$$

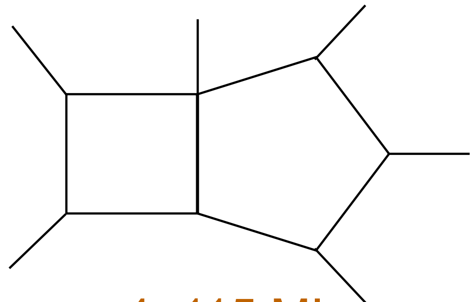


- For $D_{ext} = 4$, Gram determinant constraint

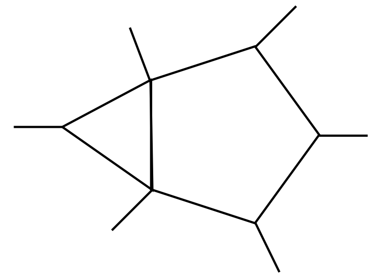
$$0 = G(p_1, p_2, p_3, p_4, p_5) = \det(p_i \cdot p_j), \qquad 1 \leq i, j \leq 5$$

GENUINE TWO-LOOP SIX-POINT MASSLESS PLANAR DIAGRAMS

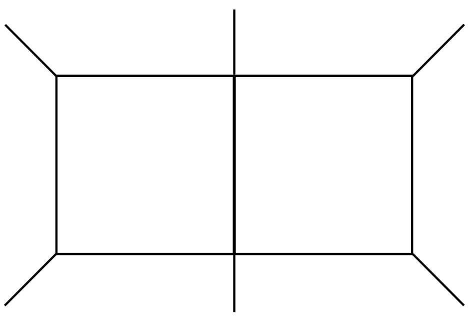
8+238 MI



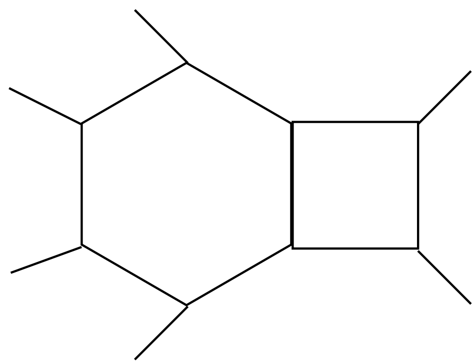
4+115 MI



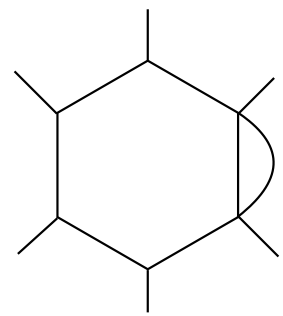
2+43 MI



9+59 MI



3+188 MI



2+31 MI

CANONICAL DIFFERENTIAL EQUATION

- By choosing a UT basis, we find canonical differential equations [Henn '13]

$$\mathrm{d}\tilde{I} = \epsilon \left[\sum_a c^a_{jk} \mathrm{d}\log(W_a) \right] \tilde{I}$$

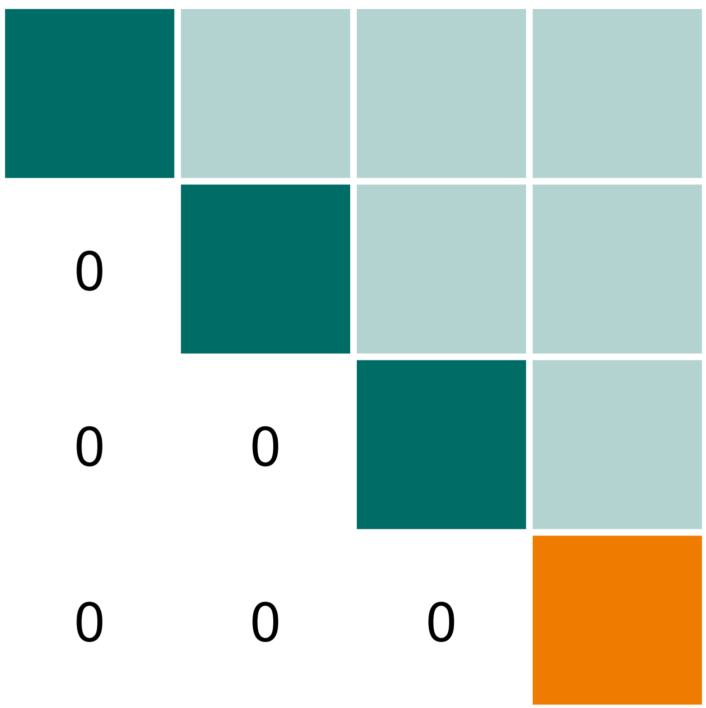
- $\mathrm{d}\tilde{I}$: vector of N basis integrals
- c^a_{jk}: constant $N \times N$ matrices
- $W_a \in \mathbb{A}$: algebraic functions of kinematic variables

- The alphabet $\mathbb{A}$ encodes the singularities and functional structure of the integrals
  - Can be decomposed into even and odd letters

$$\frac{P - \sqrt{Q}}{P + \sqrt{Q}}$$

DIFFERENTIAL EQUATION BLOCKS IN CANONICAL FORM

$$\mathrm{d}I = \epsilon \mathrm{d}AI$$



Genuine six-point blocks [Henn, Peraro, Xu, Zhang '21]

Five-particle integrals with one off-shell leg [Abreu, Ita, Moriello, Page, Tschernow, Zeng '20]

Off-diagonal blocks (to be computed)

[Work in progress with Henn, Matijašić,, Peraro, Xu, Zhang]

TWO-LOOP ALPHABET LETTERS

- From the knowledge of all even letters W_i and leading singularities $\sqrt{Q_k}$, we want to find all odd letters of the form:

W_i → 388 $\sqrt{Q_k}$ → 48

$$W_{odd} = \frac{P - \sqrt{Q}}{P + \sqrt{Q}}$$

where $Q \in \{Q_i\} \cup \{Q_i Q_j\}$

- Observation: if $W_{odd} \in \mathbb{A}$, it holds:

$$\left(P - \sqrt{Q}\right)\left(P + \sqrt{Q}\right) = c \prod_i W_i^{e_i}, \quad W_i \in \mathbb{A}_{even}$$

→

$$P(\vec{v})^2 = Q(\vec{v}) + c \prod_i W_i^{e_i}$$

EXAMPLE: ODD LETTER CONSTRUCTION

- Consider the odd letters for the three-mass one-loop triangle:

$$\frac{P - \sqrt{\lambda(s_{12}^2, s_{34}^2, s_{56}^2)}}{P + \sqrt{\lambda(s_{12}^2, s_{34}^2, s_{56}^2)}}$$

where $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$

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- Easy to see that there are three simple ways to turn λ into a perfect square:

$$\lambda(s_{12}^2, s_{34}^2, s_{56}^2) + 4 s_{12}^2 s_{34}^2 = (s_{12} + s_{34} - s_{56})^2$$

- However, for embeddings in bigger alphabets, also the following is relevant:

$$\lambda + 4(s_{12}s_{56} - s_{12}s_{123} + s_{34}s_{123} - s_{56}s_{123} + s_{123}^2) = (s_{12} - s_{34} + s_{56} - 2s_{123})^2$$

TWO-LOOP ALPHABET LETTERS

- From the knowledge of all even letters W_i and leading singularities $\sqrt{Q_k}$, we want to find all odd letters of the form:

388 48

$$W_{odd} = \frac{P - \sqrt{Q}}{P + \sqrt{Q}} \quad \text{where} \quad Q \in \{Q_i\} \cup \{Q_i Q_j\}$$

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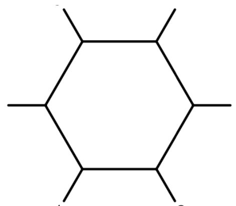
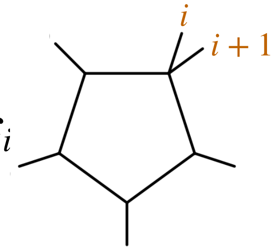
$$\left(P - \sqrt{Q}\right)\left(P + \sqrt{Q}\right) = c \prod_i W_i^{e_i}, \quad W_i \in \mathbb{A}_{even}$$

$$P(\vec{v})^2 = Q(\vec{v}) + c \prod_i W_i^{e_i}$$

- Two-loop alphabet has 884 letters so far (436 even + 448 odd)
  - 324 odd letters constructed from factorization condition

TAKING THE $D_{ext} \to 4$ LIMIT @ ONE-LOOP (2210.13505)

- Well known identity:  $= \sum_{i=1}^{6} c_i$ 

[Melrose '65; Bern, Dixon, Kosower '93; Binoth, Guillet, Heinrich '00]

$$\mathbb{A}_{\text{div}} = \{W_{40}, W_{95}, \ldots, W_{100}\}$$

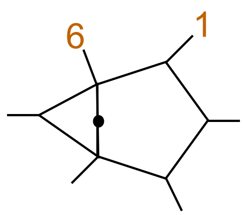
$$W_{40} \propto \frac{1}{G(p_1, p_2, p_3, p_4, p_5)}$$

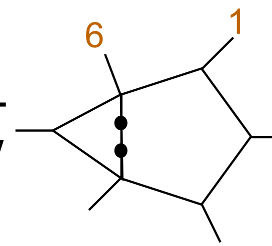
$$W_{95} = \frac{P - \sqrt{P^2 + G(p_1, p_2, p_3, p_4, p_5)}}{P + \sqrt{P^2 + G(p_1, p_2, p_3, p_4, p_5)}}$$

- Imposing vanishing of spurious divergence implies above identity!

TAKING THE $D_{ext} \to 4$ LIMIT FOR PENTA-TRI SECTOR

- On maximal cut, we have UT basis

$\tilde{I} = \{(2\epsilon - 1)\sqrt{Q_{18}}$ 

$, 4\sqrt{Q_{27}}$ 

$\}$

$$Q_{18} = G(p_1, p_2, p_3, p_5)$$
$$Q_{27} = \Delta_6$$

- With PDE matrix:

$$A = \frac{1}{2}\begin{pmatrix} -3(\log W_1 + \log W_2 + \log W_{382}) - \log W_{388} + 8\log W_{406} & \frac{1}{2}\log W_{795} \\ -6\log W_{795} & \log W_1 + \log W_2 + \log W_{382} + 3\log W_{388} - 8\log W_{415} \end{pmatrix}$$

$$\lim_{D_{ext}\to 4+\delta} W_{388} \to \delta, \qquad \lim_{D_{ext}\to 4+\delta} W_{795} \to \delta$$

- Leads to identity:

$$0 = 2\tilde{I}_1 - \tilde{I}_2$$

OPEN QUESTIONS

- Two-loop UT basis beyond the maximal cut?
- Have we constructed the full alphabet? Are all letters needed?
  - Final proof will be given by the differential equation!
- Can we use the alphabet to bootstrap interesting quantities?
  - e.g. six-point Wilson loops with Lagrangian insertion or certain six-point integrals

Thank you! Questions?

BACK-UP SLIDES

RELATION TO DUAL CONFORMAL HEXAGON ALPHABET

- The dual conformal 9-variable (A3 cluster algebra) letters from N=4 sYM can be written as

$$u_1, \quad u_2, \quad u_3, \quad 1-u_1, \quad 1-u_2, \quad 1-u_3, \quad y_1, \quad y_2, \quad y_3$$

$$u_1 = \frac{v_1 v_4}{v_7 v_9}, \quad u_2 = \frac{v_2 v_5}{v_8 v_7}, \quad u_3 = \frac{v_3 v_6}{v_9 v_8}, \quad \Delta = (1-u_1-u_2-u_3)^2 - 4u_1u_2u_3$$

$$y_1 = \frac{1+u_1-u_2-u_3-\sqrt{\Delta}}{1+u_1-u_2-u_3+\sqrt{\Delta}}, \quad y_2 = \frac{1+u_2-u_3-u_1-\sqrt{\Delta}}{1+u_2-u_3-u_1+\sqrt{\Delta}}, \quad y_3 = \frac{1+u_3-u_1-u_2-\sqrt{\Delta}}{1+u_3-u_1-u_2+\sqrt{\Delta}}$$

- They are contained in our alphabet:

$$u_1 = \frac{W_1W_4}{W_7W_9}, \quad u_2 = \frac{W_2W_5}{W_7W_8}, \quad u_3 = \frac{W_3W_6}{W_8W_9} \quad 1-u_1 = \frac{W_{31}}{W_7W_9}, \quad 1-u_2 = \frac{W_{32}}{W_7W_8}, \quad 1-u_3 = \frac{W_{33}}{W_8W_9}$$

$$y_1 = \frac{1}{W_{102}}, \quad y_2 = \frac{1}{W_{103}}, \quad y_3 = \frac{1}{W_{101}}$$