PennState

Adventures in Perturbation Theory

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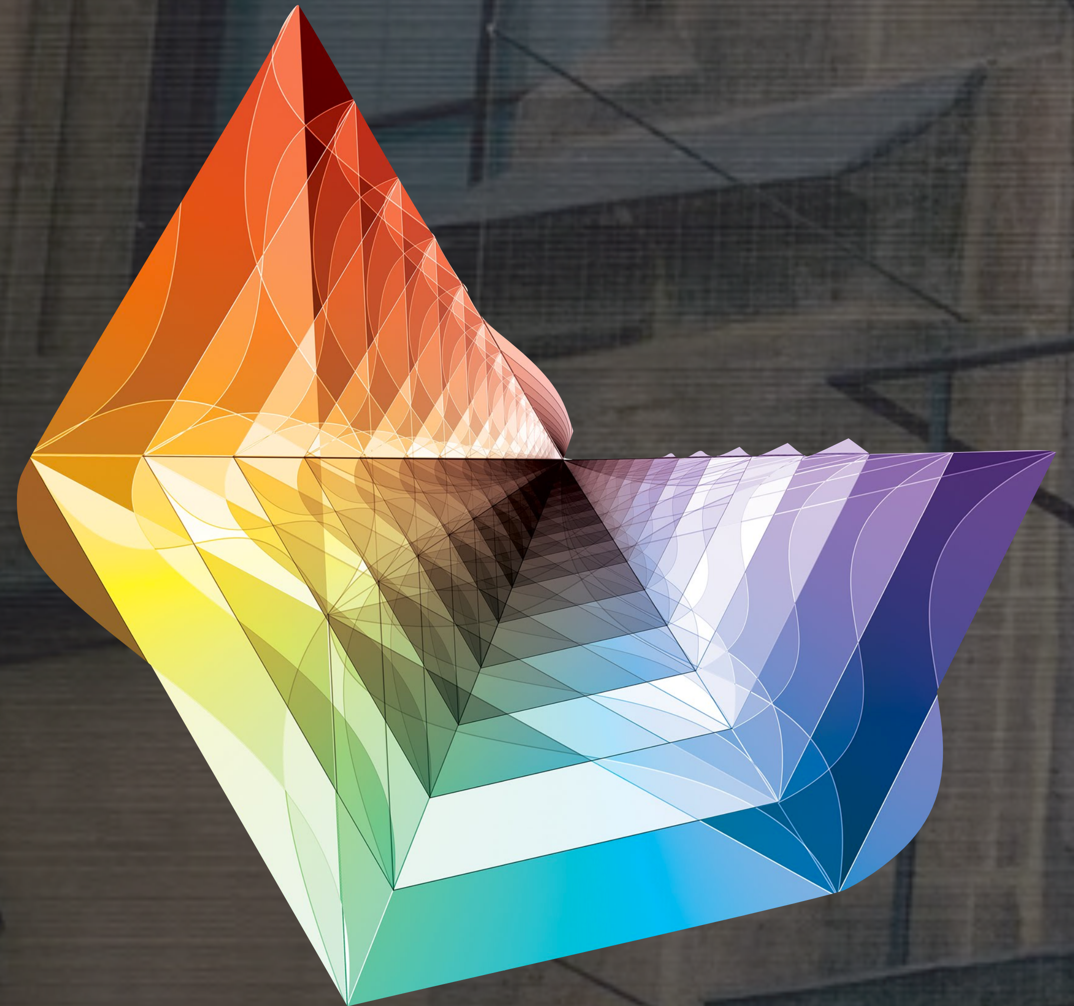


Prague Spring Amplitudes Workshop

Czech Technical University

18 May, 2023

erc

Roadmap

- ***Spiritus Movens***
  - observables in QFT *surprisingly* simple!
  - what do amplitudes *look like*—functionally?
  - why is perturbation theory so hard?

    —and how can we make it *easier*?
- **Generalized Unitarity**: ***a modern perspective***
  - *stratifying* theories and stratified Feynman integrand bases
  - what makes for a *good* Feynman integral (basis)?
- **Prescriptivity, Purity, and Polylogarithmicity**
  - Impurities, Calabi-Yau Geometries,…; tensions and resolutions

What Form do Observables Take?

- In a general (say, 4d) QFT, it would have ***recently been*** expected by most that observables took the following general form:

(general dimension d: $2 \mapsto \lfloor d/2 \rfloor$)

$$\mathcal{A} = \mathcal{A}^{\text{tree}} + \hbar\,\mathcal{A}^{(L=1)} + \hbar^2\,\mathcal{A}^{(L=2)} + \ldots + \hbar^L\,\mathcal{A}^{(L)} + \ldots$$

rational	+	(weight-2 polylogs)	+	(weight-4 polylogs)	+	⋯	+	(weight-2L polylogs)	+ ⋯
	+	(weight-1 polylogs)	+	⋮	+	⋯	+	⋮	
	+	rational	+	(weight-1 polylogs)	+	⋯	+	⋮	
			+	rational					

planar $\mathcal{N}=4$?

coefficients: weight-0, on-shell (gauge-invariant, etc.)

leading singularities

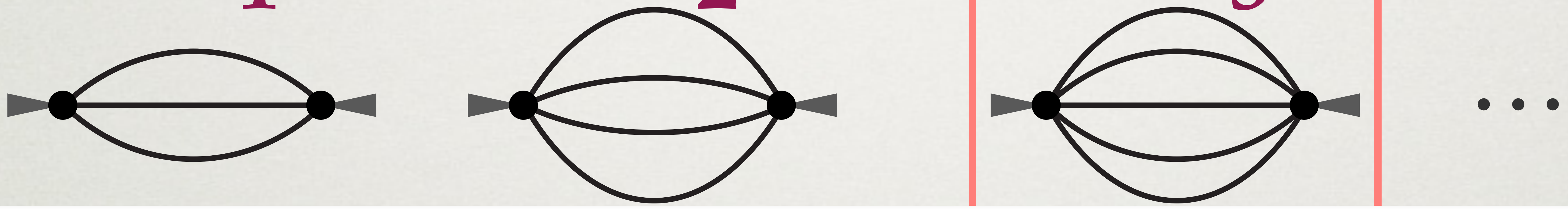
What Form do Observables Take?

- Unfortunately, many pesky counterexamples were to be found:

[Bloch, Kerr, Vanhove; Broadhurst;…]
[Doran, Harder, Thompson (2019)]

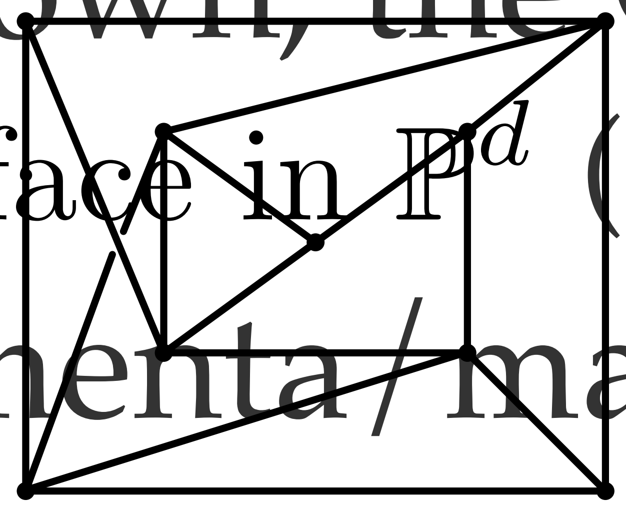
elliptic(-log) CY_1 K3 CY_2 CY_3

sunrises:



} 2d, masive

- In every instance known, the Calabi-Yau itself is **very** simple:
  - degree-$(d+1)$ hypersurface in $\mathbb{P}^d$ (or multiple-cover thereof)
- kinematic data (momenta/masses) control the moduli
  - often **extremely** *singular*

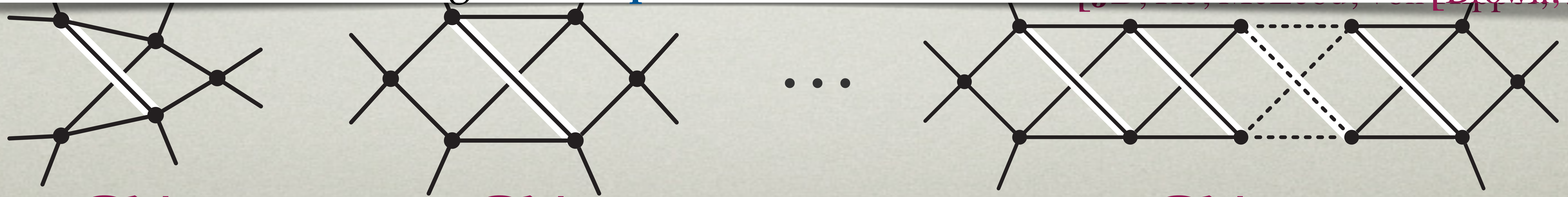


an 8-loop vacuum graph evaluating to a **K3 period**

(2018)]

[Brown, Schnetz (2018)]

tardigrades:



CY_2 CY_4 $CY_{2(L\text{-}1)}$

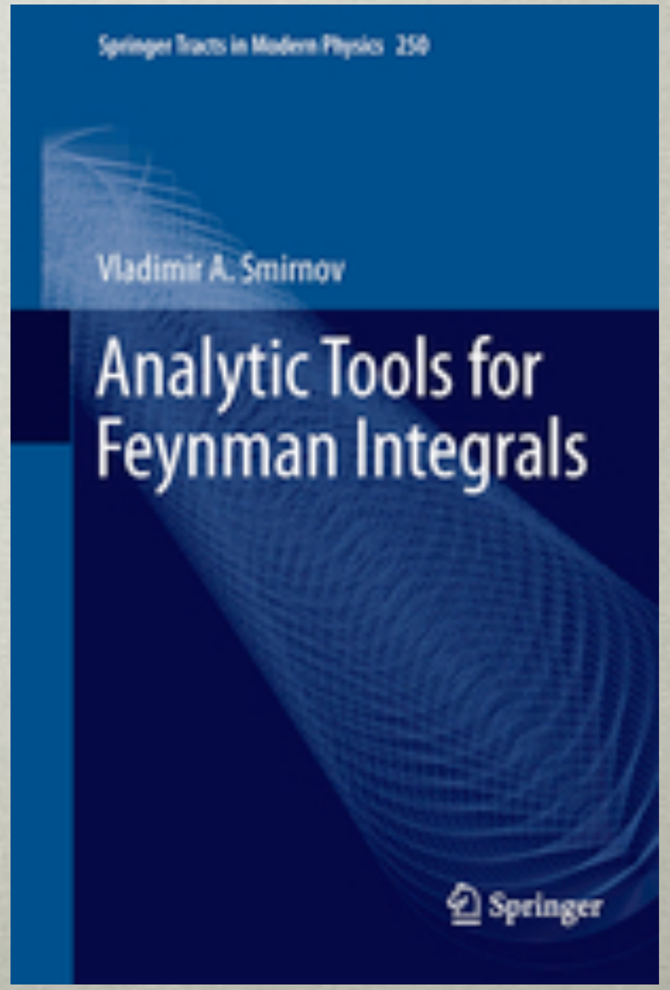
Why is Perturbation Theory so Hard?

- **Feynman integrals** (esp. with *scalar* numerators) are ***horrible***
  - difficult to integrate, explosive in number, non-physical,…

- **Regularization** obscures symmetries (+is technically difficult)

- Most familiar **mater integrand bases** are the ***unnecessarily bad*:**
  - don't satisfy nice/canonical differential equations
  - contain ***multiple*** **elliptic**(+worse(!)) geometries,
  - …

$$\mathrm{I}_{a_1a_2a_3a_4a_5a_6a_7a_8a_9} = e^{2\gamma_E\epsilon}\iint\frac{d^Dk_1d^Dk_2}{(i\pi^{d/2})^2}\frac{P_8^{-a_8}\,P_9^{-a_9}}{P_1^{a_1}P_2^{a_2}P_3^{a_3}P_4^{a_4}P_5^{a_5}P_6^{a_6}P_7^{a_7}}$$



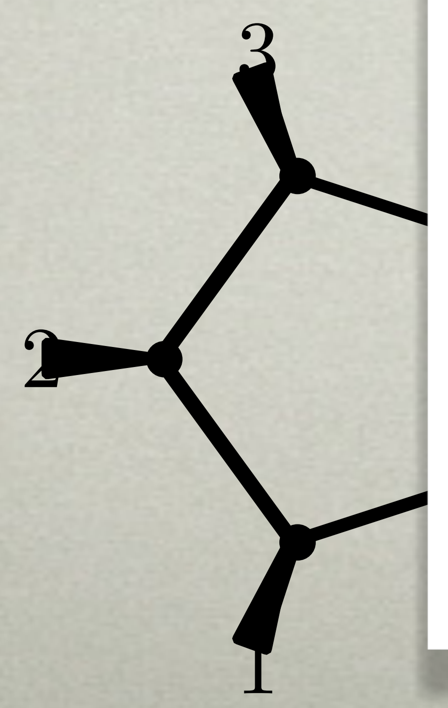
How can We Make it Easier?

- **Use unitarity** to choose the ***nicest / easiest*** integrals to integrate (*of course,* integration "ease" changes with time and new methods)
  - search for as many ***pure*** integrals as you can —those which satisfy nice (canonical) differential equations

Definition: a function *f(s)* is called ***pure*** if:
- there exists a grading of functions by "transcendental" ***weight***
- any derivative of *f(s) is* ***strictly lower in weight***

(Answer: sometimes)

…er, Patatoukos (2021); …]



e.g. $g(s)\log\big(f(s)\big)$ would be ***impure***

$$\frac{\partial}{\partial s}\Big[g(s)\log\big(f(s)\big)\Big] = g'(s)\log\big(f(s)\big) + g(s)f'(s)/f(s)$$

a sum of …rithms & …*elliptic curves*

Unitarity-Based Strategies

& the stratification of loop integrands

Generalized Unitarity (in brief)

- The basic idea behind **unitarity-based methods** is that any ***Feynman integrand*** is a *rational differential form on loop momenta*
  - as such, it can be expanded into a **basis** $\mathfrak{B}$ of such forms:

$$\mathcal{A} = \sum_{\mathfrak{b}^i \in \mathfrak{B}} a_i \mathfrak{b}^i$$

- For any fixed QFT (spacetime dimension, particle content), the space of **all amplitude integrands** is **finite-dimensional**
  - ***all-multiplicity amplitudes*** can be expressed in a ***finite basis***!
- **Key observation:** viewed as a potential element of *some* basis, *every* **Feynman** integrand can be interesting!
  - Why not try to find the ***best/easiest*** integrands—and use these?

Stratifying Integrand Bases

- Suppose that a basis could be carved up into subspaces (by any, arbitrary means):

$$\mathcal{A} = \sum_{\mathfrak{b}^i \in \mathfrak{B}} a_i \mathfrak{b}^i$$

$$\mathfrak{B} =: \bigoplus_{p=0}^{\infty} \mathfrak{B}_p \qquad \mathcal{A} = \sum_p \mathcal{A}_p \quad \text{with} \quad \mathcal{A}_p \coloneqq \mathcal{A} \bigcap \mathfrak{B}_p \coloneqq \sum_{\mathfrak{b}_p^i \in \mathfrak{B}_p} a_i \, \mathfrak{b}_p^i$$

- Such a stratification could be given by "*power-counting*" (some proxy for) ultraviolet behavior
  - recently, we gave an intrinsically graph-theoretic definition of power-counting for ***non-planar*** integrand bases

[**JB**, Herrmann, Langer, Trnka (2020)]

Stratifying Integrand Bases

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- ¿Is it possible to stratify integrand bases by ***physical** structure*?

$$(\text{max-weight})\oplus(\text{next-to-max-weight})\oplus\ldots\oplus(\text{rational})$$

$$(\text{polylogs})\oplus(\text{elliptic-logs})\oplus(\text{K3's})\oplus\ldots$$

$$(\text{finite})\oplus(\text{UV-divergent})\oplus(\text{IR-divergent})\oplus\ldots$$

$$(\text{finite})\oplus(1/\epsilon\text{-divergent})\oplus(1/\epsilon^2\text{-divergent})\oplus\ldots$$

$$\oplus(\log(m)\text{-divergent})\oplus(\log(m)^2\text{-divergent})\oplus\ldots$$

[**JB**, Kalyanapuram]
[**JB**, Langer, Zhang]
[**JB**, Herrmann, Langer, Patatoukos, *et al*]

Prescriptive *Integrand Bases*

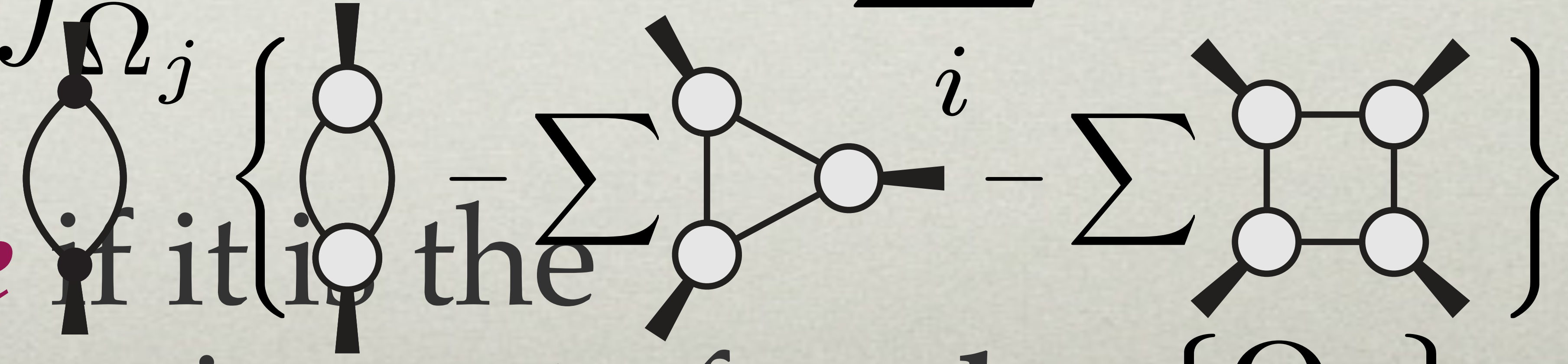
- How ***generalized unitarity*** has been used to match amplitudes:

$$\mathcal{A} = \sum_i a_i \mathcal{I}_i^0$$

with coefficients c_i determined by **cuts**: a spanning set of cycles $\{\Omega_j\}$

$$\mathcal{I}_j \coloneqq \sum_i \mathcal{I}_i^0 (\mathbf{M}^{-1})_{i,j} \qquad \oint_{\Omega_j} \mathcal{I}_i^0 \eqqcolon \mathbf{M}_{i,j} \qquad \oint_{\Omega_j} \mathcal{I}_i = \delta_{i,j}$$

$$a_j \coloneqq \oint_{\Omega_j} \mathcal{A} = \sum_i c_i \mathbf{M}_{i,j} \Rightarrow c_j = \sum_i a_i (\mathbf{M}^{-1})_{i,j}$$



- A basis is called ***prescriptive*** if it is the **cohomological dual** of a spanning set of **cycles** $\{\Omega_j\}$

Strategies for Building Bases

- Given ***some*** integrand basis (or strata thereof), one should ***diagonalize*** the space of integrands according to a

  homological/cohomological pairing:

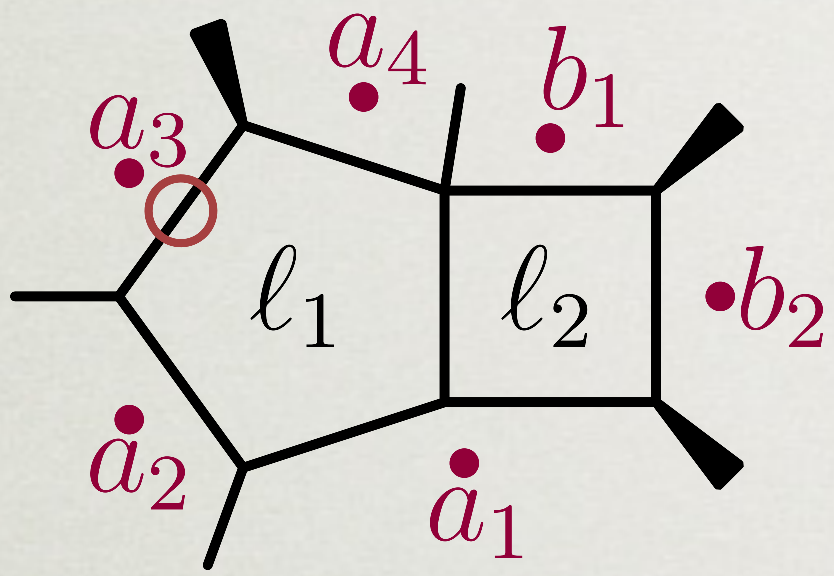
  - choose a *spanning-set* of **compact, max-dimensional** contours Ω_j
  - normalize and diagonalize the basis by the requirement

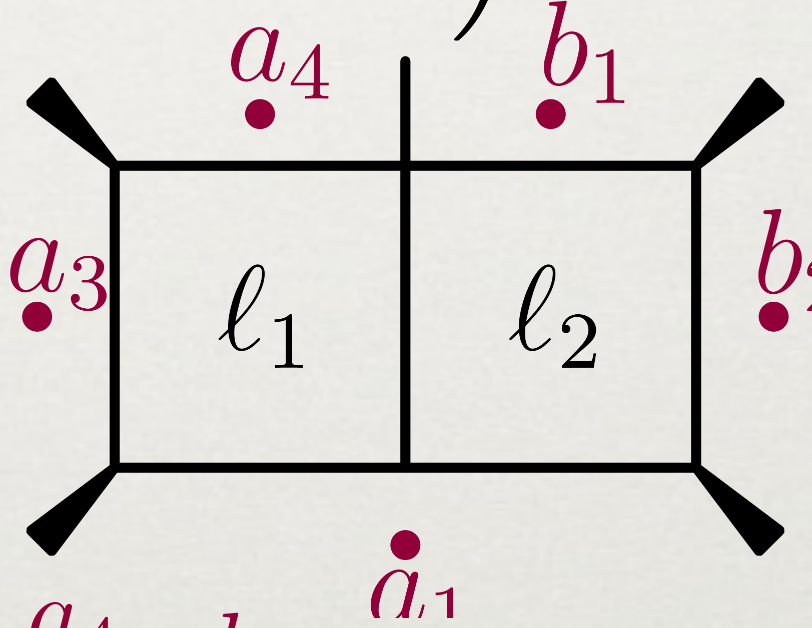
$$\int_{\Omega_j} \mathfrak{b}_i = \delta_{ij} \qquad \Omega_j : 4L\text{-dimensional compact contours} \begin{cases} \text{"residues"} \\ \text{elliptic periods} \\ \text{K3 periods, etc.} \end{cases}$$

- This ***trivializes*** the representation of amplitudes:
  - the **coefficient** of any amplitude in this basis will simply be the *on-shell function* evaluated on the contour (a **leading singularity**)
- Choosing a maximal set of IR/UV-divergence-probing contours ensures(?) that the basis is split into finite/divergent subspaces

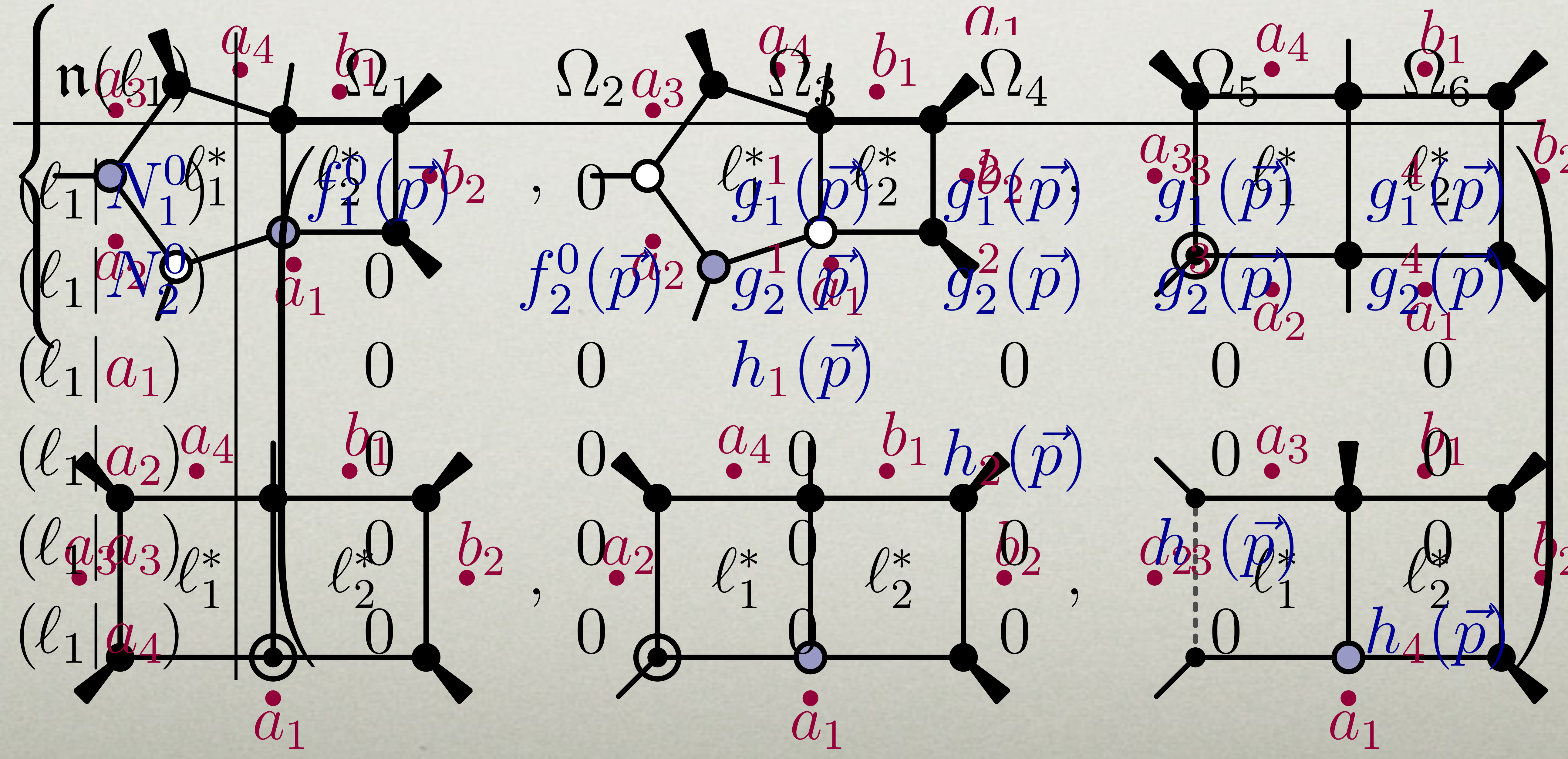
Prescriptivity and Purity

- **Prescriptive integrand bases** are naturally *pure*

$\mathcal{I}(N) \Leftrightarrow$  $= \Big(\text{pure polylogarithms}\Big) + f(\vec{p}) \times \Big(\text{other pure polylogarithms}\Big)$

$\displaystyle\int\limits_{\ell_i \in \mathbb{R}^4}$  $\propto \Big(\text{other pure polylogarithms}\Big)$

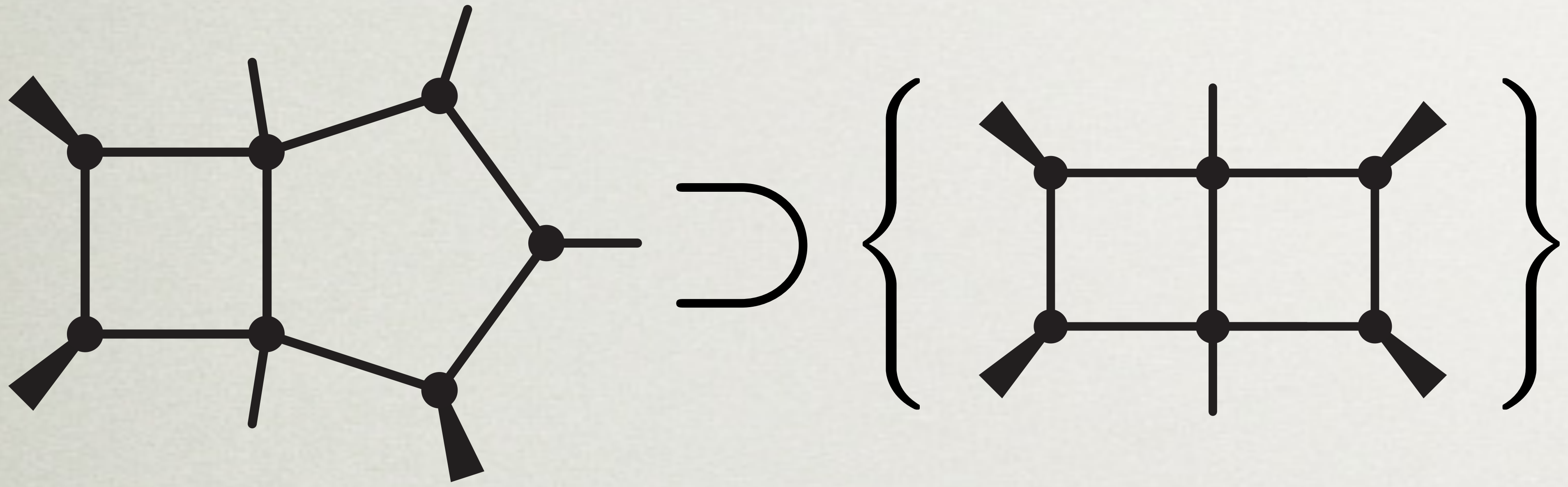
$$\{\Omega_1, \ldots, \Omega_6\} := \left\{ \begin{array}{c|cccccc} \mathfrak{n}(\ell_1) & \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & \Omega_5 & \Omega_6 \\ \hline (\ell_1|N_1^0) & f_1^0(\vec{p}) & 0 & g_1^1(\vec{p}) & g_1^2(\vec{p}) & g_1^3(\vec{p}) & g_1^4(\vec{p}) \\ (\ell_1|N_2^0) & 0 & f_2^0(\vec{p}) & g_2^1(\vec{p}) & g_2^2(\vec{p}) & g_2^3(\vec{p}) & g_2^4(\vec{p}) \\ (\ell_1|a_1) & 0 & 0 & h_1(\vec{p}) & 0 & 0 & 0 \\ (\ell_1|a_2) & 0 & 0 & 0 & h_2(\vec{p}) & 0 & 0 \\ (\ell_1|a_3) & 0 & 0 & 0 & 0 & h_3(\vec{p}) & 0 \\ (\ell_1|a_4) & 0 & 0 & 0 & 0 & 0 & h_4(\vec{p}) \end{array} \right\} =: \mathbf{M}^0$$



Stratifying *Rigidity*

- Is it possible to ***stratify*** integrands according to ***rigidity?***

[**JB**, Kalyanapuram]



$\mathfrak{n}(\ell_1)$	Ω_a	Ω_b	Ω^e_{poles}
$(\ell_1\|N_1)$	J^a_1	J^b_1	0
$(\ell_1\|N_2)$	J^a_2	J^b_2	0
$(\ell_1\|N_3)$	J^a_3	J^b_3	1

$$\begin{pmatrix} J^a_1 & J^b_1 & 0 \\ J^a_2 & J^b_2 & 0 \\ J^a_3 & J^b_3 & 1 \end{pmatrix} =: \mathbf{M}$$

$$|\overline{N_1}) := \frac{1}{\det(\mathbf{M})}\Big[J^b_2|N_1) - J^b_1|N_2)\Big]$$

$$|\overline{N_2}) := \frac{1}{\det(\mathbf{M})}\Big[-J^a_2|N_1) + J^a_1|N_2)\Big]$$

$$|\overline{N_3}) := \frac{1}{\det(\mathbf{M})}\Big[(J^a_2\,J^b_3 - J^b_2\,J^a_3)|N_1) + (J^b_1\,J^a_3 - J^a_1\,J^b_3)|N_2)\Big] + |N_3)$$

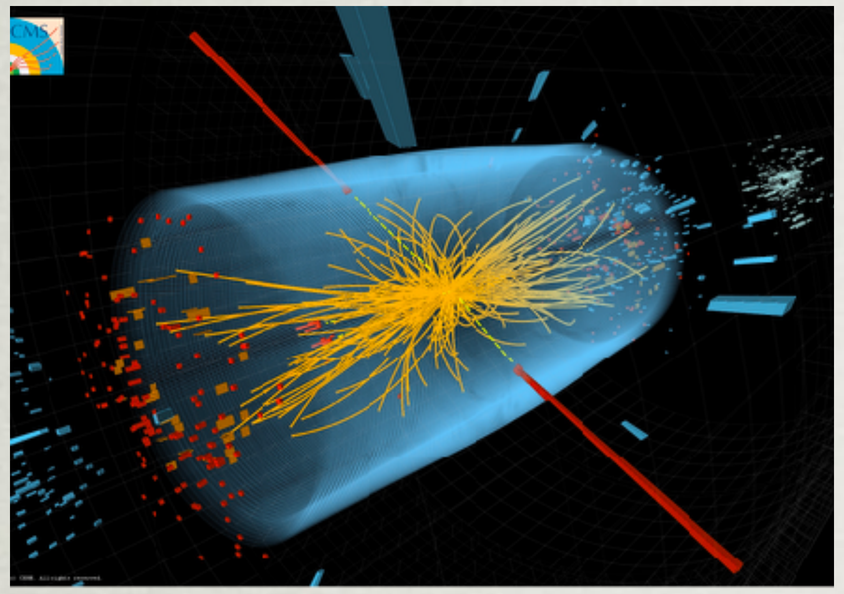
$$J^a_1 := \oint_{\text{a-cycle}} \omega_1(\alpha) = -\frac{1}{\pi\, y_4\sqrt{r_{14}\, r_{23}}}K[\varphi]$$

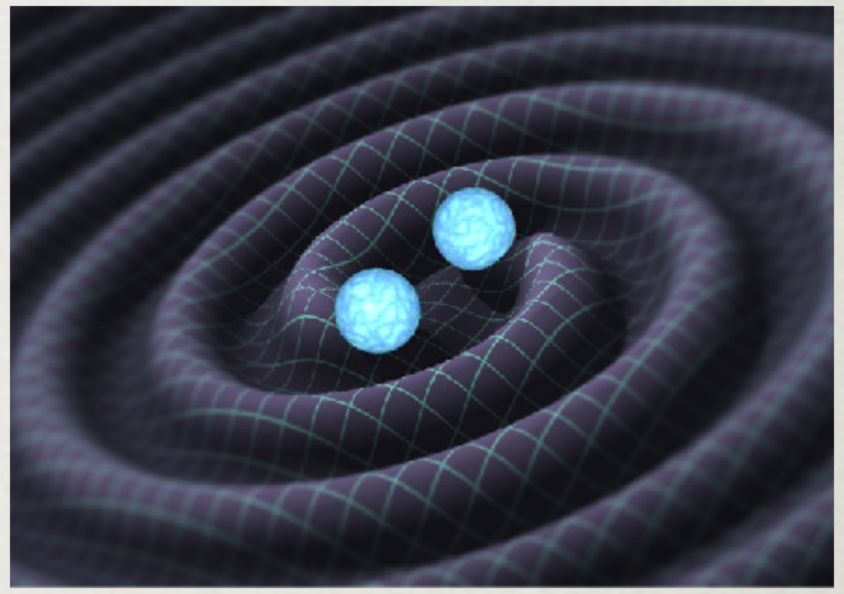
$$J^a_2 := \oint_{\text{a-cycle}} \omega_2(\alpha) = \frac{1}{\pi\, y_4\sqrt{r_{14}\, r_{23}}}\frac{y(q_+)}{q_4^+}\left(K[\varphi] + \frac{r_{24}}{q_2^+}\Pi\left[\frac{q_4^+ r_{12}}{q_2^+ r_{14}}, \varphi\right]\right) - (q_+ \leftrightarrow q_-)$$

$$J^a_3 := \oint_{\text{a-cycle}} \omega_3(\alpha) = \frac{1}{\pi\, y_4\sqrt{r_{14}\, r_{23}}}\frac{y(q_+)}{q_4^+}\left(K[\varphi] + \frac{r_{24}}{q_2^+}\Pi\left[\frac{q_4^+ r_{12}}{q_2^+ r_{14}}, \varphi\right]\right) + (q_+ \leftrightarrow q_-)$$

$$\det(\mathbf{M}) = \frac{i}{\pi^2 y_4^2}\frac{y(q_+)\, r_{34}}{q_3^+\, q_4^+\, r_{14} r_{23}}K[\varphi]\,\Pi\left[\frac{q_4^+ r_{13}}{q_3^+ r_{14}}, 1-\varphi\right] - (q_+ \leftrightarrow q_-) - (r_2 \leftrightarrow r_3)$$

Amplitudes: a Virtuous Cycle



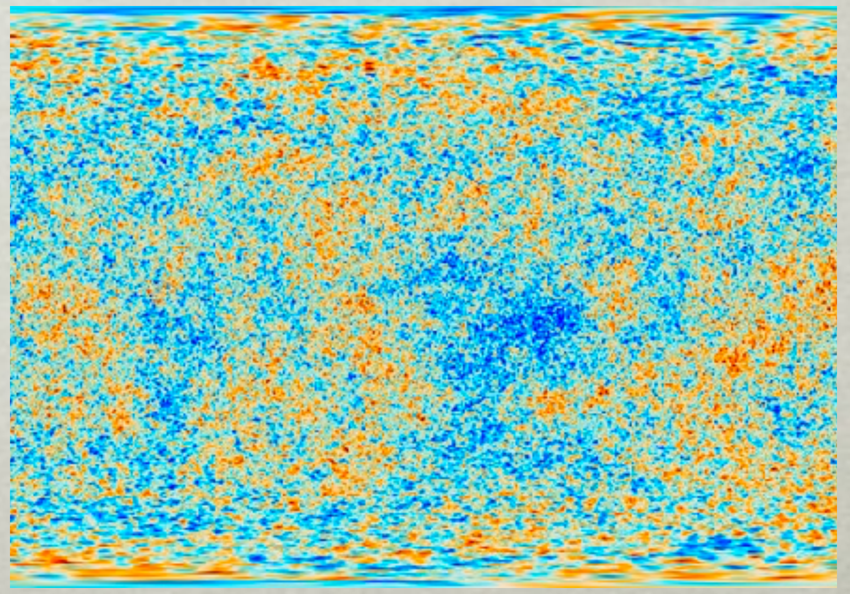


Compute Something
beyond the reach of
recent imagination

Discover Simplicity
beyond expectations

Understand Why
study it, understand it,
& explore consequences

Exploit Simplicity
to build **more powerful**
computational technology



Thank you!