Classical Dynamics of Vortex Solitons from Perturbative Scattering Amplitudes

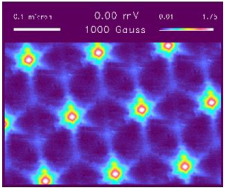
Callum R. T. Jones

[2305.08902]

UCLA

Vortices and Vortex Strings

Phenomenological model of superconductivity
[Ginzburg, Landau 1950]
[Abrikosov 1957]



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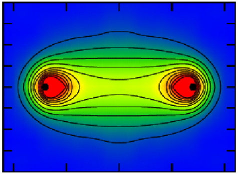

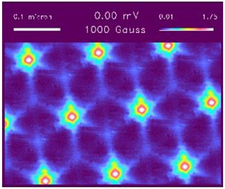

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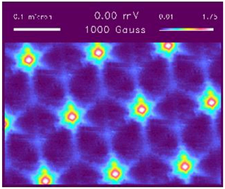

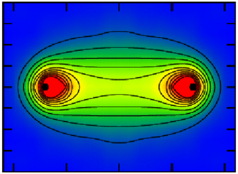

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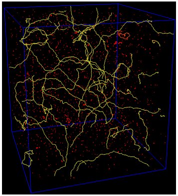

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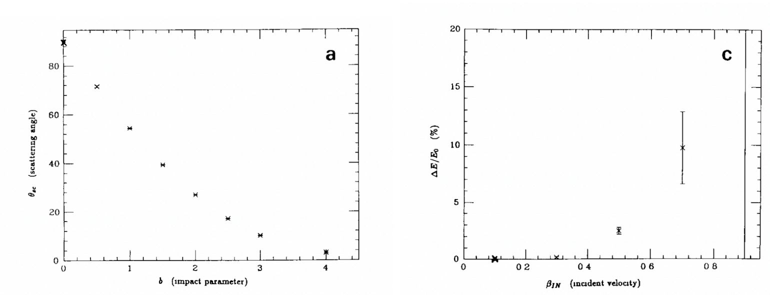

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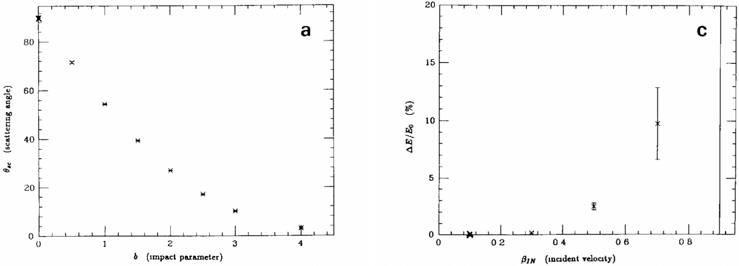

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Goal for this talk: construct alternative, *analytic*, *perturbative* approximation. Calculate vortex-vortex interactions using **scattering amplitudes**.

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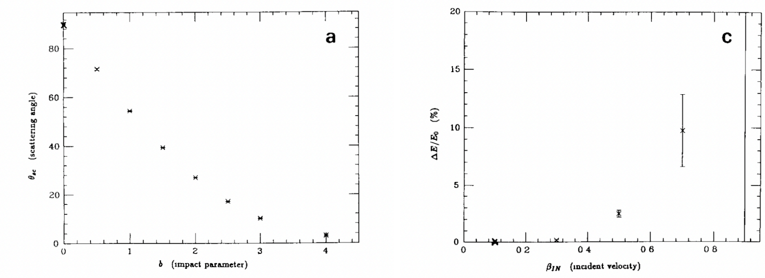

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How is a vortex like a black hole?

Abelian Higgs Model in $d = 2 + 1$

$$S = \int \mathrm{d}^3 x \left[-\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 - \frac{\mu^2}{8} \left(|\phi|^2 - v^2 \right)^2 \right], \qquad D_\mu = \partial_\mu + ieA_\mu .$$

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$$\phi(x) = \left(v + \frac{\sigma(x)}{\sqrt{2}} \right) e^{i\pi(x)/v}.$$

Elementary particle spectrum: *massive photon* $A_\mu(x)$ and *Higgs boson* $\sigma(x)$

$$m_\gamma = \sqrt{2} e v, \qquad m_\sigma = \frac{v\mu}{\sqrt{2}}.$$

Abrikosov-Nielsen-Olesen (ANO) Vortex

$$\phi(\mathrm{x}) = \rho(r)e^{i\mathrm{N}\theta},$$

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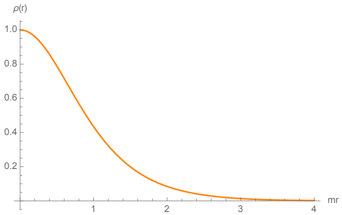

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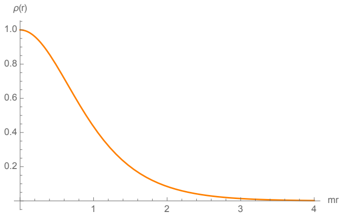

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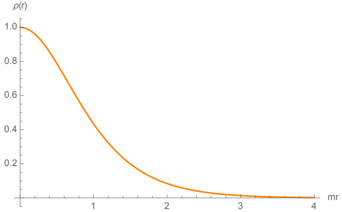

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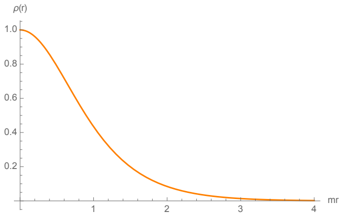

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- **Type-I Superconductor:** $\mu^2 < 4e^2$ identical vortices *attract*.
- **Type-II Superconductor:** $\mu^2 > 4e^2$ identical vortices *repel*.

Critical Abelian Higgs Model

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Leading asymptotic BPS vortex solution (unitary gauge):

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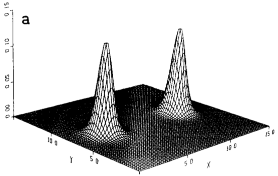
Static multi-vortex solution exists (unknown analytically) [Taubes 1980].



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Key idea: continue from $N \in \mathbb{Z}$ to $N \in \mathbb{R}$; expand around $N \to 0$ [Ohashi 2015]

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$$\left(\frac{d^2}{d\xi^2} - \frac{1}{\xi}\frac{d}{d\xi} - 1\right)a(\xi) = -\frac{2a(\xi)}{\xi}\frac{da(\xi)}{d\xi}, \qquad a(0) = -N, \qquad a(\xi \to \infty) = 0.$$

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Perturbative expansion in N:

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$$Z_1^{\text{perturbative}} = 1.7078629\ldots, \qquad Z_1^{\text{numerical}} = 1.707864175\ldots$$

Separation of Scales

Relevant scales in the problem

$$R_{\text{Compton}} \sim \frac{1}{M}, \qquad R_{\text{interaction}} \sim \frac{1}{m}, \qquad R_{\text{core}} \sim \frac{\sqrt{N}}{m}$$

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Physical picture: classical point-particle vortices interacting by exchanging mediators over distance $r \sim m^{-1}$.

An Effective Field Theory of Vortex Solitons

UV Theory:

Critical AHM

A_μ, σ

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$\downarrow$

Relativistic EFT:

Critical AHM + *point-particle* vortex

$A_\mu, \sigma, \Phi, \Phi^*$

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Non-Relativistic EFT:

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Φ, Φ^*

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$v \ll c$

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$$\begin{aligned} S_{\text{Vortex}}[\mathbf{\Phi}, \mathbf{\Phi}^*, \sigma, A_\mu] = \int \mathrm{d}^3x \Bigg[& |\partial_\mu \mathbf{\Phi}|^2 - M^2|\mathbf{\Phi}|^2 \\ & + \underbrace{i g_e A^\mu \mathbf{\Phi}^* \partial_\mu \mathbf{\Phi}}_{\mathcal{M}_3 = g_e (p\cdot\varepsilon_\pm(q))} + \underbrace{i g_m \epsilon^{\mu\nu\rho} F_{\mu\nu} \mathbf{\Phi}^* \partial_\rho \mathbf{\Phi}}_{\mathcal{M}_3 = \mp g_m (p\cdot\varepsilon_\pm(q))} + g_s \sigma |\mathbf{\Phi}|^2 + \text{c.c.} \Bigg], \end{aligned}$$

$+$ finite size effects (*Love numbers*).

No Chern-Simons coupling in AHM $\Rightarrow$ no "electric" charge $g_e = 0$.

Relativistic Effective Field Theory (REFT)

$$S_{\text{REFT}}[\boldsymbol{\Phi}, \boldsymbol{\Phi}^*, \sigma, A_\mu] = S_{\text{AHM}}[\sigma, A_\mu] + S_{\text{Vortex}}[\boldsymbol{\Phi}, \boldsymbol{\Phi}^*, \sigma, A_\mu].$$

$$S_{\text{AHM}}[\sigma, A_\mu] = \int d^3x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu + \frac{1}{2} (\partial_\mu \sigma)^2 - \frac{1}{2} m^2 \sigma^2 + \sqrt{\frac{\pi}{2}} M^{3/2} \left(\frac{m}{M}\right)^2 N^{1/2} \sigma A_\mu A^\mu - \sqrt{\frac{\pi}{8}} M^{3/2} \left(\frac{m}{M}\right)^2 N^{1/2} \sigma^3 + \frac{\pi}{4} M \left(\frac{m}{M}\right)^2 N \sigma^2 A_\mu A^\mu - \frac{\pi}{16} M \left(\frac{m}{M}\right)^2 N \sigma^4 \right],$$

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Consistent truncation of $\mathcal{N} = 2$ supersymmetry $\Rightarrow g_s = 4M g_m$.

Matching UV to REFT

Calculate g_m and g_s by matching a *probe amplitude* between UV and the REFT.

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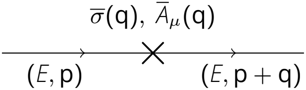
UV: Expand around classical backgrounds $\overline{\sigma}$ and $\overline{A}_\mu$



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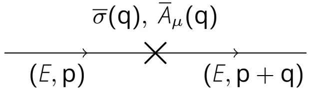

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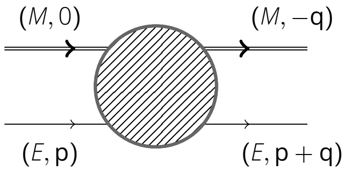

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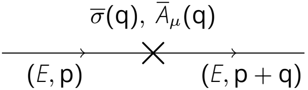

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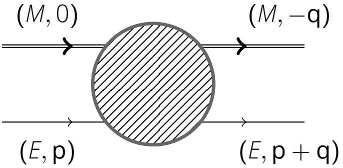

$$\overline{\sigma}(\mathrm{x}) = \frac{1}{2M} \int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} e^{i\mathrm{q}\cdot\mathrm{x}} \mathcal{M}^{(\text{probe})}_{\text{REFT}}(\mathrm{q}) \Big|_{\mathfrak{g}_s}, \quad \overline{A}_i(\mathrm{x}) = -\frac{i}{8\mathfrak{m}M} \int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} e^{i\mathrm{q}\cdot\mathrm{x}} \frac{\epsilon_{ij} q_j}{\mathrm{q}^2} \mathcal{M}^{(\text{probe})}_{\text{REFT}}(\mathrm{q}) \Big|_{\mathfrak{g}_m}.$$

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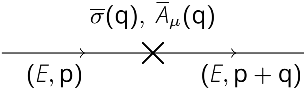

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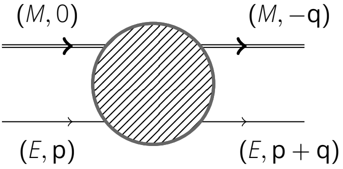

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Tree-level matching: $g_s = -4\sqrt{2\pi}M^{3/2}N^{1/2}$, $g_m = -\sqrt{2\pi}M^{-1/2}\left(\frac{m}{M}\right)^{-1}N^{1/2}$.

Classical Solitons from the S-Matrix

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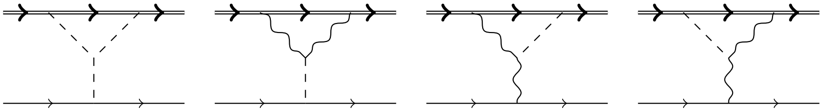

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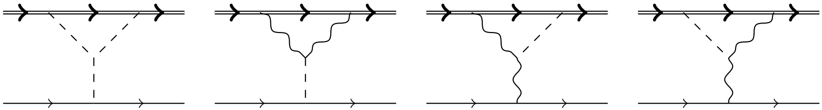

Extract classical solution from matching:

$$\overline{\sigma}(\mathrm{x}) = \sqrt{\frac{2M}{\pi N}}\left[-\left(N+\frac{\pi}{3\sqrt{3}}N^2\right)K_0(mr) - \frac{N^2}{2}\left[K_0(mr)\right]^2 \right.$$
$$\left. +2N^2\left(K_0(mr)\int_{mr}^{\infty}\mathrm{d}\xi\ \xi I_1(\xi)K_0(\xi)K_1(\xi) + I_0(mr)\int_{mr}^{\infty}\mathrm{d}\xi\ \xi K_0(\xi)\left[K_1(\xi)\right]^2\right)\right] + \mathcal{O}\left(N^{5/2}\right),$$

$$\overline{A}_i(\mathrm{x}) = \frac{1}{m}\sqrt{\frac{2M}{\pi N}}\frac{\epsilon_{ij}x_j}{r^2}\left[-\left(N+\frac{\pi}{3\sqrt{3}}N^2\right)mrK_1(mr)\right.$$
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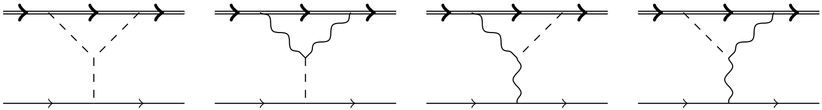

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Agrees with perturbative solution of BPS equations [de Vega, Schaposnik 1976] ✓

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Integrate out *potential modes* of σ and A_μ and construct an effective potential.

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Calculate $V(\mathbf{p}, \mathbf{p}')$ by matching amplitudes :

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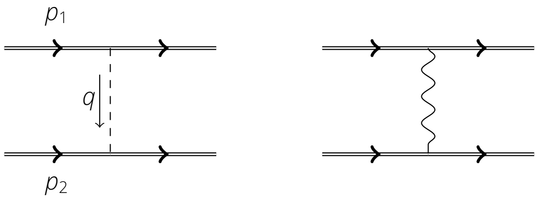
2. Expand in (non-relativistic) *potential region*: $\qquad \mathbf{p} \sim v, \quad \omega \sim v.$

Tree-Level Potential



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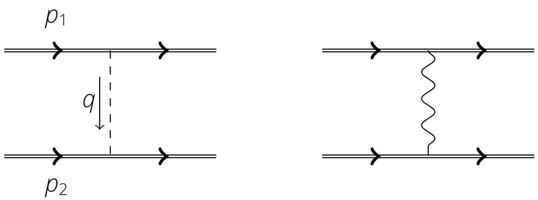

$$\mathcal{M}^{\text{tree}} = \underbrace{-\frac{32\pi MN\left(M^2-(p_1\cdot p_2)\right)}{q^2-m^2}}_{\text{classical potential}} + \underbrace{\frac{8\pi MNm^2}{q^2-m^2}}_{\text{quantum}} + \underbrace{\frac{32\pi MN}{m^2}\left(p_1\cdot p_2+\frac{1}{4}\left(q^2+m^2\right)\right)}_{\text{short-range “\textit{Darwin}” terms} \;\sim\; \delta^{(2)}(\mathbf{x})}.$$

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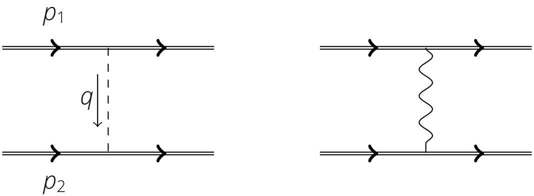

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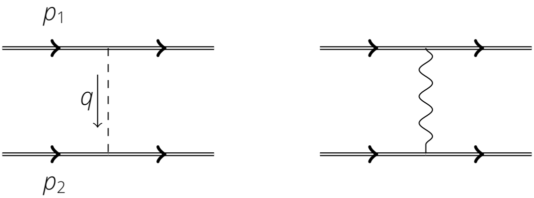

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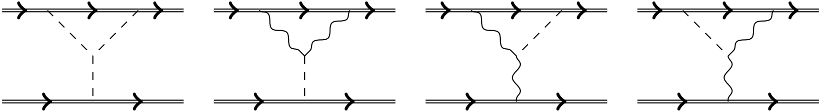
$$V(\mathbf{p},\mathbf{x}) = \frac{8MN\mathbf{p}^2}{\mathbf{p}^2+M^2}K_0(mr)+\mathcal{O}\left(N^2\right).$$

Vanishes as $\mathbf{p}\to 0$ as expected for BPS vortices.

1-loop Potential



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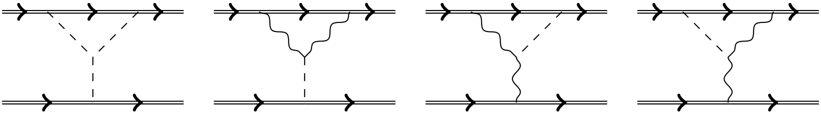
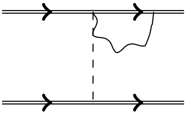


Renormalize UV divergent *pinch* contributions:



$\longrightarrow 0.$

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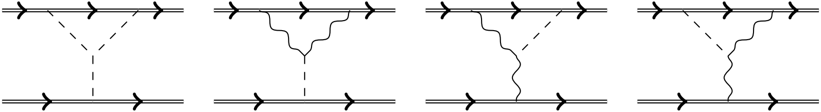
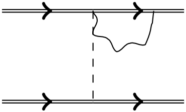


Renormalize UV divergent *pinch* contributions:  $\longrightarrow \quad 0.$

Expand in soft region: $\quad q^\mu \sim \hbar, \quad m \sim \hbar, \quad l^\mu \sim \hbar$

$$i\mathcal{M}^{\text{1-loop}}_{\Upsilon} =$$

$$128\pi^2 M^2 N^2 \left[\left(M^2 - 2(p_1 \cdot p_2) \right) + \frac{2m^2 \left(M^2 - (p_1 \cdot p_2) \right)}{q^2 - m^2} \right] \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{[l^2 - m^2][(l+q)^2 - m^2] \underbrace{[p_1 \cdot l + i0]}_{\textit{Eikonal}}} .$$

Seagull Contributions

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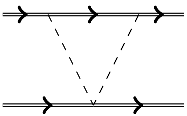

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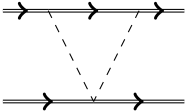

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Velocity Expansion and Resummation

Expand master soft integrals in potential region: $\mathbf{p} \sim v, \quad \omega \sim v$

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and *resum* the complete velocity dependence

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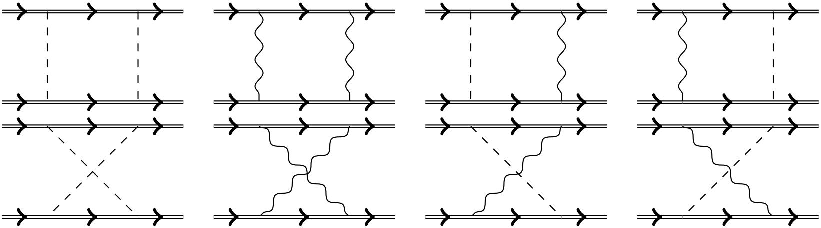
Box Diagrams and Iteration

Add boxes and crossed-boxes:



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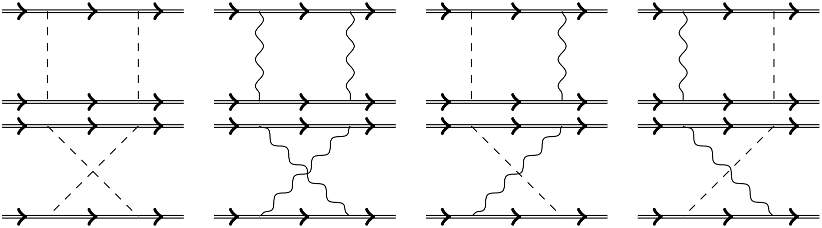
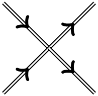
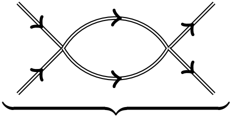


Match to amplitude in NREFT:

$M_{\text{NREFT}} =$  $+$  $+ \quad \dots$

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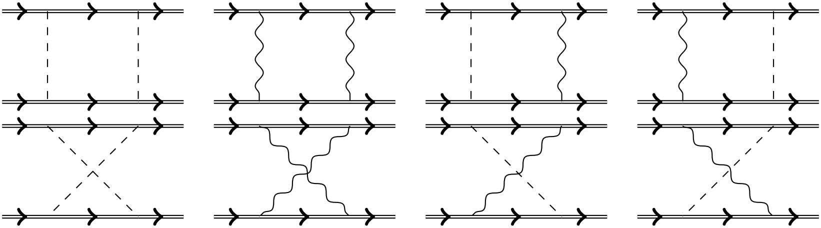

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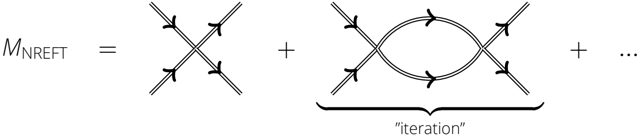

Spurious branch cut must match between REFT boxes and NREFT iteration. ✓

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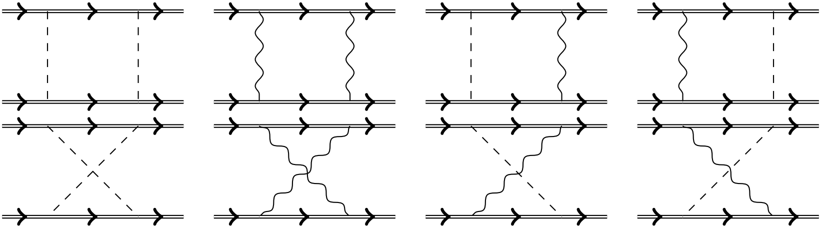

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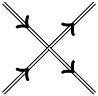
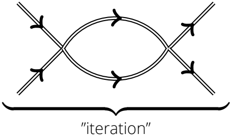
$M_{\text{NREFT}} =$  $+$  $+ \quad \ldots$

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Box, crossed-box and iteration contributions vanish in static limit. ✓

Main Result: 1-loop Vortex-Vortex Potential

$$V(\mathrm{p},\mathrm{x}) = \frac{8MN\mathrm{p}^2}{\mathrm{p}^2+M^2}K_0(mr) + \frac{16\pi MN^2\mathrm{p}^2}{3\sqrt{3}\,(\mathrm{p}^2+M^2)}K_0(mr) + \frac{16M^2N^2\mathrm{p}^2(\mathrm{p}^2+4M^2)}{(\mathrm{p}^2+M^2)^{5/2}}K_0(mr)^2$$

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- Metric on moduli space should be *Kähler* [Samols 1991] ✓

Moduli Space Metric

Truncate at $\mathcal{O}\left(\mathrm{p}^2\right)$; calculate effective Lagrangian

$$L(\dot{\mathrm{x}}_1, \mathrm{x}_1; \dot{\mathrm{x}}_2, \mathrm{x}_2) = \frac{1}{2}M\dot{\mathrm{x}}_1^2 + \frac{1}{2}M\dot{\mathrm{x}}_2^2 - \tilde{U}\left(r_{12}\right)\left|\dot{\mathrm{x}}_1 - \dot{\mathrm{x}}_2\right|^2 + \mathcal{O}\left(v^4\right),$$

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Interpret as a $0+1$d sigma model [Manton 1982] read off moduli space metric:

$$ds^2 = \left(\frac{1}{2}M - \tilde{U}\left(r_{12}\right)\right)d\mathbf{x}_1^2 + \left(\frac{1}{2}M - \tilde{U}\left(r_{12}\right)\right)d\mathbf{x}_2^2 - 2\tilde{U}\left(r_{12}\right)d\mathbf{x}_1\cdot d\mathbf{x}_2.$$

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Thank you!

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