The q-Schwarzian and its gravitational dual

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Introduction, motivation and summary

Consider the SYK model

$$H = \sum_{i_1 \dots i_p} J_{i_1 \dots i_p} \psi_{i_1} \dots \psi_{i_p} \,, \quad \log q = \frac{p^2}{N}$$

Its low energy physics is governed by the Schwarzian, which is holographically dual to $R + 2 = 0$ 2d JT dilaton gravity.
This duality has learned us many things about quantum gravity in AdS.

Interestingly there is another limit of SYK known as **double scaled** SYK with **q finite** for $N \to \infty$ which is also **exactly solvable**.
We know all amplitudes of interesting operators **analytically** (Berkooz, Isachenkov, Narovlansky, Torrents, Narayan, Simon).

This raises the question whether DSSYK also has some tractable dual **bulk gravitational description**?
One reason to pursue this, and perhaps the reason I am here today, there are hints of duality between DSSYK and **dS quantum gravity** (Susskind, Lin, Rahman).

Will try to convince you that DSSYK does **indeed** have a **simple** bulk **gravitational dual** (with features of dS).

To motivate this I will introduce and study the **q-Schwarzian** theory.

Deformation of Schwarzian 1d path integral that depends on (complex) parameter q.

Depending on q will argue q-Schwarzian has the following **bulk dual** 2d **dilaton gravity** description

$q > 1$	$\Leftrightarrow$	$V(\Phi) = \sin(2\Phi)$	
$\lvert q \rvert = 1$	$\Leftrightarrow$	$V(\Phi) = \sinh(2\Phi)$	Liouville gravity
$q = 1$	$\Leftrightarrow$	$V(\Phi) = 2\Phi$	JT gravity

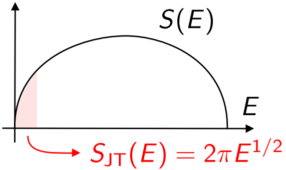
Furthermore the $q > 1$ q-Schwarzian is **equivalent** to the chord diagram description of DSSYK.
Will give more detailed action of gravity theories soon.

There are several reasons **why** these new dualities seem **interesting**.

1. Solvable examples of non AAdS holography. As we'll see.

2. For **DSSYK** classical bulk has **cosmological horizon** and $R = 2$ dS region. Certain ranges of temperature bulk becomes pure dS. Potential **dS holography** from a microscopic model.
Currently certainly not well enough understood to make strong claims.

3. DSSYK is **UV completion** (spectrum caps off) of JT gravity



For instance partition function remains finite for $\beta = 0$. So UV complete bulk gravitational theory by construction.

Outline

In the remainder will provide **evidence for dualities**.

1. Classical solutions (and introduction to) 2d **dilaton gravity** models
2. Summary of results from the **q-Schwarzian** path integral
3. From dilaton gravity to q-Schwarzian via **gauge theory**
4. Matching **classical solutions** for $|q| = 1$

Intermezzo. Lightning introduction to **DSSYK**

5. Matching classical solutions for $q > 1$ and **fake temperature**

Emphasize **work in progress** so many things not completely understood!
Unfamiliar audience please **stop me** when something is unclear.

Classical solutions dilaton gravity models

First we discuss **classical solutions** of proposed **gravity** duals.
Reasons

1. Clarify previous motivation about dS and non AAdS holography
2. Set **benchmarks** for **q-Schwarzian** (and DSSYK) to compare with

Dilaton gravity in 2d Euclidean (definition)

$$\exp\left(\frac{1}{2}\int \mathrm{d}x\sqrt{g}\Big(\Phi R + V(\Phi)\Big) + \int \mathrm{d}\tau\sqrt{h}K + \text{counter}\right)$$

Classical solution (Witten) gauge $\Phi = r$

$$\mathrm{d}s^2 = F(r)\mathrm{d}\tau^2 + \frac{1}{F(r)}\mathrm{d}r^2\,, \quad 0 < \tau < \beta$$

$$F(r) = \int_{\Phi_h}^{r} \mathrm{d}\Phi V(\Phi)$$

$$\sqrt{g}R = -V' \qquad \sqrt{h}K = \frac{1}{2}V$$

Solutions with fixed Φ_h area and β have **conical defect** at horizon

$$\sqrt{g}R = -V' + (4\pi - \beta V(\Phi_h))\,\delta(x - x_h)$$

This follows from Gauss-Bonnet for disk shaped topology

$$\frac{1}{4\pi}\int \mathrm{d}x\sqrt{g}R + \frac{1}{2\pi}\int \mathrm{d}\tau\sqrt{h}K \stackrel{!}{=} 1$$

Including the singular piece finds **on-shell action**

$$\exp\left(2\pi\Phi_h - \frac{\beta}{2}\Phi_h V(\Phi_h) + \frac{1}{2}\int \mathrm{d}\tau \int_{\Phi_h}^{\Phi_b} \mathrm{d}\Phi\Big(-\Phi V'(\Phi) + V(\Phi)\Big) + \frac{1}{2}\int \mathrm{d}\tau\, V(\Phi_b) + \text{counter}\right)$$

$$= \boxed{\exp\left(2\pi\Phi_h + \beta\int_{\Phi_h}^{\infty} \mathrm{d}\Phi\, V(\Phi) + \text{counter}\right)}$$

Should think of this as partition function with **dilaton as horizon area**

$$Z(\beta) = \int \mathrm{d}A\, \exp\left(\frac{A}{4G} - \beta E(A)\right)$$

DSSYK gravity $q > 1$

For $q > 1$ we **propose** to consider periodic potential

$$\boxed{V(\Phi) = \frac{\sin(2\log q\Phi)}{\log q}}$$

Black hole horizon $r = \Phi_h$ and **cosmological horizon** $r = \pi/\log q - \Phi_h$ (where Φ is maximal)

$$F(r) = -\frac{\cos(2\log qr)}{2\log q^2} + \frac{\cos(2\log q\Phi_h)}{2\log q^2} > 0$$

Classical **thermodynamics** (from previous slide)

$$\exp\left(2\pi\Phi_h + \beta\frac{\cos(2\log q\Phi_h)}{2\log q^2}\right)$$

Will match this later with q-Schwarzian on shell action!

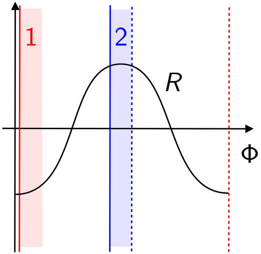
Actually mismatch with DSSYK around which **end of talk** centers.

$$R = -2\cos(2\log q\,\Phi)$$

Interesting regimes

1. $\Phi_h \sim 0$ and IR $r - r_h \sim 0$ we get $R \sim -2$ **AdS quantum gravity**.

2. $\Phi_h \sim \pi/2\log q$ reduces to $R \sim +2$ **dS quantum gravity**.



Low energies zooms in close to black hole horizon (full lines).

Notice also that there is **no AAdS** regime.

Liouville gravity $|q| = 1$

For $q = e^{\mathrm{i}\pi b^2}$ we consider **analytic continuation** of DSSYK gravity

$$V(\Phi) = \frac{\sinh\left(2\pi b^2 \Phi\right)}{\pi b^2}$$

Black hole horizon $r = \Phi_h$ and metric

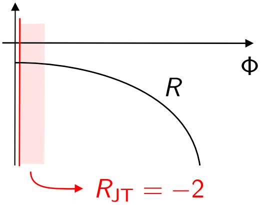
$$F(r) = \frac{\cosh\left(2\pi b^2 r\right)}{2\pi^2 b^4} - \frac{\cosh\left(2\pi b^2 \Phi_h\right)}{2\pi^2 b^4}$$

Classical **thermodynamics**

$$\boxed{\exp\left(2\pi\Phi_h - \beta\frac{\cosh\left(2\pi b^2 \Phi_h\right)}{2\pi^2 b^4}\right)}$$

Will also match this later with q-Schwarzian!

$$R = -2\cos(2\log q\,\Phi)$$



Notice again this is **not AAdS** for instance geodesic length from horizon to boundary is finite.

Using field redefinition this dilaton gravity is rewritten as two decoupled Liouville systems (Stanford, Seiberg, Mertens, Turiaci. . .). Amplitudes in Liouville can be computed **exactly** using Virasoro bootstrap.
By **quantizing** the **q-Schwarzian** one recovers those same amplitudes!

The quantization is technical, will say some words about this but focus mostly on **matching the classical solutions**.
The latter teaches us more about the gravity picture for DSSYK.

Summary of results from (and intro to) the q-Schwarzian

Lorentzian **q-Schwarzian** is following 6d phase space path integral

$$\int_0^T \mathrm{d}t \left(p_\phi \phi' + \cdots + \frac{1}{2\log q^2}\cos(\log q p_\phi) - \frac{1}{\log q^2}\mu_\beta \mu_\gamma e^{-2\phi - \mathrm{i}\log q p_\phi} \right)$$

with

$$\mu_\beta = \frac{e^{-2\mathrm{i}\beta p_\beta} - 1}{-2\mathrm{i}\beta}$$

Perhaps surprisingly this system has 6 **conserved currents** spanning two copies of the algebra

$$\{h, e\} = e\,, \quad \{h, f\} = -f\,, \quad \{e, f\} = \frac{q^{2\mathrm{i}h} - q^{-2\mathrm{i}h}}{2\mathrm{i}\log q}\,,$$

with for instance

$$h = -\frac{1}{2} p_\phi + \beta p_\beta$$

Upon quantization becomes the $U_q(SU(1,1))$ **quantum group** algebra.
Important for quantization see below.

Remember we claim the following **dualities**

$$\begin{array}{rcll} q > 1 & \Leftrightarrow & V(\Phi) = \sin(2\Phi) & \\ |q| = 1 & \Leftrightarrow & V(\Phi) = \sinh(2\Phi) & \text{Liouville gravity} \\ q = 1 & \Leftrightarrow & V(\Phi) = 2\Phi & \text{JT gravity} \end{array}$$

where we furthermore claim $q > 1$ q-Schwarzian is identical to DSSYK.

Now I **summarize** which new **evidence** we found for these **dualities**.
Then detail pieces of this evidence (rest of talk).

1. Heuristic derivation dilaton gravity equals q-Schwarzian through **first order formulation** $\sim$ topological gauge theory

2. The **classical solutions** q-Schwarzian **match** solutions dilaton gravity.

3. Classical solution q-Schwarzian matches solutions DSSYK (large p).
Naively there is **mismatch** but this can be resolved (end of talk).
Resolution is physically crucial explaining origin of "**fake temperature**" (Stanford, Lin, Susskind). Bulk resolution to be understood...

Taste of quantization

4. The q-Schwarzian can be **quantized exactly** and the results **match** exactly with the DSSYK amplitudes.
Similarly for $|q| = 1$ they match the Liouville gravity amplitudes.

Will focus on 1-3 but first small taste of **how quantization works** $q > 1$.

It is **not essential to follow** this technical intermezzo for rest of talk.
Key point is upon quantization currents satisfy $U_q(SU(1,1))$ algebra

$$[H, E] = E\,, \quad [H, F] = -F\,, \quad [E, F] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}$$

and q-Schwarzian Hamiltonian equals Casimir of **quantum group**

$$\mathbf{H} = \frac{q^{2H+1} + q^{-2H-1}}{(q - q^{-1})^2} + FE$$

Moreover ϕ, β, γ are coordinates of quantum group with group elements

$$g = e_{q^{-2}}^{\gamma F}\, e^{2\phi H}\, e_{q^2}^{\beta E} \overset{2d}{=} \begin{pmatrix} e^{\phi} & e^{\phi}\beta \\ \gamma e^{\phi} & e^{-\phi} + \gamma e^{\phi}\beta \end{pmatrix}$$

Thus q-Schwarzian is "quantum mechanics on a quantum group".
As usually this means the **wavefunctions** of this quantum mechanics are the **representation matrices** of the quantum group

$$\psi_{E\mu_1\mu_2}(g) = \langle g|E\,\mu_1\mu_2\rangle = R_{E\,\mu_1\mu_2}(g)$$

Can be computed by hand by solving differential equations and **match** wavefunctions of DSSYK for $q > 1$ (Blommaert, Mertens, Yao) and Liouville gravity for $|q| = 1$ (Mertens, Yale)

Extra important ingredient for this **match with gravity** and DSSYK is q-Schwarzian should be viewed as having **constraint**. Classically

$$e^{\mathrm{i}\log q h} f = \frac{1}{2\log q}$$

These become analogous to Brown-Henneaux **boundary conditions** in **gravity**. More on this later.

OT Some details for folks familiar with DSSYK?

Dealing with constraints can argue wavefunctions become independent of β, γ

$$\psi_E(\phi) = \langle \phi | E \rangle = R_{E\,\mathrm{ii}}(\phi)$$

Hamiltonian becomes DSSYK transfer matrix

$$\mathbf{H} = q e^{i \log q p_\phi} + (q^{-1} - q e^{-2\phi}) e^{-i \log q p_\phi}$$

One can furthermore argue that ϕ is effectively **discretized**

$$\boxed{\phi = n \log q}$$

then the solutions for the wavefunctions indeed **match DSSYK** answer

$$\psi_E(n) = H_n(\cos(\theta)|q^2)\,, \quad E(\theta) = -\frac{\cos(\theta)}{2 \log q^2}$$

From dilaton gravity to q-Schwarzian via gauge theory

Remainder of the talk will focus on points 1-3 slide 13.

First **gauge theory derivation** from dilaton gravity to q-Schwarzian.
This too can be skipped (independent of classical solution discussion).

Introducing torsion constraints and appropriate boundary/counterterms dilaton gravity action slide 6 **vielbein formulation**

$$\exp\left(\int\left(-\Phi\mathrm{d}\omega+\frac{\sin(2\log q\Phi)}{2\log q}e^0\wedge e^1+\Phi_0\mathrm{d}e^0-\Phi_0\omega\wedge e^1+\dots\right)\right.$$
$$\left.+\int\mathrm{d}\tau\left(-\Phi\omega_\tau+\Phi_0e^0_\tau+\Phi_1e^1_\tau-\Phi_0^2+\Phi_1^2+\frac{\cos(2\log q\Phi)}{2\log q^2}\right)\right)$$

Field redefinition

$$\Phi=-h\,,\quad\Phi_0=x_0\,,\quad\Phi_1=x_1\,,\quad A^H=\omega\,,\quad A^0=e^1\,,\quad A^1=e^0$$

and introducing **notation**

$$\{h,x_0\}=-x_1\,,\quad\{h,x_1\}=-x_0\,,\quad\{x_0,x_1\}=\frac{\sin(2\log qh)}{2\log q}$$

this becomes **Poisson sigma model**

$$\boxed{\int_0^\infty \mathrm{d}r \int \mathrm{d}\tau \Big(-(A_r)^A \dot{J}_A + (A_\tau)^A \Big(J'_A + \{J_A, J_B\}(A_r)^B \Big)\Big) - \int \mathrm{d}\tau \, \mathbf{H}(J_A)}$$

topological gauge theory generalization of BF (weakly coupled 2d Yang-Mills) to quantum groups (and beyond).

Integrate out Lagrange multiplier A_τ^A gives **constraint** (radial DE)

$$J'_A = -\{J_A, J_B\}(A_r)^B$$

Known (Cattaneo, Felder) that resulting **classical phase space** is 6d and includes $J_A(0)$ with induced Poisson brackets matching with q-Schwarzian

$$\{h(0), x_0(0)\} = -x_1(0)\,, \quad \{x_0(0), x_1(0)\} = \frac{\sin(2\log q h(0))}{2\log q}\,, \ldots$$

We found simple way to **determine** full 6d **classical phase space** (solve constraints)

1. Introduce fields x_A along with J_A.

2. Define fields p_A by q-Schwarzian expression for currents in terms of momenta and coordinates $J_A(x_B, p_B)$.

3. Solutions to constraint with $J'_A = 0$ can be constructed as follows
a. Choose $x_A(r)$ and $p_B(r)$ **classical solution radial** Hamiltonian **evolution** with $\mathbf{H}(J_A(x_B, p_B))$ q-Schwarzian Hamiltonian.
b. The solution to the constraint equation is now

$$(A_r)^C = -\sum_A x'_A \frac{dp_A}{dJ_C} = -\frac{d\mathbf{H}}{dJ_C} \quad \text{for Lie groups } A_r = g'g^{-1}$$

Indeed on these configurations we have

$$-\{J_A, J_B\}(A_r)^B = \{J_A, J_B\}\frac{d\mathbf{H}}{dJ_B} = J'_A = 0$$

Thus 6d classical phase space can be spanned by $x_A(0)$ and $p_A(0)$ classical **initial conditions** radial evolution and can show they inherit canonical Poisson brackets

$$\{x_A(0), p_B(0)\} = \delta_{AB}$$

Hamiltonian on this phase space comes from boundary term $\mathbf{H}(J_A(0))$ on slide 18 thus one obtains **q-Schwarzian quantum mechanics**. □

More **heuristic** but quicker way to appreciate duality is that inserting

$$A^C = \sum_A \frac{dp_A}{dJ_C} \, \mathrm{d}x_A$$

with $J_C(x_A, p_A)$ in Poisson sigma model

$$\int_0^\infty \mathrm{d}r \int \mathrm{d}\tau \Big(-(A_r)^A \dot{J}_A + (A_\tau)^A \Big(J_A' + \{J_A, J_B\}(A_r)^B \Big) \Big) - \int \mathrm{d}\tau \, \mathbf{H}(J_A)$$

one immediately recovers q-Schwarzian

$$\int \mathrm{d}\tau \Big(\sum_A p_A \dot{x}_A - \mathbf{H}(J_A(x_B, p_B)) \Big)$$

□-ish
For Lie groups rigorous because bulk localizes to $A = g^{-1}\mathrm{d}g$.
Here localization is less obvious therefore this slide is heuristic.

Boundary conditions

One **result** of derivation is concrete **mapping** bulk to boundary variables.

Can be used to translate q-Schwarzian **constraint**

$$e^{\mathrm{i}\log q h} f = \frac{1}{2\log q}$$

to the gravitational **boundary condition** (case $q = e^{\mathrm{i}\pi b^2}$)

$$\boxed{F(\Phi_{\text{bdy}})^{1/2} e^{-\pi b^2 \Phi_{\text{bdy}}} = \frac{1}{2\pi b^2}}$$

Inspecting the classical solutions Liouville gravity on slide 10 one sees this determines the **boundary location** as

$$\Phi_{\text{bdy}} = \infty$$

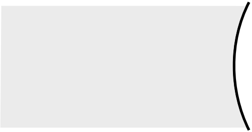
In the JT limit $q = 1$ this becomes the familiar boundary condition

$$F(\Phi_{\text{bdy}})^{1/2} = \frac{1}{\varepsilon}, \quad \varepsilon \to 0$$

Summary **holographic duality** between q-Schwarzian and dilaton gravity

dilaton gravity

vielbein formulation



gauge theory

$A^C = \frac{dp_A}{dJ_C}\,\mathrm{d}x_A$

holography

particle on $U_q(SU(1,1))$

$V(\Phi) = \frac{\sin(2\log q\Phi)}{2\log q}$

constraint

$\sqrt{h}e^{-\mathrm{i}\log q\Phi} = \frac{1}{2\log q}$

$e^{\mathrm{i}\log qh} f = \frac{1}{2\log q}$

boundary conditions

constraints

q-Schwarzian

Remainder: discuss **classical solutions q-Schwarzian** and **compare** with Liouville gravity and DSSYK.

Classical solutions $|q| = 1$ q-Schwarzian

Remember Lorentzian **q-Schwarzian** action

$$\int_0^T \mathrm{d}t \left(p_\phi \phi' + \cdots + \frac{1}{2\log q^2} \cos(\log q p_\phi) - \frac{1}{\log q^2} \mu_\beta \mu_\gamma e^{-2\phi - \mathrm{i} \log q p_\phi} \right)$$

with

$$\mu_\beta = \frac{e^{-2\mathrm{i}\beta p_\beta} - 1}{-2\mathrm{i}\beta}$$

Rescaling $\mathrm{i}\pi b^2 p_\alpha \to p_\alpha$ and going to Euclidean times $\mathrm{d}t = \mathrm{i}\pi b^2 \mathrm{d}\tau$

$$\exp\left(\frac{1}{\pi b^2} \int_0^{\beta/\pi b^2} \mathrm{d}\tau \left(p_\phi \phi' + \cdots + \frac{1}{2} \cos(p_\phi) - \mu_\beta \mu_\gamma e^{-2\phi - \mathrm{i} p_\phi} \right) \right)$$

For $b \ll 1$ the theory behaves semiclassically and **saddle** is dominant.
Same regime where bulk action has large prefactor.

The classical EOM are just **Hamilton equations**.

Playing around with the 6 Hamilton equations can reduce to one equation

$$(1 - (p_\phi''/p_\phi')^2)^{1/2} = -\frac{1}{p_\phi'}(p_\phi''/p_\phi')'$$

with **solution**

$$p_\phi' = \frac{\sqrt{1+c^2} + \sin(c\tau + d)}{1 + 1/c^2 \cos^2(c\tau + d)}$$

introducing

$$c = \sinh(\alpha)$$

this simplifies

$$p_\phi' = \frac{1}{2}\frac{\sinh^2(\alpha)}{\sinh^2(\alpha) + \cos^2(\sinh(\alpha)\tau/2 + b)}$$

$$\boxed{e^{-2\phi} = \frac{\sin(\mathrm{i}\alpha)^2}{\sin^2(\sinh(\alpha)\tau/2 + b + \mathrm{i}\alpha/2)}}$$

Can compare to **classical** limit **two-point function** Liouville gravity $e^{2\phi\Delta}$.

Superficial calculation suggests **match plausible** but we still have to work out the details.

Looking at solutions integration constant α determines their **period**

$$0 < \tau < 2\pi/\sinh(\alpha)$$

The **energy** of the solution depends on period α as

$$E(\alpha) = \frac{1}{\pi^2 b^4}\left(-\frac{1}{2}\cos(p_\phi) + \mu_\beta\mu_\gamma e^{-2\psi}\right) = \frac{\cosh(\alpha)}{2\pi^2 b^4}$$

Classical **entropy** for fixed energy α computed as

$$S(\alpha) = \frac{\mathrm{i}}{\pi b^2}\int_0^{2\pi/\sinh(\alpha)} \mathrm{d}\tau \sum_i p_i q_i' = -\frac{1}{b^2}\log(2) + \frac{\alpha}{b^2}$$

Relabeling α **match exactly** partition function **Liouville gravity** slide 10

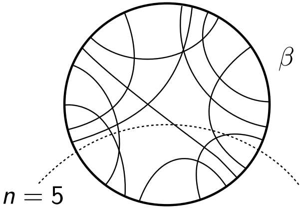
$$\boxed{\exp\left(-\frac{1}{b^2}\log(2) + 2\pi\Phi_h - \beta\frac{\cosh\left(2\pi b^2\Phi_h\right)}{2\pi^2 b^4}\right)}$$

Okay good.

Intermezzo. Lightning introduction to DSSYK

Now what about (arguably) more interesting case $q > 1$ DSSYK?

First, some relevant fact about DSSYK to compare with.
Observables computed exactly via **chord diagram** (Berkooz, Isachenkov, Narovlansky, Torrents, Narayan, Simon).



$$= \int_0^{\pi} \mathrm{d}\theta \, e^{S(\theta) - \beta E(\theta)}$$

Number **n chords** on time slice and remember from quantization

$$\phi = n \log q$$

Crucially **number** of **chords positive** (obvious), therefore

$$\boxed{\phi > 0}$$

Remember this constraint for later!

In **classical regime** $\log q \ll 1$ one finds (Goel, Narovlansky, Verlinde)

$$Z(\beta) = \int_0^\pi \mathrm{d}\theta \exp\left(\frac{\pi\theta}{\log q} - \frac{\theta^2}{\log q} + \beta \frac{\cos(\theta)}{2\log q^2}\right)$$

This **mismatches** naive **gravity calculation** slide 8 $\theta = 2\log q\Phi_h$?!

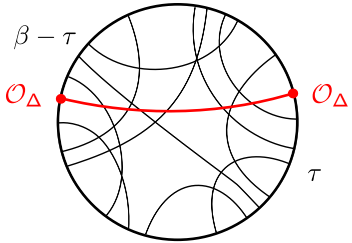
$$Z_{\text{grav}}(\beta) \overset{?}{=} \int_0^\pi \mathrm{d}\theta \exp\left(\frac{\pi\theta}{\log q} + \beta \frac{\cos(\theta)}{2\log q^2}\right)$$

Help comes because will actually find **similar discrepancy** in DSSYK versus naive **q-Schwarzian**.
In case of q-Schwarzian we found **resolution**. Which I will soon present.

Translation of **resolution** to **gravity not** yet **understood**.

But first second data point DSSYK to compare with q-Schwarzian...

Can compute **correlators of operators** product s fermions with $\Delta = s/q$



$$= \int_0^\pi \mathrm{d}\theta_1\, e^{S(\theta_1)-(\beta-\tau)E(\theta_1)} \int_0^\pi \mathrm{d}\theta_2\, e^{S(\theta_2)-\tau E(\theta_2)} \sum_{n=0}^{+\infty} \psi_{E(\theta_1)}(n)\psi_{E(\theta_2)}(n) q^{-2n\Delta}$$

$$= \int_0^\pi \mathrm{d}\theta\, e^{S(\theta)-\beta E(\theta)} \frac{\sin^{2\Delta}(\theta)}{\sin^{2\Delta}(\sin(\theta)\tau/2+\theta)}$$

remembering $\phi = n \log q$ slide 16 this computes expectation value $e^{-2\phi\Delta}$
so one finds classically (Goel, Narovlansky, Verlinde, Stanford, Lin)

$$\boxed{e^{-2\phi} = \frac{\sin^2(\theta)}{\sin^2(\sin(\theta)\tau/2+\theta)}}$$

Back to the q-Schwarzian now!

Classical solutions $q > 1$ DSSYK q-Schwarzian

Obtains **analytic continuation** solutions for $|q| = 1$ slide 24 $\mathrm{i}\alpha \to \theta$ and $\tau \to \mathrm{i}\tau$

$$e^{-2\phi} = \frac{\sin^2(\theta)}{\sin^2(\sin(\theta)\tau/2 + \theta)}$$

which **exactly matches** DSSYK calculation (quite nontrivial check). Integration constant b choice of time origin.

The **energy** of classical orbit **matches** with DSSYK and with gravity

$$E(\theta) = \frac{1}{\log q^2}\left(-\frac{1}{2}\cos(p_\phi) + \mu_\beta \mu_\gamma e^{-2\psi} \right) = -\frac{\cos(\theta)}{2\log q^2}$$

The naive **period** of this solution is

$$0 < \tau < 2\pi / \sin(\theta)$$

resulting in **entropy**

$$S(\theta) = \frac{\mathrm{i}}{\log q} \int_0^{2\pi/\sin(\theta)} \mathrm{d}\tau \sum_i p_i q_i' = \frac{\mathrm{i}\pi}{\log q}\log(2) + \frac{\pi\theta}{\log q}$$

which by construction reproduces thermodynamics

$$\frac{\beta}{\log q} \stackrel{?}{=} \frac{2\pi}{\sin\theta}$$

this **matches** with naive **gravity calculation** slide 8

$$Z_{\text{qsch}}(\beta) = Z_{\text{grav}}(\beta) \stackrel{?}{=} \int_0^{\pi} \mathrm{d}\theta \exp\left(\frac{\pi\theta}{\log q} + \beta\frac{\cos(\theta)}{2\log q^2}\right)$$

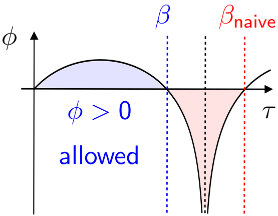
but **period mismatches** with DSSYK calculation slide 27?!

Then what is **resolution** of this tension?
We still must **impose constraint** from exact quantum theory

$$\boxed{\phi > 0}$$

on our classical solutions!

Plot classical solution



So $\phi > 0$ constrains solution to **shorter interval** (endpoints identified)!

$$0 < \tau < \frac{2\pi - 4\theta}{\sin(\theta)}$$

Therefore **correct classical period** aka temperature is actually

$$\boxed{\frac{\beta}{\log q} \stackrel{!}{=} \frac{2\pi - 4\theta}{\sin(\theta)}}$$

This does **match DSSYK** thermodynamics!
Known classical solution two-point DSSYK also limited to this interval!

Some comments about this

1. For $q > 1$ regardless of constraining $\phi > 0$ this is **periodic path**.
For $|q| = 1$ no other (real) identification except naive one

$$e^{-2\phi} = \frac{\sin(\mathrm{i}\alpha)^2}{\sin^2(\sinh(\alpha)\tau/2 + \mathrm{i}\alpha/2)}$$

Fundamental reason? Want to understand better (Stanford, Lin).
Maximal vs sub-maximal chaos. Periodic orbit interpretation?

2. This **shorter period** reason for distinction between temperature

$$\frac{\beta}{\log q} = \frac{2\pi - 4\theta}{\sin(\theta)}$$

and **fake temperature** or **tomperature** (or your favorite) (Susskind, Lin, Rahman)

$$\frac{\beta_{\text{naive}}}{\log q} = \frac{2\pi}{\sin(\theta)}$$

The latter determines **characteristic time** for **physics** for instance decay

$$e^{-2\phi\Delta} \sim \langle \mathcal{O}_\Delta(0)\mathcal{O}_\Delta(t)\rangle \sim \exp\left(-\frac{2\pi \log q}{\beta_{\text{naive}}}\Delta\, t\right)$$

notice also for $\beta = 0$ decay time remains finite $\beta_{\text{naive}} = 2\pi \log q$.
The latter feature is **expected of dS space**.

Main **open question** how shorter period **constraint** $\phi > 0$ q-Schwarzian translates back to **gravity description**??
Here any resemblance of factual statements from my side stops...

In some sense having a **smooth horizon** implies $\beta_{\text{naive}} = \beta$ since Rindler boost identification, hence fundamental tension with black hole horizon?
Some **defect** sourcing offset between β and β_{naive}?
Note role of **observers in dS** important there to understand max mixed $\beta = 0$ state but finite $\beta_{\text{naive}} = \beta_{\text{dS}}$ (Chandrasekaran, Longo, Pennington, Witten).
Our **boundary observer** (q-Schwarzian) measures **similar** physics...
Maybe similar logical makes our (semi)classical gravity description work?
Stay tuned...

Summary

Depending on q **q-Schwarzian** has the following bulk dual 2d **dilaton gravity** description

$q > 1$	$\Leftrightarrow$	$V(\Phi) = \sin(2\Phi)$	
$\lvert q \rvert = 1$	$\Leftrightarrow$	$V(\Phi) = \sinh(2\Phi)$	Liouville gravity
$q = 1$	$\Leftrightarrow$	$V(\Phi) = 2\Phi$	JT gravity

Furthermore the $q > 1$ q-Schwarzian is **equivalent** to the chord diagram description of DSSYK.

Match q-Schwarzian with DSSYK because **restriction** $\phi > 0$.
Implies difference between temperature and effective temperature!

How does restriction or different temperatures **translate to** gravity?
Or how does the gravity description explain temperature?
Once duality understood could attempt probing $R > 0$ regions.

Thanks.