New Calabi-Yau manifolds from Genetic Algorithms

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based on 2306.06159

Outline

- Introduction
- Genetic algorithms
- Reflexive polytopes
- Methodology for polytope search
- Results
- Conclusion

Introduction

Numbers in string theory

Set $h := h^{1,1}(X)$ for a CY manifold X.

CY 3-folds

- # of 4d reflexive polytopes: 473,800,776, where $1 \leq h \leq 491$
  (M. Kreuzer, H. Skarke, hep-th/0002240)
- # of triangulation for each 4d reflexive polytopes: $\sim 10^h$
  (R. Altman, J. Gray, Y.-H. He, V. Jejjala, B. D. Nelson, JHEP 02 (2015) 158)

this talk

CY 4-folds

- # of 5d reflexive polytopes: perhaps $\sim 10^{18}$

heterotic line bundle models

- # of line bundle sums on CY 3-fold: $\sim 10^{4h}$
- # of heterotic CY models with SM spectrum: $\sim 10^{h-3}$
  (A. Constantin, Y.-H. He, A. Lukas, 1810.00444)

Andrei Constantin
Thomas Harvey,
in parallel session

. . . .

Exploration requires efficient and targeted search methods!

Search algorithms

- MCMC with schedules annealing ("classical annealing")

- Quantum annealing

- Reinforcement learning (RL)

- Genetic algorithms (GAs) ⟵ this talk

. . . .

GA in particle/string theory pioneered by:

(S. Abel, J. Rizzos, 1404.7359, search for free fermionic models)

O(10) papers applying GA to various classes of models since.

-> GA is impressively efficient searching classes of physics models

Recent proposal to combine GA and quantum annealing (GQAA).

(S. Abel, L. Nutricati, M. Spannowsky, 2209.07455)

Questions

- Can GAs find reflexive polytopes?
- Can GAs find complete lists of reflexive polytopes, for $d<5$?
- Can GAs be used to find new $d=5$ reflexive polytopes?
- Do GAs facilitate a targeted search of reflexive polytopes with specific properties?

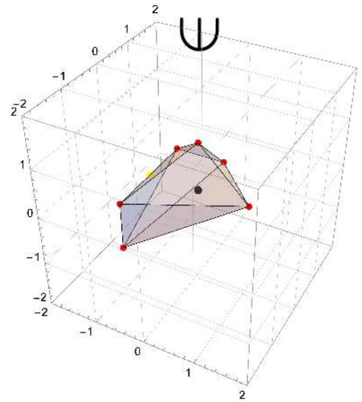
Genetic Algorithms

Basic ingredients of GAs

probability distribution → p_{in}; genotypes → $\mathbb{F}_2^{n_{\mathrm{bits}}}$; genotype/phenotype map → φ; environment= phenotypes → E; fitness function → f

$$[0,1] \xleftarrow{\;p_{\mathrm{in}}\;} \mathbb{F}_2^{n_{\mathrm{bits}}} \xrightarrow{\;\varphi\;} E \xrightarrow{\;f\;} \mathbb{R}$$

$$\mathbb{F}_2^{n_{\mathrm{bits}}} \ni (1,0,0,1,\cdots,0,1)$$



$E \ni$ (polytope shown above)

"quality" of model ↑ (f)

Genetic evolution

(1) Sample with p_{in} to get initial population $P_0 \subset \mathbb{F}_2^{n_{\mathrm{bits}}}$ with size $n_{\mathrm{pop}} = |P_0|$.

(2) Evolve P_0 by combining (i) selection, (ii) cross-over and (iii) mutation:

$$P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots \longrightarrow P_{n_{\mathrm{gen}}}$$

(3) Select all $b \in \bigcup_i P_i$ with $f \circ \varphi(b) \geq f_{\mathrm{term}}$.
These are the terminal states b which lead to "good models" $\varphi(b)$.

Evolving $P_k \to P_{k+1}$

(i) selection:

- define probability distribution $p_k : \mathbb{F}_2^{n_{\rm bits}} \to [0,1]$ by (roulette selection)

$$p_k(b) = \frac{1}{n_{\rm pop}} \frac{(\alpha-1)(f(\varphi(b)) - \bar{f}) + f_{\rm max} - \bar{f}}{f_{\rm max} - \bar{f}} \qquad \alpha \in [2,5]$$

- based on p_k, select $n_{\rm pop}/2$ pairs of individuals from P_k

(ii) cross-over:

- for each pair from (i), pick a random position $k \in \{1, \dots, n_{\rm bits}\}$
- swap tails of two individuals:

$k^{\rm th}$

$$b_1 = (1,0,\dots,1,0,\boxed{1,1,1,0,\dots,1,1,0,0}) \qquad \tilde{b}_1 = (1,0,\dots,1,0,\boxed{0,0,1,0,\dots,0,1,0,1})$$

$$b_2 = (0,0,\dots,0,1,\boxed{0,0,1,0,\dots,0,1,0,1}) \qquad \tilde{b}_2 = (0,0,\dots,0,1,\boxed{1,1,1,0,\dots,1,1,0,0})$$

- the $n_{\rm pop}$ new individuals obtained in the way form a population $\tilde{P}_{k+1}$

(iii) mutation:

- randomly flip a small fraction $r \sim 0.01$ of bits in $\tilde{P}_{k+1}$ to obtain P_{k+1}

Reflexive Polytopes

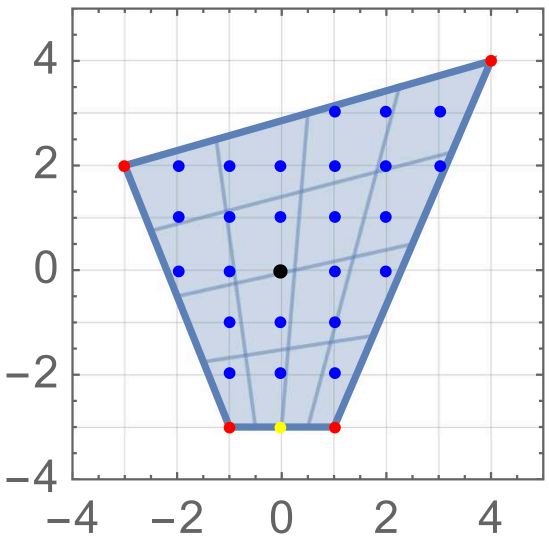
Lattice polytopes

lattice polytope Δ in n dimensions:

convex hull in $\mathbb{R}^n$ of lattice points $x_1, \ldots, x_m \in \mathbb{Z}^n$

represented by $n \times m$ matrix $\mathcal{X} = (x_1, \ldots, x_m)$ of generators

Example



generators x_i:

$$\mathcal{X} = \begin{pmatrix} 4 & 1 & -3 & 0 & -1 & 1 \\ 4 & 2 & 2 & 2 & -3 & -3 \end{pmatrix}$$

red points: vertices v_i

$$V = \begin{pmatrix} -3 & -1 & 1 & 4 \\ 2 & -3 & -3 & 4 \end{pmatrix}$$

Vertices

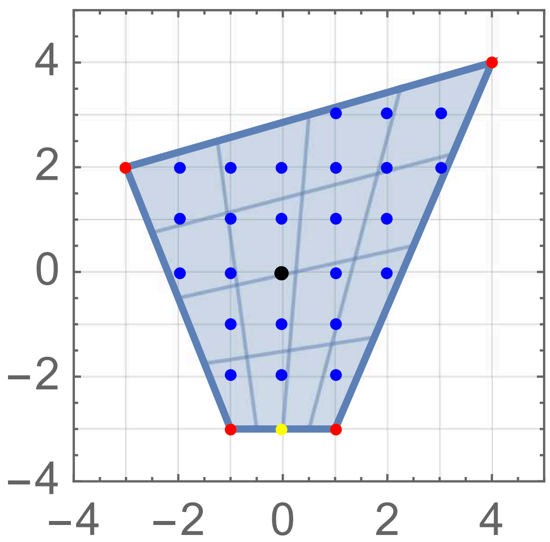
minimal subset, $V = (v_1, \ldots, v_{N_\mathrm{v}})$, of generators whose convex hull is Δ.

N_v, N_p : number of vertices and lattice points of Δ

Faces and facets

face: intersection of Δ with a hyperplane whose negative half-space contains Δ

facets $F(\Delta)$: dimension n-1 faces of Δ <-> supporting hyperplanes $u \cdot x = d$



equations for supporting hyperplanes:

$$\begin{aligned} -2x_1 + 7x_2 &= \boxed{20} \\ 7x_1 - 3x_2 &= \boxed{16} \\ -x_2 &= \boxed{3} \\ -5x_1 - 2x_2 &= \boxed{11} \end{aligned}$$

$$= d_\Delta(\varphi)$$

Lattice distance

Facet $\varphi \in F(\Delta)$, supporting hyperplane $u \cdot x = d$.

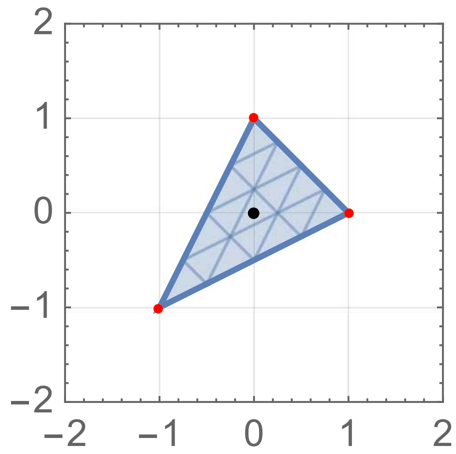
Then $d_\Delta(\varphi) = d$ is called the lattice distance of φ.

Reflexive polytopes

A lattice polytope Δ is reflexive iff

(i) the origin is the only interior lattice point of Δ (IP property)

(ii) all facets $\varphi \in F(\Delta)$ are at lattice distance 1, so $d_\Delta(\varphi) = 1$



equations for supporting hyperplanes:

$$\begin{aligned} x_1 + x_2 &= \boxed{1} \\ x_1 - 2x_2 &= \boxed{1} \\ -2x_1 + x_2 &= \boxed{1} \end{aligned}$$

$= d_\Delta(\varphi)$

Equivalence

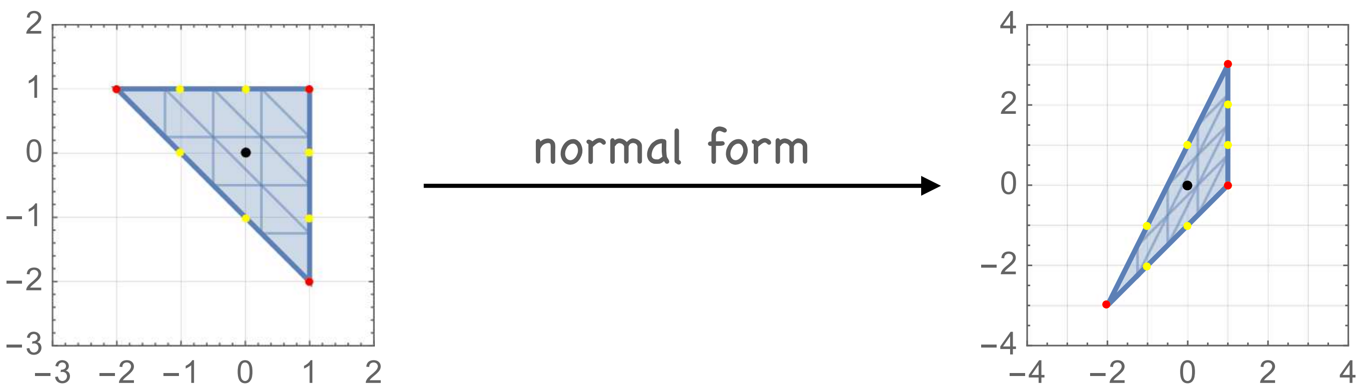
Two polytopes $\Delta, \tilde{\Delta}$ with vertex matrices $V, \tilde{V}$ (same number N_v of vertices) are equivalent iff

$$\widetilde{V} = G\,V\,P$$

for a permutation matrix P and $G \in \mathrm{GL}(n, \mathbb{Z})$.

Eliminate by computing normal form of V.

$$V = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \qquad \tilde{V} = \begin{pmatrix} -2 & 1 & 1 \\ -3 & 0 & 3 \end{pmatrix}$$



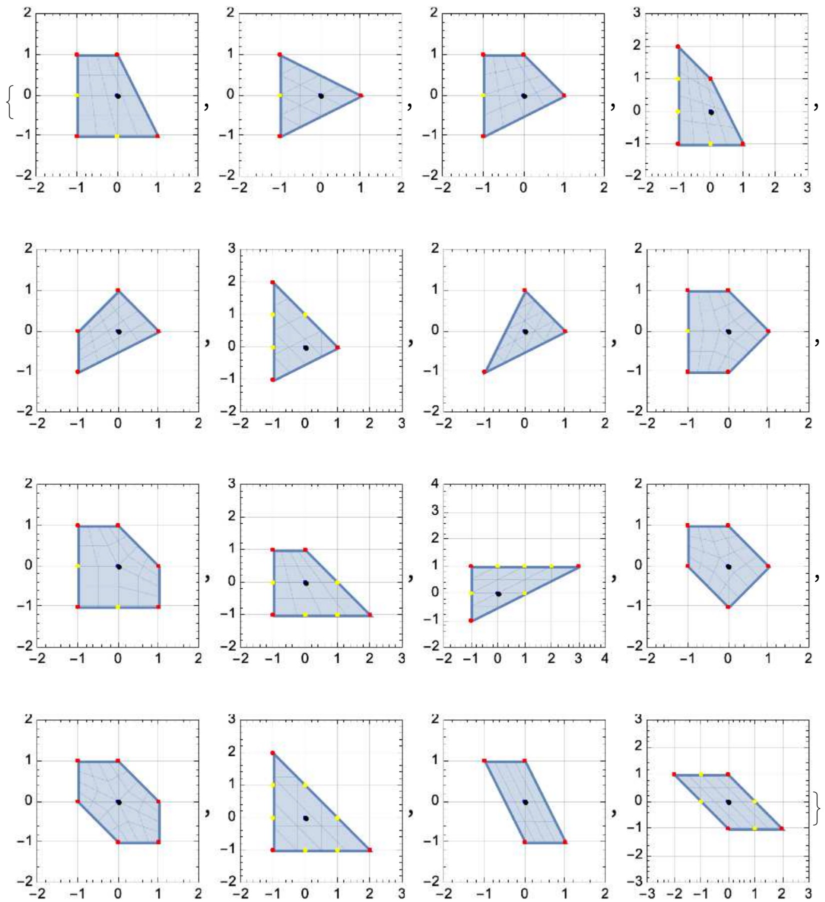
CY manifolds

A pair (Δ, Δ°) of an n-dim reflexive polytope Δ and its dual Δ° (plus triangulation) defines a mirror pair of CY $(n-1)$ - folds.

Number of reflexive polytopes

After modding out equivalence, number is finite in any dimension.

n=2: 16 reflexive polytopes



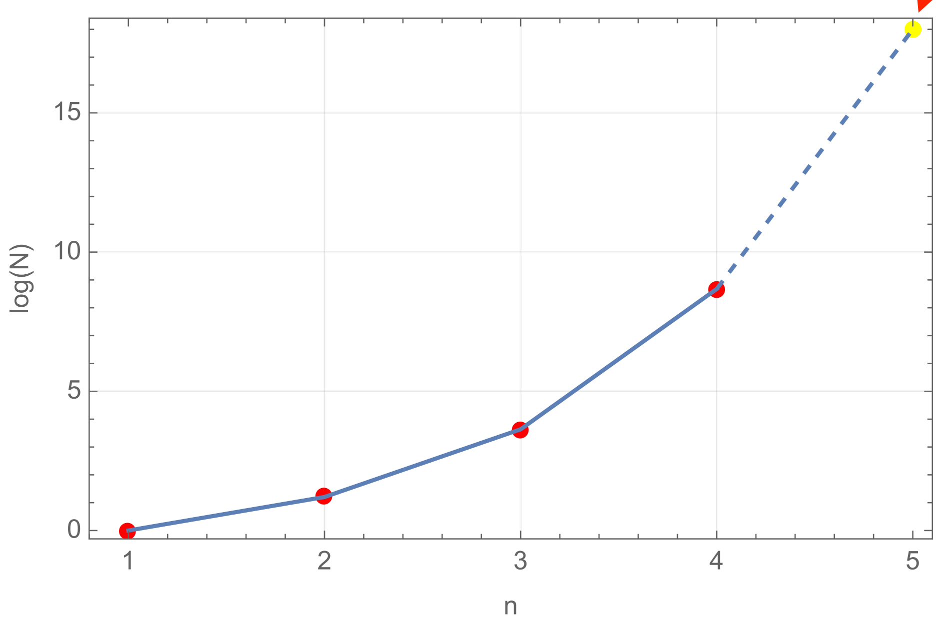
(as found by GA)

n=3: 4319 reflexive polytopes (M. Kreuzer, H. Skarke, hep-th/9805190)

n=4: 473,800,776 reflexive polytopes (M. Kreuzer, H. Skarke, hep-th/0002240)

n=5: partial classification
-> 185,269,499,015 reflexive polytopes,
obtained from maximal polytopes,
estimate: $\sim 10^{18}$ reflexive polytopes (F. Schoeller, H. Skarke, CMP 372 (2019) 657)

n=5 estimate



Methodology for polytope search

Basic set-up

Fix dimension n and (maximal) number of generators m.

Describe polytope by n x m integer matrix $\mathcal{X} = (x^i_a)$

Constrain $x^i_a \in \{x_{\rm min}, x_{\rm min}+1, \ldots, x_{\rm min}+2^\nu-1\}$ -> ν bits per integer

Genotype-phenotype map

Matrix $\mathcal{X}$ described by $n_{\rm bits} = n\, m\, \nu$ bits.

bijective

$$\mathbb{F}_2^{n_{\rm bits}} \xrightarrow{\ \varphi\ } \{\mathcal{X}\} = E$$

Size of environment

Say, $m = n+1$ (minimal choice) and $\nu = 3$:

$$2^{n_{\rm bits}} \simeq \begin{cases} 10^7 & , \quad n=2 \\ 10^{14} & , \quad n=3 \\ 10^{24} & , \quad n=4 \\ 10^{36} & , \quad n=5 \end{cases}$$

Initial probability distribution

$p_{\rm in}$ either flat or with bias towards small $|x_a^i|$

Fitness

weights

$$f(\Delta) = w_1\,(\mathrm{IP}(\Delta) - 1) - \frac{w_2}{|F(\Delta)|} \sum_{\varphi \in F(\Delta)} |d_\Delta(\varphi) - 1| \leq 0$$

IP property — lattice distance

$$\Delta \text{ reflexive} \quad \Longleftrightarrow \quad f(\Delta) = 0$$

Fitness for targeted searches

Search for polytopes Δ with target numbers $N_{\rm v,0}$, $N_{\rm p,0}$ of points and vertices:

Modify fitness function to:

$$\widetilde{f}(\Delta) = f(\Delta) - w_3\,|N_{\rm p}(\Delta) - N_{\rm p,0}| - w_4\,|N_{\rm v}(\Delta) - N_{\rm v,0}|$$

Realisation

GA code, realised in c: https://github.com/harveyThomas4692/GA-C

polytope environment,
PALP plus c code: https://github.com/elliheyes/Polytope-Generation

Compute all relevant properties of Δ from generator matrix $\mathcal{X}$

For ease of use, there is also a python interface . . .

Removing reduncancies

For all reflexive polytopes found by GA, compute normal form of V with PALP

Results

Common settings: $n_{\rm gen} = 500$, $r_{\rm mut} = 0.005$, $\alpha = 3$.

Two-dimensional polytopes, n=2

Settings: $x_a^i \in [-3, 4]$, $\nu = 3$, $m = 6$, $n_{\rm pop} = 200$ -> #states $\simeq 10^{11}$

Finds all 16 reflexive polytopes in a single GA run (few seconds on single CPU)

Three-dimensional polytopes, n=3

Settings: $x_a^i \in [-7, 8]$, $\nu = 4$, $m = 14$, $n_{\rm pop} = 450$ -> #states $\simeq 10^{51}$

Finds all 4319 reflexive polytopes in 117251 GA runs.

Fraction of the environmental states visited $\sim 10^{-40}$

Evidence that GA can find (nearly) complete lists of reflexive polytopes

Four-dimensional polytopes, n=4

Have not attempted to reproduce the entire Kreuzer-Skarke list.

Instead, targeted searches for fixed number of vertices and points.

$N_{\mathrm{v},0} = 5$

Settings: $x_a^i \in [-15, 16]$, $\nu = 5$, $m = 5$, $n_{\mathrm{pop}} = 200$ -> #states $\simeq 10^{30}$

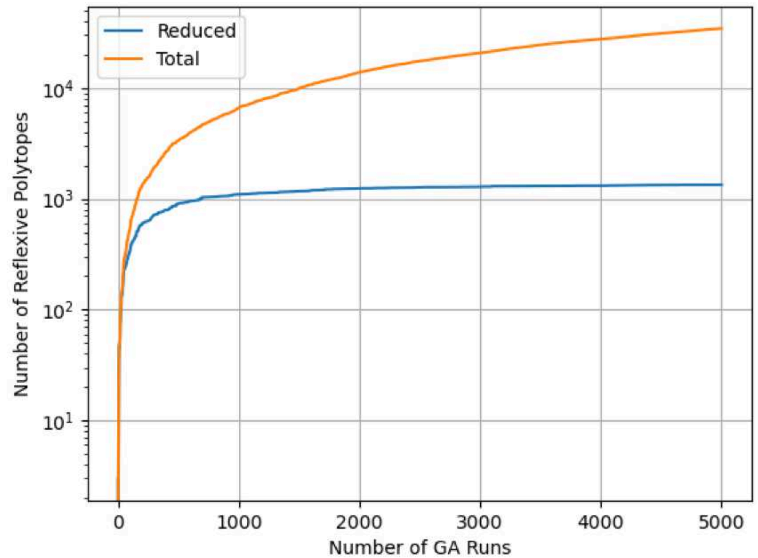


Figure 1: Log plot of total number of generated four-dimensional reflexive polytopes with five vertices against number of genetic algorithm evolutions. The total before and after removing redundancy are shown in orange and blue, respectively.

Find all but 6 of the 1561 reflexive polytopes with $N_{\mathrm{v}} = 5$.

$N_{p,0} \in \{6, 7, 8, 9, 10\}$

Settings: $x_a^i \in [-3, 4]$, $\nu = 3$, $m = N_{p,0} - 1$

Results:

# points	# states	n_{pop}	# refl. poly.	# GA runs	states visited
6	$\sim 10^{19}$	400	3	5	$\sim 10^{-13}$
7	$\sim 10^{22}$	300	25	30	$\sim 10^{-16}$
8	$\sim 10^{26}$	400	168	60	$\sim 10^{-19}$
9	$\sim 10^{29}$	300	892	9378	$\sim 10^{-20}$
10	$\sim 10^{33}$	350	3838	9593	$\sim 10^{-24}$

GA finds complete lists of 4d reflexive polytopes with small N_p

Five-dimensional polytopes, n=5

Proceed in analogy with n=4 case: small number of vertices/points

$N_{\mathrm{v},0} = 6$

Settings: $x^i_a \in \{-15, \ldots, 16\}$, $\nu = 5$, $m = 6$, $n_{\mathrm{pop}} = 500$. -> #states $\simeq 10^{46}$

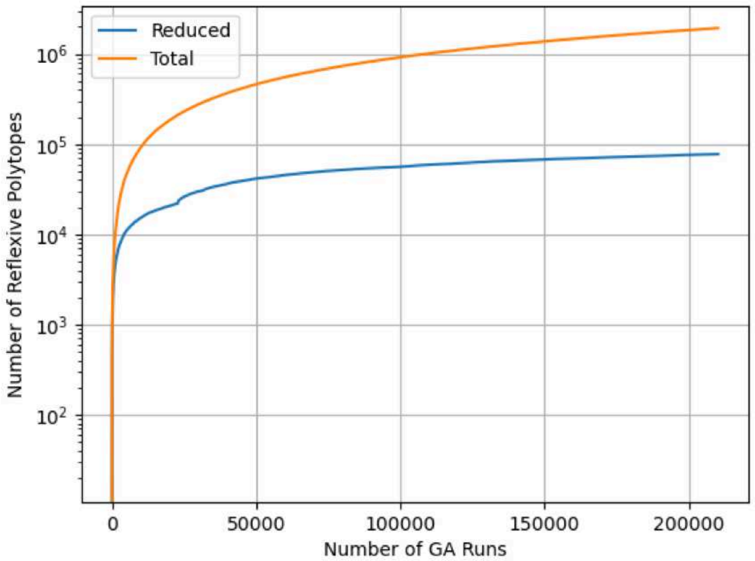


Figure 2: Total number of generated five-dimensional reflexive polytopes with six vertices against number of genetic algorithm evolutions.

Find 77470 n=5 reflexive polytopes with $N_{\mathrm{v}} = 6$ -> likely strong lower bound

$N_{\mathrm{p},0} \in \{7, 8, 9, 10, 11\}$

Settings: $x_a^i \in \{-3, \ldots, 4\}$, $\nu = 3$, $m = N_{\mathrm{p},0} - 1$

Results:

# points	# states	n_{pop}	# refl. poly.	# GA runs	states visited
7	$\sim 10^{28}$	350	9	36	$\sim 10^{-22}$
8	$\sim 10^{32}$	350	115	1278	$\sim 10^{-24}$
9	$\sim 10^{37}$	450	1385	7520	$\sim 10^{-28}$
10	$\sim 10^{41}$	750	12661	31857	$\sim 10^{-31}$
11	$\sim 10^{46}$	650	87907	67382	$\sim 10^{-36}$

Likely strong lower bounds on numbers.

#polytopes with $h^{1,1}(\Delta) = \begin{cases} 1 & : \quad 15 \\ 2 & : \quad 195 \end{cases}$

Many of these not in Schoeller-Skarke list, even new Hodge numbers.

Conjecture 5.1 *There are precisely* 15 *five-dimensional reflexive polytopes that give rise to four complex dimensional Calabi–Yau hypersurfaces with Hodge number* $h^{1,1} = 1$.

A search for specific Euler numbers

M-theory compactification on CY 4-fold with flux, N=2 -> N=1 breaking:

Requires Euler number $\chi(\Delta)$ divisible by $\delta \in \{24, 224, 504\}$.

(M. Berg, M. Haack, H. Samtlebel, hep-th/0212225)

Modify fitness function to:

$$\widetilde{f}(\Delta) = f(\Delta) - w_5 \sum_{\delta} \chi(\Delta) \bmod \delta$$

Settings: $x_a^i \in \{-3, \ldots, 4\}$, $\nu = 3$, $m = 10$, $n_{\text{pop}} = 550$

After 10 GA runs, 21 new polytopes with such an Euler number are found.

Conclusions

- GAs are efficient in generating (nearly) complete lists of reflexive polytopes, as comparison with classifications in n=2,3,4 dimensions shows.

- We have used GAs to find (nearly) all five-dimensional reflexive polytopes with low numbers of vertices and points.

- From these we have extracted cases with $h^{1,1}(\Delta) = 1, 2$. There are many new ones, compared to the Schoeller-Skarke list.
  There are even cases with new Hodge numbers.

- Conjecture: There are precisely 15 reflexive polytopes Δ in five dimensions with $h^{1,1}(\Delta) = 1$.

Outlook:

- Triangulations with GAs?

- Targeted search of reflexive polytopes (and triangulations) for cases with prescribed properties.

감사합니다