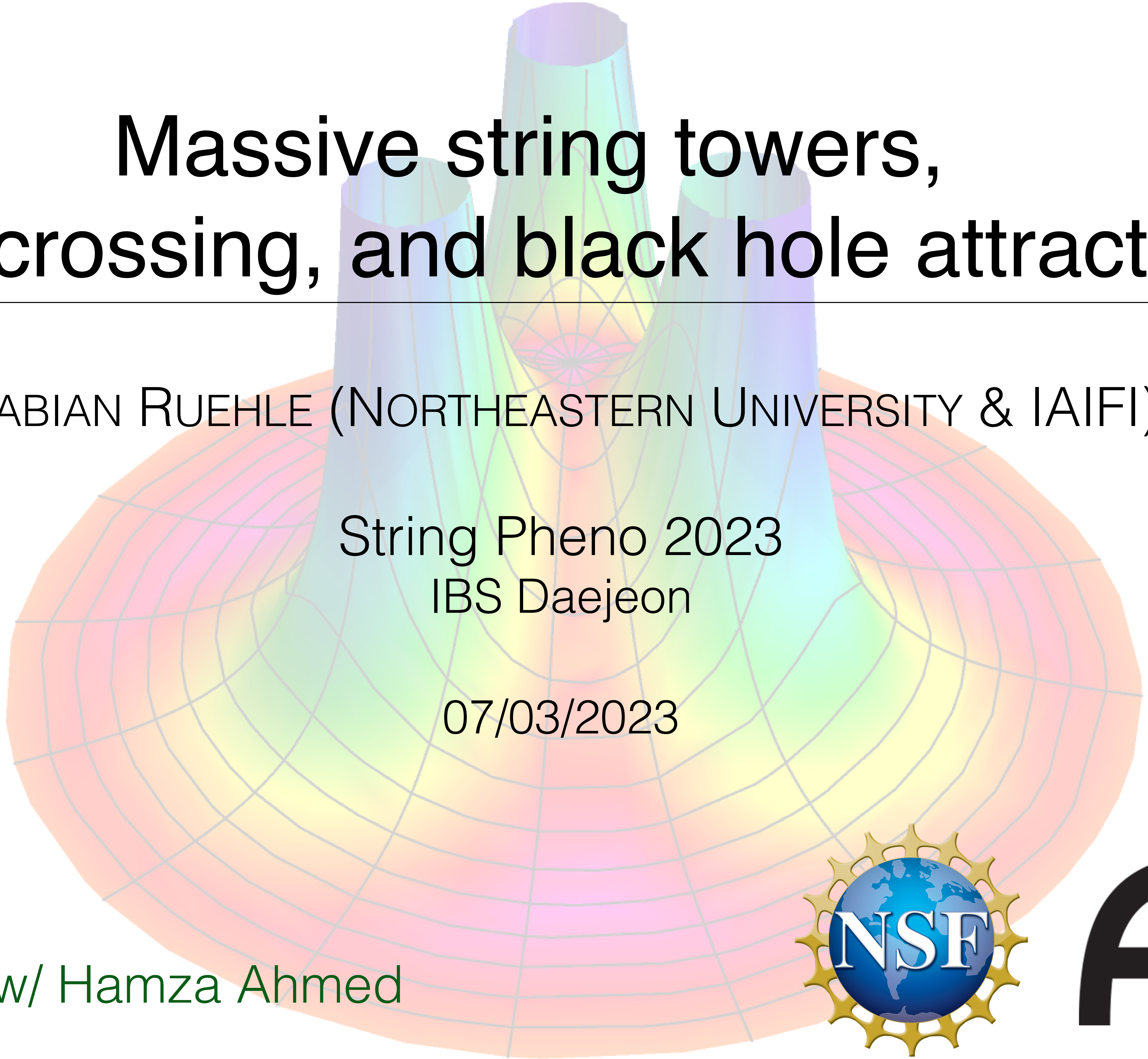

Massive string towers, level crossing, and black hole attractors

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FABIAN RUEHLE (NORTHEASTERN UNIVERSITY & IAIFI)

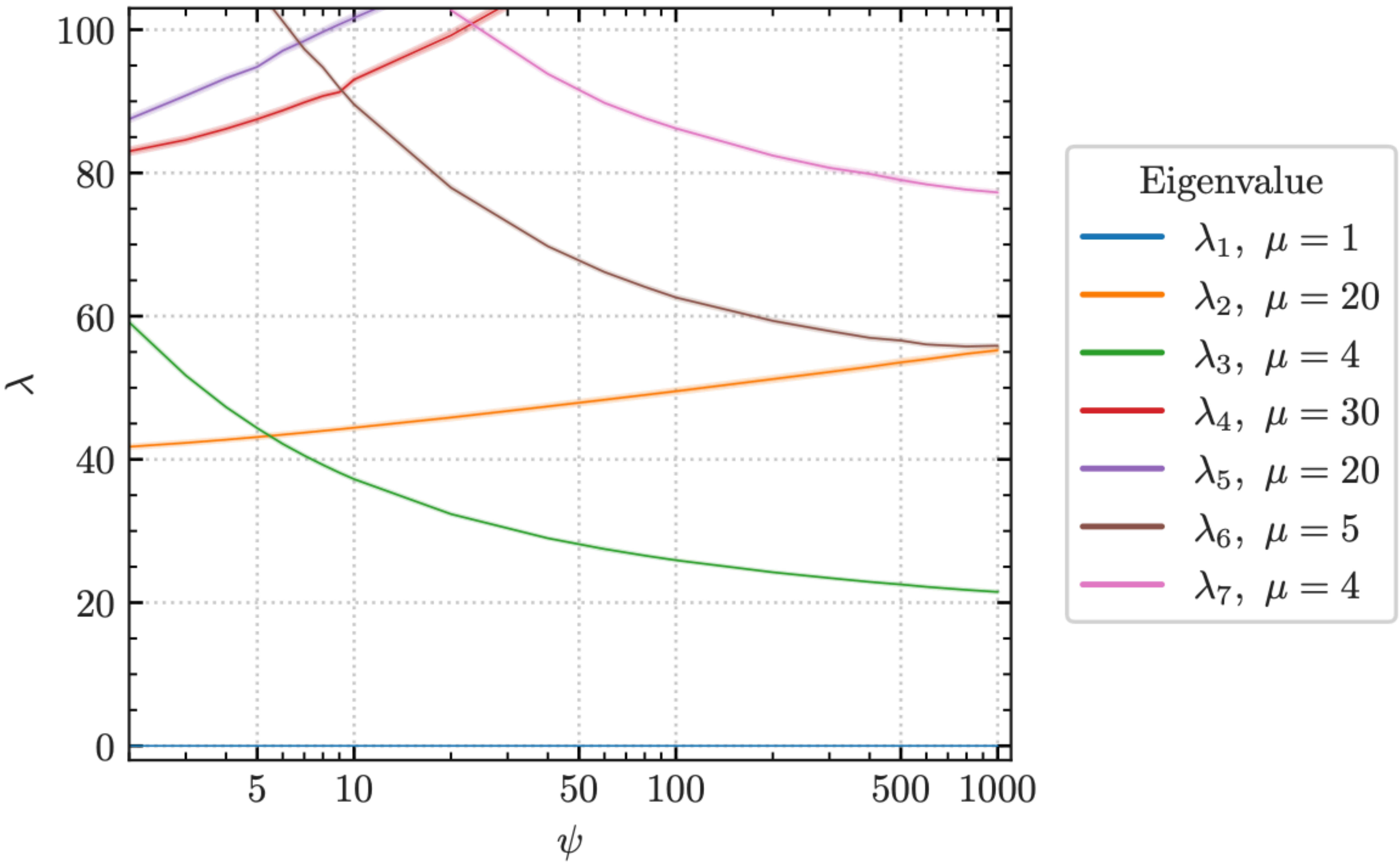
String Pheno 2023
IBS Daejeon

07/03/2023

Based on

2304.00027 w/ Hamza Ahmed

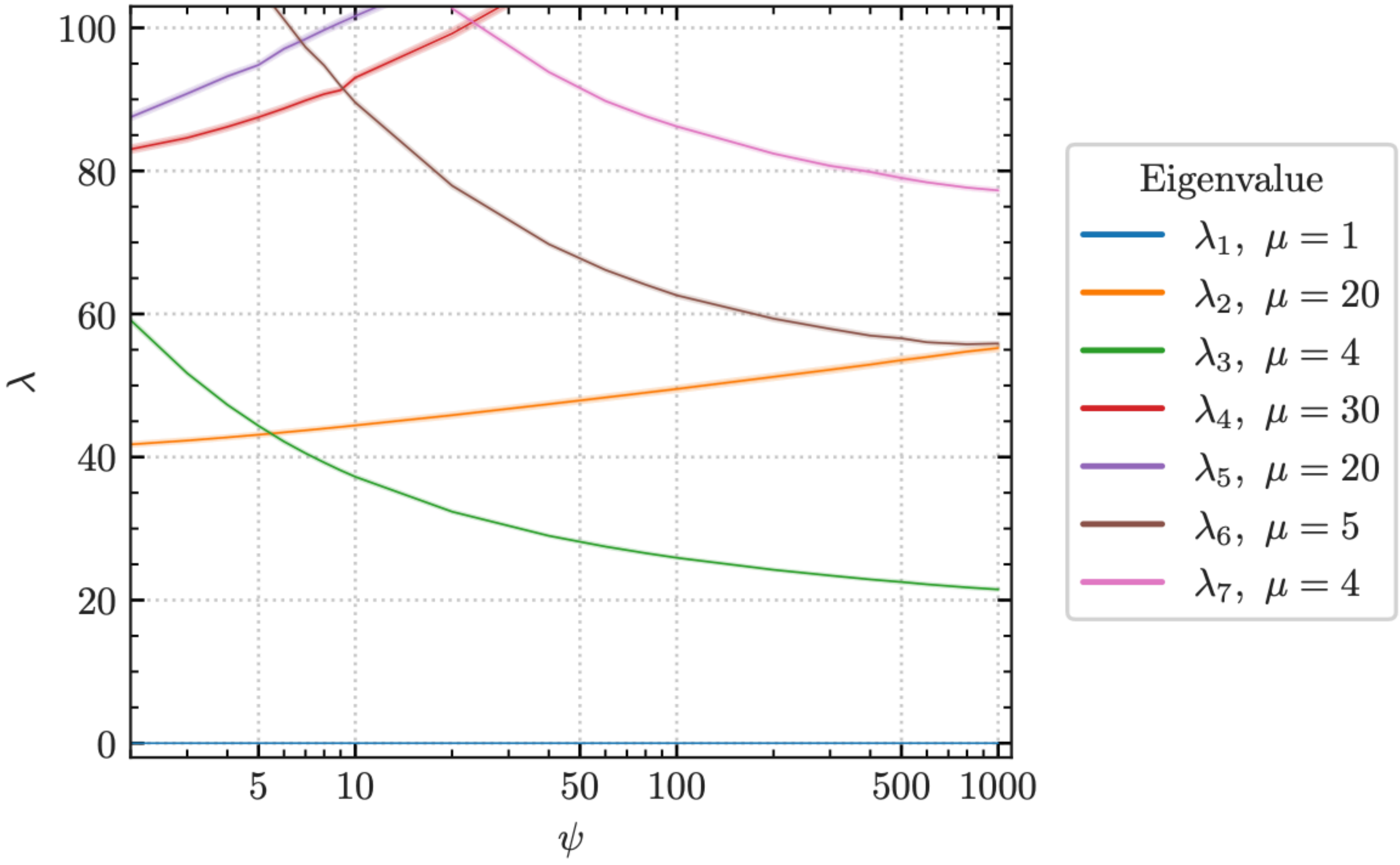
Motivation



$$z_0^5+z_1^5+z_2^5+z_3^5+z_4^5-5\psi z_0z_1z_2z_3z_4=0\subset\mathbb{P}^4$$

[Ashmore, Ruehle `21]

Motivation



Questions:

- Why do some modes become heavier and others lighter?
- Is there something special about the crossing points

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4 = 0 \subset \mathbb{P}^4$$

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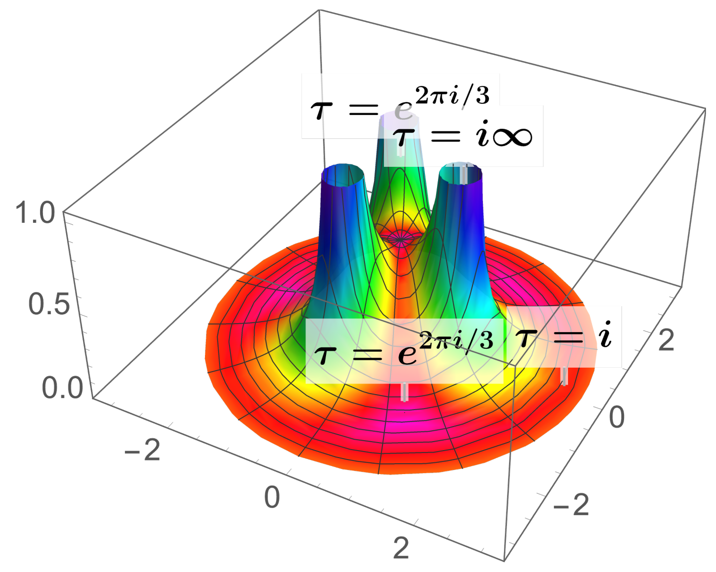
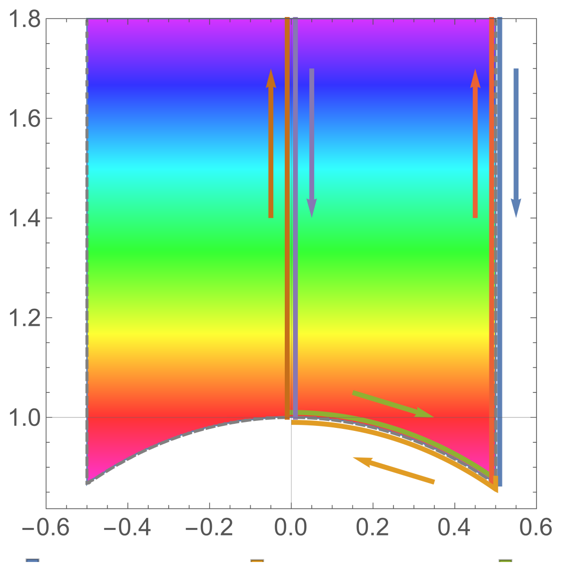
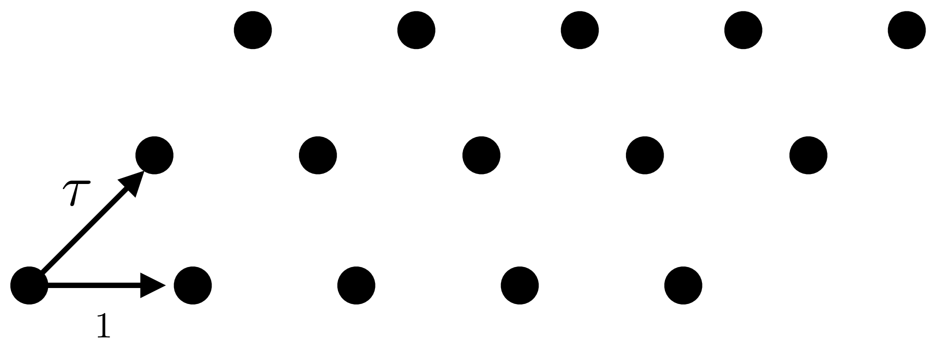
Outline

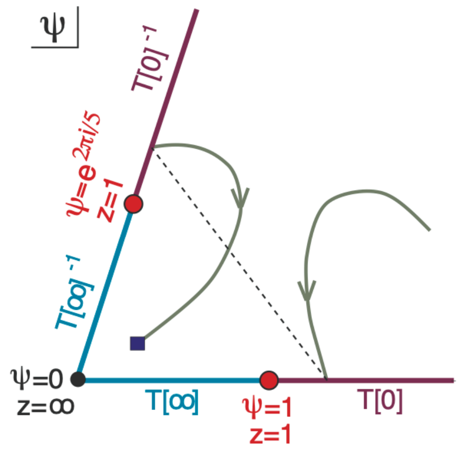
1 Toroidal compactifications



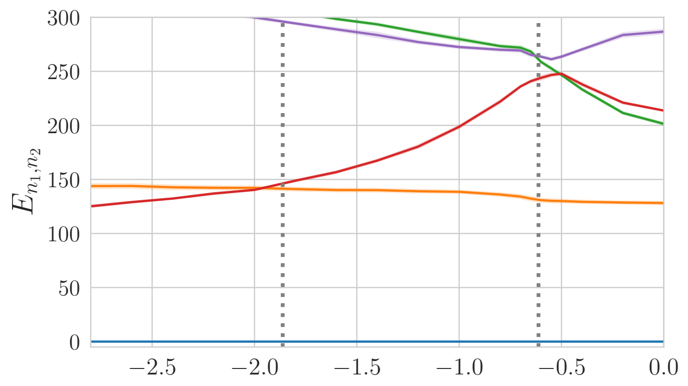


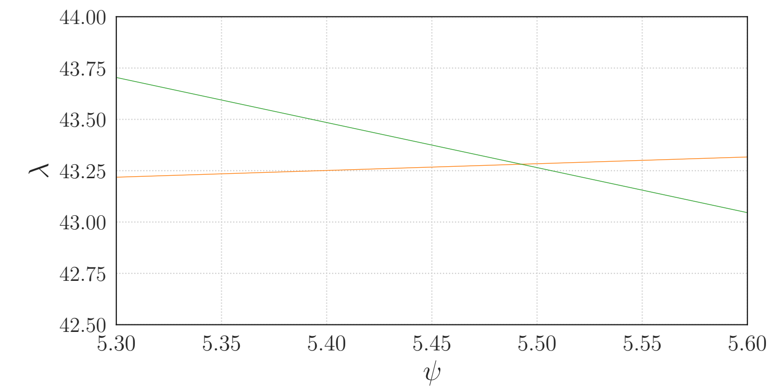
2 Attractors and CM points



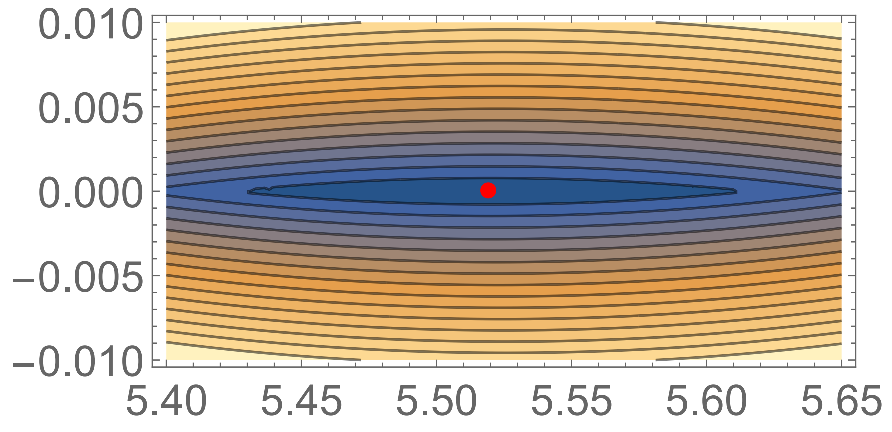


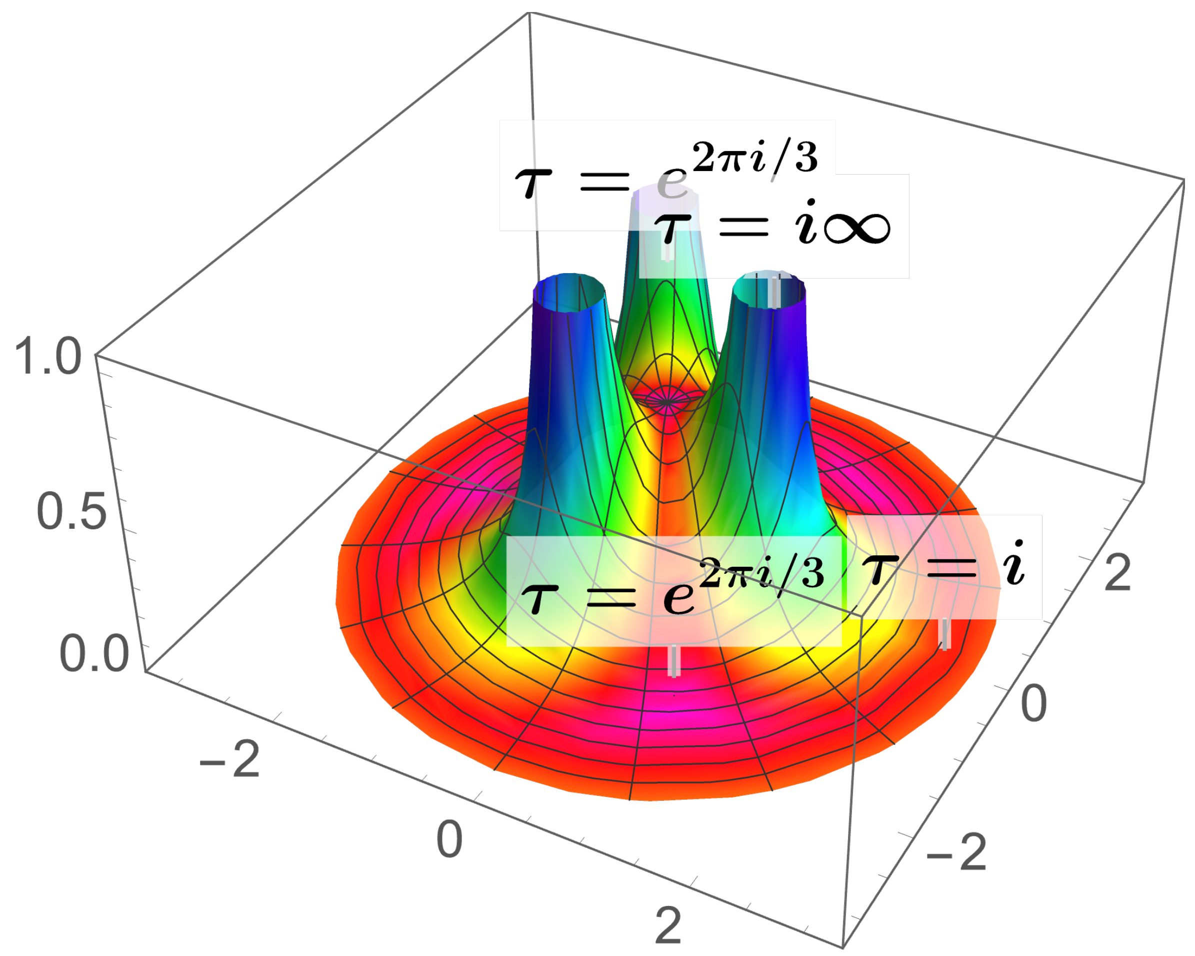
3 Generalizations

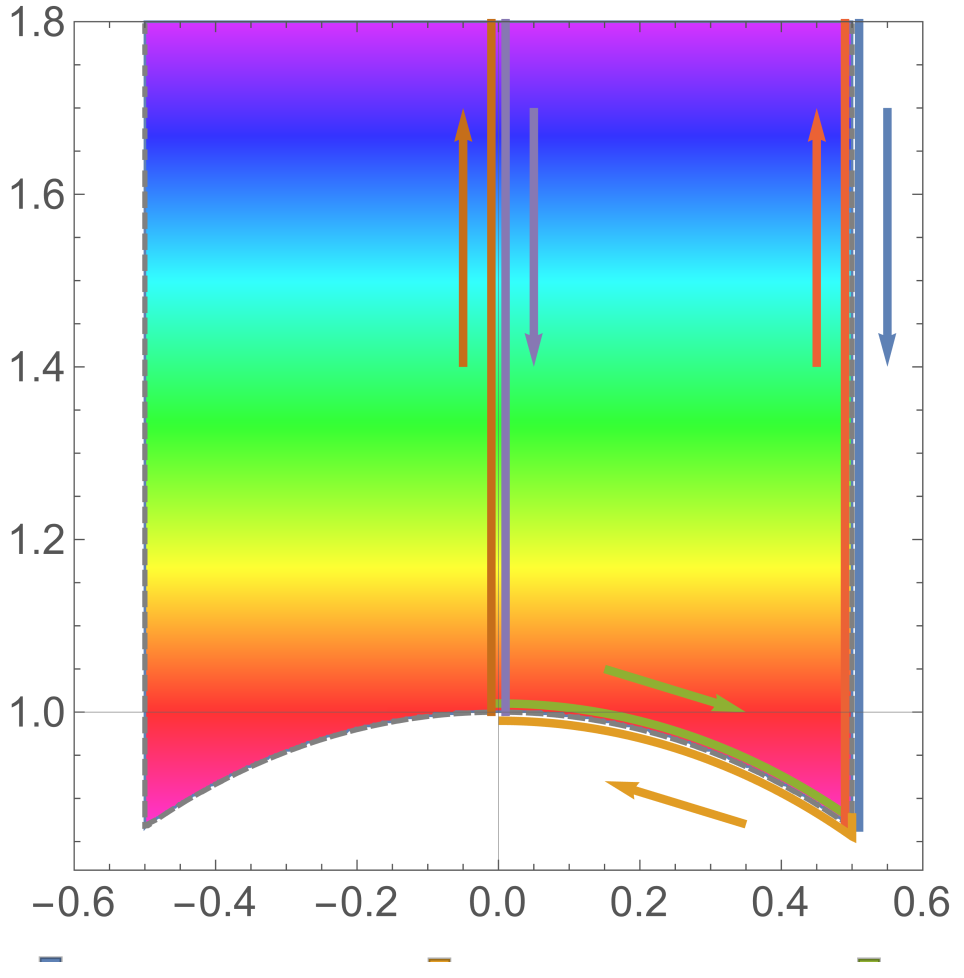


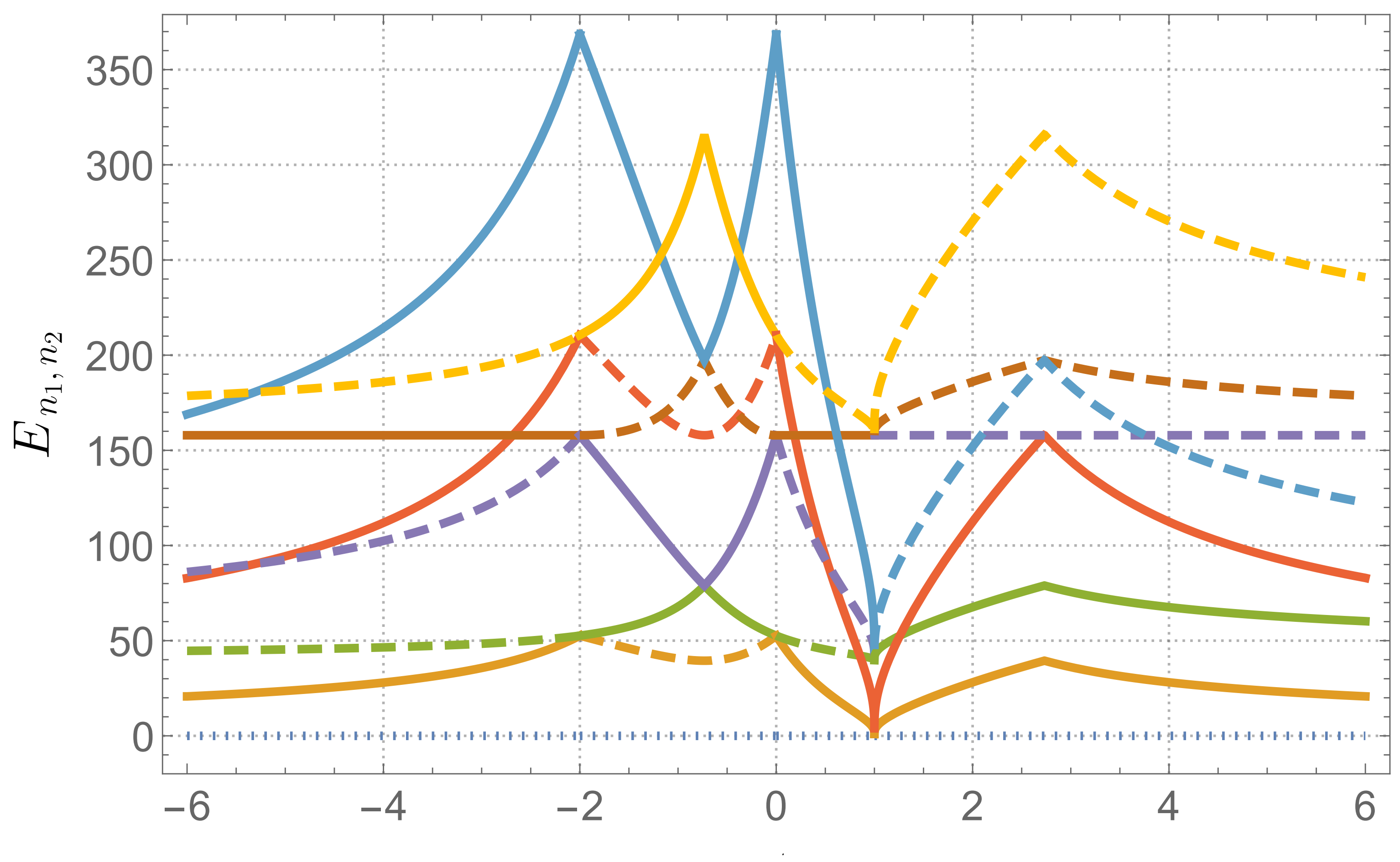


4 Conclusions









Toroidal Compactifications

One parameter families

We want to study CY n - folds $X \subset \mathbb{P}^{n+1}$ of the form

$$(z_0)^{n+2} + (z_1)^{n+2} + \ldots + (z_{n+1})^{n+2} - (n+2)\psi \; z_0 z_1 \cdots z_{n+1} = 0$$

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$n = 1$

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↓ $n = 1$

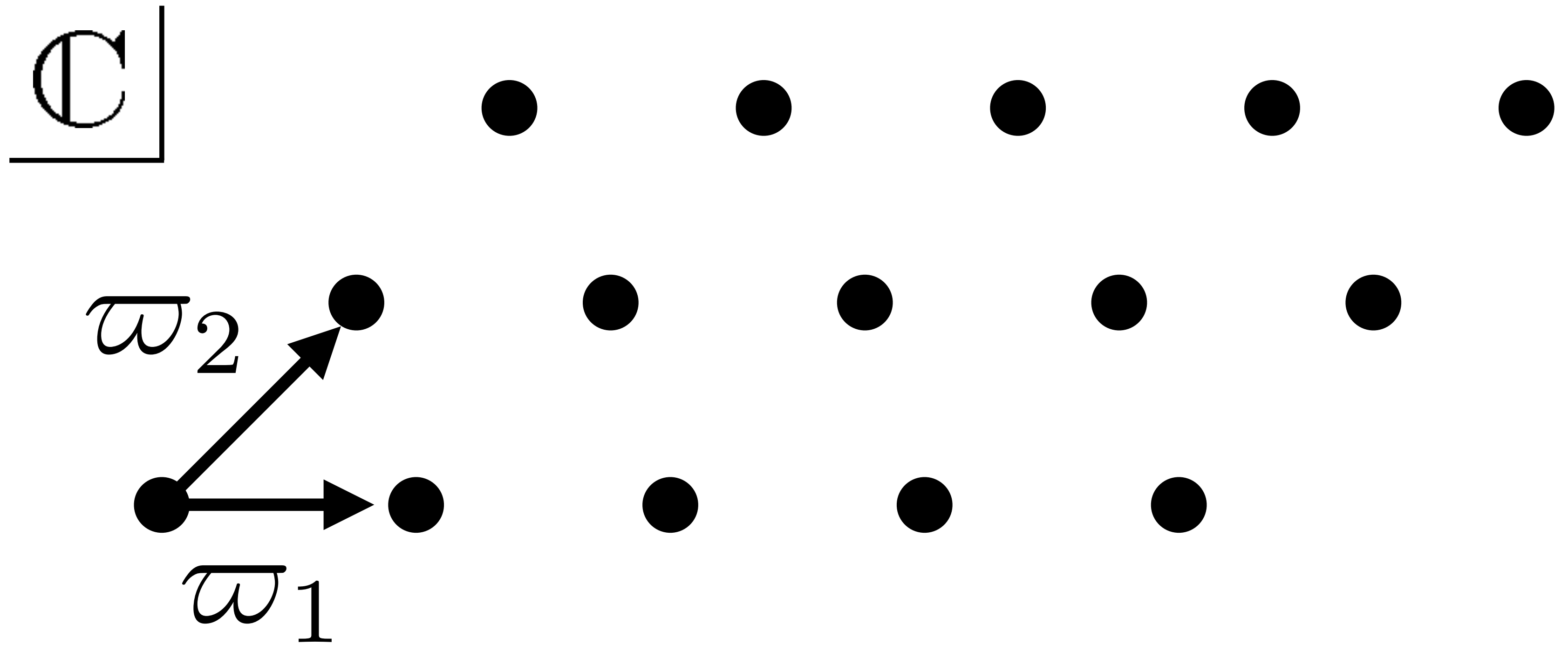
$$z_0^3 + z_1^3 + z_2^3 - 3\psi z_0 z_1 z_2 = 0$$

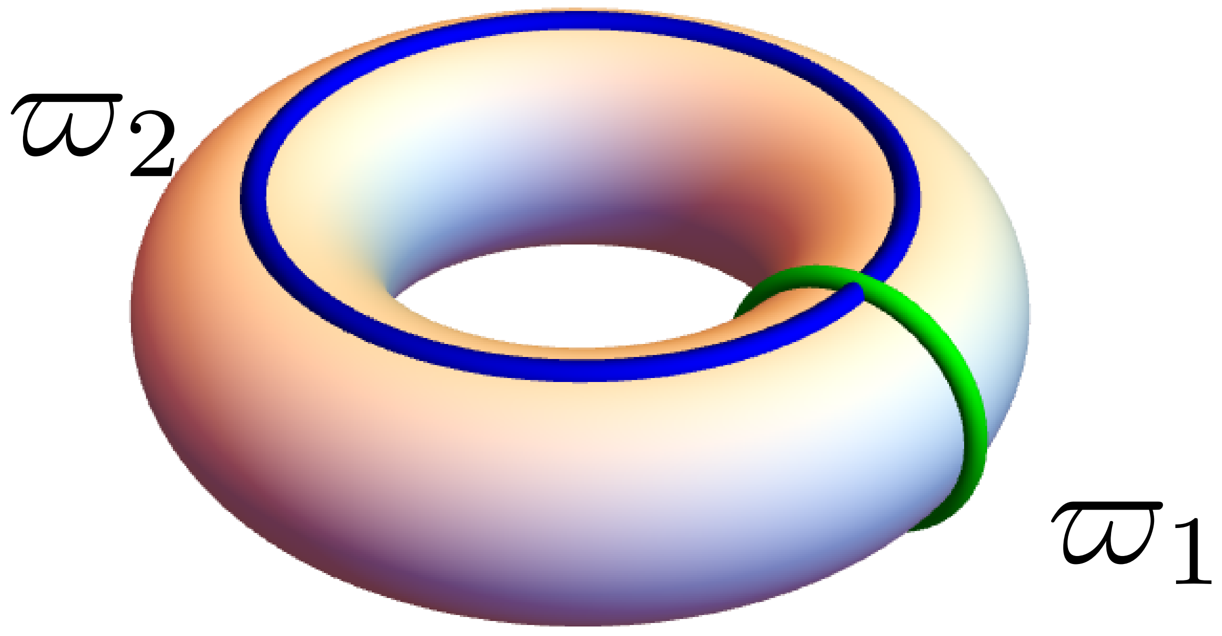
↓ WSF

$$y^2 z = x^3 + g_2(\psi) x z^2 + g_3(\psi) z^3 \quad \text{w/} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \wp(w) \\ \wp'(w) \\ 1 \end{pmatrix}$$

Torus parametrization

- Given $\wp(w)$, we know the exact CY metric: $g \sim dw \otimes d\overline{w}$, $w \in \mathbb{C}/\Gamma$

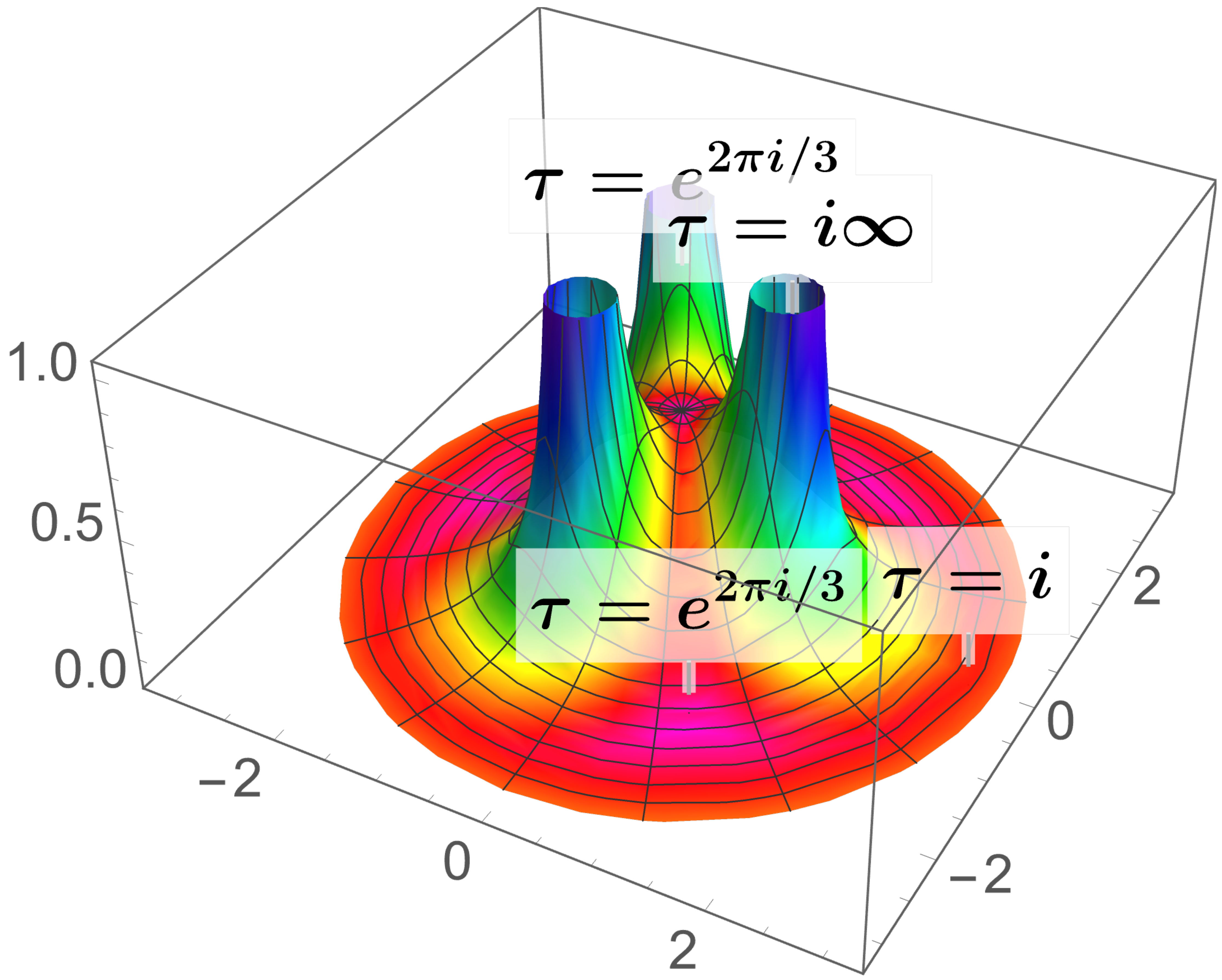


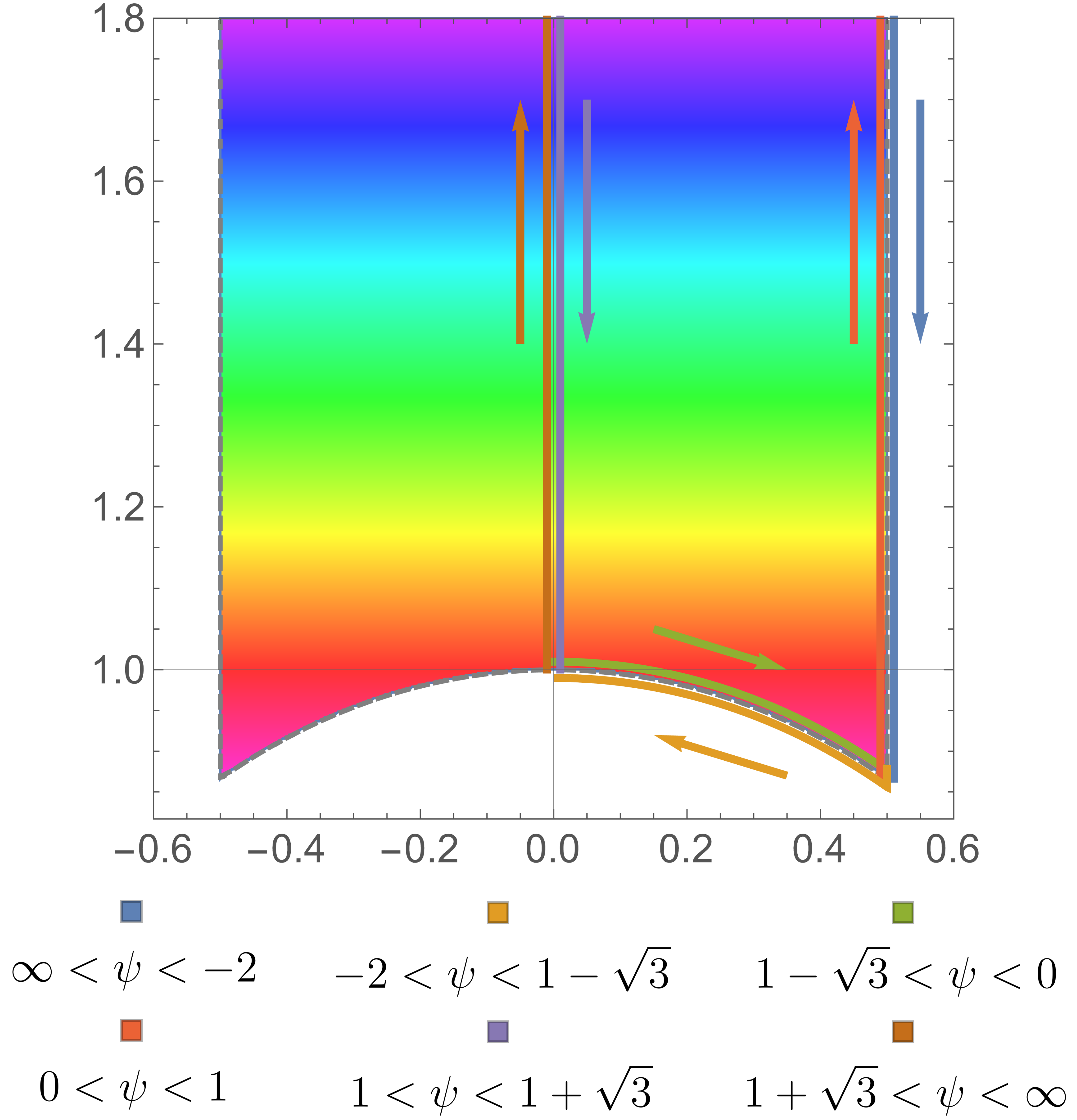


- We can also compute the CS (modular) parameter

$$\tau = \tau(\psi) = \frac{\varpi_2}{\varpi_1} = \frac{i}{\sqrt{3}}\,\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\frac{1}{\psi^3})}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\frac{1}{\psi^3})} = j^{-1} \quad \text{w/} \quad j(\tau(\psi)) = \frac{g_2^3}{g_2^3 - 27g_3^2}$$

Moduli Space





Spectrum on the torus

- Next, we compute the Eigenspectrum of the scalar Laplacian w.r.t. the (flat) CY metric as a function of $\tau(\psi)$

$$E_{n_1,n_2} = \frac{4\pi^2}{A\tau_2}|n_1 - n_2\tau|^2 = \frac{4\pi^2}{\tau_2^2}|n_1 - n_2\tau|^2$$

$$F_{n_1,n_2}(w,\bar{w}) = \frac{1}{\sqrt{A}}e^{2\pi i(cw+\bar{c}\bar{w})} = \frac{1}{\sqrt{\tau_2}}e^{2\pi i(cw+\bar{c}\bar{w})}\,, \qquad c = \frac{1}{2\tau_2}\left(n_1(1+i\tau_1) - in_2\right)$$

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- Note: $z_0^3 + z_1^3 + z_2^3 - 3\psi z_0 z_1 z_2 = 0$ has additional symmetries

$$\psi \in \mathbb{C} - \mathbb{R}: \qquad S_3 \rtimes \mathbb{Z}_3; \longleftarrow (z_0, z_1, z_2) \mapsto (\xi z_0, \xi^2 z_1, z_2)$$

permutation

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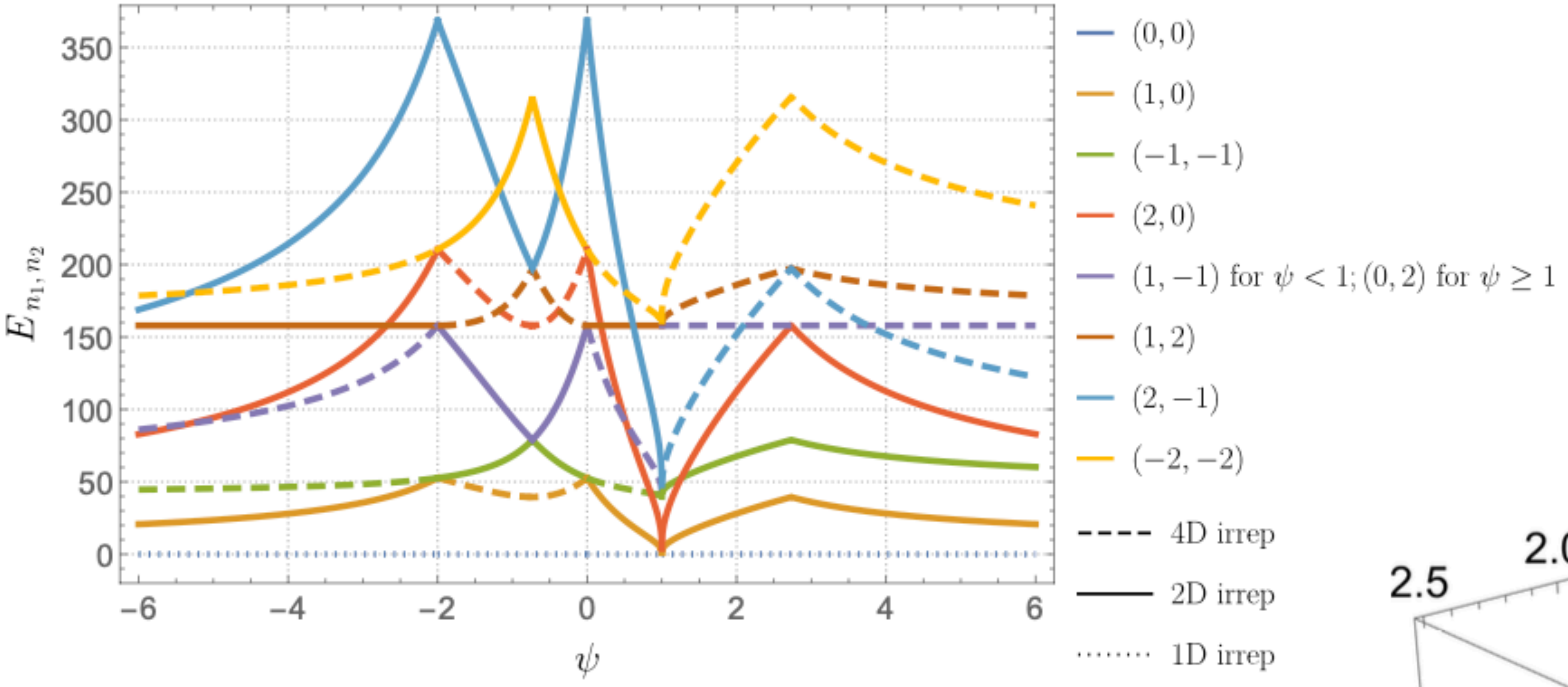
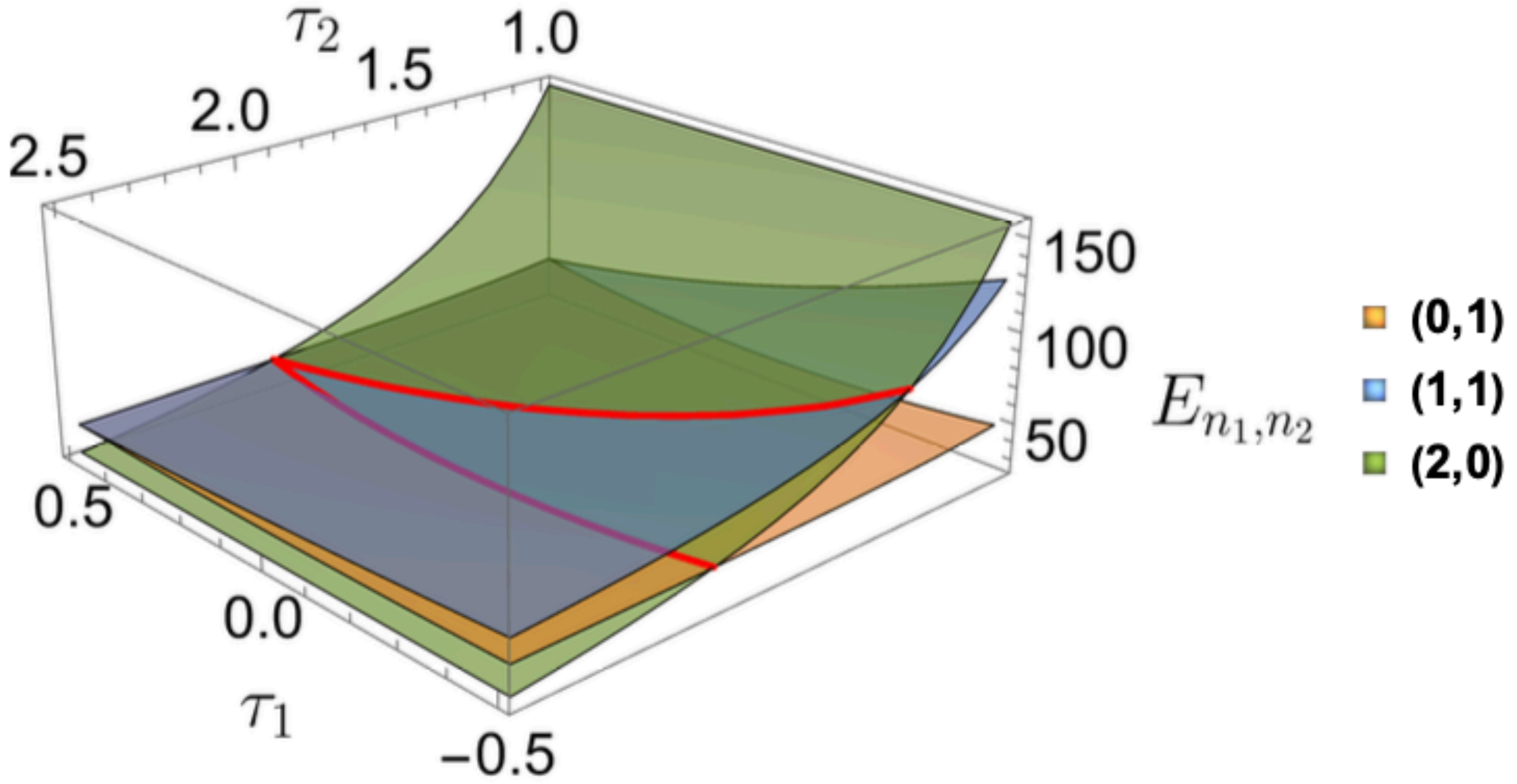
$$\begin{aligned} \psi \in \mathbb{C} - \mathbb{R}: &\qquad S_3 \rtimes \mathbb{Z}_3\,, \\ \psi \in \mathbb{R}: &\qquad (S_3 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3\,, \\ \psi = 0: &\qquad (S_3 \times \mathbb{Z}_2) \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_3) \end{aligned}$$

$(z_0, z_1, z_2) \mapsto (\xi z_0, z_1, z_2)$

CC

Spectrum on the torus





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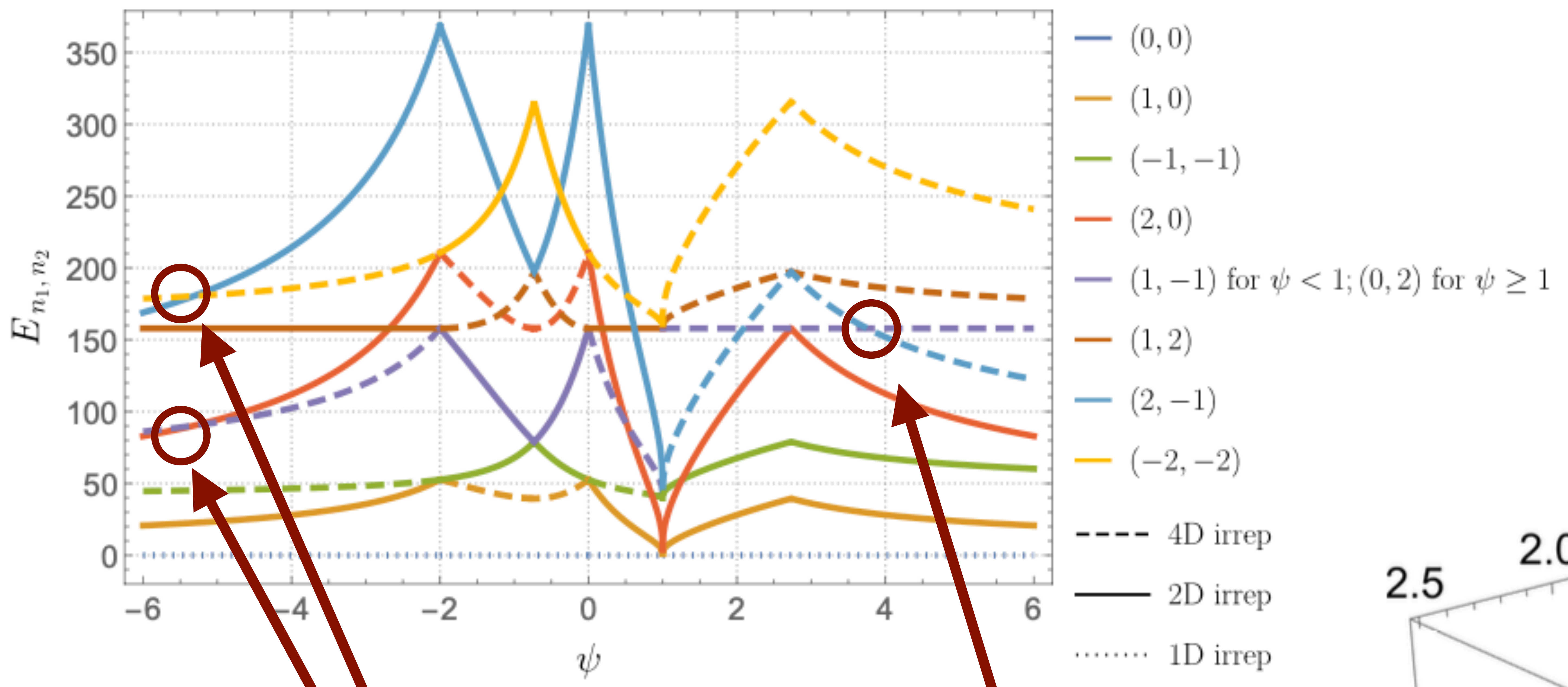
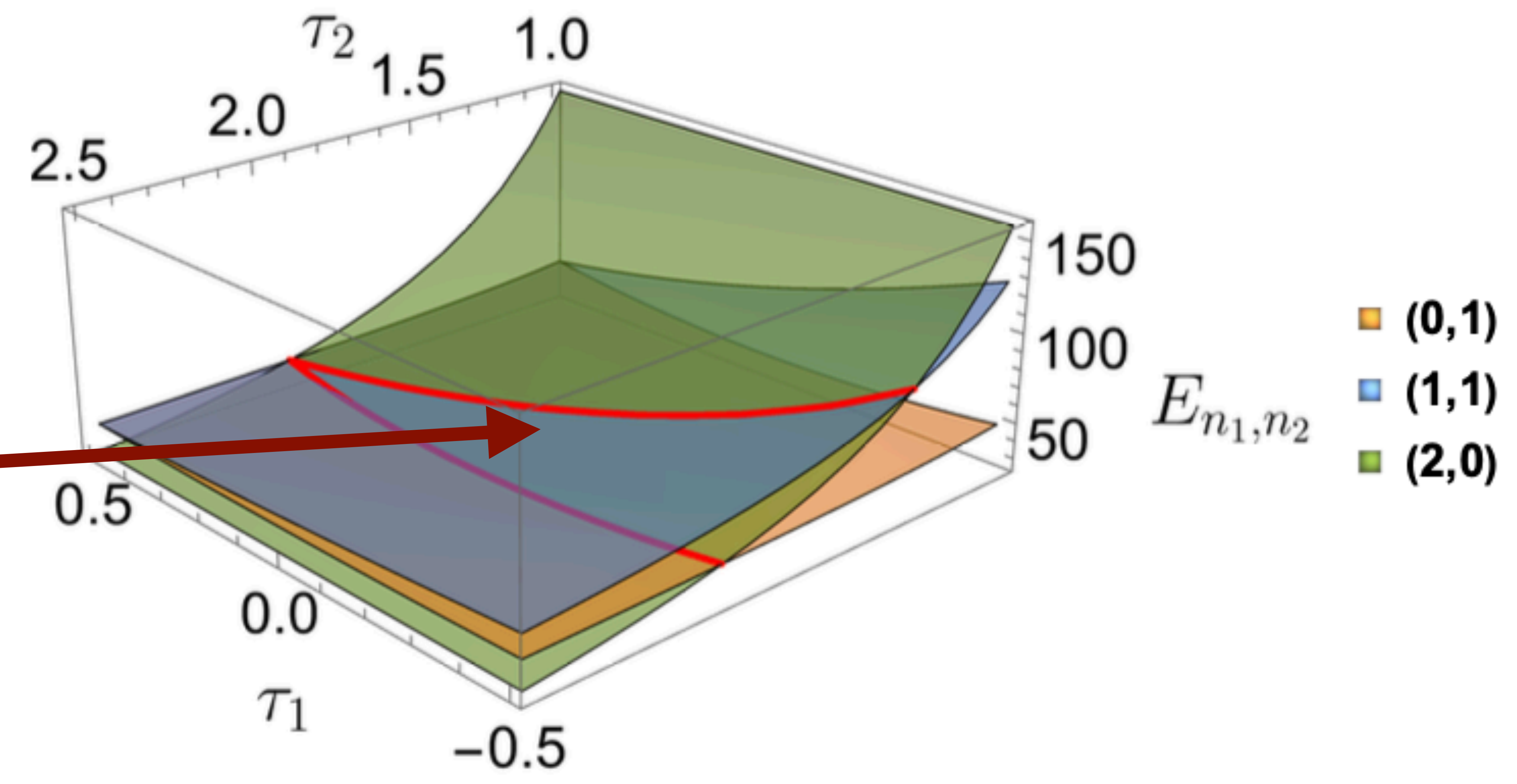




6D irrep??

8D irrep??

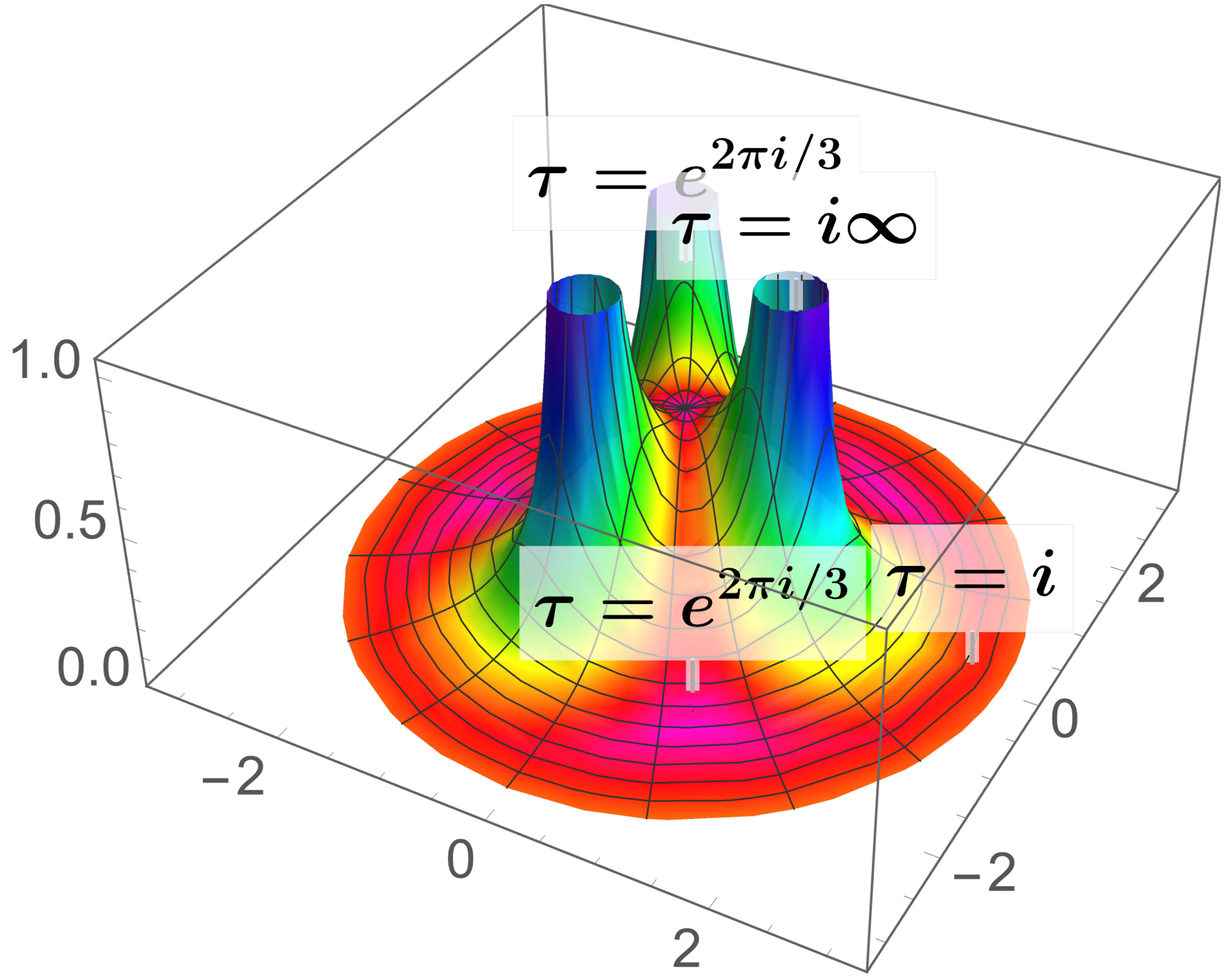
What is special about these points?

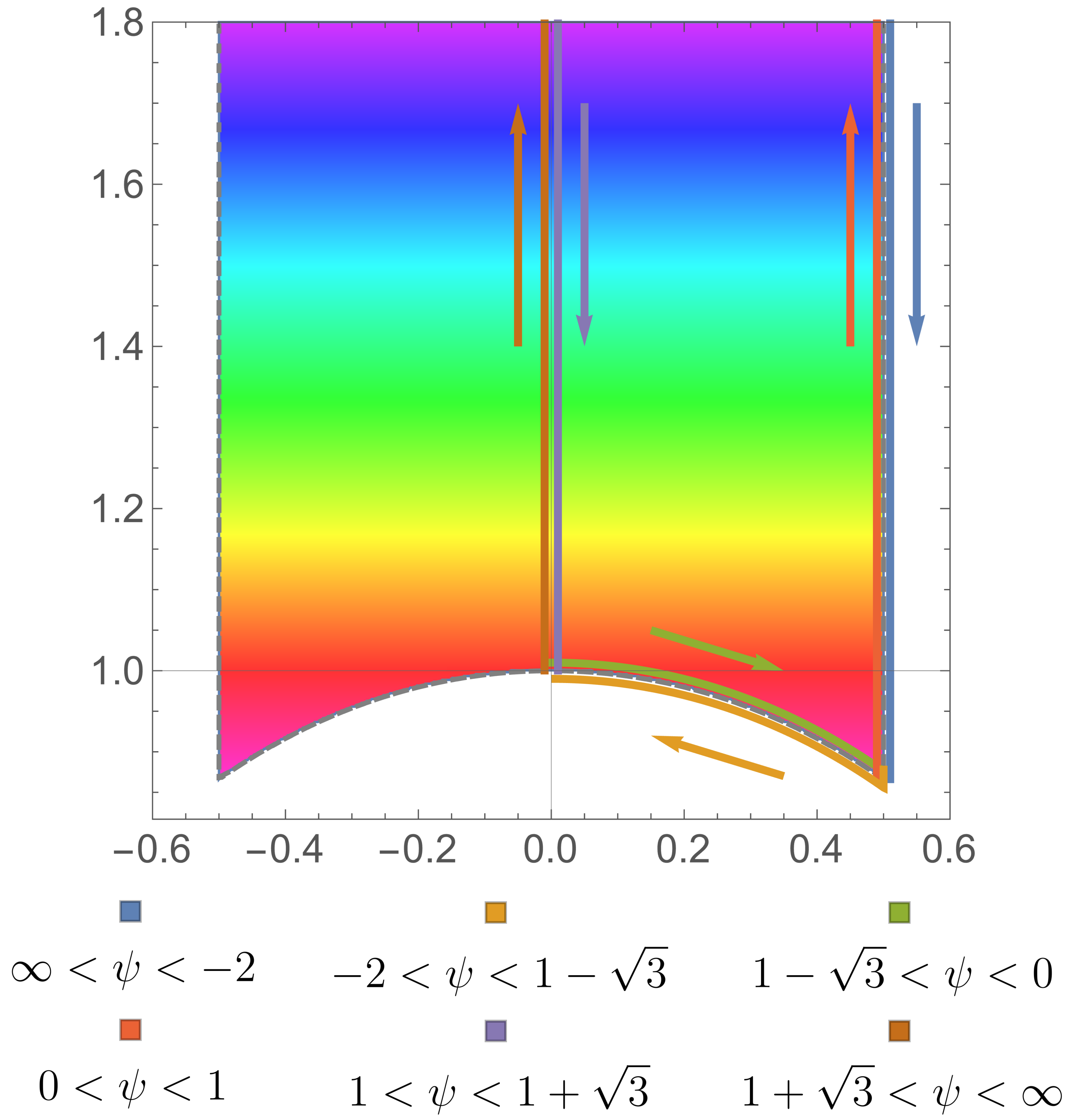
Crossing Points - Defining Eqn

$$z_0^3 + z_1^3 + z_2^3 - 3\psi z_0 z_1 z_2 = 0$$

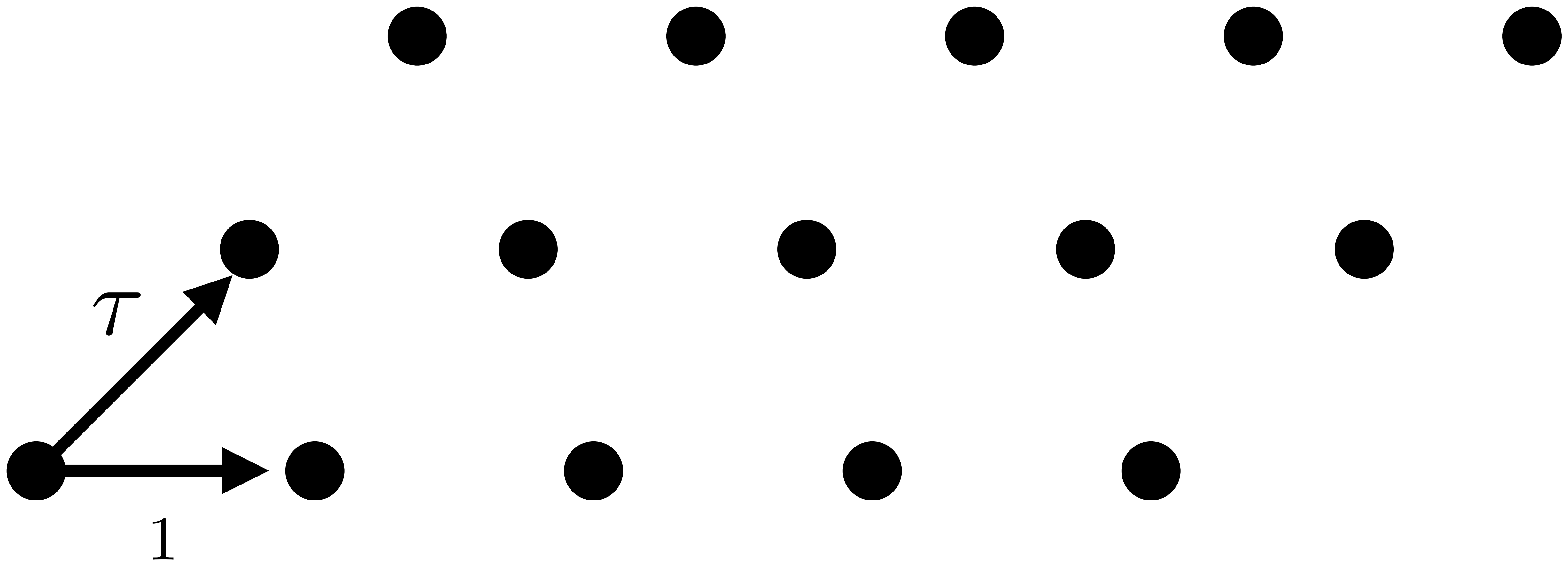
No obvious symmetry enhancement in defining eqn at these values of ψ

Crossing Points - Moduli Space

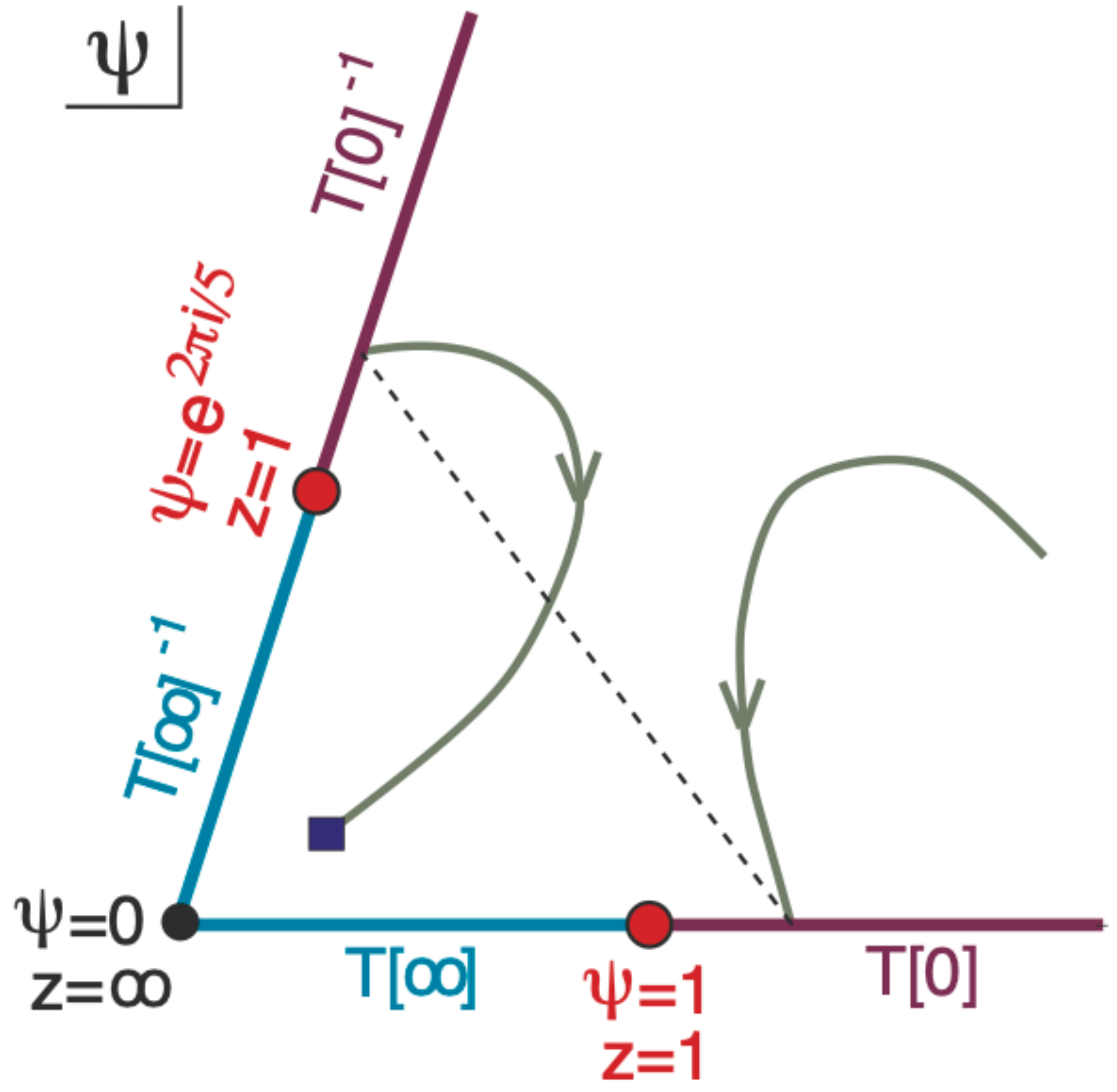




Nothing obvious in moduli space either



[Figure: Denef, Greene, Raugas `01]



Attractors and CM points

Crossing Points - Moduli Dependence

- Let us study the values of τ where eigenmodes E_{n_1,n_2} and E_{m_1,m_2} cross:

$$\frac{4\pi^2}{\tau_2^2}|n_1 - n_2\tau|^2 \overset{!}{=} \frac{4\pi^2}{\tau_2^2}|m_1 - m_2\tau|^2 \quad \Leftrightarrow \quad \tau_1^2 + \tau_2^2 = \frac{n_1^2 - m_1^2}{m_2^2 - n_2^2} + 2\tau_1 \frac{n_1 n_2 - m_1 m_2}{m_2^2 - n_2^2}$$

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- Focus on $\psi > 1 \Leftrightarrow \tau \in i\mathbb{R}$: $\tau = i\tau_2 = i\sqrt{\frac{n_1^2 - m_1^2}{m_2^2 - n_2^2}}$, i.e. $\tau_2 \in \mathbb{Q}$ or $\tau \in \mathbb{Q}(\sqrt{D})$

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- Such τ are special! The torus lattice has a special (number-theoretic) property known as complex multiplication:
  - Usually $\Lambda = \mathbb{C}/\Gamma = \{a + b\tau \mid 0 \leq a, b \leq 1\}$ has only scaling: $a\Lambda \subset \Lambda$, $a \in \mathbb{Z}$
  - At these special points, there exists an additional symmetry: $a'\Lambda \subset \Lambda$, $a' = \frac{D + \sqrt{D}}{2} \in \mathbb{C}$

Complex Multiplication and attractors

- In two absolutely remarkable papers, Moore pointed out a relation between CM points and solutions to BH attractor equations [Moore `98]

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- Roughly, this goes as follows:
  - Look at supersymmetric dyonic BH. The vector multiplets follow a gradient flow to fixed points in their target space, so-called attractors [Ferrara, Kallosh, Strominger `95]
  - In the context of IIB, the vector multiplet moduli space is the CS moduli space, and attractor points are special CS moduli points
  - The central charge of the BH, $|Z(\Omega(\psi),\gamma)|^2 = \frac{\left|\int_\gamma \Omega(\psi)\right|}{i\int \Omega(\psi)\wedge\bar{\Omega}(\psi)}$, decreases along the flow

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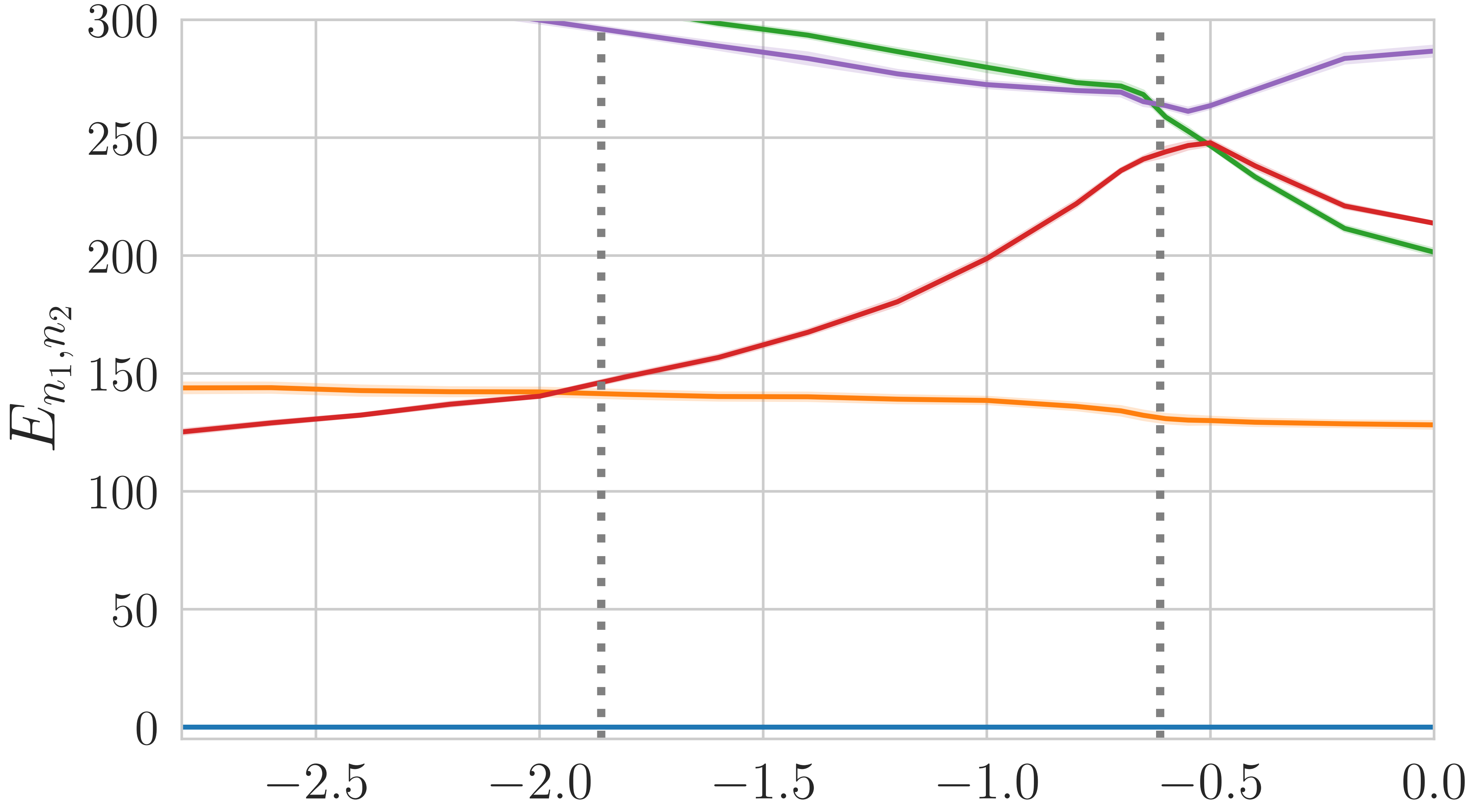
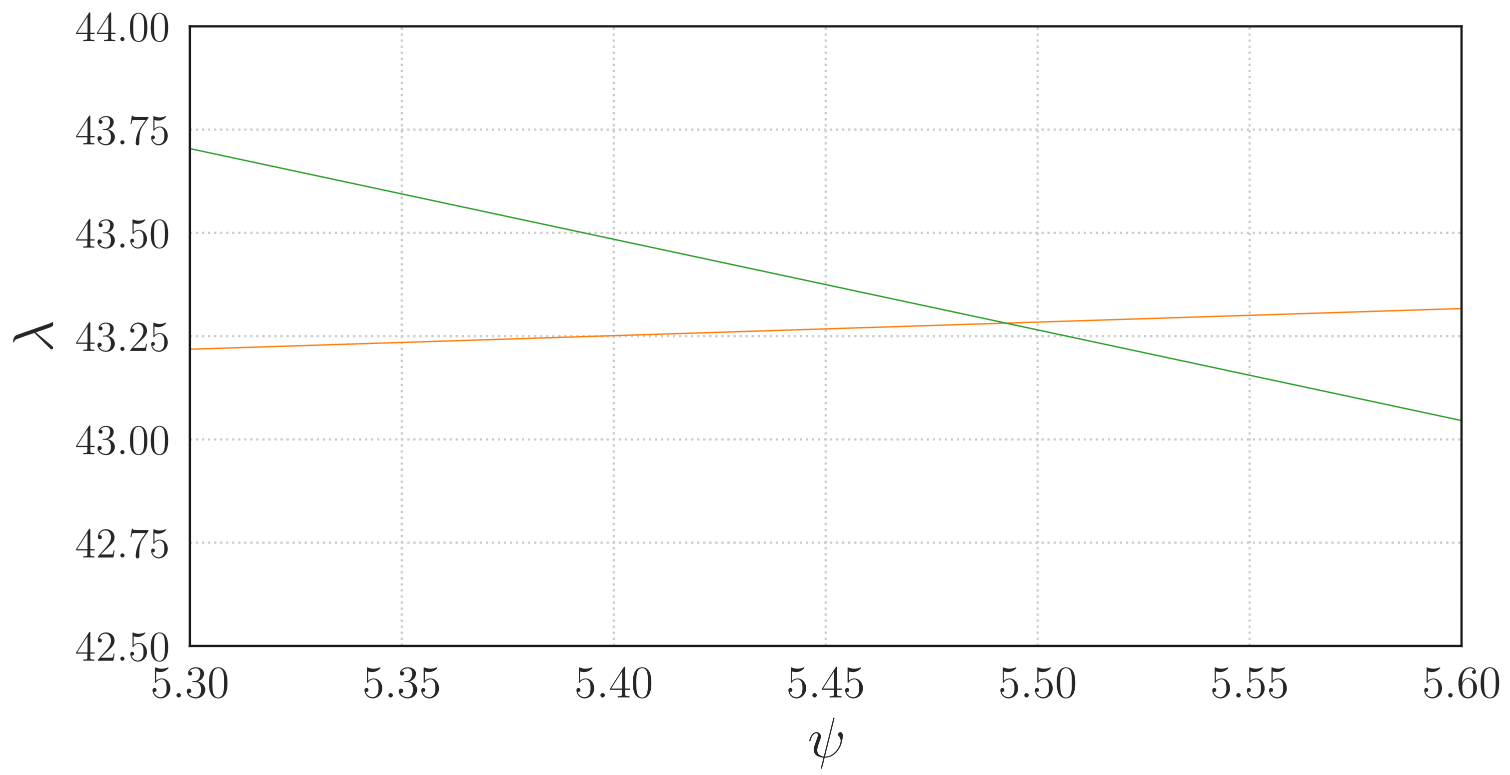
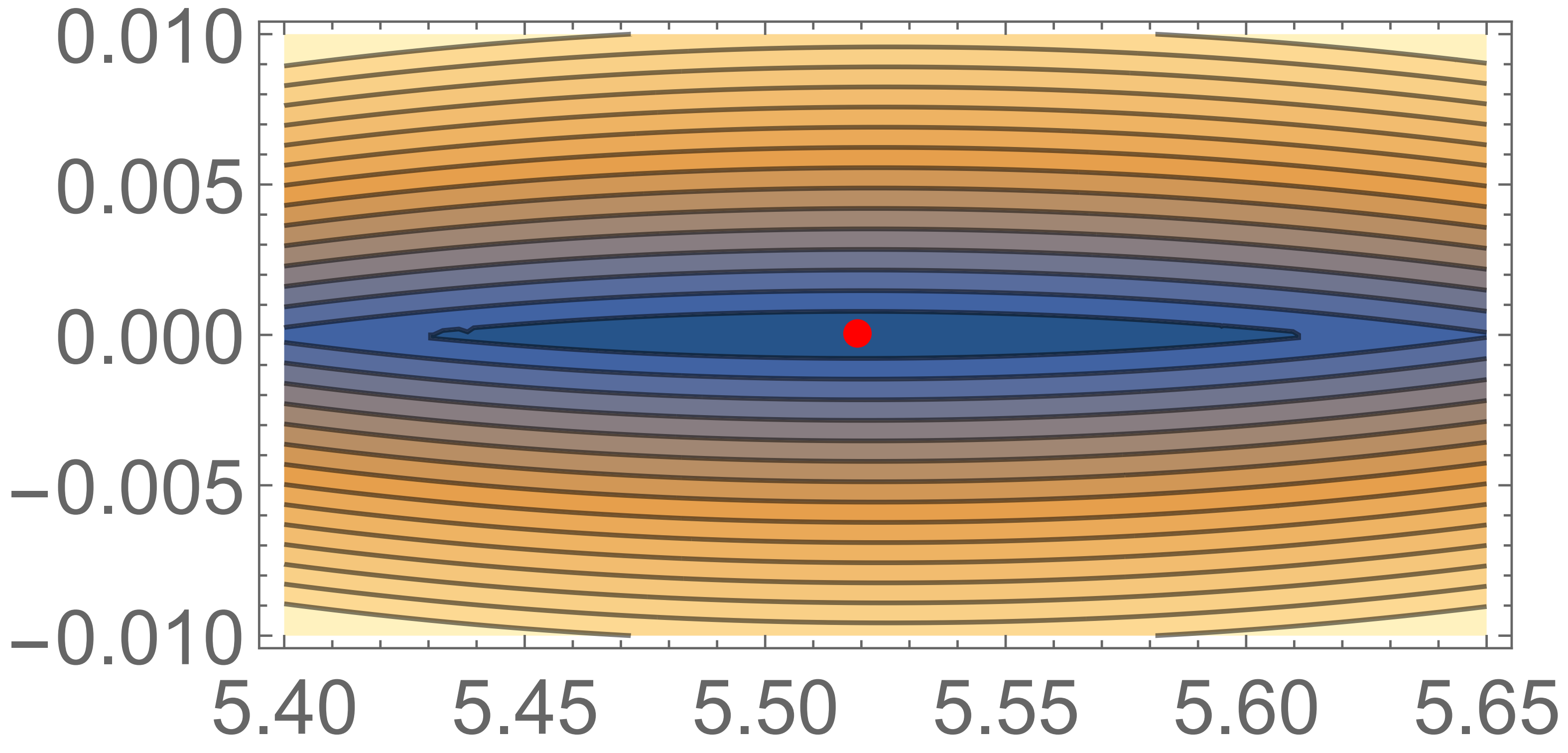
- Moore showed that for T^6 and $K3 \times T^2$, the solution to the attractor equations are CM points







Generalizations

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  - The generalization is to attractors that do not correspond to BPS objects
  - This is an accident for tori and does not generalize
- The notion of CM can be generalized to any CY 3-fold (where it becomes a condition on the middle cohomology lattice)

Crossing Points - Generalizations

- Beyond T^6 or $K3 \times T^2$, there is likely no relation between attractors and CM points:
  - Gukov and Vafa argued that attractors are dense, while CM points are rare (based on a relation of (dense) RCFTs and the (recently proven) Andre-Oort conjecture [Gukov, Vafa `98]
  - Attractors are not algebraic ($\neq$ Moore's conjecture) for higher-dim CYs (they construct counter-examples for CY $2n+1$ folds for $2n+1 \geq 7$) [Lam, Tripathy `20]
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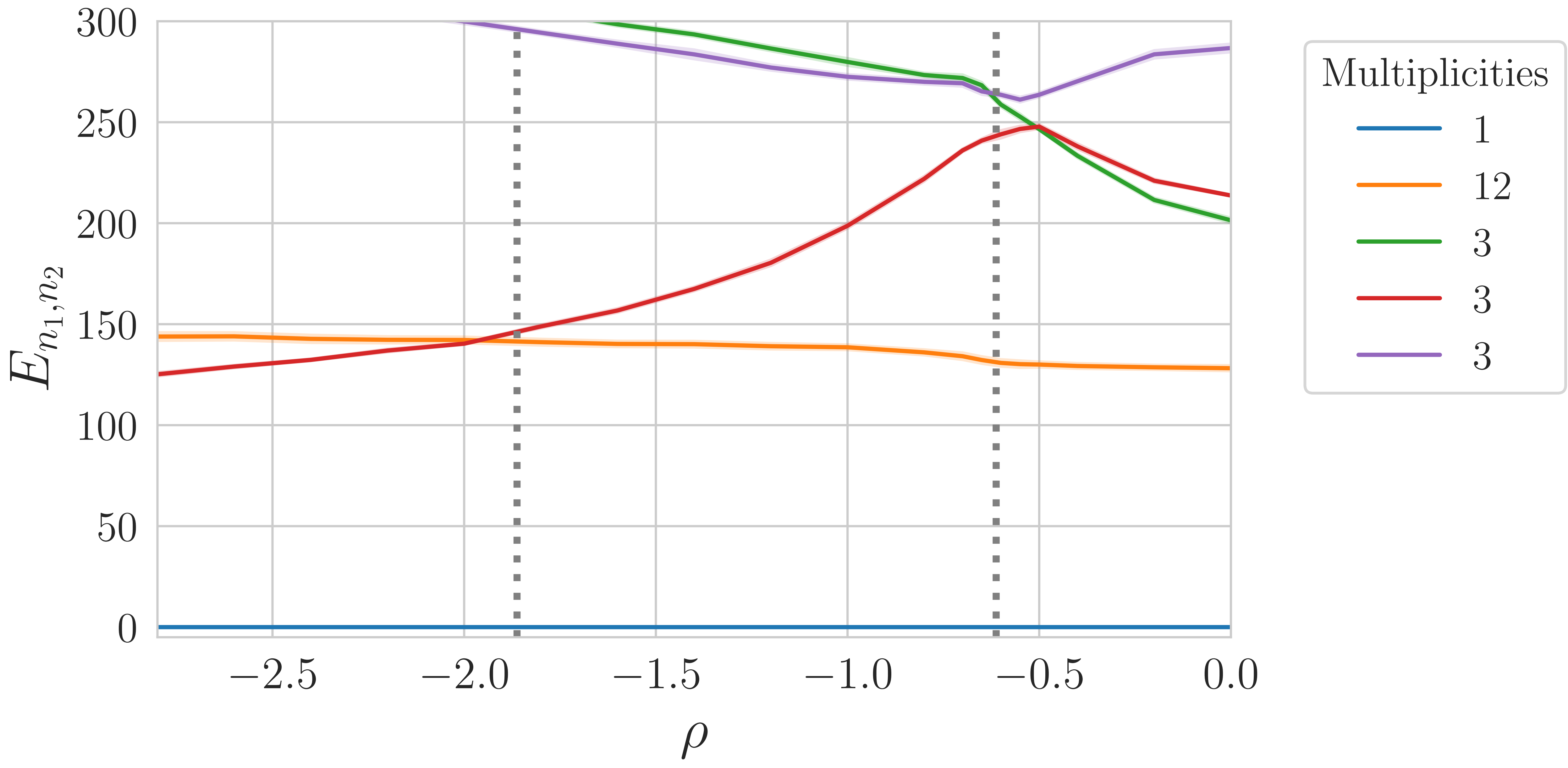
- Since crossings are also ubiquitous (dense?), the generalization (if any) is to a relation between attractors and crossings, not CM and crossings
- The appearance of CM was an accident of the 1:1 correspondence between CM and attractors for tori

Crossings on 1-parameter Quartic K3

- For the K3, there is still a correlation between CM points and attractors
- The CM points for the quartic are again at special values of the periods

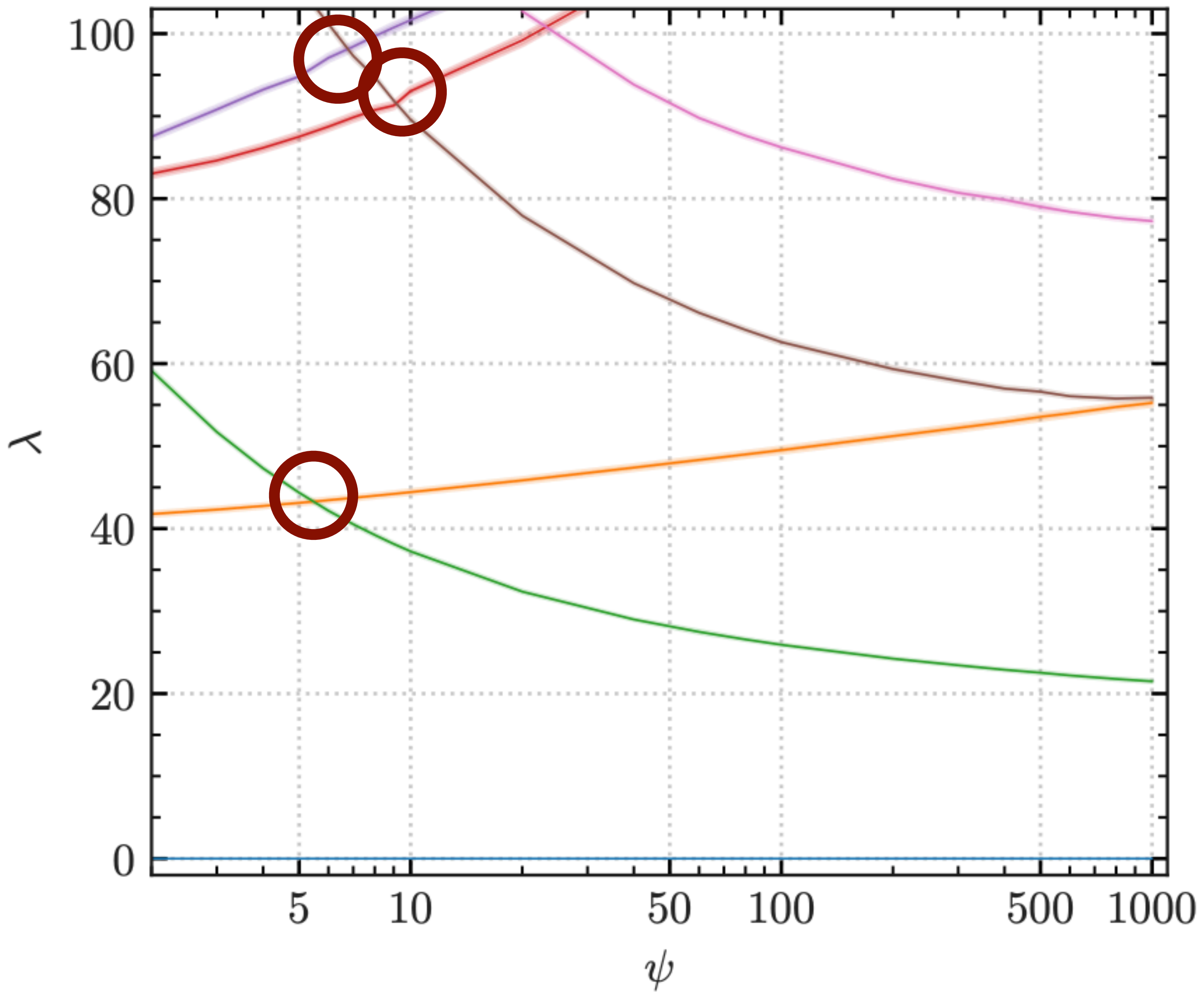
$$ {}_3F_2\left(\frac{1}{2},\frac{1}{4},\frac{3}{4};1,1;\rho(\tau)\right) = \left(\frac{\eta(\tau)^{16}}{\eta(2\tau)^{8}} + 64\frac{\eta(2\tau)^{16}}{\eta(\tau)^{8}}\right)^{1/2}, \qquad \rho(\tau) = \frac{256\,\eta(\tau)^{24}\,\eta(2\tau)^{24}}{(\eta(\tau)^{24}+64\eta(2\tau)^{24})^{2}} $$

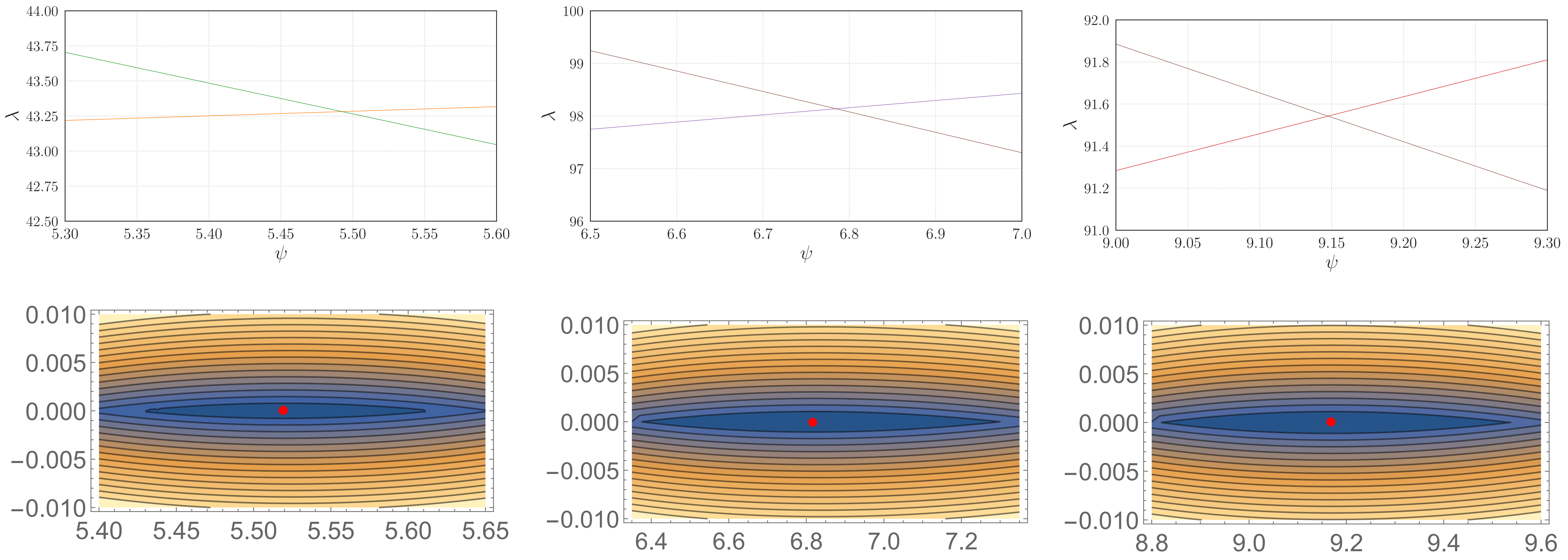
- We compute the spectrum and compare crossings to these CM points



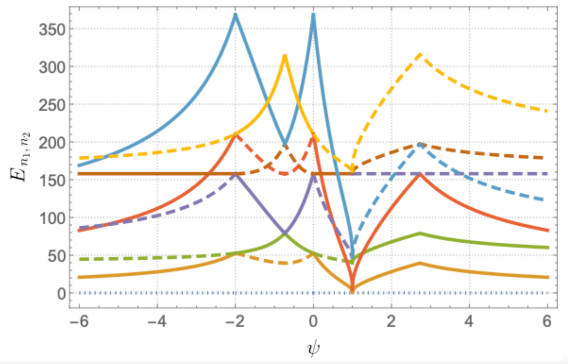
Crossings on 1-parameter Quintic CY3

- For the Quintic, we need to compare crossings to attractors:
  - Compute the CY spectrum
  - Read off crossings
  - Compute the attractors in the mirror-dual (D0,D2,D4,D6) brane system and look for minima of the central charge at the crossing points [Denef, Greene, Raugas `01]





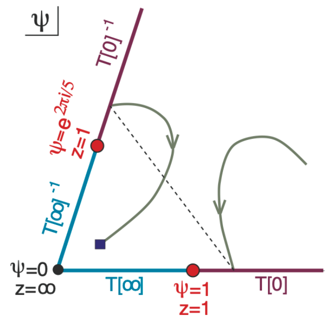
Conclusions



$$\tau_1^2+\tau_2^2=\frac{n_1^2-m_1^2}{m_2^2-n_2^2}+2\tau_1\frac{n_1n_2-m_1m_2}{m_2^2-n_2^2}$$

T^2 Eigenvalue crossings ↔ $|\tau|^2$ rational

↕



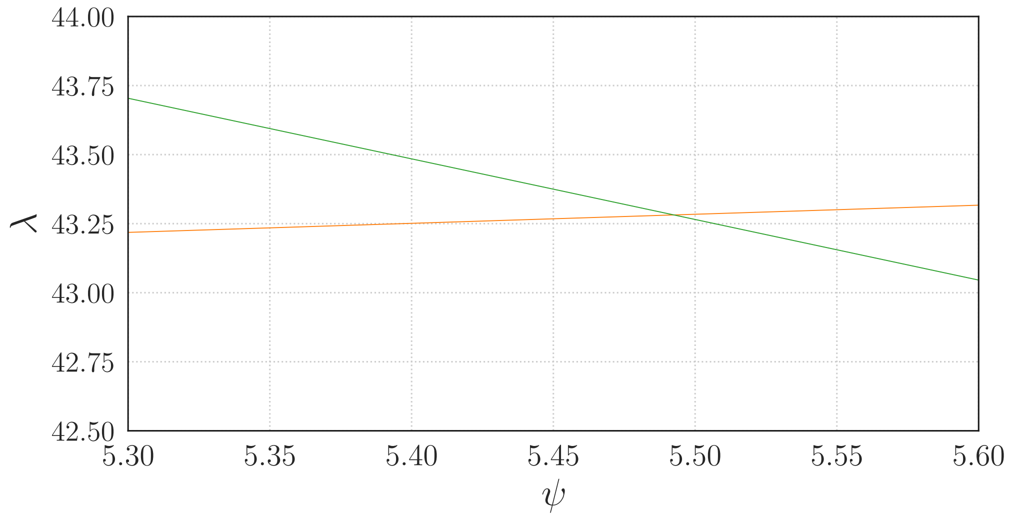
Attractors ↔ CM points

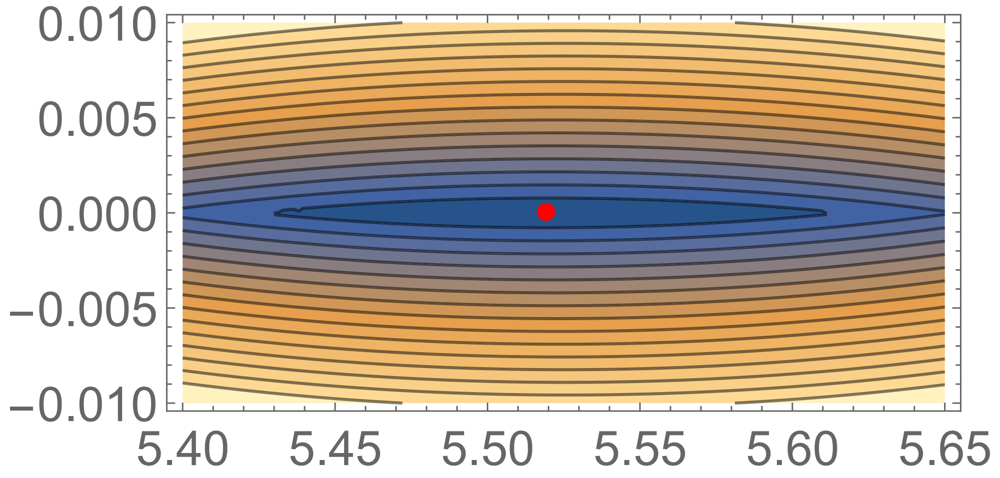
$$a'\Lambda\subset\Lambda$$

$$a'=\frac{D+\sqrt{D}}{2}\in\mathbb{C}$$

↕

General CY





Announcements

Aspen Winter Conference

Image credit: wikimedia commons

Jan 14 – 19, 2024
Aspen Center for Physics
US/Mountain timezone

- Overview
- Timetable
- Registration and Accommodation
- Travel and Visas
- Local information, Transportation, Skiing
- Block Award
- Poster

Fields, Strings, and Deep Learning

Progress in deep learning has traditionally involved experimental data, but in recent years it has impacted our understanding of formal structures arising in theoretical high energy physics and pure mathematics, via both theoretical and applied deep learning. This conference will bring together high energy theorists, mathematicians, and computer scientists across a broad variety of topics at the interface of these fields. Featured topics include the interface of neural network theory with quantum field theory, lattice field theory, conformal field theory, and the renormalization group; theoretical physics for AI, including equivariant, diffusion, and other generative models; ML for pure mathematics, including knot theory and special holonomy metrics, and deep learning for applications in string theory and holography.

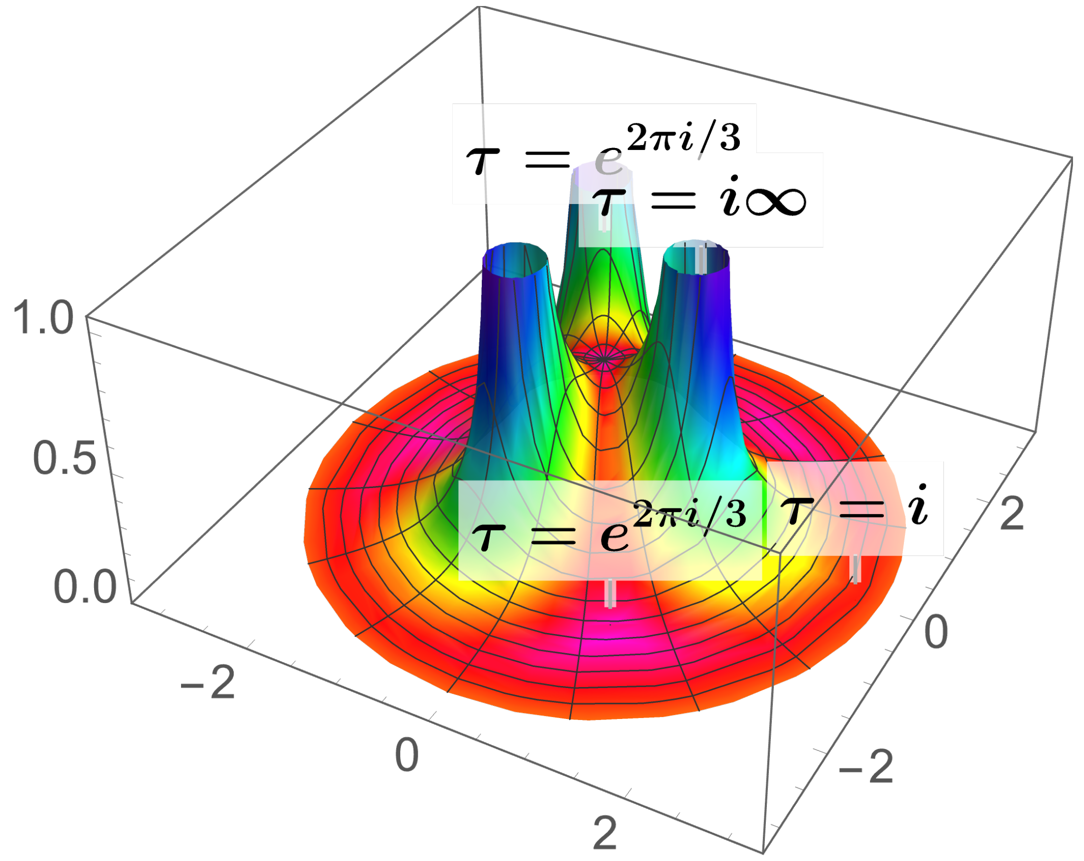
ML Meetings @ Caltech

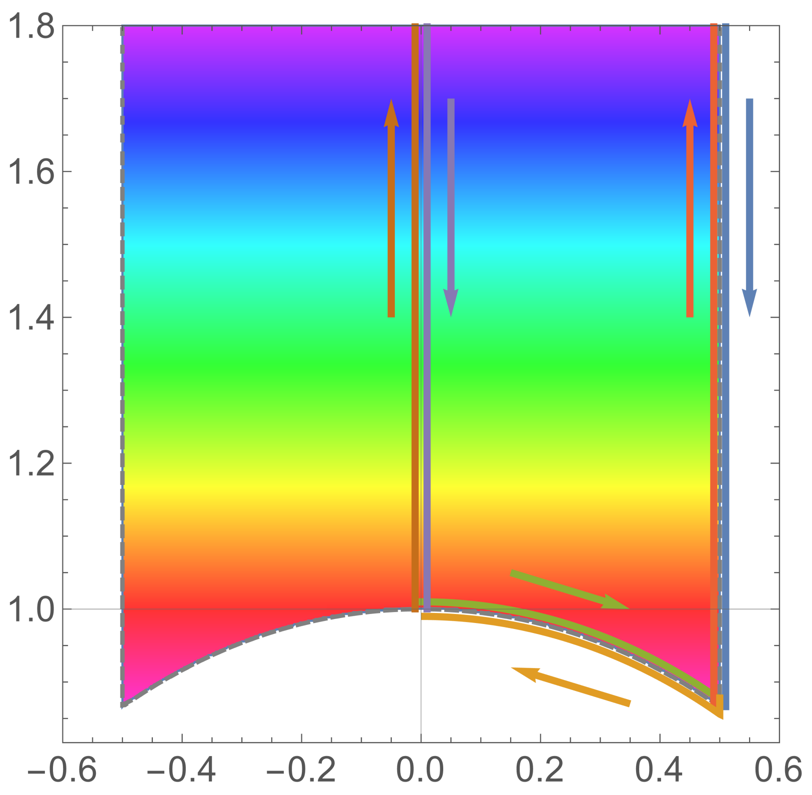
- Dec 10-12: Mathematics and ML 2023
- Dec 13-15: string_data 2023

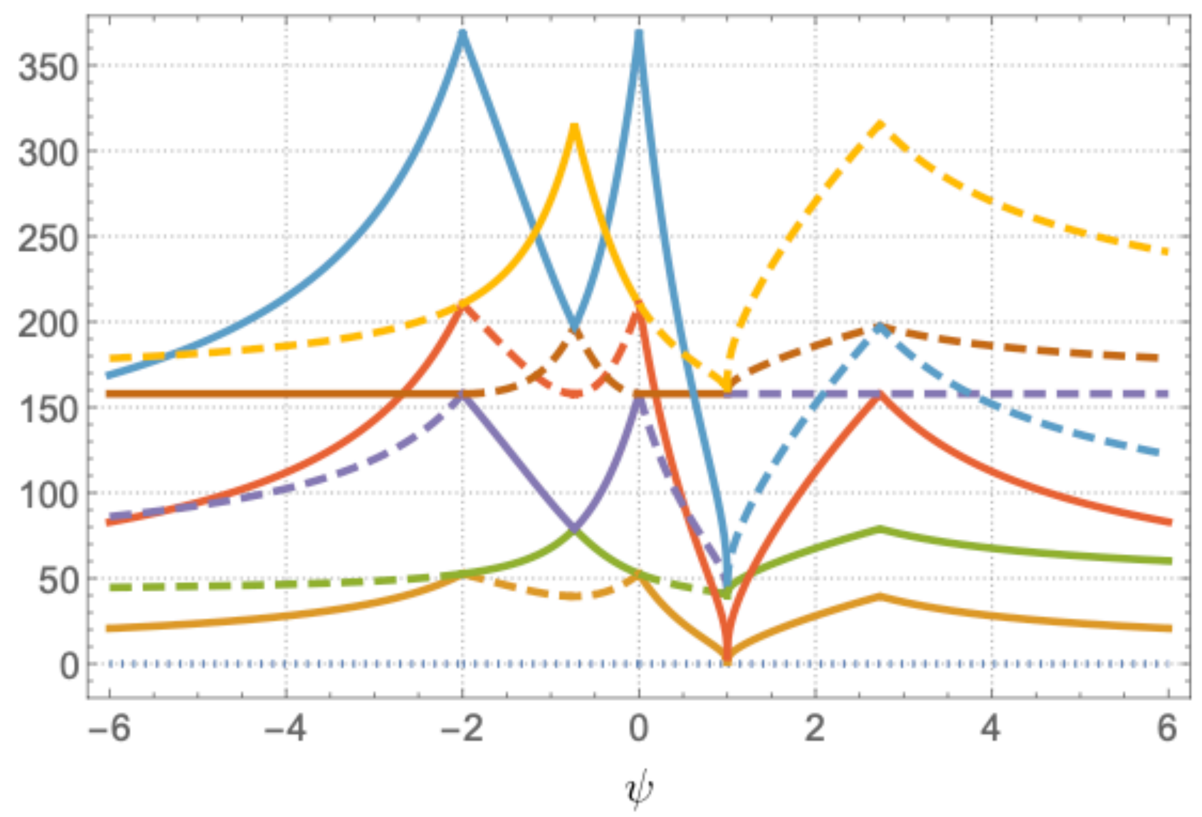
Aspen Winter Conference

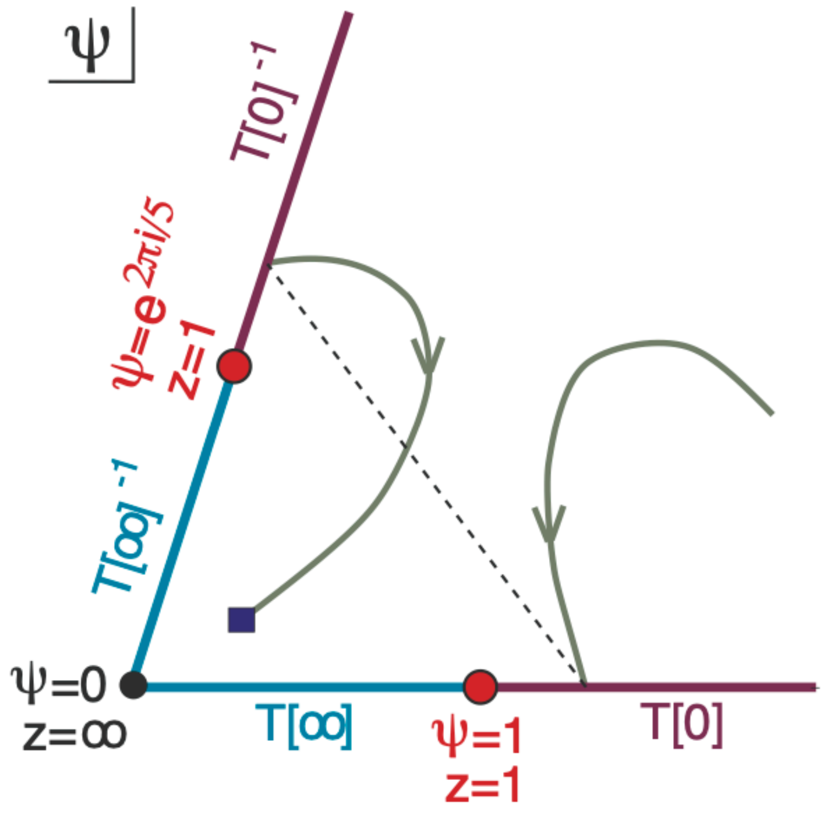
- Jan 14-19: Fields, Strings, and Deep Learning

Application deadline: Aug 31

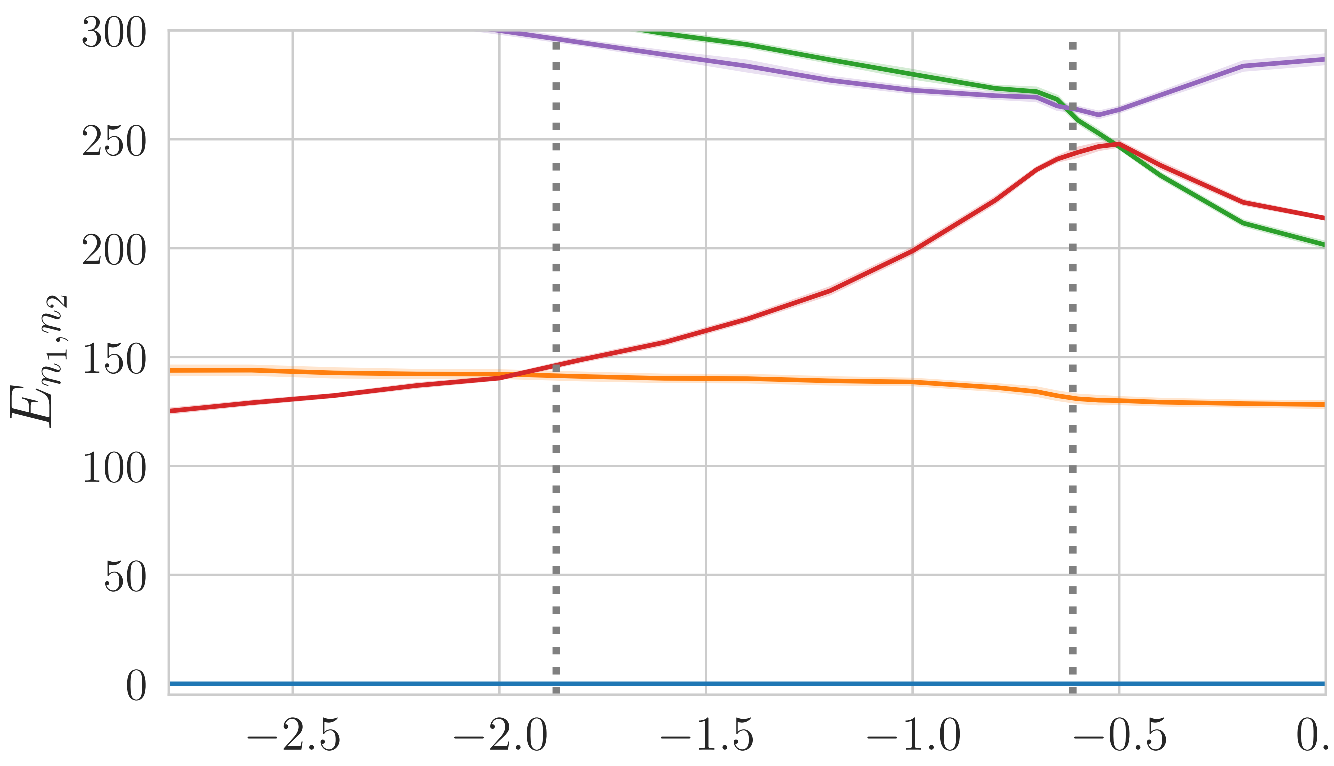

$\tau = e^{2\pi i/3}$
$\tau = i\infty$
$\tau = e^{2\pi i/3}$
$\tau = i$
1.0
0.5
0.0
−2
0
2
2
0
−2

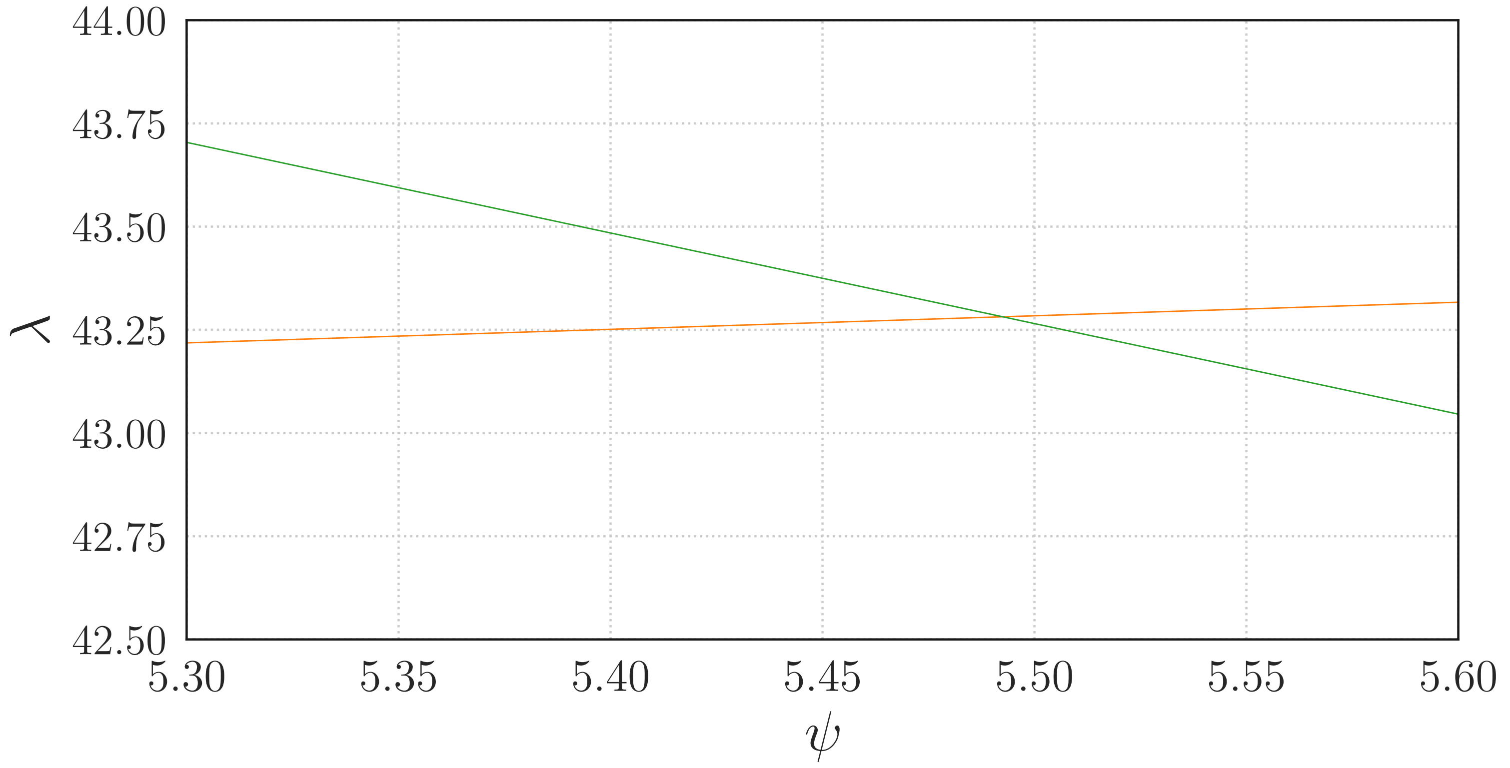

1.8
1.6
1.4
1.2
1.0
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6

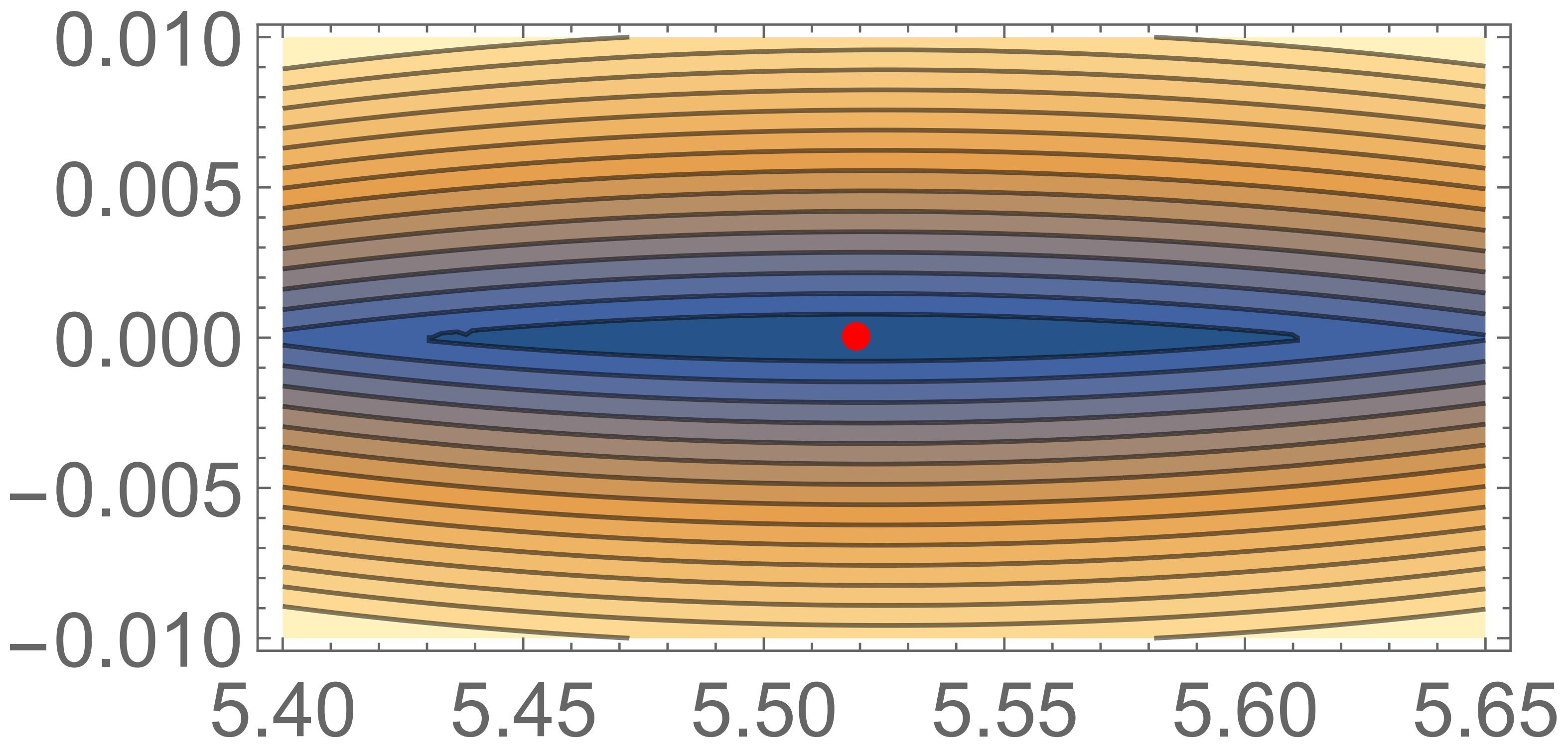

350
300
250
200
150
100
50
0
−6
−4
−2
0
2
4
6
ψ


ψ
$T[0]^{-1}$
$\psi=e^{2\pi i/5}$
z=1
$T[\infty]^{-1}$
ψ=0
z=∞
T[∞]
ψ=1
z=1
T[0]

Thank You!

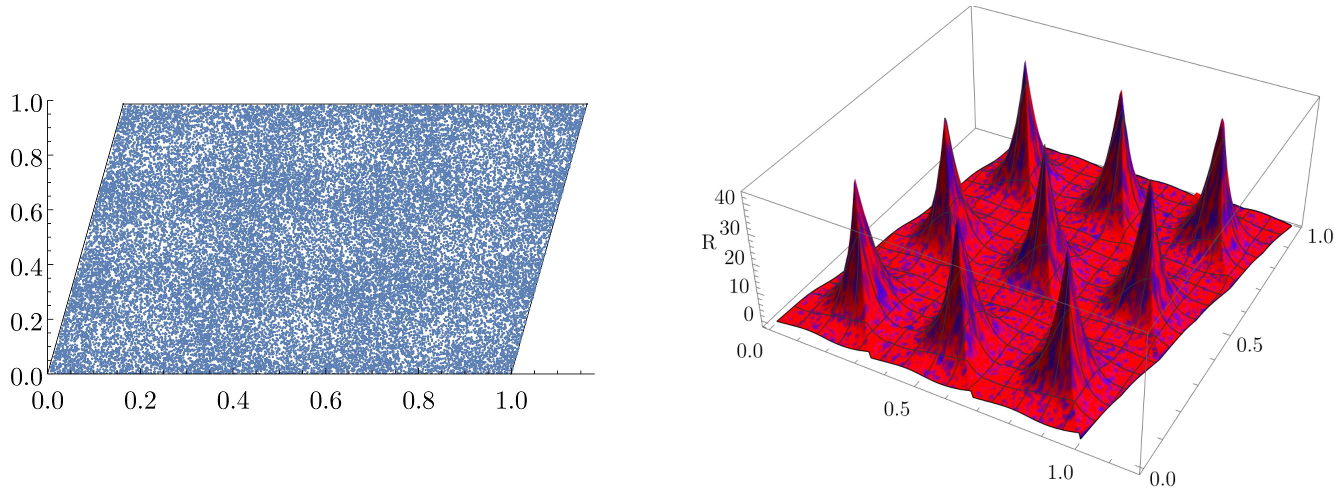

E_{n_1,n_2}
300
250
200
150
100
50
0
−2.5
−2.0
−1.5
−1.0
−0.5
0.0


λ
44.00
43.75
43.50
43.25
43.00
42.75
42.50
5.30
5.35
5.40
5.45
5.50
5.55
5.60
ψ

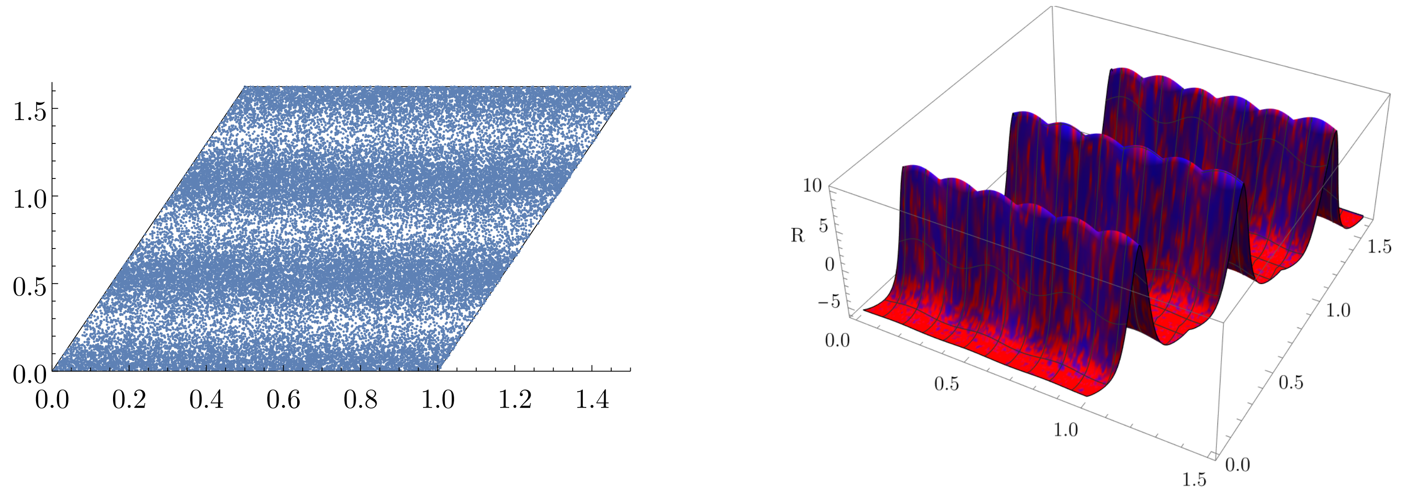

0.010
0.005
0.000
−0.005
−0.010
5.40
5.45
5.50
5.55
5.60
5.65

Additional Material

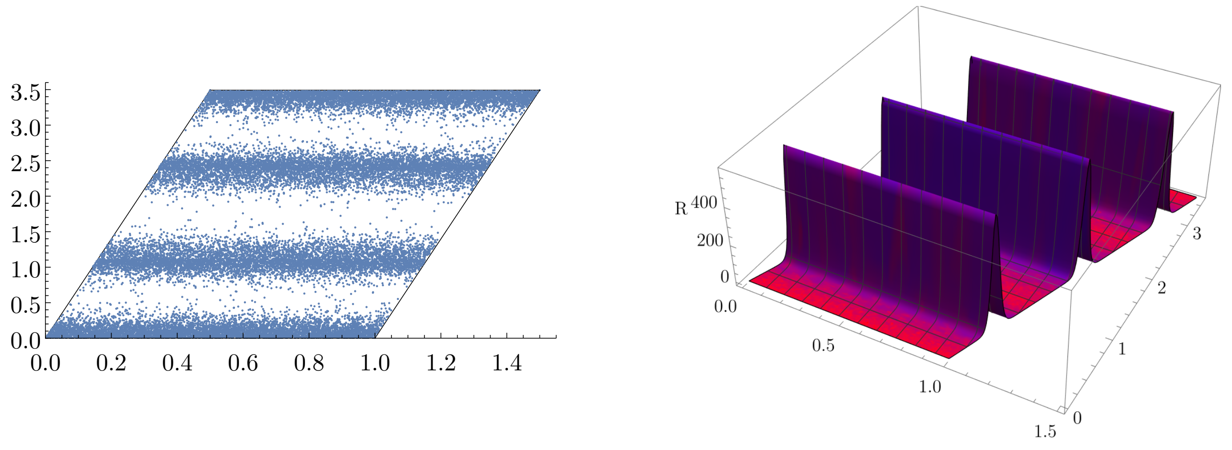
Numerical Spectrum - Discretizing the CY



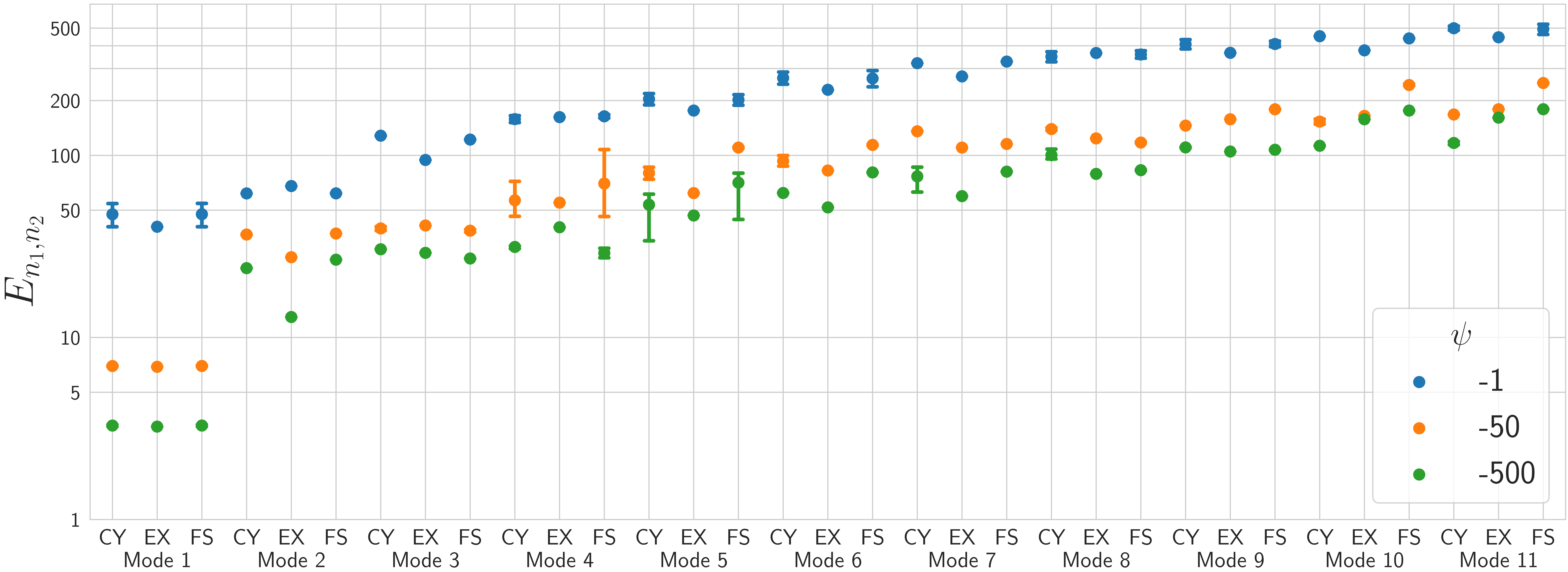
$\psi = -1$



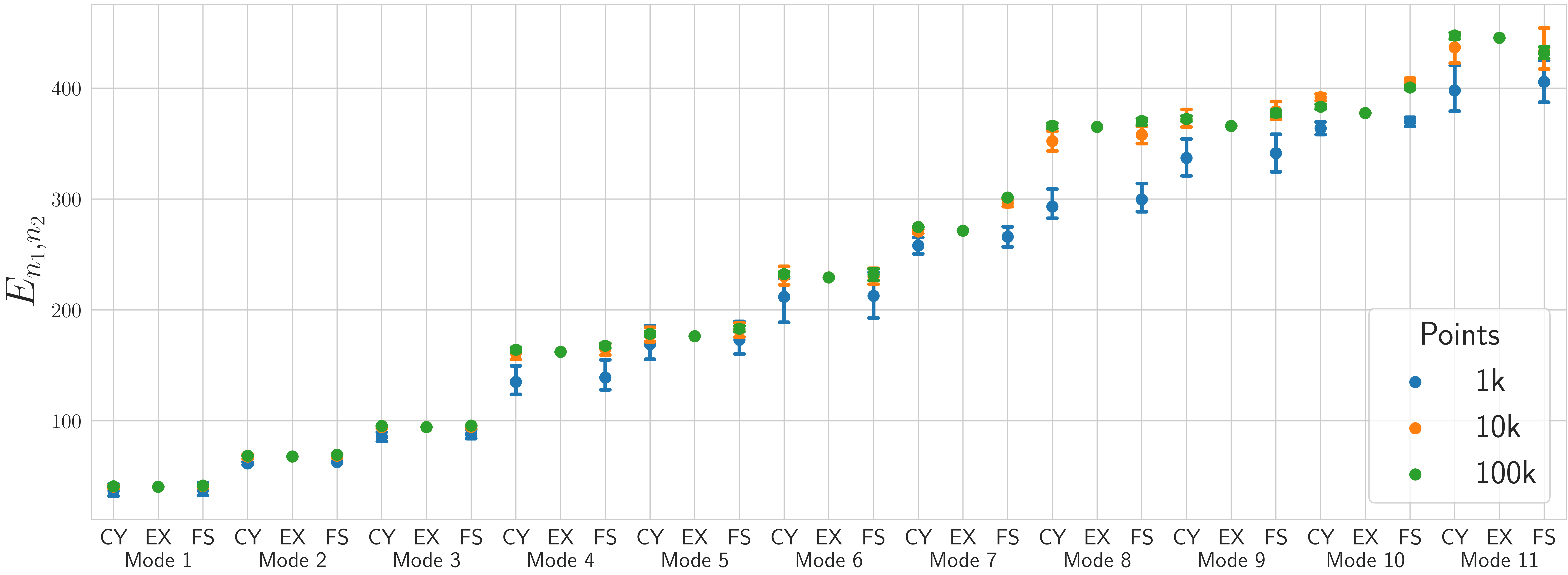
$\psi = -50$



$\psi = -500$



Numerical Spectrum - Discretizing the CY



Numerical Spectrum - Discretizing the Eigenfunctions