Instituto de Física Teórica presents:

Torsion in Cohomology and Dimensional Reduction

by Fernando Marchesano

based on 2306.14959
w/ Gonzalo F. Casas & Matteo Zatti

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Parallel talk on Tuesday

Motivation: Discrete Gauge Symmetries

In a 4d EFT coupled to Einstein gravity, we expect to realise a $\mathbb{Z}_k$ gauge symmetry in terms of a Stückelberg Lagrangian or BF coupling

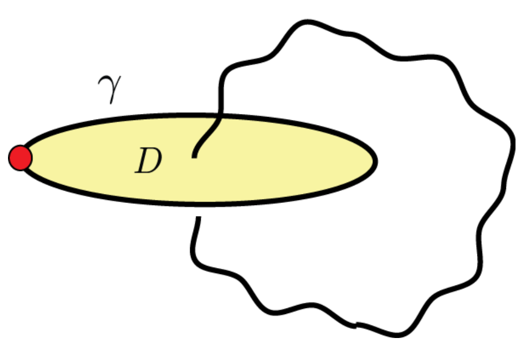
Banks & Seiberg '11

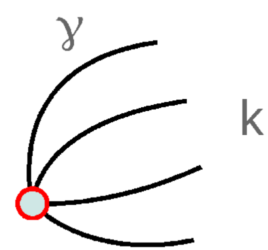
$$f^2\left(d\phi - kA\right)^2 + g^{-2}F^2 \xrightarrow[d\phi = \star_4 dB]{} kB \wedge F$$

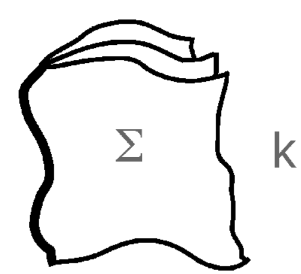
$$m_{\mathrm{st}} = fgk$$

Direct consequences:

- Forbidden couplings (applications to MSSM-like models) *Ibanez & Ross '92*
- Aharanov-Bohm strings and particles, with a $\mathbb{Z}_k$ charge *Alford, Krauss, Wilczek '89*







DGS in type II compactifications

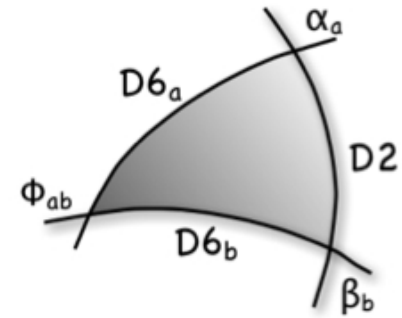
◉ Orientifold models:

k = g.c.d. of D-brane multi-wrapping numbers

Berasaluce-González et al. '11

- Charged chiral matter
- Stückelberg scale above EFT cut-off $m_{\rm st} \sim m_s$

Ghilencea et al. '02



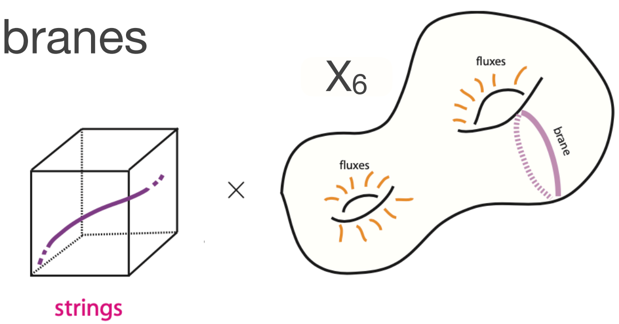
◉ Flux compactifications:

k = flux quanta

$B_2 \wedge F_p \wedge F_{8-p} \rightarrow kB \wedge F$

Berasaluce-González et al. '12

- Charged objects: D-branes
- $m_{\rm st} \sim m_{\rm flux} < m_{\rm KK}$



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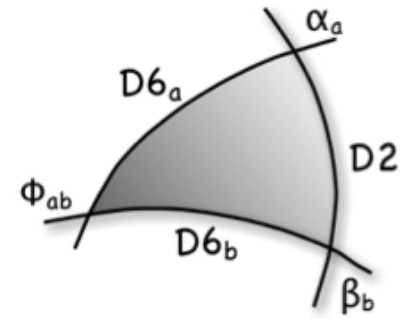

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  k = flux quanta

  $$B_2 \wedge F_p \wedge F_{8-p} \to kB \wedge F$$

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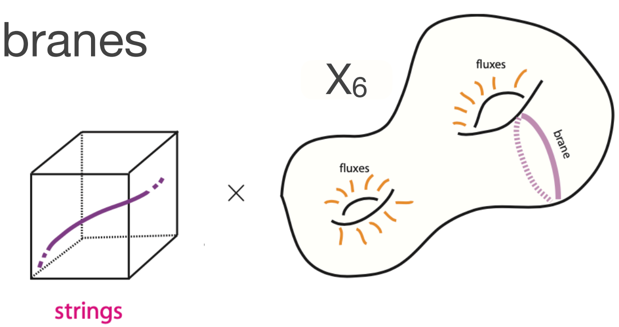


- Torsion in (co)homology:

  $$\text{DGS} \to \text{Tor}\, H_p(X_6, \mathbb{Z})$$

- Related to the other two cases by dualities

Non-abelian DGS:

Berasaluce-González et al. '12

Grimm, Pugh, Regalado '15

Braun et al. '17

Camara, Ibanez, F.M. '11 *Grimm et al. '11* *Mayrhofer et al. '14*

Torsion in (co)homology

The integral homology groups of a manifold have the form:

$$H_p(X_n, \mathbb{Z}) = \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{b_p} \oplus \underbrace{\mathbb{Z}_{k_1} \oplus \ldots \mathbb{Z}_{k_r}}_{\mathrm{Tor}\, H_p(X_n, \mathbb{Z})} \qquad k\Pi_p^{\mathrm{tor}} = \partial\Sigma_{p+1}$$

- Torsion cycles cannot be detected by closed forms $\displaystyle\int_{\Pi_p^{\mathrm{tor}}} \omega_p = \frac{1}{k}\int_{\Sigma_{p+1}} d\omega_p = 0$

- As a consequence, branes wrapping torsion cycles of a CY are necessarily non-BPS objects of the EFT, although stable mod k.

- In general hard to compute the volume and embedding of a torsion cycle

- Similarly, torsion cohomology classes $\mathrm{Tor}\, H^p(X_n, \mathbb{Z})$ can only be represented by exact p-forms → different from conventional dimensional reduction

AB strings from torsion

Despite all, we know that branes on torsion cycles correspond to Aharanov-Bohm particles and strings:

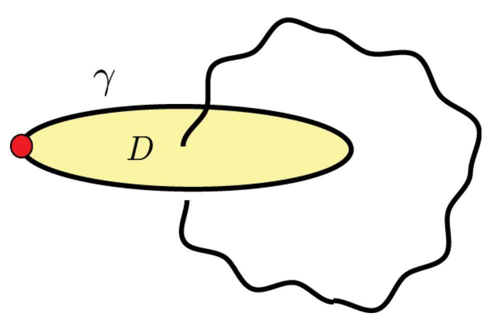
- AdS/CFT — Gukov, Rangamani, Witten '98
- CY orientifold compact. — Camara, Ibanez, F.M. '11

Let us focus on type IIA 4d compactifications. The spectrum of torsion cycles is constrained by Poincaré duality and the UCT:

$$\mathrm{Tor}\, H_2(X_6, \mathbb{Z}) \simeq \mathrm{Tor}\, H_3(X_6, \mathbb{Z}) \qquad \mathrm{Tor}\, H_1(X_6, \mathbb{Z}) \simeq \mathrm{Tor}\, H_4(X_6, \mathbb{Z})$$

D2-brane wrapping $\Pi_2^{\mathrm{tor}} \Longrightarrow$ 4d EFT particle

D4-brane wrapping $\Pi_3^{\mathrm{tor}} \Longrightarrow$ 4d EFT string



$$\frac{1}{2\pi i} \log[\mathrm{hol}(\gamma)] = L(\Pi_2^{\mathrm{tor}}, \Pi_3^{\mathrm{tor}}) = \frac{p}{k} \mod 1$$

Torsion linking number

RR U(1)'s in type IIA

U(1) gauge sym	Discrete gauge sym
Massless U(1)	Massive U(1)
$U(1)^{b_2}$	$\mathrm{Tor}\, H_2(X_6, \mathbb{Z})$
$H_2(X_6, \mathbb{R}) \simeq H_4(X_6, \mathbb{R})$	$\mathrm{Tor}\, H_2(X_6, \mathbb{Z}) \simeq \mathrm{Tor}\, H_3(X_6, \mathbb{Z})$
Elec. charge: D2 (4d part.)	Elec. charge: D2 (4d AB part.)
Mag. charge: D4 (4d part.)	Mag. charge: D4 (4d AB string)
Intersection number	Torsion linking number

Towards dimensional reduction

Question: How do we recover this physics from the procedure of dimensional reduction?

Proposal: Include massive p-form modes that correspond to the generators of the torsion cohomology groups $\mathrm{Tor}\, H^3(X_6, \mathbb{Z}) \simeq \mathrm{Tor}\, H^4(X_6, \mathbb{Z})$

CY-like basis $\{\omega_a, \alpha_A, \beta^A, \tilde{\omega}^a\}$ with relations:

$$d\omega_a = k_{aA}\beta^A,$$

$$d\alpha_A = k_{aA}\tilde{\omega}^a,$$

$$L^{-1} \simeq k_{aA} \in \mathbb{Z}$$

Camara, Ibanez, F.M. '11

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Camara, Ibanez, F.M. '11

Expanding the RR potential:

$$C_3 = \phi_A \beta^A + A^a \wedge \omega_a \longrightarrow k_{aA} B^A \wedge F^a$$

Basis necessary in SU(3) structure manifold comp. leading to 4d N=2 EFTs

Gurrieri et al. '02 D'Auria et al. '04 Grana, Louis, Waldram '05 Kashani-Poor & Minasian '06

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Camara, Ibanez, F.M. '11

New Question: How do we determine such a basis?

- Knowing the spectrum of light p-form modes not enough. Besides a mass matrix, these p-forms specify the quantisation of axions and gauge bosons
- Wall & Žubr theorems classify X_6 up to diffeom. in terms of its massless spectrum, so we need to go beyond diff. geom. to compute $\mathrm{Tor}\, H^p(X_6, \mathbb{Z})$

Tomasiello '05

Recap

- From the physics of D-branes, we know that torsion cohomology groups in a compactification manifold translate into EFT discrete gauge symmetries
- From dualities, we know that some of these DGS should be captured by a Stückelberg Lagrangian or BF coupling below the EFT cut-off
- From consistency of dimensional reduction in 4d N=2 compactifications, we know this physics should be captured by a CY-like basis of p-forms $\{\omega_a, \alpha_A, \beta^A, \tilde{\omega}^a\}$ involving harmonic, exact and co-exact p-forms
- What all this does not tell us is how to determine $\{\omega_a, \alpha_A, \beta^A, \tilde{\omega}^a\}$ in terms of the spectrum of light p-forms of the compact manifold
- Equivalently, we don't know how to compute some couplings in the EFT Lagrangian:

$$f^2(d\phi - kA)^2 + g^{-2}F^2 \qquad\qquad m_{\mathrm{st}} = fgk$$

missing

Smeared deltas

To each p-cycle $\Pi_p \subset X_n$, we can associate a bump delta *(n-p)*-form

$$\int_{X_n} \omega_p \wedge \delta_{n-p}(\Pi_p) = \int_{\Pi_p} \omega_p$$

Given a metric on X_n, one can decompose such a delta as a sum of Laplace eigenforms $\Delta b^i_{n-p} = \lambda_i^2 b^i_{n-p}$, and then smear it by truncating the series:

$$\delta_{n-p}(\Pi_p) = \sum_i c_i\, b^i_{n-p}\,, \qquad c_i = \int_{\Pi_p} \star\, b^i_{n-p} \qquad \longrightarrow \qquad \delta^{\rm sm}_{n-p}(\Pi_p) = \sum_{\lambda_i \ll m_{\rm KK}} c_i\, b^i_{n-p}$$

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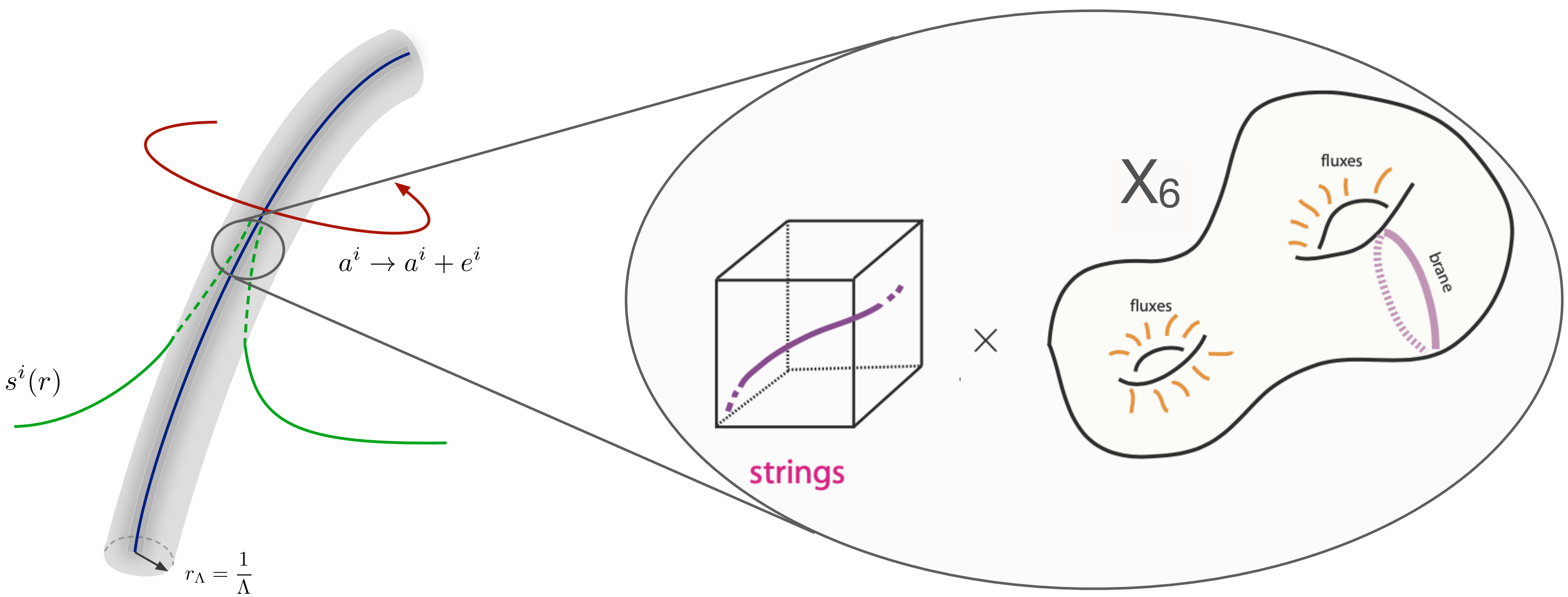
- Π_p non-trivial in de Rham $\Longrightarrow$ $\delta^{\mathrm{sm}}(\Pi_p) \in$ Poincaré dual of $[\Pi_p]$
- Π_p torsion or trivial $\Longrightarrow$ $\delta^{\mathrm{sm}}(\Pi_p)$ exact, non-vanishing if there are massive *p*-form modes in the 4d EFT coupling to Π_p

Smeared deltas

Intuition: $\delta^{\rm sm}(\Pi_p)$ encodes the 4d EFT long-wavelength description of a backreacted brane source wrapping Π_p

Example: D4-brane on $\Pi_3 \subset X_6 \rightarrow$ 4d solution of an *EFT string* *Lanza et al. '21*



Same solution for AB strings near the string core, and with an IR regulator at $m_{\rm st}^{-1}$

$\delta^{\rm sm}(\Pi_3^{\rm tor}) \neq 0 \implies$ non-trivial 4d backreaction $\implies$ Stückelberg term in 4d EFT

Smeared deltas and calibrations

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Example: D4-brane on $\Pi_3 \subset X_6$ → 4d solution of an *EFT string* Lanza et al. '21

In N=1 vacua, EFT strings and membranes capture info of the 4d EFT Lagrangian thanks to being 1/2 BPS objects $\Longrightarrow$ Π_p calibrated submanifold of X_6

Harvey & Lawson '82 Koerber '05, Martucci & Smyth '05

Intuition: smeared backreaction under control for BPS objects Blaback et al. '10

smeared data of BPS objects capture their tension and physical charge, and the latter translate into EFT Lagrangian terms

$$f^2 (d\phi - kA)^2 + g^{-2} F^2$$

smeared BPS data

Smeared deltas and linking numbers

Idea: compute X₆ topological data using $\delta^{\rm sm}(\Pi_p)$ of calibrated/BPS cycles

Torsion Linking number:

$$\Delta b^i_{n-p} = \lambda_i^2 b^i_{n-p}$$

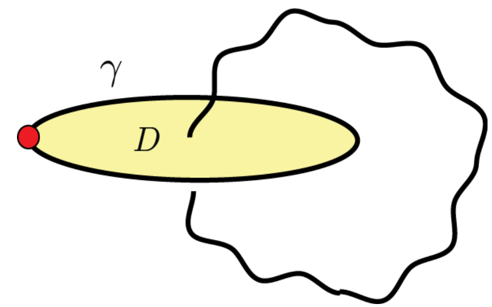
$$L(\Pi^{\rm tor}_{n-p-1}, \Pi^{\rm tor}_p) = \int_{X_n} d^{-1}\delta(\Pi_{n-p-1}) \wedge \delta(\Pi_p) \mod 1 = \sum_i \frac{c_i e_i}{\lambda_i} \mod 1$$

$$\delta_{n-p}(\Pi_p) = \sum_i c_i\, b^i_{n-p}\,,$$

$$\delta(\Pi_{n-p-1}) = \sum_i \frac{e_i}{\lambda_i} d \star b^i_{n-p}$$

Smeared version:

$$L^{\rm sm}(\Pi^{\rm tor}_{n-p-1}, \Pi^{\rm tor}_p) = \sum_{\lambda_i \ll m_{\rm KK}} \frac{c_i e_i}{\lambda_i} \mod 1$$



Conjecture: If non-vanishing, $L^{\rm sm} = L$ for mutually BPS cycles

Extension: $[\Pi^{\rm tor}_p] = [\Pi'_p] - [\Pi_p]$

BPS representatives

Caveats

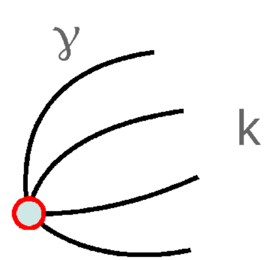
How can a torsion cycle be calibrated?

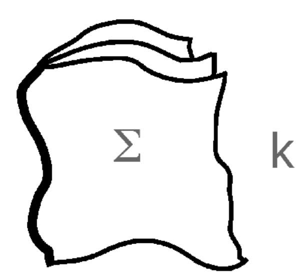
In a manifold with a special holonomy metric it cannot, because calibrations are closed *p*-forms. But in manifolds with G-structure metrics $d\Phi_{\pm} \neq 0$

Simple class: SU(3)-structure manifolds (J, Ω) such that $dJ \neq 0 \neq d\Omega$

How can a $\mathbb{Z}_k$-charged object be BPS?

The nucleation process is allowed topologically, but not favoured energetically





A simple example

We consider a 1/2 BPS domain-wall solution in a 4d N=2 EFT *Gurrieri et al. '02*

Type IIA on half-flat X_6

Fibered over interval to X_7
G_2 manifold *Hitchin '01*

mirror symmetry

$\longleftrightarrow$

Type IIB on a CY_3

Backreacted NS5 on Π_3

$$ds^2 = ds^2_{\mathbb{R}^{1,2}} + V(d\xi)^2 + ds^2_{X_6}(\xi) \longrightarrow \big(J(\xi), \Omega(\xi)\big)$$

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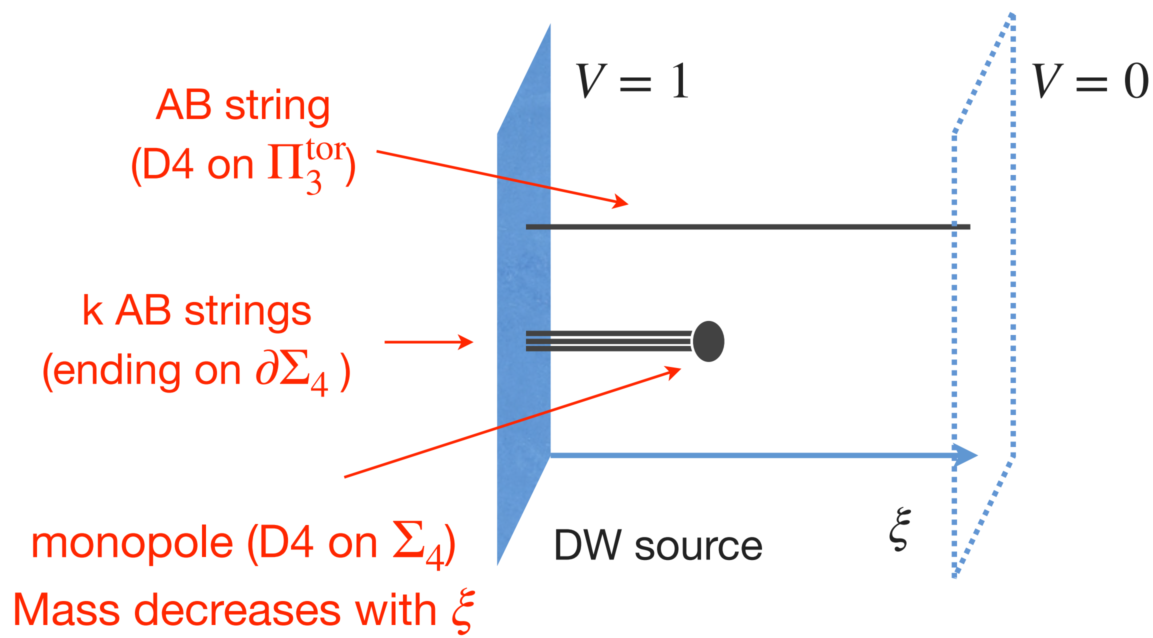
V monot. decreasing to 0

Hitchin flow eqs:

$$V^{1/2} dJ = \frac{1}{2}\partial_\xi \mathrm{Re}\Omega$$
$$V^{1/2} d\mathrm{Im}\Omega = -\frac{1}{2}\partial_\xi (J \wedge J)$$



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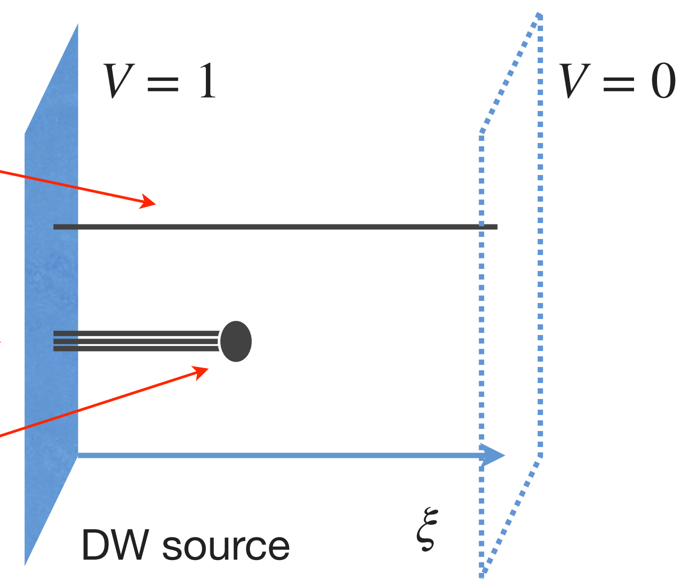
Total energy independent of ξ

AB string (D4 on Π_3^{tor})

k AB strings (ending on $\partial\Sigma_4$)



monopole (D4 on Σ_4)
Mass decreases with ξ

A simple example

We consider a 1/2 BPS domain-wall solution in a 4d N=2 EFT *Gurrieri et al. '02*

Type IIA on half-flat X_6

Fibered over interval to X_7
G_2 manifold *Hitchin'01*

mirror symmetry ⟷

Type IIB on a CY_3

Backreacted NS5 on Π_3

$$ds^2 = ds^2_{\mathbb{R}^{1,2}} + V(d\xi)^2 + ds^2_{X_6}(\xi) \longrightarrow \big(J(\xi), \Omega(\xi)\big)$$

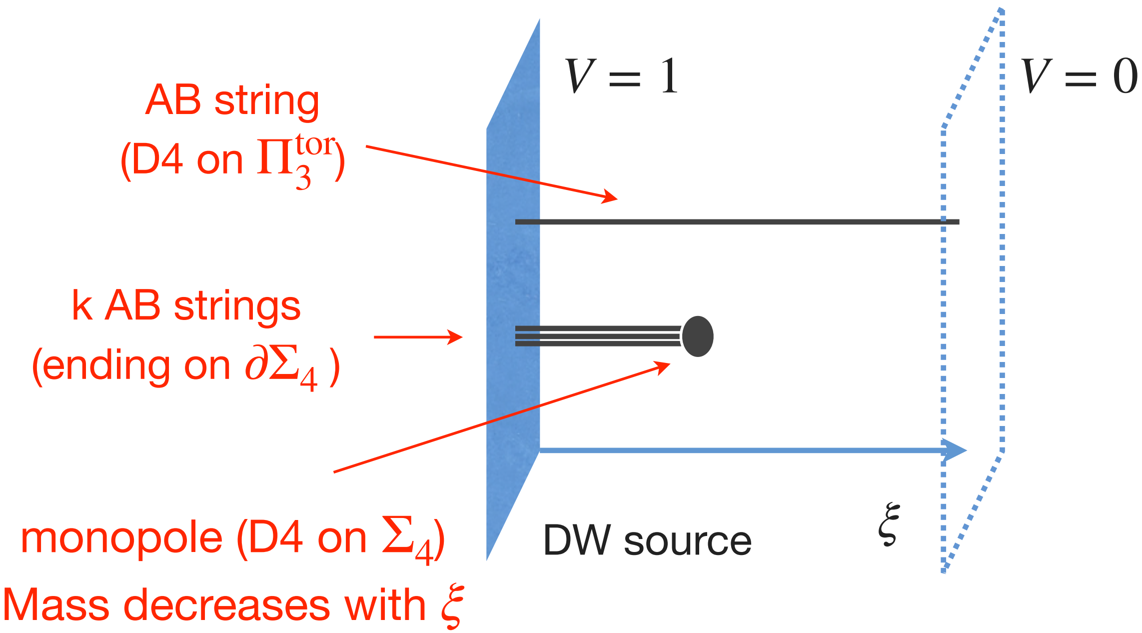
Simplest example:

$$X_6 = \tilde{T}^3 \times T^3$$

$$V = 1 - kC\xi \qquad C = m_{\rm st}|_{\xi=0}$$

Conjecture can be checked explicitly

M. Zatti's talk



More general 4d N=2 EFTs

Grana, Louis, Waldram'05

Kashani-Poor & Minasian'06

General framework for dimensional reduction:

CY-like basis of p-forms $\{\omega_a, \alpha_A, \beta^A, \tilde{\omega}^a\}$:

$$\int_{X_6} \omega_a \wedge \tilde{\omega}^b = \delta_a^b \qquad \int_{X_6} \alpha_A \wedge \beta^B = \delta_A^B$$

$$J_c = (b^a + it^a)\,\omega_a \longrightarrow K_J = -\log \frac{4}{3}\int_{X_6} -J^3$$

$$\Omega = Z^A \alpha_A - \mathscr{F}_B \beta^B \longrightarrow K_\rho = -\log \int_{X_6} i\bar{\Omega} \wedge \Omega$$

$$\boxed{\begin{aligned} d\omega_a &= m_a{}^A \alpha_A + k_{aA}\beta^A, \\ d\alpha_A &= k_{aA}\tilde{\omega}^a, \\ d\beta^B &= -m_a{}^B \tilde{\omega}^a, \\ d\tilde{\omega}^a &= 0, \end{aligned}}$$

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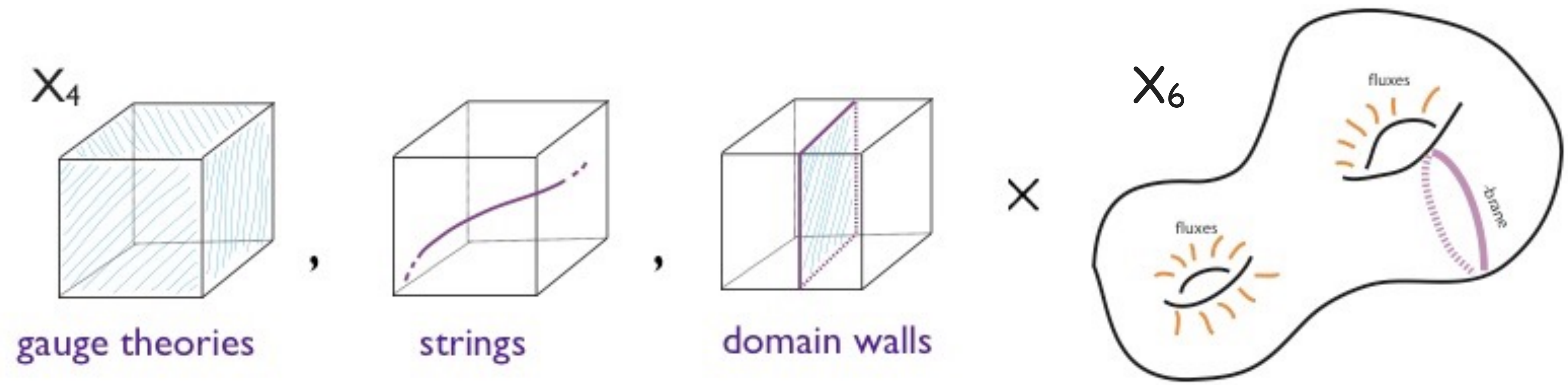
Proposal: basis elements $\{\omega_a, \alpha_A, \beta^A, \tilde{\omega}^a\}$ are smeared deltas of calibrated cycles, and generate a discrete cone of BPS objects that includes the torsion generators

- Motivated by the EFT string completeness conjecture extended to AB strings
- Reproduces automatically the tension/mass of 4d BPS strings and particles

Torsion BPS objects in 4d N=1 vacua

In 4d N=1 vacua particles cannot be BPS, and strings are calibrated by closed p-forms, so torsion data are related to different BPS objects



Space-time filling branes ending on membranes: $\hat{g}^2\left(dD_3 - kA_4\right)^2$

Membranes ending on strings: $\hat{f}^2\left(dB_2 - kC_3\right)^2 + \hat{g}^{-2}(dC_3)^2$

The same ideas and machinery work for torsion cycles on SU(3)-structure vacua

$\delta^{\rm sm}(\Pi_p^{\rm tor})$ appear in the smeared flux background

Conclusions

- Torsion in (co)homology implies a discrete gauge symmetry. This may or may not be realised as a Stückelberg-like Lagrangian in a lower-dimensional EFT. If it does, it means that we can compute torsion data in terms of certain smooth p-forms.

- The key objects to consider are smeared delta forms $\delta^{\text{sm}}(\Pi_p)$. These encode if the backreaction of a localised source is non-trivial at EFT wavelengths.

- We have argued that when Π_p is calibrated, $\delta^{\text{sm}}(\Pi_p)$ encodes EFT Lagrangian couplings. Intuitively, the smeared solution of a BPS localised object is under control. This applies even for torsion cycles Π_p^{tor}.

- This leads to the conjecture $L = L^{\text{sm}}$ for computing torsion linking numbers using smeared data. Intuitively: $m_{\text{st}} = fgk$

$$\underbrace{f^2}_{\delta^{\text{sm}}(\Pi_3^{\text{tor}})} \left(d\phi - kA\right)^2 + \underbrace{g}_{\delta^{\text{sm}}(\Pi_2^{\text{tor}})}{}^{-2} F^2$$

- Additionally, it gives us an integral basis of p-forms to obtain 4d N=2 EFTs from dimensional reduction in SU(3)-structure manifolds.


Thank you!