On Classical de Sitter Solutions and Quantum Transitions

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S. Céspedes, S. de Alwis, F. Muia, FQ (to appear)
(also previous work with V. Pasquarella)
S. Bogojevic, C.P. Burgess, F. Muia, FQ (to appear)

Two Related Questions

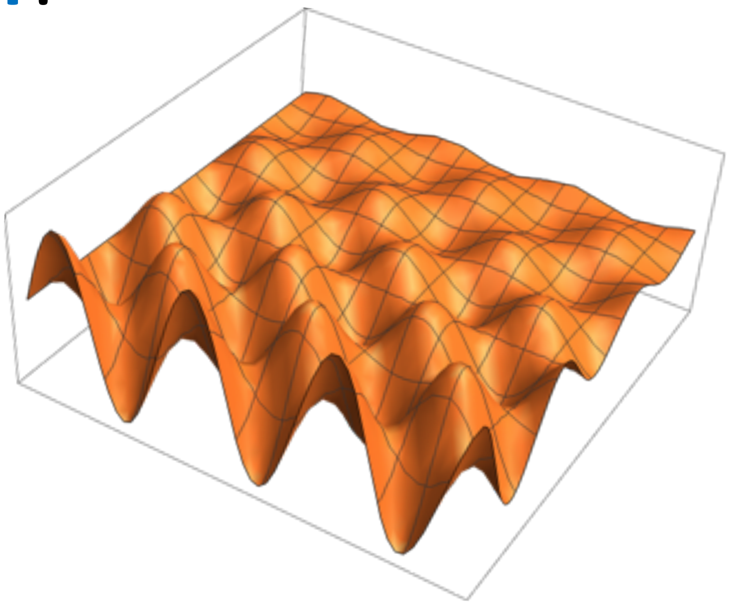
- Classical de Sitter from supergravity and string theory.
- Quantum transitions in the landscape.

Transitions among dS/Minkowski/AdS and nothingness

Predictions from the landscape?

- Bubble nucleations imply open universe!

- Not possible to tunnel up from Minkowski nor anti de Sitter.



Early History

- Coleman de Luccia (1980)
- Witten (1981)
- Vilenkin + Hartle-Hawking (1982-3)
- Brown-Teitelboim (1987)
- Farhi-Guth-Guven (1990)
- Fischler-Morgan-Polchinski (1990)

Wave functions of the universe

Mini-superspace

$$ds^2 = -N^2(t)dt^2 + a^2(t)(dr^2 + \sin^2 r d\Omega_2^2)$$

Hartle-Hawking vs Vilenkin (tunneling to dS from nothing)

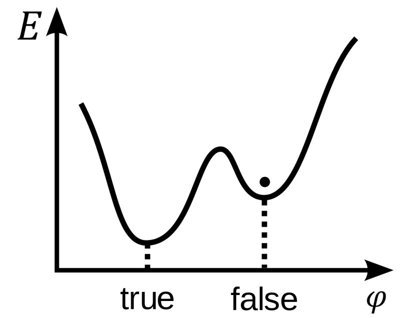
entropy

$$\mathcal{P}_{\mathrm{HH}}(\mathrm{Nothing} \to \mathrm{dS}) = \|\Psi_{\mathrm{HH}}\left(\mathrm{H}_{\mathrm{dS}}\right)\|^2 \propto e^{\frac{\pi}{GH_{\mathrm{dS}}^2}} = e^{+S_{\mathrm{dS}}}$$

$$\mathcal{P}_{\mathrm{T}}(\mathrm{Nothing} \to \mathrm{dS}) = \|\Psi_{\mathrm{T}}\left(\mathrm{H}_{\mathrm{dS}}\right)\|^2 \propto \mathrm{e}^{-\frac{\pi}{\mathrm{GH}_{\mathrm{dS}}^2}} = e^{-S_{\mathrm{dS}}}$$

(Question: What about Minkowski and AdS Entropy?)

Two types of vacuum transitions

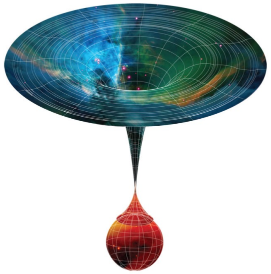


1. **Transition between two minima of scalar potential**

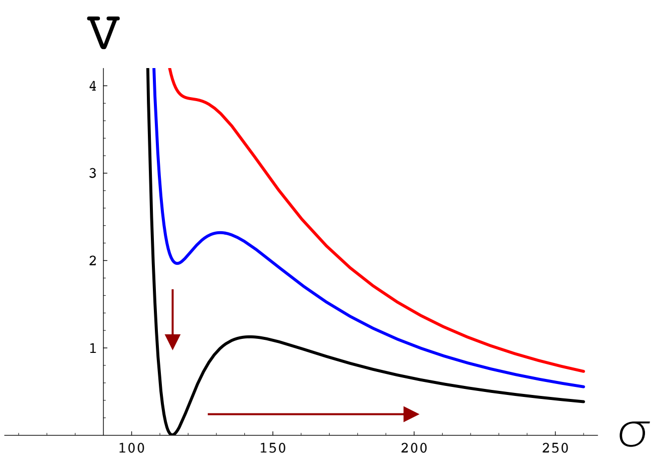
   Coleman-De Luccia 1980

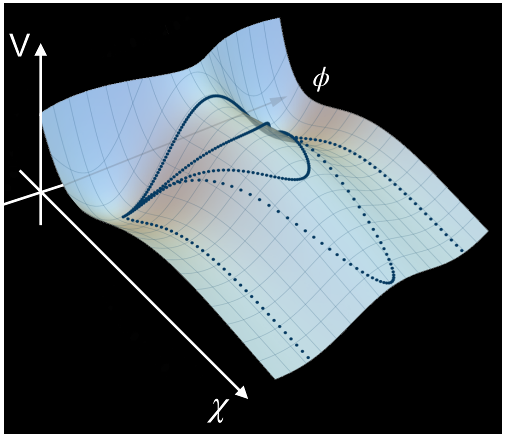
2. **No scalar field: M_1 to M_1+Wall+M_2**

   Brown-Teitelboim 87



Both realised in string landscape





Approximate picture

dS to dS

Euclidean approach (Coleman-de Luccia, Lee-Weinberg, Brown-Teitelboim) :

$$\Gamma \sim e^{-B}, \qquad B = S[instanton] - S[background]$$

$$B = \frac{\pi}{2G}\left[\frac{\left[(H_O^2 - H_I^2)^2 + \kappa^2(H_O^2 + H_I^2)\right] R_o}{4\kappa H_O^2 H_I^2} - \frac{1}{2}\left(H_I^{-2} - H_O^{-2}\right)\right]$$

$$R_o^2 = \frac{4\kappa^2}{(H_O^2 - H_I^2)^2 + 2\kappa^2(H_O^2 + H_I^2) + \kappa^4}$$

Analytic continuation from Euclidean to Lorentzian implies open universe but just a "guess" (O(4) symmetry)

Up-Tunneling and Minkowski limit

Detailed balance

entropies

$$\Gamma_{\rm up} = \Gamma_{\rm down} \exp\left[\frac{\pi}{G}\left(\frac{1}{H_{\rm I}^2} - \frac{1}{H_{\rm O}^2}\right)\right] = \Gamma_{\rm CDL} \exp\left(S_{\rm I} - S_{\rm O}\right)$$

For HH sign only!

De Sitter to Minkowski ?

$$H_I \to 0, \qquad \Gamma_{\rm down} \to \exp\left[-\frac{\pi}{2G}\frac{\kappa^4}{H_O^2\left(H_O^2+\kappa^2\right)^2}\right]$$

$$H_O \to 0, \qquad \Gamma_{\rm up} \to 0$$

Hamiltonian Approach

Fischler, Morgan, Polchinski 1990

Metric $$ds^2 = -N_t^2(t,r)dt^2 + L^2(t,r)(dr + N_r dt)^2 + R^2(t,r)d\Omega_2^2\,,$$ **Spherically symmetric**

Action

$$S_{\rm tot} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{g}\,\mathcal{R} + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3y\sqrt{h}\,K + S_{\rm mat} + S_{\rm W}$$

$$S_{\rm W} = -4\pi\sigma\int dtdr\,\delta(r-\hat{r})[N_t^2 - L^2(N_r + \dot{\hat{r}})^2]^{1/2}$$

$$S_{\rm mat} = -4\pi\int dtdr\,LN_tR^2\,\rho(r)\,,\qquad \boxed{\rho = \Lambda_{\rm O}\,\theta(r-\hat{r}) + \Lambda_{\rm I}\,\theta(\hat{r}-r)}$$

Conjugate variables

$$\pi_L = \frac{N_rR' - \dot{R}}{GN_t}R,\qquad \pi_R = \frac{(N_rLR)' - \partial_t(LR)}{GN_t}\,,$$

$$\mathcal{H}_g = \frac{GL\pi_L^2}{2R^2} - \frac{G}{R}\pi_L\pi_R + \frac{1}{2G}\left[\left(\frac{2RR'}{L}\right)' - \frac{R'^2}{L} - L\right]$$

$$P_g = R'\pi_R - L\pi_L'\,.$$

Constraints

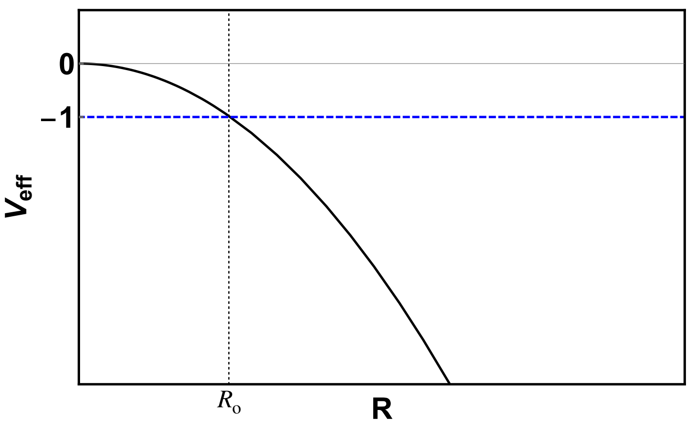
$$\mathcal{H} = \mathcal{H}_g + 4\pi LR^2\rho(r) + \delta(r-\hat{r})E = 0\,,$$

$$P = P_a - \delta(r-\hat{r})\hat{p} = 0\,,$$

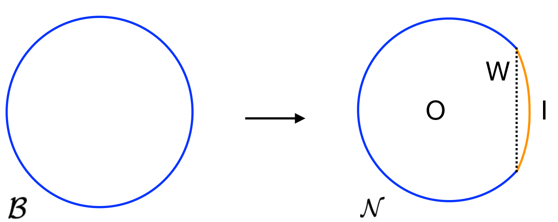
$$E = \sqrt{\frac{\hat{p}^2}{\hat{L}^2} + m^2}\,,\qquad m = 4\pi\sigma\hat{R}^2\,,\qquad \hat{p} = \partial\mathcal{L}/\partial\dot{\hat{r}}$$

De Sitter to de Sitter

$$\mathcal{P}(\mathrm{dS} \to \mathrm{dS/dS} \oplus \mathrm{W}) = \frac{|\Psi(\mathrm{dS/dS} \oplus \mathrm{W})|^2}{|\Psi(\mathrm{dS})|^2}$$



$$V = -\frac{1}{4\kappa^2}\hat{R}^2\left[(H_{\mathrm{O}}^2 - H_{\mathrm{I}}^2)^2 + 2\kappa^2(H_{\mathrm{O}}^2 + H_{\mathrm{I}}^2) + \kappa^4\right]$$



$$A_{\mathrm{O}} = 1 - H_{\mathrm{O}}^2 R^2\,, \qquad A_{\mathrm{I}} = 1 - H_{\mathrm{I}}^2 R^2$$

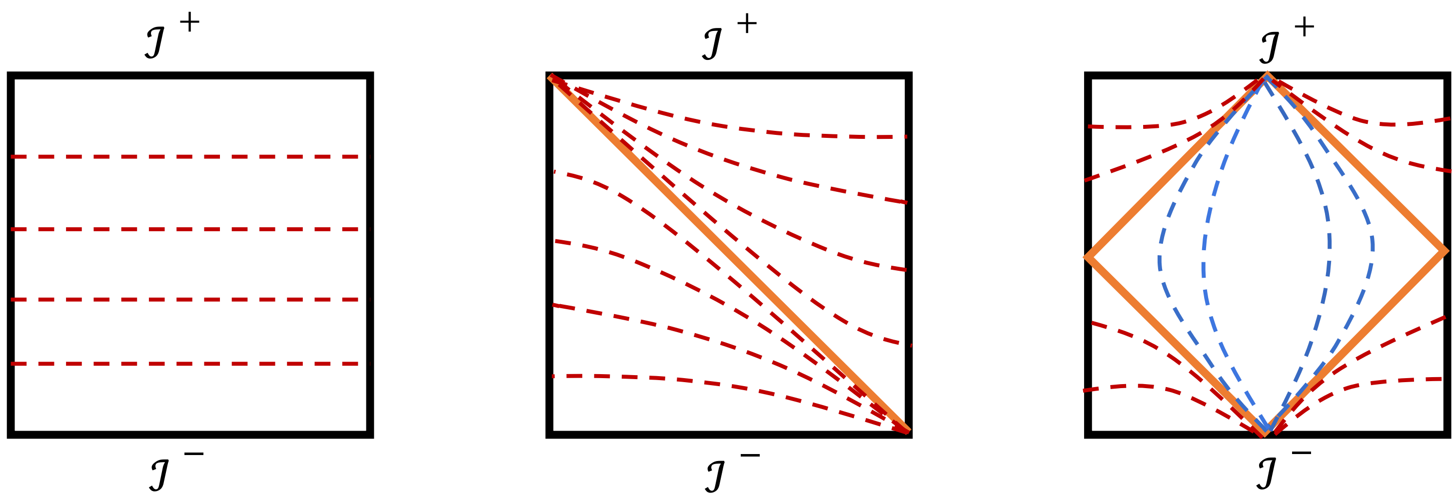
$$I_{\mathrm{tot}}\Big|_{\mathrm{tp}} - \bar{I} = -\frac{\eta\pi}{G}\left[\frac{\left[(H_{\mathrm{O}}^2 - H_{\mathrm{I}}^2)^2 + \kappa^2(H_{\mathrm{O}}^2 + H_{\mathrm{I}}^2)\right] R_{\mathrm{o}}}{8\kappa H_{\mathrm{O}}^2 H_{\mathrm{I}}^2} - \frac{1}{4}\left(H_{\mathrm{I}}^{-2} - H_{\mathrm{O}}^{-2}\right)\right]$$

Same result as Euclidean approach

$\eta = +1$ **Background Hartle-Hawking**

$\eta = -1$ **Background Vilenkin**

De Sitter Slicings



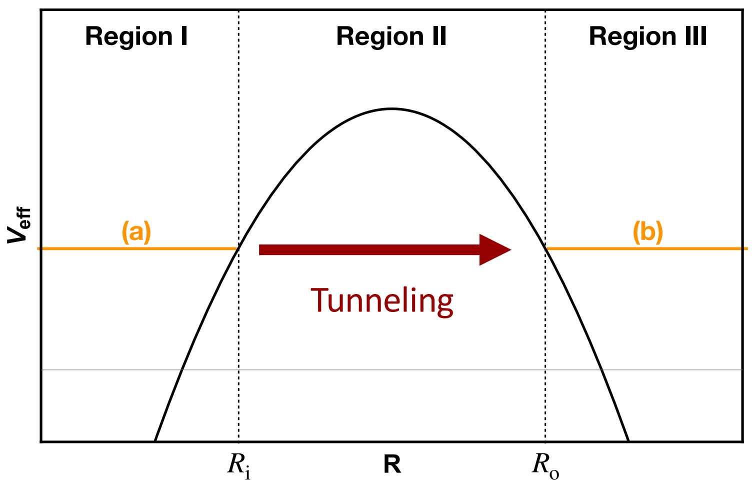
Closed | Flat | Open

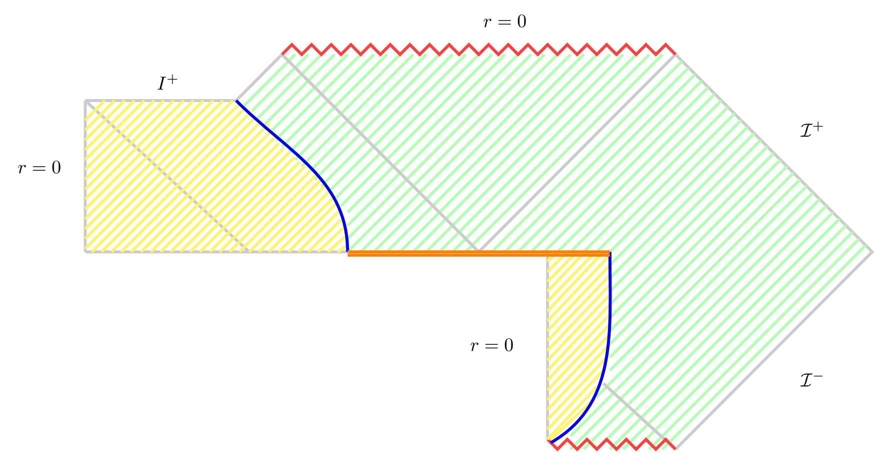
From Hamiltonian approach: O(3) symmetry, closed slicing.
Universe inside the bubble is closed for global slicing.

Schwarzschild to de Sitter

(H_O=0)

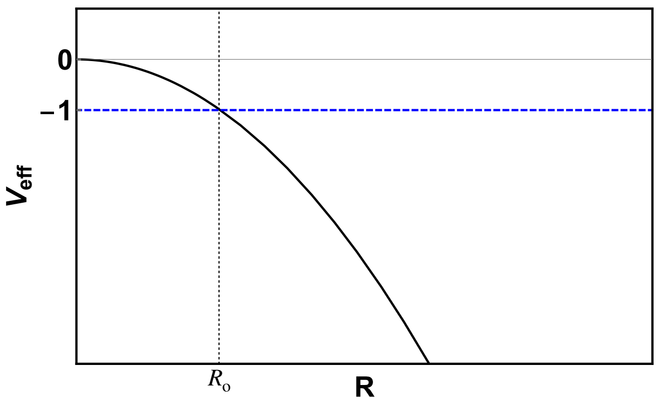
Farhi,Guth, Guven (Euclidean) + Fischler, Morgan, Polchinski (Hamiltonian)





Zero Schwarzschild mass limit

(Minkowski ≈ Schwarzschild in the M=0 limit)



$$V=-\frac{(H^2+\kappa^2)^2}{4\kappa^2}\hat{R}^2$$

$$\mathcal{P}(\mathcal{M}\to\mathcal{M}/\mathrm{dS}\oplus\mathrm{W})=\exp\left[\frac{\eta\pi}{GH^2}\left(1-\frac{\kappa^4}{(H^2+\kappa^2)^2}\right)\right]$$

Up-tunneling

$$\mathcal{P}(\mathrm{dS}\to\mathrm{dS}/\mathcal{M}\oplus\mathrm{W})=\exp\left[\frac{\eta\pi}{GH^2}\left(-\frac{\kappa^4}{(H^2+\kappa^2)^2}\right)\right]$$

Down-tunneling

Detailed Balance

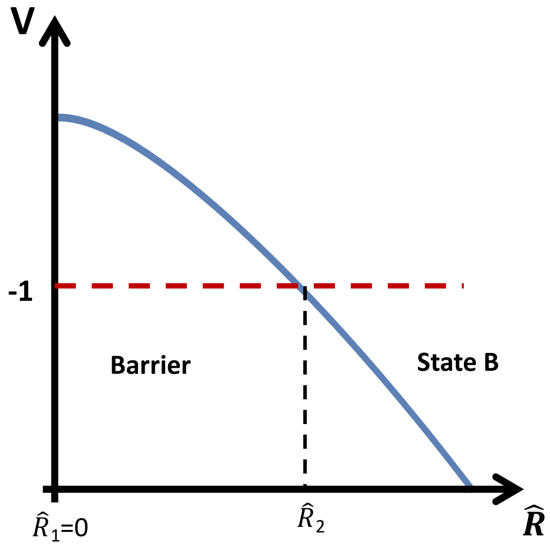
$$\frac{\mathcal{P}(\mathcal{M}\to\mathcal{M}/\mathrm{dS}\oplus\mathrm{W})}{\mathcal{P}(\mathrm{dS}\to\mathrm{dS}/\mathcal{M}\oplus\mathrm{W})}=\exp\left[\eta\frac{\pi}{G}\frac{1}{H^2}\right]$$

Entropy

M=0 Schwarzschild ≠ H=0 de Sitter

(Difference on background wave function)

AdS to AdS



$$\kappa < \left|\sqrt{|H_I^2|} - \sqrt{|H_O^2|}\right|, \qquad \text{or} \qquad \kappa > \left|\sqrt{|H_I^2|} + \sqrt{|H_O^2|}\right|,$$

$$B = -\frac{\eta\pi}{2G}\left[\frac{\left(\left|H_I^2\right| - \left|H_O^2\right|\right)^2 - \kappa^2\left(\left|H_I^2\right| + \left|H_O^2\right|\right)}{2\kappa\left|H_I^2\right|\left|H_O^2\right|}R_0 - \left(\frac{1}{\left|H_O^2\right|} - \frac{1}{\left|H_I^2\right|}\right)\right]$$

$$\mathcal{P}_{\text{up}}^{\text{AdS}\to\text{AdS}} = \mathcal{P}_{\text{down}}^{\text{AdS}\to\text{AdS}},$$

Detailed balance if Entropy of AdS = 0 !

AdS to dS

$$B^{\mathrm{AdS}->\mathrm{dS}} = \frac{\eta\pi}{G}\left\{\frac{\left\{(|H_B^2| + H_A^2)^2 + \kappa^2(-|H_B^2| + H_A^2)\right\} R_{\mathrm{o}}}{4\kappa|H_B^2|H_{\mathrm{A}}^2} + \frac{1}{2}\left(\frac{1}{H_A^2} - \frac{1}{|H_B^2|}\right)\right\},$$

$$\frac{P^{\mathrm{AdS}->\mathrm{dS}}}{P^{\mathrm{dS}->\mathrm{AdS}}} = \frac{e^{B^{\mathrm{AdS}->\mathrm{dS}}}}{e^{B^{\mathrm{dS}->\mathrm{AdS}}}} = \frac{\exp\left(\frac{\eta\pi}{2G}\frac{1}{H_A^2}\right)}{\exp\left(-\frac{\eta\pi}{2G}\frac{1}{H_A^2}\right)} = e^{\eta(S_{\mathrm{dS}}-(S_{\mathrm{AdS}}=0))},$$

Detailed balance if AdS entropy=0!

Minkowski limit from dS blows-up but from AdS is finite!?

To Nothingness and Back?

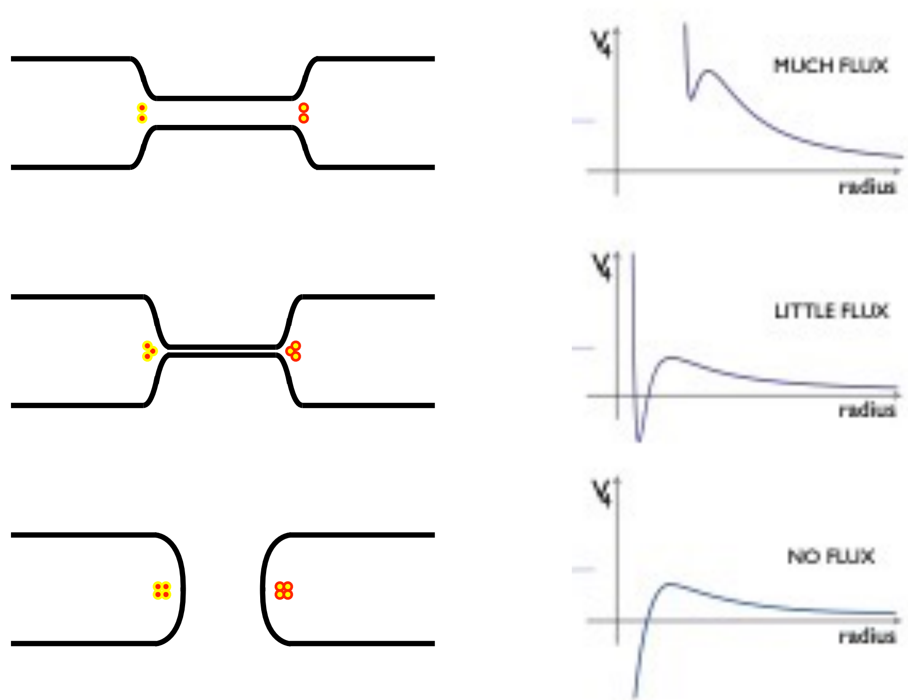
For SAdS to dS $H_{\rm O} \gg H_{\rm I}, M, \kappa$

$$B^{\rm AdS->dS} \to \frac{\eta\pi}{G}\left\{ \frac{\{(|H_B^2|)^2\}\, 2\kappa/|H_B^2|}{4\kappa|H_{\rm B}^2|H_A^2} + \frac{1}{2}\left(\frac{1}{H_A^2}+0\right)\right\} = \frac{\eta\pi}{2G}\frac{1}{H_A^2}.$$

The same as Vilenkin, Hartle-Hawking wave functions!

≈ Brown-Dahlen:
Nothing as AdS

$H_{\rm O} \to \infty$



SAdS to dS

$$I_{\rm B}\Big|_{\rm tp} \equiv I_{\rm B}\Big|_{R_{\rm I}}^{R_{\rm O}} = \begin{cases} \frac{\eta\pi}{2G}(R_{\rm O}^2 - R_{\rm I}^2)\,, & M > M_{\rm S}\,, \\ \frac{\eta\pi}{2G}(R_{\rm O}^2 - R_{\mathcal{S}}^2)\,, & M_{\rm S} > M > M_{\rm D} \\ \frac{\eta\pi}{2G}(R_{\rm dS}^2 - R_{\mathcal{S}}^2)\,, & M_{\rm D} > M\,. \end{cases}$$

$$M_{\rm S} = \frac{H_{\rm O}^2 + H_{\rm I}^2 + \kappa^2}{2G\left(H_{\rm I}^2 + \kappa^2\right)^{3/2}}, \qquad M_{\rm D} = \frac{H_{\rm O}^2 + H_{\rm I}^2 - \kappa^2}{2GH_{\rm I}^3}.$$

Need numerical estimates for wall contribution but the transition is allowed however detailed balance is OK only for M_D>M (?)

$$\frac{P^{\rm AdS->dS}}{P^{\rm dS->AdS}} = \frac{e^{B^{\rm AdS->dS}}}{e^{B^{\rm dS->AdS}}} = \frac{\exp\left(\frac{\eta\pi}{2G}\frac{1}{H_A^2}\right)}{\exp\left(-\frac{\eta\pi}{2G}\frac{1}{H_A^2}\right)} = e^{\eta(S_{\rm dS}-(S_{\rm AdS}=0))},$$

Summary

- Schwarzschild M=0 to dS allowed
- AdS Schwarschild M=0 to (A)dS also allowed
- Entropy of Minkowski/AdS is 0 or ∞
- Transition from $\Lambda \to -\infty$ to dS same as HH/Vilenkin universe from nothing!
- Universe after transition open or closed!
- Detailed balance OK for small bh mass (?)

On classical dS solutions on 6D supergravity and their uplift

6D Supergravity (Salam-Sezgin)

$$S = -\int \mathrm{d}^D x \sqrt{-g} \left[\frac{1}{2\kappa^2} g^{MN} \Big(R_{MN} + \partial_M \varphi \, \partial_N \varphi \Big) + \frac{1}{2} \sum_r \frac{1}{(p_r+1)!} e^{-p_r \varphi} F_r^2 + \mathcal{A} \, e^{\varphi} \right] ,$$

D=6, r=2, *A*>0

- **Positive potential (evades Maldacena-Nunez theorem)**
- **Chiral**
- **No maximally symmetric solution in 6D**
- **Maximally symmetric in 4D**
- **Maximally symmetric smooth solution: Minkowski x S^2, N=1 SUSY.**

General 4D Solutions

Gibbons et al 2004
Burgess et al 2005

$$ds^2 = \hat{g}_{MN}\, dx^M\, dx^N = W^2(y)\, g_{\mu\nu}(x)\, dx^\mu\, dx^\nu + \tilde{g}_{ij}(y)\, dy^i dy^j$$

$$\hat{g}_{\mu\nu} = W^2 g_{\mu\nu}\,, \qquad \hat{R}_{\mu\nu} = R_{\mu\nu} + \frac{1}{n}(W^{2-n}\tilde{\nabla}^2 W^n)\, g_{\mu\nu} \quad \text{and} \quad \hat{\Box}\varphi = W^{-n}\tilde{\nabla}_i(W^n \tilde{g}^{ij}\partial_j\varphi)\,,$$

$$\frac{1}{n}\int_M d^d y\, \sqrt{\tilde{g}}\, W^{n-2}\, R = -\sum_\alpha \int_{\Sigma_\alpha} d^{d-1}y\, \sqrt{\tilde{g}}\, N_i \left[W^n \tilde{g}^{ij}\partial_j \left(\ln W + \frac{2\,\varphi}{D-2}\right)\right]$$

No singularities/boundaries imply $R=H^2=0$

Asymptotic Near Brane solutions

Burgess et al 2005

$$\varphi \approx q \ln r \quad \text{and} \quad \mathrm{d}s^2 \approx r^{2w}\, g_{\mu\nu}(x)\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu + \mathrm{d}r^2 + r^{2\alpha} f_{ab}(z)\, \mathrm{d}z^a \mathrm{d}z^b\,,$$

$$W(y) = r^w \qquad \text{and} \qquad \tilde{g}_{ij}\mathrm{d}y^i\mathrm{d}y^j = \mathrm{d}r^2 + r^{2\alpha} f_{ab}\mathrm{d}z^a\mathrm{d}z^b,$$

$$nw + \alpha(d-1) = 1. \qquad nw^2 + \alpha^2(d-1) + q^2 = 1.$$

Kasner constraints (BKL: Belinsky et al)

n=4, d=2

$$-\frac{1}{\sqrt{n}} \le w \le \frac{1}{\sqrt{n}}\,, \quad -\frac{1}{\sqrt{d-1}} \le \alpha \le \frac{1}{\sqrt{d-1}} \quad \text{and} \quad -1 \le q \le 1\,.$$

Flat Solutions

Gibbons et al.

$$ds^2 = \hat{g}_{MN}\, dx^M dx^N = W^2 q_{\mu\nu}\, dx^\mu dx^\nu + a^2 d\theta^2 + a^2 W^8 d\eta^2,$$

$$\begin{aligned}
e^{\varphi} &= W^{-2} e^{-\lambda_3 \eta} \\
W^4 &= \left(\frac{Q\lambda_2}{4g\lambda_1}\right) \frac{\cosh[\lambda_1(\eta - \eta_1)]}{\cosh[\lambda_2(\eta - \eta_2)]} \\
a^{-4} &= \left(\frac{gQ^3}{\lambda_1^3 \lambda_2}\right) e^{-2\lambda_3 \eta} \cosh^3[\lambda_1(\eta - \eta_1)] \cosh[\lambda_2(\eta - \eta_2)] \\
F &= \left(\frac{Qa^2}{W^2}\right) e^{-\lambda_3 \eta}\, d\eta \wedge d\theta\,.
\end{aligned}$$

Numerical de Sitter solution

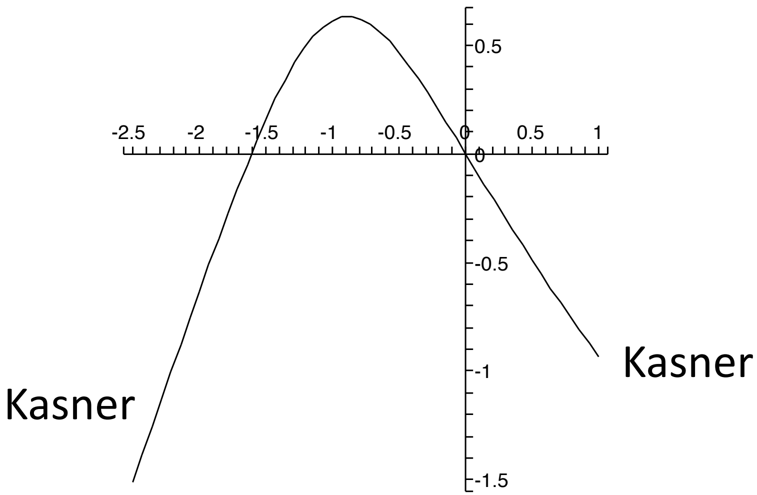
$$X'' + e^{2X} = 0$$

$$Y'' + e^{2Y} - \epsilon e^{2Y+Z} = 0$$

$$Z'' + \frac{\epsilon}{2} e^{2Y+Z} = 0\,,$$

$$e^{-X} = \lambda_1^{-1} \cosh[\lambda_1(\eta - \eta_1)].$$

X,Y,Z linear combinations of log W, log a, φ



Burgess et al 2005

6D Supergravity from F-theory

Grimm et al 2013

11D M-theory to 5D on elliptically fibred CY_3 and uplift to D=6

h_{12} +1 hypermultiplets, h_{11}-1 tensor multiplets

$$S^{(6)} = \int_{\mathcal{M}_6} \Big[\frac{1}{2}\hat{R}\hat{*}1 - \frac{1}{4}\hat{g}_{\alpha\beta}\hat{G}^{\alpha} \wedge \hat{*}\hat{G}^{\beta} - \frac{1}{2}\hat{g}_{\alpha\beta} d\hat{j}^{\alpha} \wedge \hat{*} d\hat{j}^{\beta} - \frac{1}{2}\hat{h}_{UV}\hat{D}\hat{q}^{U} \wedge \hat{*}\hat{D}\hat{q}^{V}$$
$$- 2\Omega_{\alpha\beta}\hat{j}^{\alpha} b^{\beta} C_{IJ}\hat{F}^{I} \wedge \hat{*}\hat{F}^{J} - \Omega_{\alpha\beta} b^{\alpha} C_{IJ}\hat{B}^{\beta} \wedge \hat{F}^{I} \wedge \hat{F}^{J} - \hat{V}^{(6)}\hat{*}\hat{1} \Big],$$

6D potential from D7 fluxes

$$\hat{V}^{(6)}_{\text{flux}} = \frac{1}{32\Omega_{\alpha\beta}\hat{j}^{\alpha} b^{\beta} \hat{\mathcal{V}}^2} C^{-1ij}\theta_i\theta_j \, .$$

From 6D to 4D

Field equations

$$\varphi'' = \tilde{V} e^{\varphi - 2\chi + 2\Omega + 8\Gamma} - 2C\Delta'^2 e^{-\varphi - 2\Omega + 2\Delta} \,,$$

$$\chi'' = -\frac{k^2}{4} e^{-2\chi + 8\Gamma + 2\Delta} - 4\tilde{V} e^{\varphi - 2\chi + 2\Omega + 8\Gamma} \,,$$

$$\Gamma'' = 3H^2 e^{2\Omega + 6\Gamma} - \frac{1}{2}\varphi'' \,,$$

$$\Omega'' = -4C \left(\Delta'\right)^2 e^{-\varphi - 2\Omega + 2\Delta} - \frac{1}{8} k^2 e^{-2\chi + 8\Gamma + 2\Delta} - \frac{1}{2}\varphi'' \,,$$

$$\Delta'' = \Delta'\varphi' + 2\Omega'\Delta' - \left(\Delta'\right)^2 + \frac{k^2}{32C} e^{\varphi - 2\chi + 2\Omega + 8\Gamma} \,.$$

Constraint

$$6H^2 e^{2\Omega + 6\Gamma} - 4\Omega'\Gamma' - 6\Gamma'^2 + \frac{1}{2}\varphi'^2 + \frac{1}{4}\chi'^2 + 2C e^{-\varphi - 2\Omega + 2\Delta}\Delta'^2 - \tilde{V} e^{\varphi - 2\chi + 2\Omega + 8\Gamma} - \frac{k^2}{16} e^{-2\chi + 8\Gamma + 2\Delta} = 0$$

χ =log volume, Γ =log W, Ω = log , Δ = log A

$H^2>0$ de Sitter

Asymptotic Solutions

Near brane solutions:

$$\varphi = q \ln r \,, \quad \chi = s \ln r \,, \quad ds^2 = r^{2w} g_{\mu\nu} dx^\mu dx^\nu + dr^2 + r^{2\alpha} f(z) dz dz \,,$$

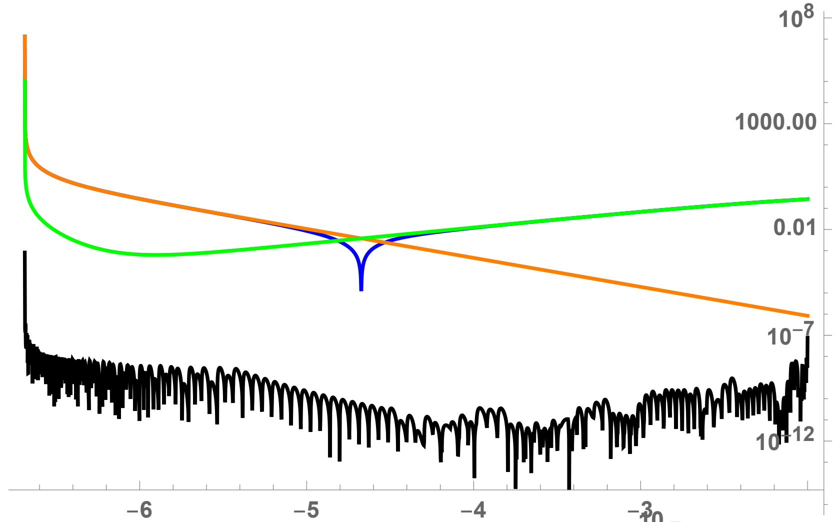
$$F^{ra} \sim r^\gamma \,.$$

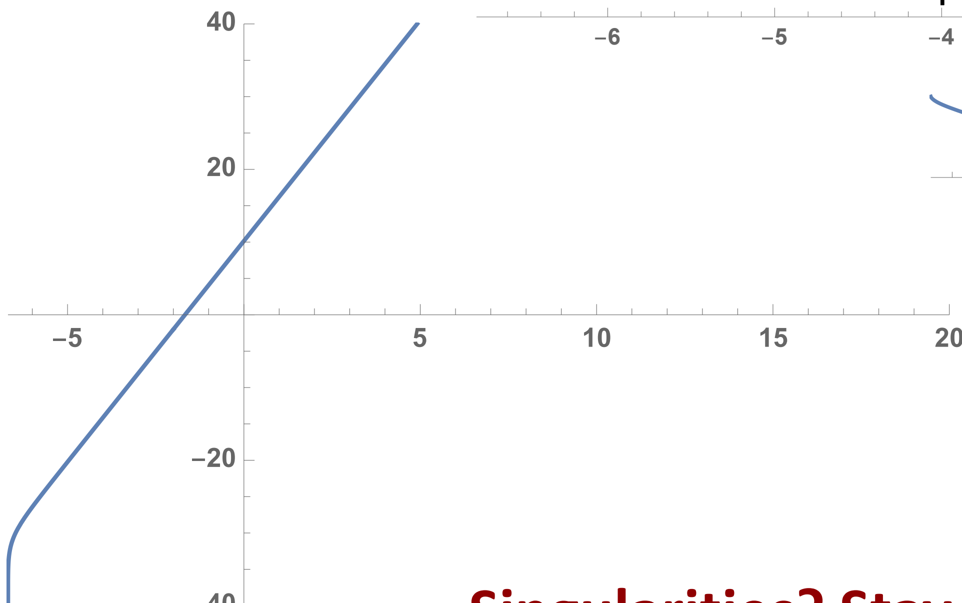
$$4w + \alpha = 1 \qquad\qquad 4w^2 + \alpha^2 + q^2 + \frac{1}{2} s^2 = 1 \,.$$

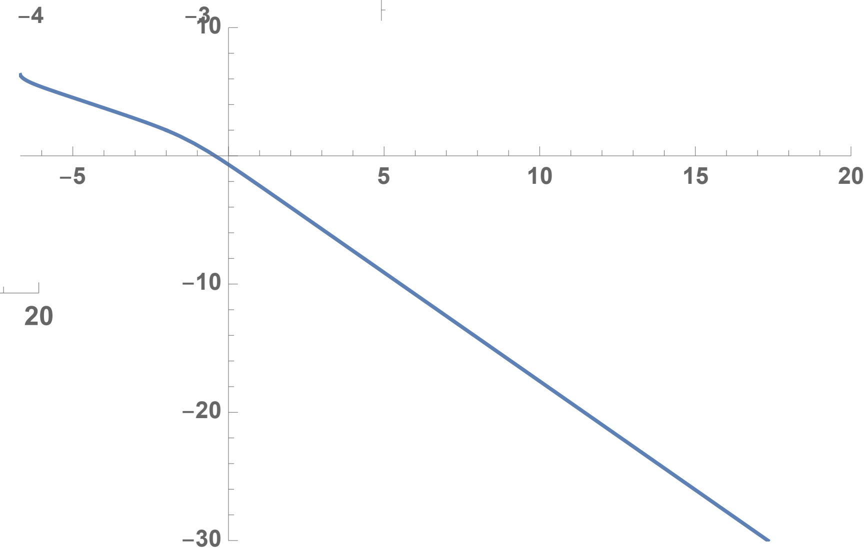
Kasner constraints (BKL: Belinsky et al)

$$-\frac{1}{2} \le w \le \frac{1}{2} \,, \quad -1 \le \alpha \le 1 \,, \quad -1 \le q \le 1 \,, \quad -\sqrt{2} \le s \le \sqrt{2} \,.$$

Numerical dS Solutions







Singularities? Stay tuned…

THANK YOU !

General 6D Equations from F-theory

Grimm et al 2013

$$
\begin{aligned}
\hat{R}_{MN} &= +\frac{1}{4}\hat{g}_{\alpha\beta}\hat{G}^{\alpha}{}_{M}{}^{RS}\hat{G}^{\beta}{}_{NRS} - \frac{1}{24}\hat{g}_{\alpha\beta}\hat{G}^{\alpha RST}\hat{G}^{\beta}{}_{RST}\hat{g}_{MN} \\
&\quad + 4\Omega_{\alpha\beta}\hat{j}^{\alpha}b^{\beta}C_{IJ}\hat{F}^{I}{}_{M}{}^{R}\hat{F}^{J}{}_{NR} - \frac{1}{2}\Omega_{\alpha\beta}\hat{j}^{\alpha}b^{\beta}C_{IJ}\hat{F}^{IRS}\hat{F}^{J}{}_{RS}\hat{g}_{MN} \\
&\quad + \hat{g}_{\alpha\beta}\partial_M\hat{j}^{\alpha}\partial_N\hat{j}^{\beta} + \hat{h}_{UV}\hat{D}_M\hat{q}^U\hat{D}_N\hat{q}^V + \frac{1}{2}\hat{V}_{(6)}\hat{g}_{MN}\,,
\end{aligned}
$$

$$
d(\hat{h}_{UV}\hat{*}\hat{D}\hat{q}^V) = \frac{1}{2}\partial_U\hat{h}_{VW}\hat{D}\hat{q}^V\wedge\hat{*}\hat{D}\hat{q}^W + \hat{h}_{VW}\partial_U\hat{k}^V_I\hat{A}^I\wedge\hat{*}\hat{D}\hat{q}^W + \partial_U\hat{V}_{(6)}\hat{*}1\,,
$$

$$
d(\Omega^{\alpha\beta}\hat{g}_{\beta\gamma}\hat{*}d\hat{j}^{\gamma}) = \hat{j}_{\beta}\hat{G}^{\alpha}\wedge\hat{*}G^{\beta} + 2\hat{j}_{\beta}d\hat{j}^{\alpha}\wedge\hat{*}d\hat{j}^{\beta} + 2b^{\alpha}C_{IJ}\hat{F}^I\wedge\hat{*}\hat{F}^J - \frac{1}{\Omega_{\beta\gamma}\hat{j}^{\beta}b^{\gamma}}b^{\alpha}\hat{V}_{(6)}\hat{*}1\,,
$$

$$
\begin{aligned}
\hat{D}(4\Omega_{\alpha\beta}\hat{j}^{\alpha}b^{\beta}\hat{*}\hat{F}^I) &= -\hat{h}_{UV}C^{-1IJ}\hat{k}^U_J\hat{*}\hat{D}\hat{q}^V - 4b^{\alpha}\hat{g}_{\alpha\beta}\hat{F}^I\wedge\hat{*}\hat{G}^{\beta} \\
&\quad - 2\Omega_{\alpha\beta}b^{\alpha}b^{\beta}C_{JK}\hat{A}^I\wedge\hat{F}^J\wedge\hat{F}^K + 4\Omega_{\alpha\beta}b^{\alpha}b^{\beta}C_{JK}\hat{F}^I\wedge\hat{\omega}^{cs}\,,
\end{aligned}
$$

$$
d(\Omega^{\alpha\beta}\hat{g}_{\beta\gamma}\hat{*}\hat{G}^{\gamma}) = 2b^{\alpha}C_{IJ}\hat{F}^I\wedge\hat{F}^J\,, \qquad \hat{g}_{\alpha\beta}\hat{*}\hat{G}^{\beta} = \Omega_{\alpha\beta}\hat{G}^{\beta}\,.
$$