Late-time Attractors and Cosmic Acceleration

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Based on work with:

Flavio Tonioni
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- G. Shiu, F. Tonioni, H.V. Tran, ``Accelerating universe at the end of time,'' [arXiv:2303.03418]
- G. Shiu, F. Tonioni, H.V. Tran, ``Late-time attractors and cosmic acceleration,” [arXiv:2306.07327]

[See also Flavio Tonioni’s parallel talk on Thursday]

A plea to the theorists

Nobel Prize 2011

Photo: Roy Kaltschmidt. Courtesy: Lawrence Berkeley National Laboratory

Saul Perlmutter

Photo: Belinda Pratten, Australian National University

Brian P. Schmidt

Photo: Homewood Photography

Adam G. Riess

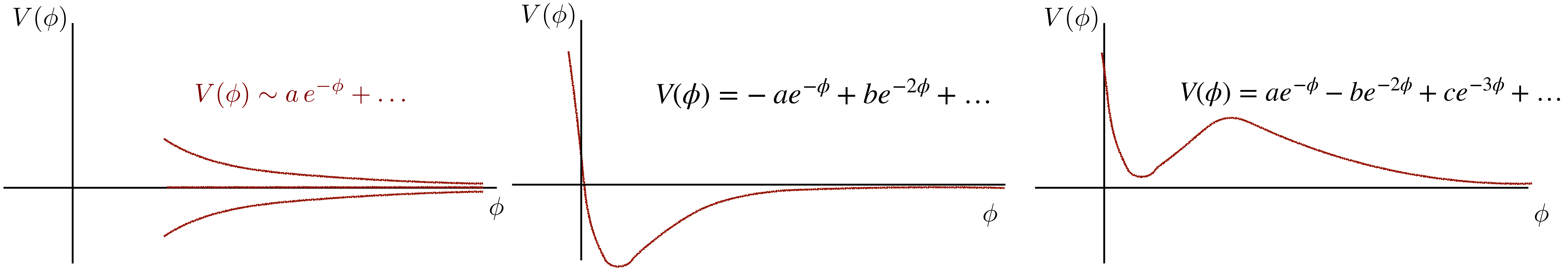
But Riess suspects that the mystery can't be solved by observations alone. "We won't really resolve it until some brilliant person, the next Einstein-like person, is able to get the idea of what's going on," he said.

So he issued **a plea to the theorists**: "Keep working," he said. "We need your help. ... It's a very juicy problem, it's hard, and **you'll win a Nobel Prize if you figure it out. In fact, I'll give you mine.**"

Oct 4, 2011 https://www.nbcnews.com/science/cosmic-log/physics-prize-highlights-puzzles-flna6c10402772

Dark Energy in String Theory

- Simplest possibility is $\Lambda > 0$. Sophisticated string theory scenarios for realizing dS vacua have been developed (KKLT, LVS, …), but a fully explicit construction remains elusive.
- Root of the challenge: source of cosmic acceleration should be ***derived*** (not just postulated) in a UV complete theory of gravity.
- It is a formidable task to demonstrate that the microphysics which stabilizes all moduli would lead to a theoretically controlled metastable de Sitter vacuum.
- The Dine-Seiberg problem highlights the difficulty in finding parametrically weakly-coupled vacua.



Asymptotic runaway potentials

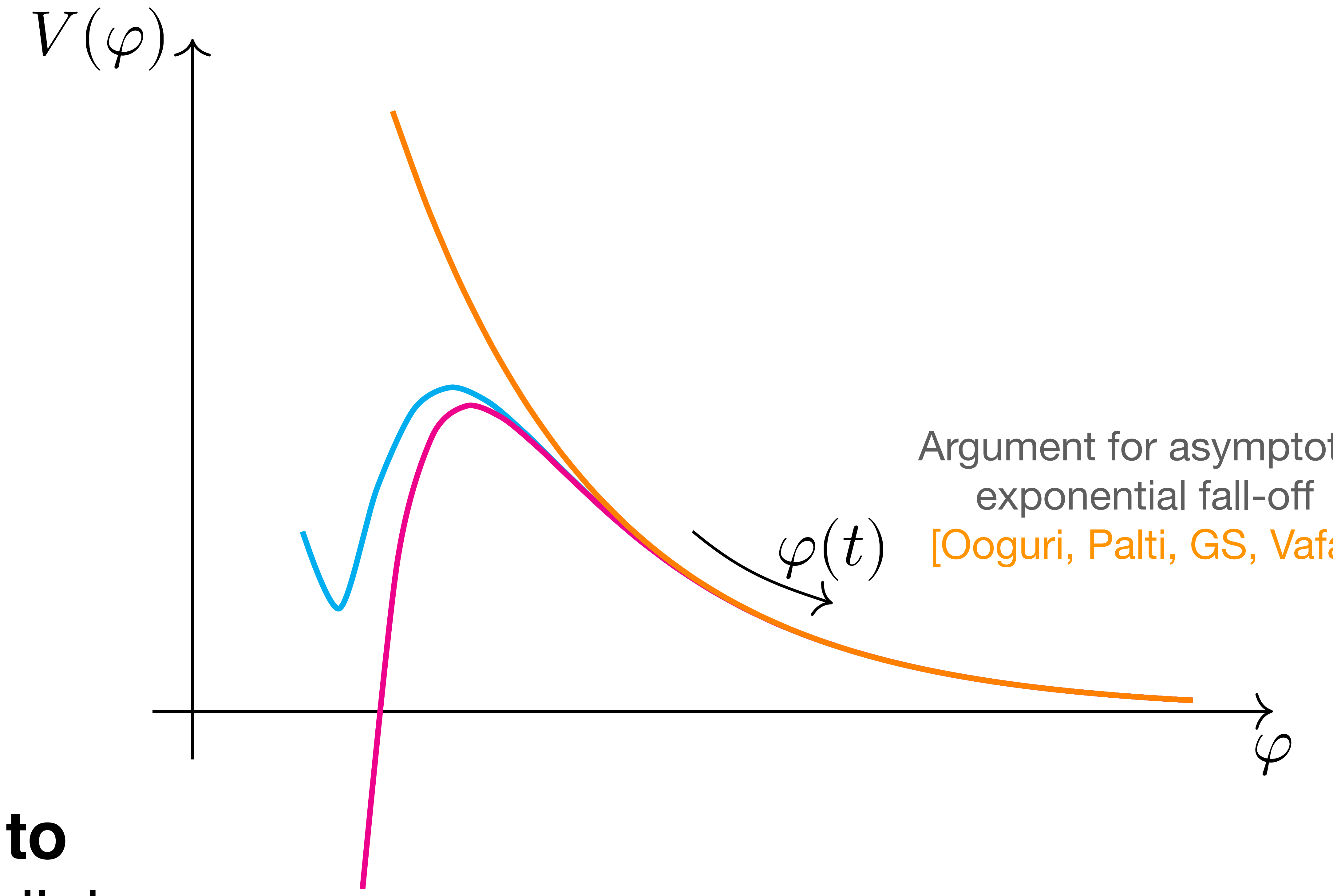
This makes runaway to the boundary of field space an interesting possibility.
[Obied, Ooguri, Spodyneiko, Vafa];[Ooguri, Palti, GS, Vafa]

Cosmic acceleration can be realized with:

- a de Sitter critical point, or
- a runaway potential with $\epsilon \equiv -\frac{\dot{H}}{H^2} < 1$

Related to the "deceleration parameter" q:

$q \equiv -\ddot{a}/\dot{a}^2$ $\qquad$ $\epsilon = 1 + q.$



Argument for asymptotic exponential fall-off
[Ooguri, Palti, GS, Vafa]

Criterion for acceleration is in general unrelated to potential gradient. An aim or our work is to find the link (& the conditions for the link to exist) [GS, Tonioni, Tran]

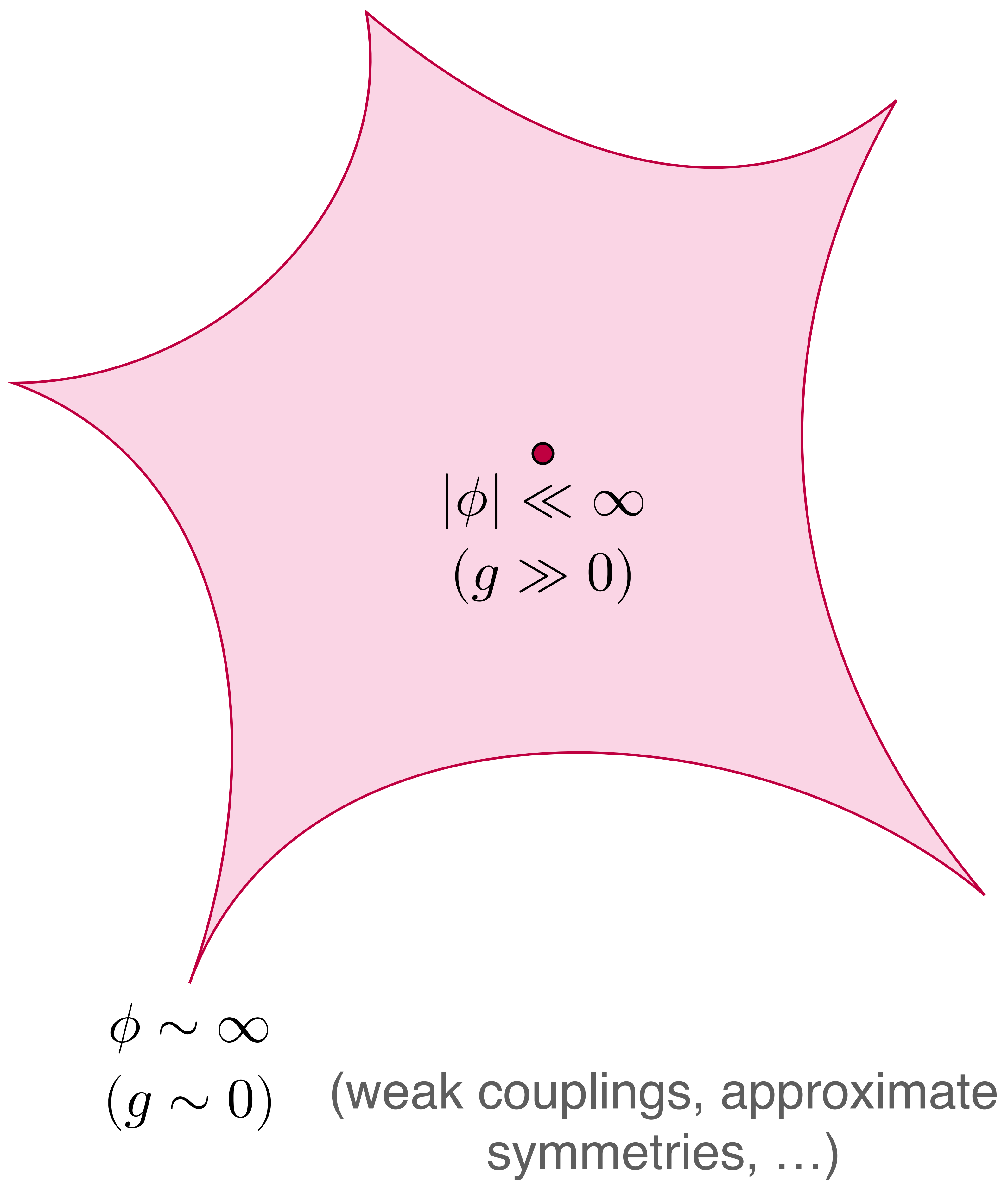
Is our current universe in the asymptotic region of the landscape?

- The small numbers & approximate symmetries observed in nature are consistent with the current universe approaching an asymptotic region where couplings are weak, global symmetries are restored, and $V \to 0$.

- This possibility has recently been explored in various forms [Montero, Vafa, Valenzuela];[Rudelius]; [Calderon-Infante, Ruiz, Valenzuela];[GS, Tonioni, Tran], [Cremonini, Gonzalo, Rajaguru, Tang, Wrase], …

- As in many dynamical systems, the late-time regime exhibits some universal behaviors. This allows us to prove bounds on acceleration [GS, Tonioni, Tran, '23]

- Like large N expansion for QCD, studying the asymptotically late-time behavior may teach us about our current (old) universe [a la Dirac].



Summary of Results

[GS, Tonioni, Tran, ’23]

- We bound the rate of time variation of the Hubble parameter at late time [GS, Tonioni, Tran, ’23, STT1] and in the recent work, we further turn this into a bound on $\gamma \equiv |\nabla V|/V$ [GS, Tonioni, Tran, ’23, STT2].

- The proper diagnostic for cosmic acceleration should be stated in terms of ϵ rather than potential gradient commonly used in Swampland criteria.

- Our bound when applied to string theoretic constructions imposes a generic obstacle to acceleration if the dilation is one of the rolling fields. We also suggest several ways out.

- We prove the conditions under which scaling solutions are **late-time attractors**. Moreover, we prove that scaling solutions **saturate** our bound on ϵ.

- For scaling solutions: 1) we can express $\epsilon = -\dot{H}/H^2$ in terms of a directional derivative γ_* of the potential, w/o assuming that a single potential term dominates or whether the kinetic or potential term dominates; 2) $\gamma = \gamma_* = 2\sqrt{\epsilon/(d-2)}$. But in general, γ and γ_* are unrelated to acceleration.

- Our results go beyond previous no-goes as we allow for quantum effects and we encompass vacua and rolling solutions (irrespective of whether the kinetic term is negligible or not).

Asymptotic late-time cosmologies

Multi-exponential potentials

- In the asymptotic region, the non-compact scalars when canonically normalized to ϕ^a, $a = 1,\ldots,n$ have a potential that takes the form (also argument by [Ooguri, Palti, GS, Vafa]):

$$V = \sum_{i=1}^{m} \Lambda_i \, \mathrm{e}^{-\kappa_d \gamma_{ia} \phi^a} .$$

where Λ_i, γ_{ia} depend on the microscopic origin of V_i, κ_d = d-dim. gravitational coupling. The sources of potential include e.g. internal curvature, fluxes, branes/O-planes, Casimir-energy.

- The set of scalars ϕ^a includes minimally the d-dimensional dilaton $\tilde{\delta}$ and a radion $\tilde{\sigma}$ that controls the string-frame volume unless these fields are stabilized at high energy scales.

- The field space metric is not always flat, e.g., the axio-dilaton and the Kahler modulus have constant curvature. Obstruction to canonical normalization?

- Asymptotically, saxion-axion mixings & axion kinetic terms are exponentially suppressed in the EOMs. Consistent to stabilize the axions and work with canonically normalized saxions.

- More involved arguments for asymptotic limit of complex structure moduli [Grimm, Li, Valenzuela]; [Calderon-Infante, Ruiz, Valenzuela].

Cosmological Equations

- Non-compact d-dim. spacetime is characterized by the FLRW metric:

$$d\tilde{s}_d^2 = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}l^2_{\mathbb{R}^{d-1}}\,,$$

- Hubble parameter: $H \equiv \frac{\dot{a}}{a}$. The proper diagnostic for cosmic **acceleration** is $\epsilon \equiv -\frac{\dot{H}}{H^2} < 1$ to be **distinguished from the slow-roll parameter** $\epsilon_V = \frac{d-2}{4}\left(\frac{\nabla V}{V}\right)^2$.

- Scalar field equations and Friedmann equations:

$$\ddot{\phi}^a + (d-1)H\dot{\phi}^a + \frac{\partial V}{\partial \phi_a} = 0,$$

$$\frac{(d-1)(d-2)}{2}H^2 - \kappa_d^2\left[\frac{1}{2}\dot{\phi}_a\dot{\phi}^a + V\right] = 0,$$

$$\dot{H} = -\frac{\kappa_d^2}{d-2}\left[\frac{1}{2}\dot{\phi}_a\dot{\phi}^a - V\right] - \frac{d-1}{2}H^2,$$

Cosmological Autonomous System

- It is convenient to work with the rescaled variables:

$$x^a = \frac{\kappa_d}{\sqrt{d-1}\sqrt{d-2}} \frac{\dot{\phi}^a}{H}, \; y_i = \frac{\kappa_d \sqrt{2}}{\sqrt{d-1}\sqrt{d-2}} \frac{\sqrt{V_i}}{H}$$

- The cosmological equations can be formulated in terms of an autonomous system of ODEs given schematically as follows:

$$\frac{d\vec{z}}{dt} = g(\vec{z}) \;, \qquad \text{where } \vec{z} \equiv (x^1, \ldots, x^n, y^1, \ldots, y^m, H)$$

- Among the above ODEs is $\epsilon = -\dot{H}/H^2 = (d-1)x^2$; strategy is to bound the kinetic energy.

- Friedmann equation also takes a simple form:

$$(x)^2 + \left(y\right)^2 = 1$$

Bound on Late-time Cosmic Acceleration

- An accelerating universe can only be achieved if the total scalar potential is positive; we therefore focus on scenarios in which $V > 0$ at least asymptotically.

- Individual potential terms can be positive or negative: our proof covers general cases but for clarity, let us first show how we bound the case when $\Lambda_i > 0$ [General case in STT1].

- Rank order the exponents:

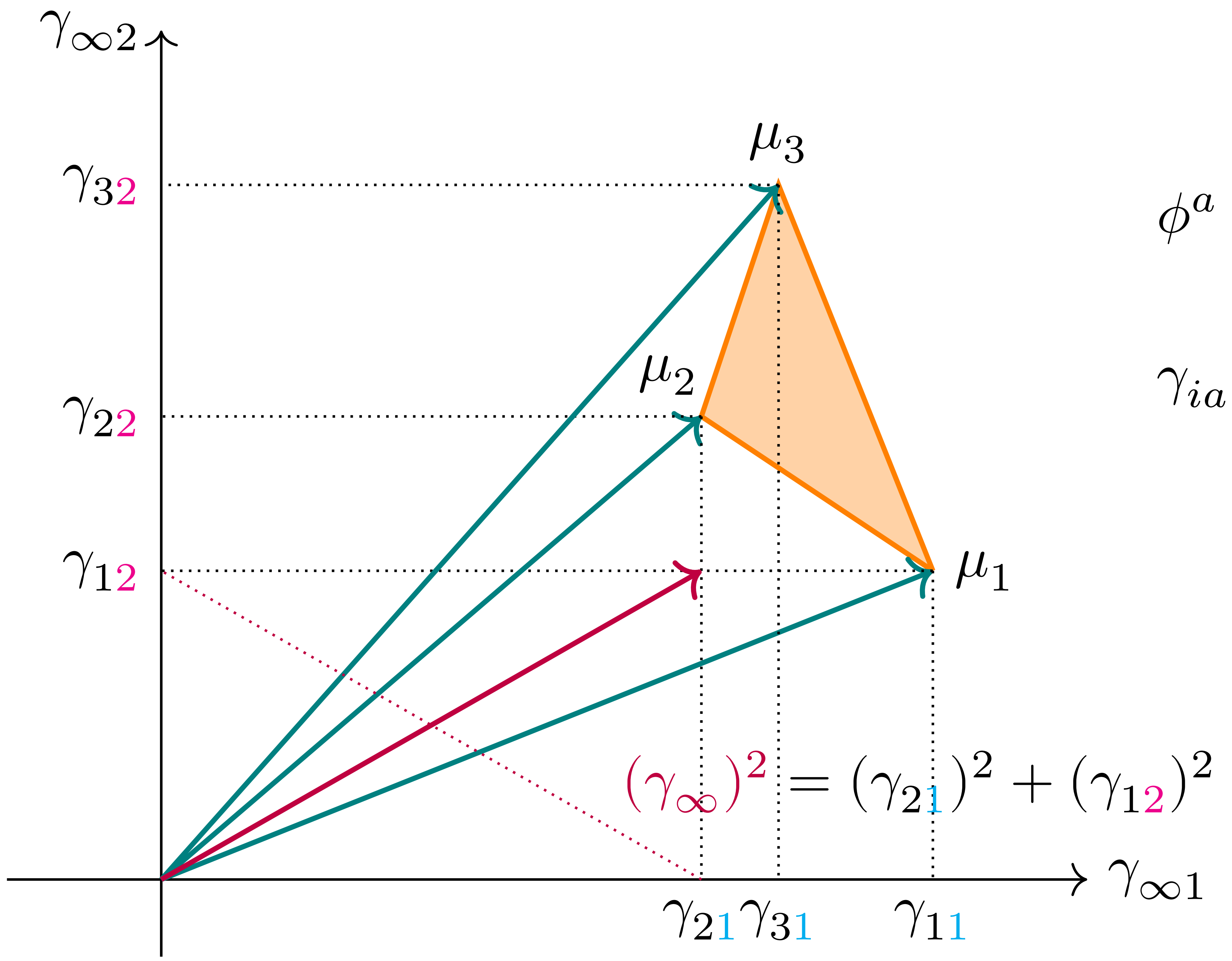
$$\gamma_\infty^a = \begin{cases} \gamma^a, & \gamma^a = \min_i {\gamma_i}^a > 0 \\ 0, & \gamma^a \leq 0 \end{cases}$$

- Then we derived analytically a late-time acceleration bound:

$$d - 1 \geq \epsilon \geq \frac{d-2}{4} (\gamma_\infty)^2$$

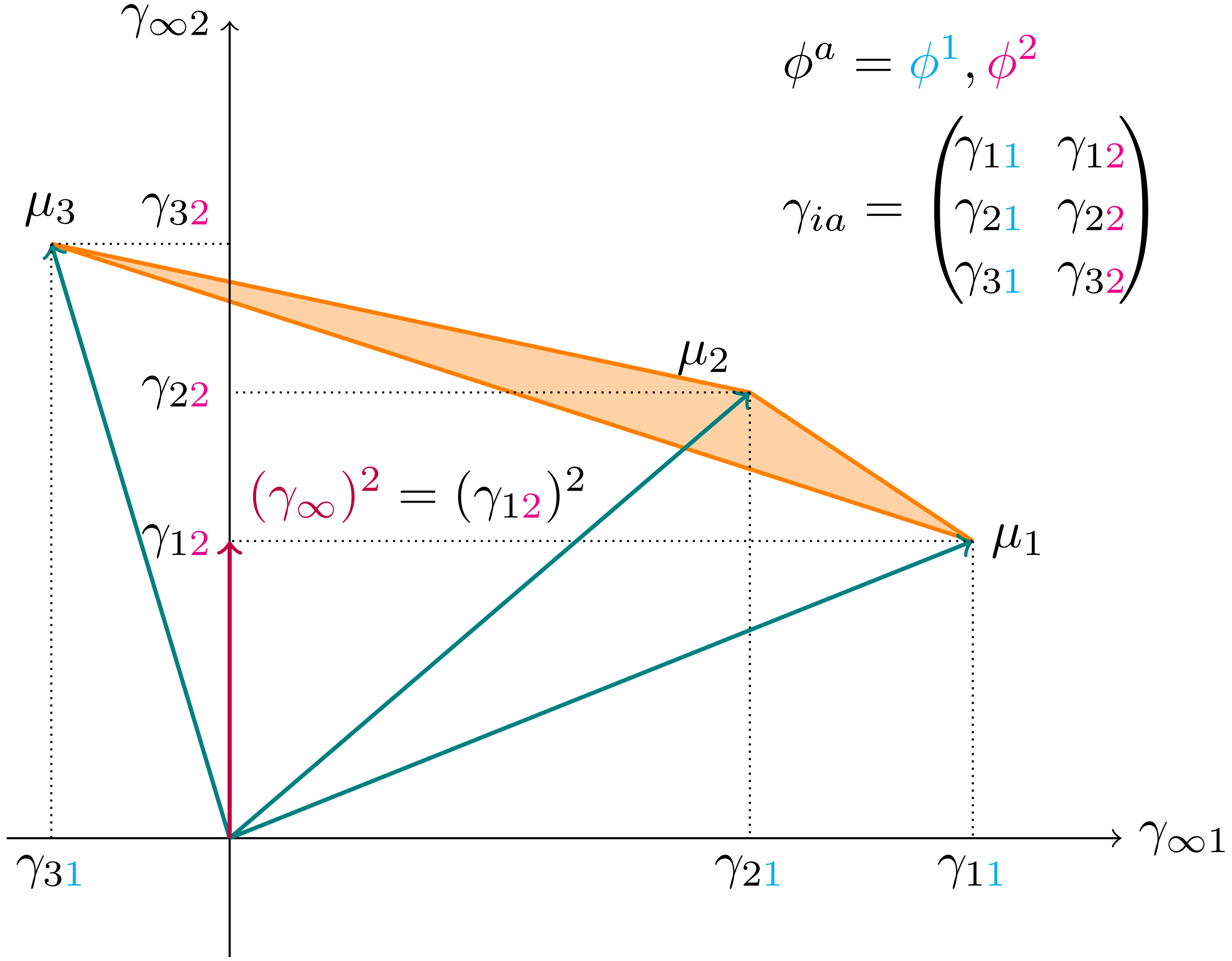
Visualizing the Acceleration Bound

- Define vectors m vectors μ_i, one for each potential term with components $(\mu_i)_a = \gamma_{ia}$



$$\phi^a = \phi^1, \phi^2$$

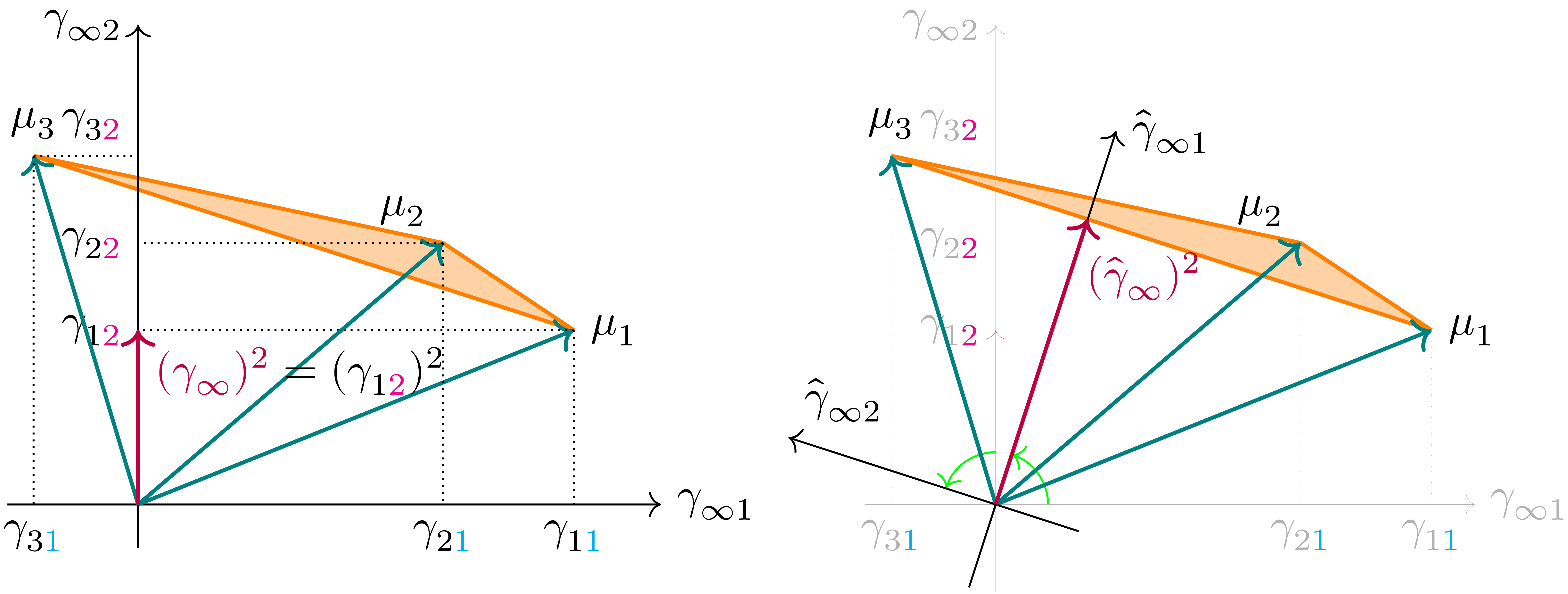
$$\gamma_{ia} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \end{pmatrix}$$



Optimizing the Acceleration Bound

- The bound in [GS, Tonioni, Tran, '23] as stated is naively basis-dependent, but it is clear that we can find an **optimal bound** by an $O(n)$ rotation:

$$\epsilon \geq \frac{d-2}{4} \max_{\mathrm{R}\in\mathrm{O}(n)} [\gamma_\infty(\mathrm{R})]^2$$



Dilaton Obstruction and General Remarks

- String-theoretical potentials take the form:

$$S = -\int_{X_{1,9}} [A_r \wedge \star_{1,9} A_r]\, \Lambda_{10,r}\, \mathrm{e}^{-k\sigma - \chi_E \Phi} = -\int_{X_{1,d-1}} \tilde{*}_{1,d-1} \Lambda\, \mathrm{e}^{\kappa_d [\gamma_{\tilde{\delta}}(\chi_E)\tilde{\delta} - \gamma_{\tilde{\sigma}}(\chi_E, r, k)\tilde{\sigma}]}$$

RR fields are not weighed by $e^{-\chi_E \Phi}$ but would not affect our argument.

- Universal couplings for the canonically normalized d-dimensional dilaton $\tilde{\delta}$:

$$\gamma_{\tilde{\delta}} = \frac{d}{\sqrt{d-2}} - \frac{1}{2}\chi_E \sqrt{d-2} \geq \frac{2}{\sqrt{d-2}} \quad \Rightarrow \quad \epsilon \geq \frac{d-2}{4}(\gamma_\infty)^2 \geq \frac{d-2}{4}\gamma_{\tilde{\delta}}^2 \geq 1$$

- Ways out: 1) $\tilde{\delta}$ is stabilized; 2) $\tilde{\delta}$ is rolling but not in the asymptotic regions; 3) V contains at least three terms, not all of the same sign (e.g., from loop corrections).

- Non-universal couplings for other moduli: can use our bound to **constrain compactifications**.

- **Non-negligible kinetic terms** unless $\gamma_\infty^2 \approx 0$. Hence slow-roll generally does not hold.

Scaling Solutions

Scaling Solutions

- The cosmological autonomous system admits **scaling solutions** (ϵ = constant > 0):
  - scale factor takes a power law form: $a(t) \sim t^p$
  - critical points of the autonomous system: $\dot{x}^a = 0$
- **Analytic solution:** for rank $\gamma_{ia} = m$
  - field space trajectory: $\phi_*^a(t) = \phi_0^a + \frac{2}{\kappa_d}\left[\sum_{i=1}^{m}\sum_{j=1}^{m}\gamma_i{}^a (M^{-1})^{ij}\right]\ln\frac{t}{t_0},$ $\qquad M_{ij} = \gamma_{ia}\gamma_j{}^a$
  - scale factor: $p = \frac{4}{d-2}\sum_{i=1}^{m}\sum_{j=1}^{m}(M^{-1})^{ij}.$
- The kinetic term & every potential term have the same parametric dependence in time:

No slow-roll: $T(t) = T(t_0)\left(\frac{t_0}{t}\right)^2, \quad V_i(t) = V_i(t_0)\left(\frac{t_0}{t}\right)^2$

Scaling Solutions: Relevance

- Late time scale factor is bounded by power-law behavior [GS, Tonioni, Tran, '23, STT1]:

$$d-1 \geq \epsilon \geq \frac{d-2}{4}(\gamma_\infty)^2 \quad \text{or } \epsilon = d-1$$

- Scaling solutions are **perturbative late-time attractors** (linear stability) [Copeland, Liddle, Wands]; [Bergshoeff, Collinucci, Gran, Nielsen, Roest];[Hartong, Ploegh, Van Riet, Westra]

- **New result** [GS, Tonioni, Tran, '23, STT2]: we can analytically prove that if
  - all potential terms are positive definite, i.e., $\Lambda_i > 0$, and
  - $\lambda^i = \sum_{j=1}^{m} (M^{-1})^{ij} \geq 0,$ subject to $\sum_{i=1}^{m} \lambda^i > 0.$ [no apparent subleading terms]

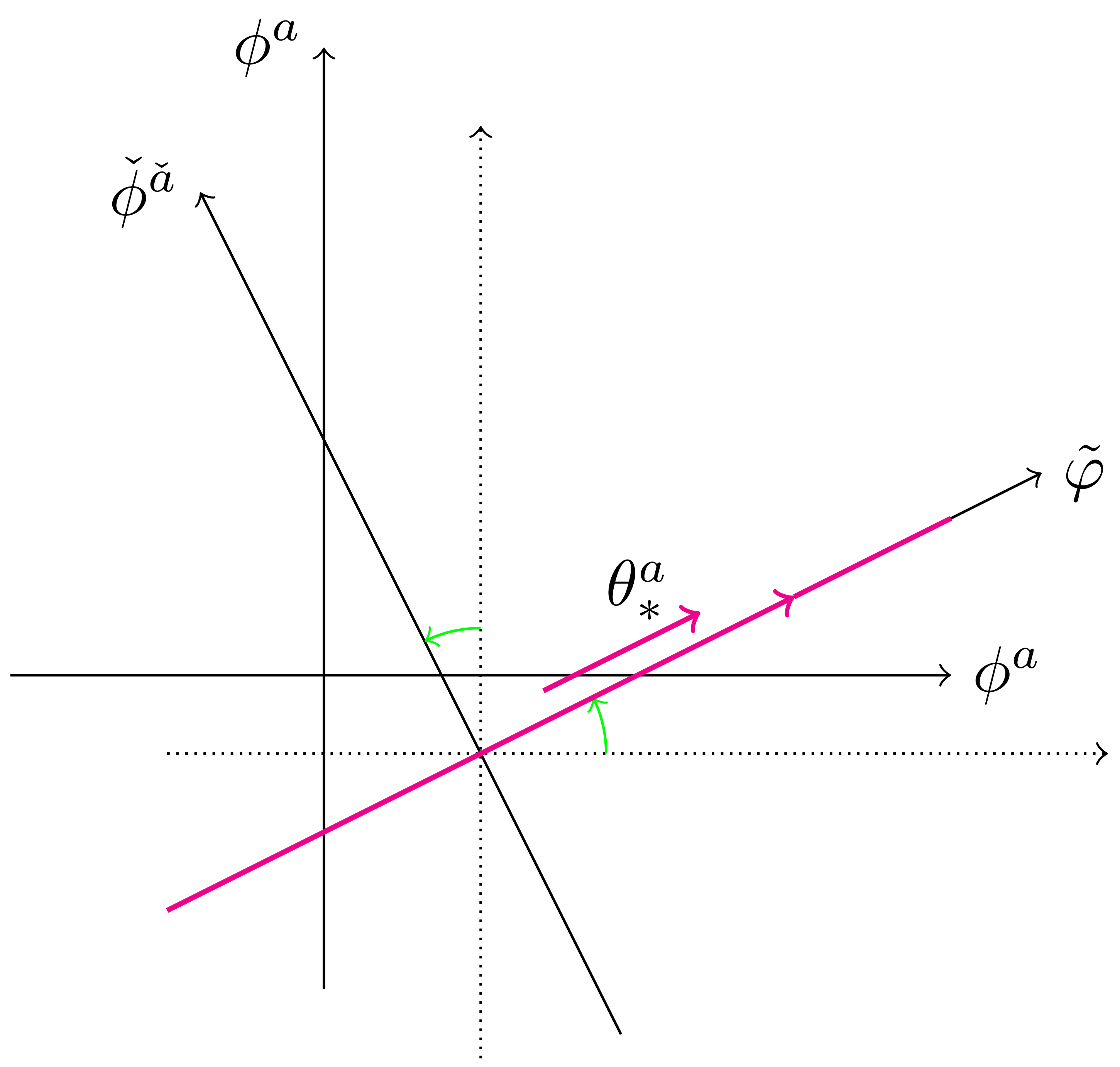
then scaling solutions are late-time attractors, irrespective of initial conditions!

Scaling Solutions: Trajectory

- Straight line in field space:

$$\phi_*^a(t) = \phi_\infty^a + \frac{1}{\kappa_d}\,\alpha^a\,\ln\frac{t}{t_\infty}$$

$$\theta_*^a = \frac{\alpha^a}{\sqrt{\alpha_b\alpha^b}}$$



- Field space rotation such that $\check{\phi}_*^{\check{a}}(t) = \check{\phi}_\infty^{\check{a}}$ $\quad \tilde{\varphi}_*(t) = \tilde{\varphi}_\infty + \frac{1}{\kappa_d}\frac{2}{\gamma_*}\ln\frac{t}{t_\infty}$

- Normalized directional derivative: $\gamma_* = -\left[\frac{1}{V(\phi_*)}\,\theta_*^a\,\frac{\partial V}{\kappa_d\,\partial\phi_*^a}(\phi_*)\right] = \frac{2}{\sqrt{d-2}}\sqrt{\epsilon}.$

Criteria for Cosmic Acceleration

Criteria for Cosmic Acceleration

[GS, Tonioni, Tran, '23]

- The proper criterion for acceleration is **time variation of Hubble**: $\epsilon = -\dot{H}/H^2 < 1$

- For scaling solutions, we found an exact relationship of ϵ with the **directional derivative**:

$$\gamma_* = -\left[\frac{1}{V(\phi_*)}\,\theta^a_*\,\frac{\partial V}{\kappa_d\,\partial\phi^a_*}(\phi_*)\right] = \frac{2}{\sqrt{d-2}}\,\sqrt{\epsilon}.$$

- **Potential gradient norm** is often used in Swampland studies:

$$\gamma = \frac{\sqrt{\delta^{ab}\,\partial_a V\,\partial_b V}}{\kappa_d V},$$

- Scaling solutions are special $\gamma = \gamma_* = 2\sqrt{\epsilon}/\sqrt{d-2}$ $\Rightarrow$ ϵ measures potential gradient.

General Case

[GS, Tonioni, Tran,'23]

- In general γ_* and γ are **unrelated to acceleration** though $\gamma \geq \gamma_*$ is general due to triangle inequality [Andriot, Horer, Tringas, '22].

- **Directional derivative**:

$$\gamma_\star = \frac{2\sqrt{\epsilon}}{\sqrt{d-2}}\left[1 - \frac{1}{2}\frac{\eta}{(d-1)-\epsilon}\right] \qquad \eta = -\dot{\epsilon}/(\epsilon H)$$

- Introduce a normal vector $\nu_\star^a = -\dot{\theta}_\star^a / \sqrt{\dot{\theta}_{\star b}\dot{\theta}_\star{}^b}$ and define a **non-geodesity factor Ω**:

$$\dot{\theta}_\star^a = -\Omega\, \nu_\star^a. \qquad \text{rate of turning of field space trajectory}$$

- **Potential gradient norm:**

$$\gamma^2 = \gamma_\star^2 + \frac{4\epsilon}{d-2}\frac{1}{[(d-1)-\epsilon]^2}\frac{\Omega^2}{H^2}.$$

[Achucarro, Palma, '18]

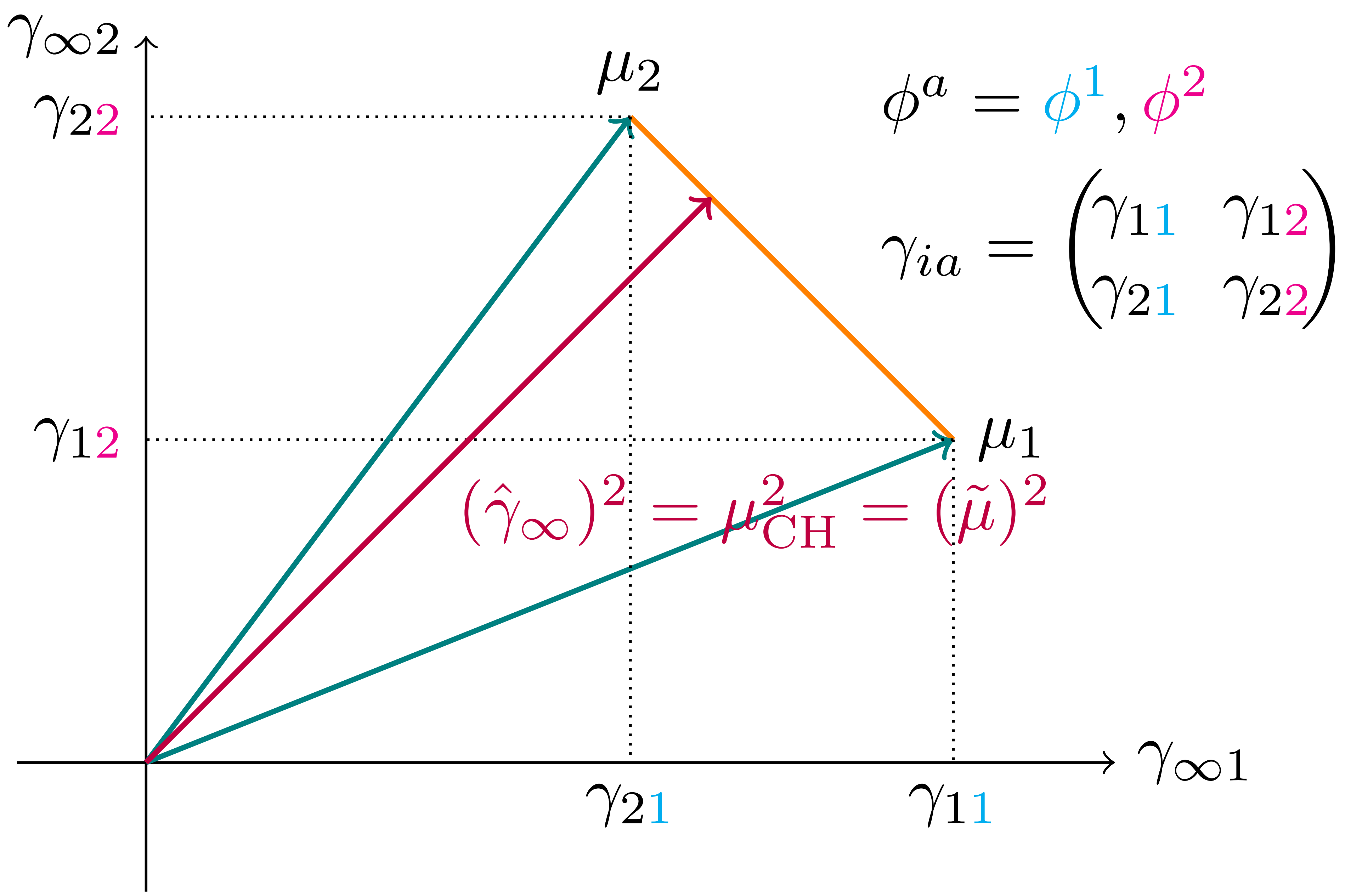
Convex Hull

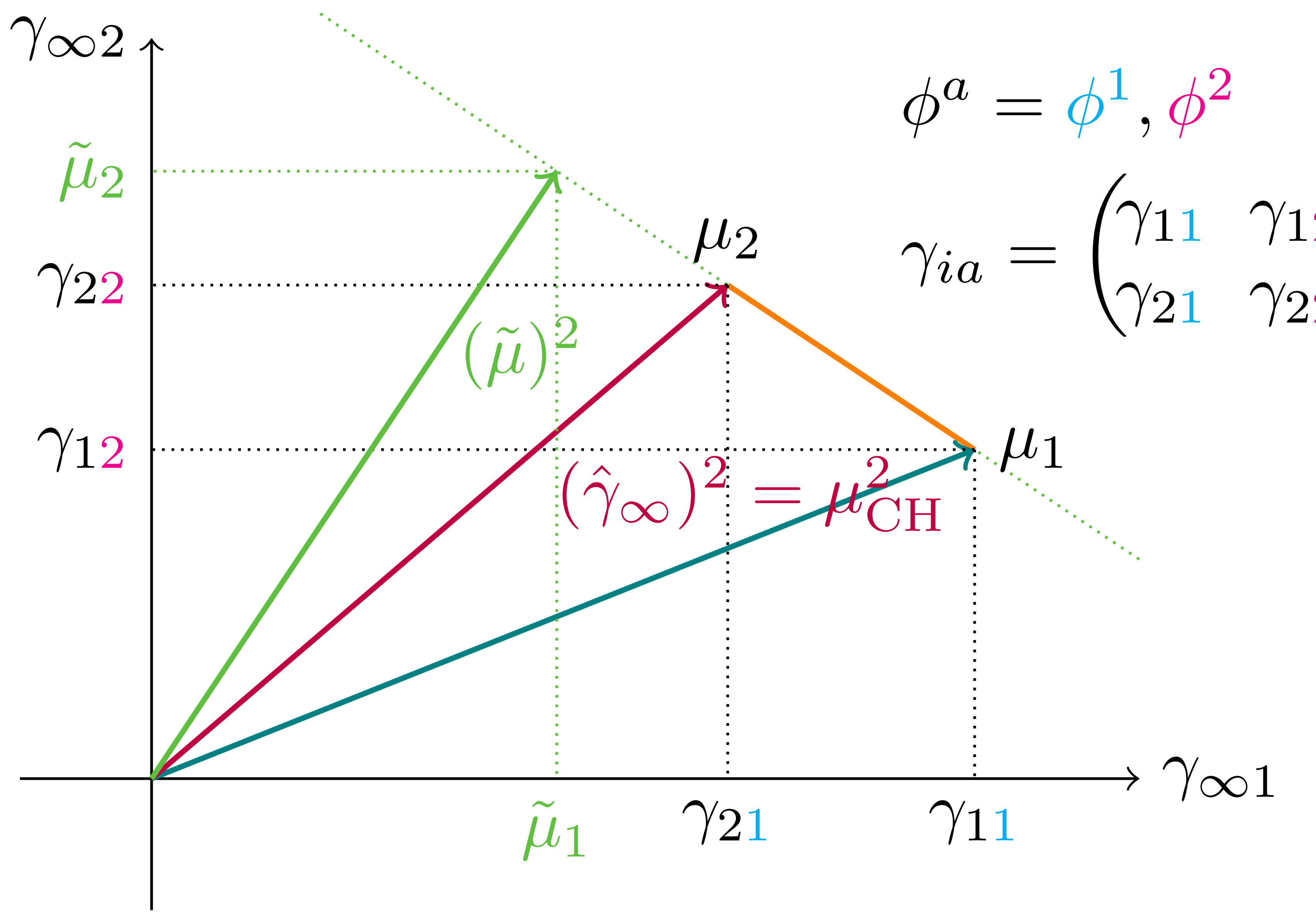
[GS, Tonioni, Tran, '23]

- Given the set of vectors $(\mu_i)_a = \gamma_{ia}$, define the convex hull of exponential couplings

$$\mathrm{CH}(\{\mu_i\}_{i=1}^m) = \left\{\nu_a = \sum_{i=1}^m \xi_i (\mu_i)_a : \ (\xi_i)_{i=1}^m \in (\mathbb{R}_0^+)^m, \ \sum_{i=1}^m \xi_i = 1\right\}$$

- Similar to how it arises in multi-field generalization of the WGC and the distance conjecture.
- Two notion of distances: CH distance $\mu_{\mathrm{CH}} = \inf_{\nu \in \mathrm{CH}} \sqrt{\nu_a \nu^a}$ and distance of the CH hyperplane: $\tilde{\mu}$





Convex Hull Criterion

[GS, Tonioni, Tran, '23]

- If all potential terms are positive, our optimal bound on ϵ coincides with the CH distance:

$$\epsilon \geq \frac{d-2}{4} (\hat{\gamma}_\infty)^2 = \frac{d-2}{4} \mu_{\rm CH}^2 .$$

- Note that for potentials involving terms of both signs (but overall potential is positive), our optimal bound still holds, but the **CH distance may be an overestimation:**

$$\mu_{\rm CH}^2 \geq (\hat{\gamma}_\infty)^2 \geq (\tilde{\mu})^2$$

- What is the significance of $\tilde{\mu}$= the distance of the CH hyperplane from the origin?
  - $\tilde{\mu}$ appears in a new bound on ϵ we found in our paper 2!
  - if $\mu_{CH} = \tilde{\mu}$, we show that the above bound (in paper 1) is saturated by scaling solutions!

Saturation by Scaling Cosmologies

[GS, Tonioni, Tran, ’23]

- We show that $\tilde{\mu}$ gives the ϵ-parameter for scaling solutions (which are late-time attractors)!

$$(\tilde{\mu})^2 = \frac{1}{\sum_{i=1}^{m}\sum_{j=1}^{m} \delta_{ab}\,(\gamma^{-1})^{ai}(\gamma^{-1})^{bj}}$$

- Our (first) ϵ-bound is saturated by scaling cosmology whenever the hyperplane distance vector intersects with the CH,

$$\epsilon = \frac{d-2}{4}\,(\hat{\gamma}_\infty)^2 = \frac{d-2}{4}\,\mu_{\rm CH}^2 = \frac{d-2}{4}\,(\tilde{\mu})^2.$$

- If not, we show that there are subdominant potential terms, and no guarantee of convergence at late-time to scaling solutions.

Swampland Conjectures & String Theory Examples

Swampland Conjectures & String Examples

[GS, Tonioni, Tran, '23]

- Various values for the $\mathcal{O}(1)$ constant in the de Sitter Conjecture $\gamma \geq \gamma_{dS}$ have been proposed. Our bound informs us whether (& which of) these criteria have to do with cosmic acceleration.

- $(\nabla V)^2 = 0$ implies either a de Sitter vacuum $(V \neq 0)$ or pure kination $(V = 0)$, which **saturates** our bound:

$$\gamma \geq \gamma_\star \geq \hat{\gamma}_\infty \left[1 - \frac{1}{2} \frac{\eta}{(d-1) - \epsilon} \right]$$

  Our bound is in some sense optimal.

- We can test the conditions under which there is **separation of scales**, by tracking the time-dependence of moduli.

- We applied our scaling solution results and bounds to **string theory examples** (F-theory and type II string with fluxes, heterotic O(16)xO(16) string with curvature & Casimir energy).

Summary

Summary of Results

[GS, Tonioni, Tran, '23]

- We bound the rate of time variation of the Hubble parameter at late time [GS, Tonioni, Tran, '23, STT1] and in the recent work, we further turn this into a bound on $\gamma \equiv |\nabla V|/V$ [GS, Tonioni, Tran, '23, STT2].

- The proper diagnostic for cosmic acceleration should be stated in terms of ϵ rather than potential gradient commonly used in Swampland criteria.

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Backup Slides

String Theory Example 1

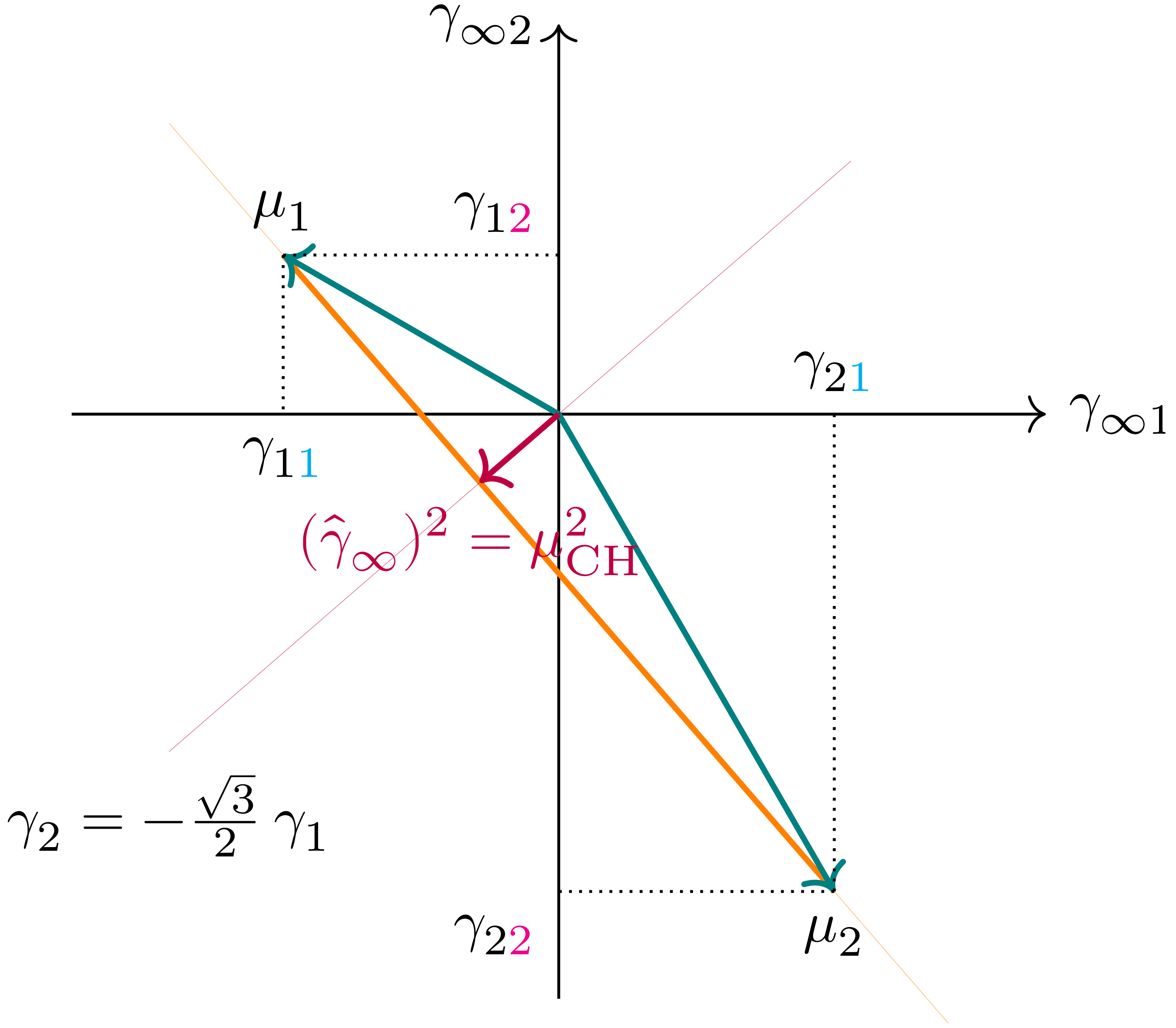
[GS, Tonioni, Tran]

Example previously considered in [Calderon-Infante, Ruiz, Valenzuela]

- Asymptotic limit of the complex structure moduli space of F-theory on CY 4-fold:

$$V = \Lambda_1\, e^{\kappa_4 \sqrt{2}\, \tilde{\phi}^1 - \kappa_4 \sqrt{\frac{2}{3}}\, \tilde{\phi}^2} + \Lambda_2\, e^{-\kappa_4 \sqrt{2}\, \tilde{\phi}^1 + \kappa_4 \sqrt{6}\, \tilde{\phi}^2} \qquad \Lambda_1, \Lambda_2 > 0$$

assuming the Einstein frame volume is stabilized w/0 affecting the above potential (big if!).



$$\phi^a = \phi^1, \phi^2$$

$$\gamma_{ia} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} -\sqrt{2} & \sqrt{\frac{2}{3}} \\ \sqrt{2} & -\sqrt{6} \end{pmatrix}$$

$$\epsilon \geq \frac{d-2}{4} (\hat{\gamma}_\infty)^2 = \frac{4-2}{4} \times \left[\left(-\frac{2\sqrt{2}}{7} \right)^2 + \left(-\frac{\sqrt{6}}{7} \right)^2 \right] = \frac{1}{7}$$

$(\lambda^1, \lambda^2) = (3/4, -1/4)$ together with $\Lambda_1, \Lambda_2 > 0 \Rightarrow$

converges to scaling solutions at late-time, $\epsilon = \frac{1}{7}$

String Theory Example 2

[GS, Tonioni, Tran]

- $O(16) \times O(16)$ heterotic string with **positive** internal curvature ($\Lambda_R < 0$) & Casimir energy:

$$V = \Lambda_R \, \mathrm{e}^{\kappa_d \left[\frac{2}{\sqrt{d-2}} \tilde{\delta} - \frac{2}{\sqrt{10-d}} \tilde{\sigma}\right]} + \Lambda_C \, \mathrm{e}^{\kappa_d \left[\frac{d}{\sqrt{d-2}} \tilde{\delta} + \sqrt{10-d} \, \tilde{\sigma}\right]} .$$

$\epsilon = d - 1$ because

$$\gamma_\infty^{-\tilde{\delta}} = \frac{d}{\sqrt{d-2}}, \qquad \gamma_\infty^{-\tilde{\sigma}} = \sqrt{10-d},$$

$$\Rightarrow \quad (\gamma_\infty)^2 = 4/(d-2) + 12 > 4(d-1)/(d-2),$$

- If internal curvature is **negative**, $\Lambda_R, \Lambda_C > 0$, we have a scaling solution as an attractor

$$\tilde{\delta}_*(t) = \tilde{\delta}_0 - \frac{1}{\kappa_d} \frac{12-d}{10} \sqrt{d-2} \ln \frac{t}{t_0},$$

$$\tilde{\sigma}_*(t) = \tilde{\sigma}_0 + \frac{1}{\kappa_d} \frac{d-2}{10} \sqrt{10-d} \ln \frac{t}{t_0},$$

with

$$\epsilon = \frac{1}{1 - \frac{3(d-2)}{25}} > 1.$$

No acceleration for both cases as expected since $\tilde{\delta}$ is rolling

String Theory Example 3

[GS, Tonioni, Tran]

- Type II compactification with RR p-form and q-form fluxes:

$$V = \Lambda_1 \, \mathrm{e}^{\kappa_d \left[\frac{d}{\sqrt{d-2}}\tilde{\delta} + \frac{10-d-2q}{\sqrt{10-d}}\tilde{\sigma}\right]} + \Lambda_2 \, \mathrm{e}^{\kappa_d \left[\frac{d}{\sqrt{d-2}}\tilde{\delta} + \frac{10-d-2p}{\sqrt{10-d}}\tilde{\sigma}\right]},$$

We showed that scaling solutions are attractors if $q < (10-d)/2$ and $p > (10-d)/2$:

$$\tilde{\delta}_*(t) = \tilde{\delta}_0 - \frac{1}{\kappa_d} \frac{2\sqrt{d-2}}{d} \ln \frac{t}{t_0},$$
$$\tilde{\sigma}_*(t) = \tilde{\sigma}_0,$$

$$\epsilon = \frac{d^2}{4} > 1$$

No acceleration as expected since $\tilde{\delta}$ is rolling

- Separation of scales:

$$l_H = l_H(t_0) \, \frac{t}{t_0},$$
$$l_{\mathrm{KK},d} = l_{\mathrm{KK},d}(t_0) \left(\frac{t}{t_0}\right)^{\frac{2}{d}}, \qquad l_{s,d}(t) = l_{s,d}(t_0)\left(\frac{t}{t_0}\right)^{\frac{2}{d}}$$

KK length grows less quickly than Hubble length.