Symmetry TFTs

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It turns out that $\mathrm{Symm}[\mathcal{T}[X]]$ is significantly easier to understand than $\mathcal{T}[X]$ itself, so our goal will be to construct $\mathrm{Symm}[X] \coloneqq \mathrm{Symm}[\mathcal{T}[X]]$ directly from the geometry.

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It is precisely in the cases where we don't know a Lagrangian that the information about symmetries and anomalies is most valuable.

Why

Categorical symmetries can be used in similar ways to ordinary symmetries:

- Constraints on the spectrum of physical theories coupled to gravity. [IGE, Montero '18], [McNamara, Vafa '19], [Heidenreich, McNamara, Montero, Reece, Rudelius, Valenzuela '21], [Blumenhagen, Cribiori '21], [Blumenhagen, Cribiori, Kneissl, Makridou '22], ...

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- 't Hooft anomaly matching (from a more modern viewpoint: $\mathrm{Symm}[\mathcal{T}]$ matching), for instance for testing duality [Del Zotto, IGE, Hosseini '20], [Lee, Ohmori, Tachikawa '21], ..., and restricting IR behaviour ([Gaiotto, Kapustin, Komargodski, Seiberg '17], ...)

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- Using representation theory to constrain the Hilbert space. [Komargodski, Ohmori, Roumpedakis, Seifnashri '20], . . .
- Generalised Goldstone theorems. (For continuous symmetries.) [Lake '18], [Hofman, Iqbal '18], [IGE, Iqbal '22], [García-Valdecasas '23]

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Can we reconstruct X (modulo string dualities) given $\mathrm{Symm}[X]$?

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Geometric Landau question

Can we reconstruct X (modulo string dualities) given $\mathrm{Symm}[X]$?

There is a categorical version of this question, where we ask about some category associated to X instead. For instance, in some cases we can associate a cluster category to X. The Grothendick group of this cluster category is easy to read from $\mathrm{Symm}[X]$. [Caorsi, Cecotti '17], [Del Zotto, IGE, Hosseini '20], [Del Zotto, IGE '22].

Geometric engineering

For reasons of analytic control we want to impose restrictions on the manifolds X that we consider. These are:

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For instance, if X is a complex two-fold, we will assume that it is an ALE space of the form $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$, with $\Gamma_{\mathfrak{g}} \subset SU(2)$. This is a cone over $S^3/\Gamma_{\mathfrak{g}}$, with $\Gamma_{\mathfrak{g}}$ acting freely on S^3. On $\mathbb{C}^2$ the origin is fixed by all elements of $\Gamma_{\mathfrak{g}}$, so we have an orbifold singularity there.

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If we place IIB string theory (10d) on this geometry we obtain a $(2,0)$ SCFT $\mathfrak{g}_{(2,0)}$ in six dimensions, arising from modes at the singularity. These theories are believed to be indexed by $\Gamma_{\mathfrak{g}}$, or equivalently by an algebra $\mathfrak{g}$ of type $\mathfrak{a}_n$, $\mathfrak{d}_n$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ or $\mathfrak{e}_8$.

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One important property of the $(2,0)$ theory with algebra $\mathfrak{g}$ is that upon reduction on T^2 with complex structure τ it gives rise to 4d $\mathcal{N}=4$ SYM with algebra $\mathfrak{g}$ and complexified gauge coupling τ. Let me call this object $\mathfrak{g}_4$.

$\mathfrak{g}_4$ as a relative theory

What I have just described fully specifies the behaviour of local operators, but it does not fully fix the theory. For example it does not fully fix the partition function on K3: for a generic coupling τ, $SU(2)$ and $SO(3)$ SYM have different partition functions on K3. [Vafa, Witten '94]

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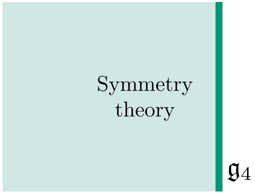
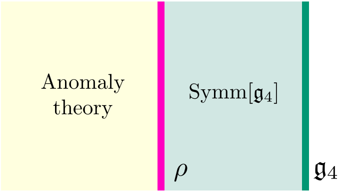
A good way of thinking about $\mathfrak{g}_4$ is as a "relative theory" [Freed, Teleman '12]: in physical terms it is a set of boundary gapless modes for a TFT in one dimension higher ($4 + 1 = 5$ here). This TFT includes information about the potential symmetries, anomalies and gaugings of all theories with local dynamics given by $\mathfrak{g}_4$. We refer to this TFT as $\mathrm{Symm}[\mathfrak{g}_4]$.



The absolute $\mathcal{N} = 4$ theories

We can obtain more familiar objects by introducing a second gapped interface ρ between $\mathrm{Symm}[\mathfrak{g}_4]$ and an invertible TFT, the anomaly theory.



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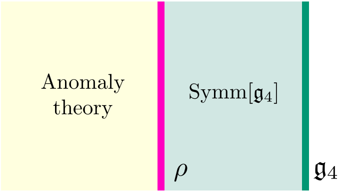

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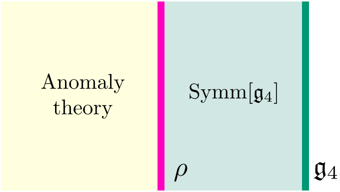

Colliding ρ and $\mathfrak{g}_4$ we obtain what we usually think of as $\mathcal{N} = 4$ SYM *theories* in $d = 4$. The symmetries of the resulting SYM theory are those topological operators of $\mathrm{Symm}[\mathfrak{g}_4]$ that do not become trivial when restricted to ρ.

An example: $\mathfrak{g}_4 = \mathfrak{so}(4k)$

Take for instance $\mathfrak{g}_4 = \mathfrak{so}(4k)$. Some of the possible "global forms" (choices of ρ) for this theory are [Witten '98], [Aharony, Seiberg, Tachikawa '13], [Tachikawa '14], [Tachikawa '17], [IGE, Heidenreich, Regalado '19], [Bhardwaj, Bottini, Schäfer-Nameki, Tiwari '22], [IGE '22], [Bergman, Hirano '22], [Etheredge, IGE, Heidenreich, Rauch '23], [Apruzzi, Bonetti, Gould, Schäfer-Nameki '23]:

- $\mathrm{Spin}(4k)$: 2-group symmetry
- $SO(4k)$: 1-form and 0-form symmetry, with mixed 't Hooft anomaly.
- $Ss(4k)$: 1-form symmetries and non-invertible 0-form symmetries.

$\mathrm{Symm}[\mathfrak{so}(4k)]$ is the same in all cases, and the problem of studying the different theories becomes a problem in representation theory. [Freed, Moore, Teleman '22], [Bhardwaj, Schäfer-Nameki], [Bartsch, Bullimore, Ferrari, Pearson]

Back to 10d

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My goal will be to derive $\mathrm{Symm}[\mathfrak{g}_4]$ (*) without using any knowledge about the Lagrangian of $\mathcal{N} = 4$.

(*) In this talk I will explain how to work out some aspects of the symmetry theories, and make a conjecture for what the general prescription is.

The geometric engineering perspective

In this talk I will work in the M-theory ($D = 11$) and IIB ($D = 10$) duality frames. In either frame, the basic picture is the same.

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We place the string theory on $X^{2n} \times \mathcal{M}^{D-2n}$, where X^{2n} is a Calabi-Yau manifold of complex dimension n, which is also a real cone with base B^{2n-1}. There is a singularity at the base of the cone, where we have a field theory $\mathcal{T}[X^{2n}]$ on $\mathcal{M}^{D-2n}$.

Dynamical states and branes

In the context of geometric engineering we treat the background geometry $X^{2n} \times \mathcal{M}^{D-2n}$ as given, and ignore its dynamics. All the dynamical field theory behaviour comes from p-branes and their associated fluxes moving in the given geometry.

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These are the dynamical states, but for understanding the symmetry of the theory we are more interested in the behaviour of extended defect operators.

Heavy branes

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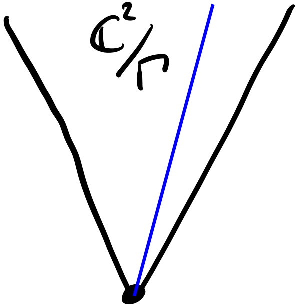

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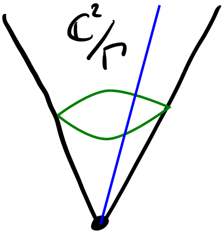

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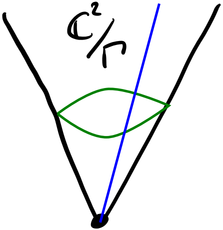

[*] The full story is more subtle: understanding non-invertible symmetries requires thinking of the symmetry generators as bona fide branes. [Apruzzi, Bah, Bonetti, Schäfer-Nameki '22], [I.G.E. '22], [Heckman, Hübner, Torres, Zhang '22], [. . .]

Boundary conditions and flux non-commutativity

In order to define the string theory fully on the non-compact space $X^{2n} \times \mathcal{M}^{D-2n}$ we need to specify the boundary conditions at infinity, which is (assuming $\mathcal{M}^{D-2n}$ compact) of the form $B^{2n-1} \times \mathcal{M}^{D-2n}$.

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We do this by giving a state in the Hilbert space associated to $B^{2n-1} \times \mathcal{M}^{D-2n}$. There are many subtleties in making this statement precise, but the crucial one for us is due to flux non-commutativity [Moore '04], [Freed, Moore, Segal '06], due to the fact that the non-compact brane wraps a torsional cycle of B^{2n-1} (the base of X^{2n}), and therefore the flux sourced by it should be measured on a torsional cycle.

Non-commutativity of fluxes in M-theory

Let us put M-theory on $\mathcal{M}_{11} = \mathcal{N}_{10} \times \mathbb{R}$. We will try to understand the Hilbert space $\mathcal{H}(\mathcal{N}_{10})$, or more precisely its grading by flux. This was done in [Freed, Moore, Segal '06].

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$$\psi(C_3 + \lambda) = e^{2\pi i \int_{\mathcal{N}_{10}} Q_e \lambda} \psi(C_3)$$

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So we cannot simultaneously measure electric and magnetic charges, if there are flat topologically non-trivial λ. This is the case iff $\mathrm{Tor}\, H^4(\mathcal{N}_{10}) \neq 0$.

Non-commutativity of fluxes in M-theory

This can be restated in terms of the flux operators, as follows: for every $\sigma \in \operatorname{Tor} H_6(\mathcal{N}_{10};\mathbb{Z}) = \operatorname{Tor} H^4(\mathcal{N}_{10};\mathbb{Z})$ there is a unitary flux operator Φ_σ. Similarly for any $\sigma' \in \operatorname{Tor}(H_3(\mathcal{N}_{10};\mathbb{Z})) = \operatorname{Tor} H^7(\mathcal{N}_{10};\mathbb{Z})$.

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These operators in general do not commute:

$$\Phi_\sigma \Phi_{\sigma'} = e^{2\pi i\, \mathsf{L}(\sigma,\sigma')} \Phi_{\sigma'} \Phi_\sigma$$

where $\mathsf{L}(\sigma,\sigma')$ is the linking pairing on $\mathcal{N}_{10}$: choose $n \in \mathbb{Z}$ such that $n\sigma = \partial D$. Then

$$\mathsf{L}(\sigma,\sigma') = \frac{1}{n} D \cdot \sigma' \mod 1\,.$$

Non-commutativity of fluxes in M-theory

The pairing $\mathsf{L}(\cdot,\cdot)$ is *perfect*, which implies that for each torsional $\sigma \neq 0$ there is some σ' such that $\mathsf{L}(\sigma,\sigma') \neq 0$, and thus

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What this all implies, it that whenever $\mathrm{Tor}(H_3(\mathcal{N}_{10};\mathbb{Z})) \neq 0$ it is not possible to simultaneously diagonalize all Φ_σ. In particular, it is not consistent to take the simple "fluxless" choice $\Phi_\sigma = 1$ for all σ. We need to turn on *some* flux at infinity!

Maximal isotropic subspaces

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We can specify a state in the Hilbert space as usual: by choosing a maximal subspace $\mathcal{I} \subset \mathrm{Tor}(H_3(\mathcal{N}_{10}); \mathbb{Z}) \times \mathrm{Tor}(H_6(\mathcal{N}_{10}); \mathbb{Z})$ such that the corresponding group of operators $\{\Phi_x\}$ for $x \in \mathcal{I}$ is abelian, and imposing that

$$\Phi_x |0; L\rangle = |0; L\rangle \qquad \forall x \in \mathcal{I}$$

In our M-theory setting, this corresponds to setting to zero on the boundary as many fluxes as possible. (This is the sector with vanishing Φ_x flux, for non-zero background flux for the higher form symmetries choose non-zero eigenvalues.)

Fluxes and global forms

Classification

The possible global forms of the $d = 7$ theories on $\mathcal{M}_7$ are given by maximal commuting subspaces of $H_2(\mathcal{M}_7; \Gamma_{\mathfrak{g}}^{\mathsf{ab}}) \times H_5(\mathcal{M}_7; \Gamma_{\mathfrak{g}}^{\mathsf{ab}})$, with commutators worked out in [IGE, Heidenreich, Regalado '19].

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Fluxes and global forms

Classification

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Related ideas apply in many other contexts: [Morrison, Schäfer-Nameki, Willett '20], [Albertini, Del Zotto, IGE, Hosseini '20], [Closset, Schäfer-Nameki, Wang '20], [Del Zotto, IGE, Hosseini '20], [Apruzzi, Dierigl, Lin '20], ...

Relative theories

A high level summary of the previous discussion is that in geometric engineering we have something like a "QFT on a singularity relative to the string theory bulk": the full theory is only defined only after specifying boundary values for the supergravity fields, even in the deep IR limit where dynamical excitations for the bulk decouple.

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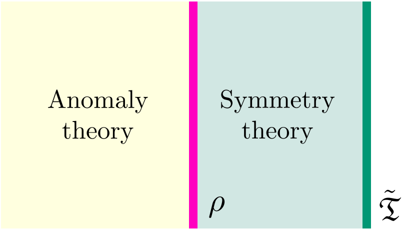
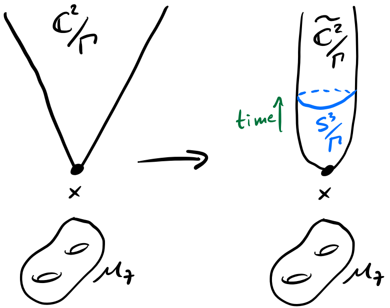
In general this relates a $D-2n$-dimensional field theory to a D-dimensional supergravity bulk, with $n>1$. I would now like to relate this picture to the "SymTFT" picture discussed before:



How symmetry theories appear in string theory

The derivation in [IGE, Heidenreich, Regalado '19] uses a modified asymptotic structure.



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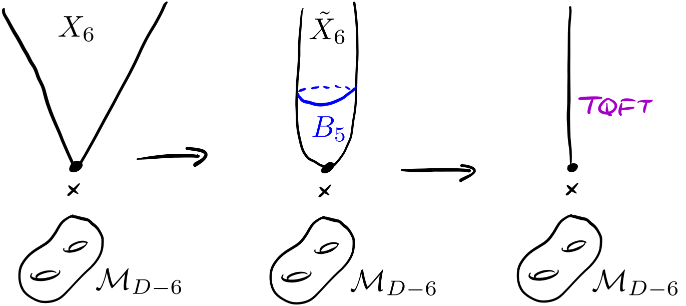

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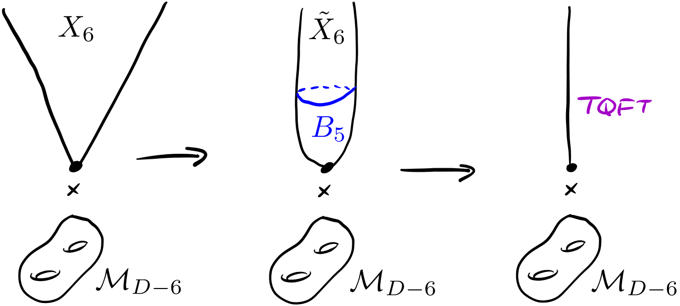

In this picture the boundary conditions at infinity that we need to specify in string theory correspond to ρ, so the object that arises naturally is the symmetry theory. ("Symmetry inflow" instead of "anomaly inflow".)

An example

As an example, for 5d SCFTs the resulting symmetry theory is:

$$S_{\mathsf{Sym}} = \int_{\mathcal{W}_6} \Big(K_{ij} B_2^{(i)} \cup \delta C_3^{(j)} + \Omega_{ijk} B_2^{(i)} \cup B_2^{(j)} \cup B_2^{(k)} + \Upsilon_{ij\alpha} B_2^{(i)} \cup B_2^{(j)} \cup F_2^{(\alpha)} \Big)$$

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$$K_{11} = \gcd(p,q) \ ; \ \Omega_{111} = \frac{q\, p\,(p-1)\,(p-2)}{6\gcd(p,q)^3} \ ; \ \Upsilon_{111} = \frac{p\,(p-1)}{2\gcd(p,q)^2}$$

in agreement with [Gukov, Pei, Hsin '20].

The SymTFT from self-dual fields

Upcoming work with S. Hosseini

This approach to deriving the SymTFT, while workable, had some unsatisfactory aspects:

- The anomaly was obtained by reducing the topological sector

$$S_{11d} = \ldots - \frac{1}{6} C_3 \wedge G_4 \wedge G_4 + C_3 \wedge X_8$$

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The problem in obtaining a unified treatment is that it is not clear how to reduce in a way that keeps both electric and magnetic degrees of freedom manifest. We want a formulation where we can think of $(F, \star F)$ as a single (self-dual) field!

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Such a framework was introduced by [Witten '96] for studying the M5 brane partition function (with important elaborations by [Belov, Moore '06] and [Hsieh, Tachikawa, Yonekura '20]): we think of the theory of the self-dual field as the set of boundary modes for a Chern-Simons like in one dimension higher, with appropriate (gapless) boundary conditions.

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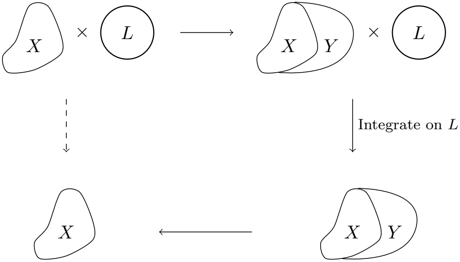

See [Apruzzi, Bah, Bonneti, Schäfer-Nameki '22], [Lawrie, Yu, Zhang '23], [Apruzzi, Bonetti, Gould, Schäfer-Nameki '23] for recent work taking a similar approach.

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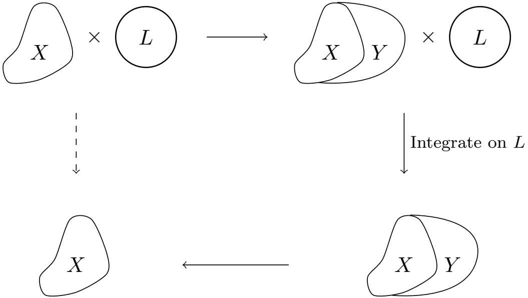

$$“\int_{Y\times L} A\wedge dA” = \int_{Y\times L} \breve{a}\cdot\breve{a} = \sum_{ij}\int_{Y\times L} (\breve{b}_i\breve{t}_i)\cdot(\breve{b}_j\breve{t}_j)$$

$$= \sum_{ij} \underbrace{\left(\int_L \breve{t}_i\cdot\breve{t}_j\right)}_{K_{ij}} \int_Y db_i db_j = \sum_{ij} K_{ij}\int_X b_i db_j\,.$$

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We can also include the anomaly sector: recall that this came from $\int_L S_{GS}$ in the original theory (without extending to a bulk).

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Conjecture (type II and type I)

Given a QFT $\mathcal{T}[X]$ arising from string theory on X, we have

$$\int_{\partial Y} \mathrm{Symm}[\mathcal{T}[X]] = \int_Y \int_{\partial X} S^K_{CS}$$

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(Perhaps the methods of [Fiorenza, Sati, Schreiber '19] together with [Hopkins, Singer '02] would allow us to define the analogue of S^K_{CS} for M-theory.)

Conclusions

SymTFTs seem to be fairly fundamental objects when talking about modern ideas of symmetry, and they appear very naturally in string theory, basically from "dimensional" reduction.

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Many threads are pointing towards (differential) K-theory being the right framework to think about symmetries in theories engineered from string theory. This entails some important conceptual differences with respect to the (now) standard picture.