Tameness and Complexity in QFTs

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based on

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Together with Michael Douglas and Thomas Grimm

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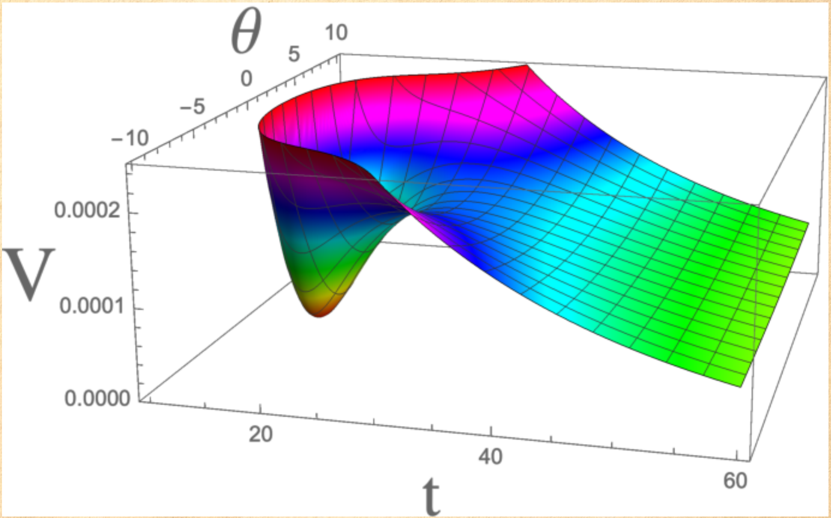
Content

- What is tameness?
- Tameness of QFT
  - perturbative results
  - non-perturbative results
- Tameness as a Swampland Conjecture
  - Tameness of the space of CFTs
  - Tameness of the observables in CFTs
  - Tameness of the observables in EFTs
- Applications
  - Complexity from tameness

What is tameness?

- Tameness is a generalized finiteness principle
- Forbids *discrete* infinities
- Idea: Functions appearing in physics should only behave in finitely many different ways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o-minimal structures

- In this talk: Tameness "=" o-minimality
- Allow only functions which are definable in an o-minimal structure S

Definition of a Structure

Collections $S = (S_n)_{\geq 1}$ of sets in $\mathbb{R}^n$ closed under $\cup, \cap, \times, /$ and linear projections containing at least all algebraic sets (= zero sets of polynomials).

Definition of o-minimality

A structure is o-minimal if the definable subsets of $\mathbb{R}$ are finite unions of intervals and points

Definable subsets of $\mathbb{R}$

- Only finitely many points and intervals.
- But the intervals can be infinitely long.
- Higher dimensional sets have to project down to these.

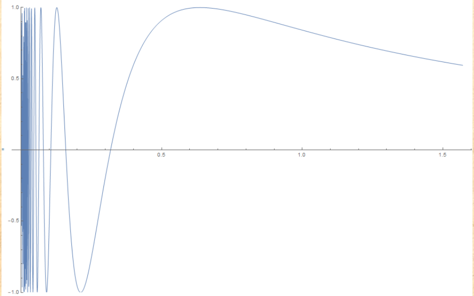
The language

- sets in o-minimal structure: tame sets
- functions whose graph is a tame set: tame functions

$\rightarrow$ tame manifolds, tame bundles, tame geometry

What does this mean in practice?

- o-minimal structures forbid anything infinite discrete
  - no integers $\mathbb{Z}$
  - no periodic functions
  - no sin(x) and cos(x) for $x \in \mathbb{R}$ (but tame on finite interval)
  - no error or gamma functions on $\mathbb{R}$
  - also restricts functions on finite intervals



Examples of o-minimal structures

- $\mathbb{R}_{\mathrm{alg}}$: semi-algebraic sets ($P(x) \geq 0$ instead of $P(x) = 0$)
- $\mathbb{R}_{\mathrm{an}}$: restricted analytic functions
- $\mathbb{R}_{\mathrm{exp}}$: real exponential function
- $\mathbb{R}_{\mathrm{an,exp}}$: combination of the two above
- $\mathbb{R}_{\mathrm{Pfaff}}$: structure of all Pfaffian functions
- ...

$$
\begin{aligned}
\partial f_1(x) &= P_1(x, f_1(x)) \\
\partial f_2(x) &= P_2(x, f_1(x), f_2(x)) \\
\partial f_3(x) &= P_3(x, f_1(x), f_2(x), f_3(x)) \\
&\dots
\end{aligned}
$$

Tameness of QFT

What is the right question?

- Basic questions
  - Which objects are tame?
  - To be able to talk about tameness we need structures, what are the right structures?
  - Do different objects live in different structures or does there exist an overarching structure like $\mathbb{R}_{\text{an,exp}}$?
  - Is every QFT tame?

Interesting objects of a QFT

- The Lagrangian/action
- The partition function Z
- The correlators
- Amplitudes/observables

Structures from QFTs

- Define the relevant structures in terms of 2 sets
  - A set of theories $\mathcal{T}$, e.g parameter space of specified Lagrangians
  - Set $\mathcal{S}$ of Euclidean spacetimes with metric (Σ, g)
  - Both are definable in some structure $\mathbb{R}^{\mathrm{def}}_{\mathcal{T},\mathcal{S}}$
- Simplest example: Polynomial Lagrangians in $\mathbb{R}^d \to \mathbb{R}^{\mathrm{def}}_{\mathcal{T},\mathcal{S}} = \mathbb{R}_{\mathrm{alg}}$
- Add the partition function and correlators to the original structure $\to \mathbb{R}_{\mathcal{T},\mathcal{S}}$
- For simplicity in this talk $\mathcal{S} = \mathbb{R}^d$ and the explicit dependence on $\mathcal{S}$ is dropped, e.g $\mathbb{R}_{\mathrm{QFT}}$, $\mathbb{R}_{\mathrm{CFT}}$, $\mathbb{R}_{\mathrm{EFT}}$.

Questions about Tameness

- If $\mathbb{R}^{\mathrm{def}}_{\mathcal{T},\mathcal{S}}$ is o-minimal, when is $\mathbb{R}_{\mathcal{T},\mathcal{S}}$ o-minimal?.
  - Are observables tame?
- Under which conditions is $\mathbb{R}^{\mathrm{def}}_{\mathcal{T},\mathcal{S}}$ o-minimal?
  - Tameness of the space of theories?

Tameness of perturbative QFT

Theorem: For any renormalizable QFT with finitely many fields and interactions all finite-loop amplitudes are tame functions of the masses, external momenta and coupling constants definable in $\mathbb{R}_{\mathrm{an,exp}}$. [Douglas,Grimm,LS - Part I]

- Feynman integrals are periods
- periods are definable in $\mathbb{R}_{\mathrm{an,exp}}$ [Bakker,Klingler,Tsimerman][Bakker,Mullane '22]

  $\rightarrow$ Feynman integrals are definable

  $\rightarrow$ Amplitudes are definable
- If the Lagrangian is tame the *perturbative* corrections will not destroy this tameness!

$\rightarrow$ perturbative QFTs are tame if the Lagrangian is tame

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More on GKZ systems and period integrals in Arno´s Talk on Thursday

What about non-perturbative effects?

- Instantons appear to produce cos potentials$\rightarrow$ appear to be dangerous.
- The Feynman diagram argument does not help due to the non-perturbative nature.
- But: Tameness is not conserved under power series expansion!

$$x^2 = \frac{\pi^2}{3} - 4cos(x) + cos(2x) + \dots$$

- Look at some examples of exactly solvable theories.

Gauged linear sigma models

- 2d theory with $\mathcal{N} = 2$ supersymmetry.
- Exactly solvable by supersymmetric localization.
- The sphere partition function is given in terms of the Kähler potential of the described geometry [Jockers et al. 12']

$$Z_{S_2} = e^{-K} = \overline{\Pi}\Sigma\Pi$$

- As the partition function is given in terms of periods it is definable in $\mathbb{R}_{\mathrm{an,exp}}$!

Solvable 0d QFTs

- On points the path integral reduces to usual integrals
- Many 0d QFTs are solvable like the Sine-Gordon model or the ϕ^4 theory

$$Z(m,\lambda) = \int_{-\infty}^{\infty} \mathrm{d}\phi \; e^{-\frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4} = \sqrt{\frac{3}{\lambda}} e^{\frac{3m^4}{4\lambda}} \; m \, K_{1/4}\left(\frac{3m^4}{4\lambda}\right) ,$$

$$Z(g) = \int_{-\pi}^{\pi} \mathrm{d}\phi \; e^{-g \; sin(\phi)^2} = 2e^{-g/2}\pi I_0(g/2) ,$$

- I_0 is a tame function, period of an explicit geometry
- $K_{1/4}$ is an exponential period, tameness of these is an open question!
- Both are part of a Pfaffian chain, definable in $\mathbb{R}_{\mathrm{Pfaff}}$

  [Van den Dries, private communication]

Other tame examples

- 1d harmonic oscillator

$$Z(\beta, m) = \frac{1}{sinh(\frac{\beta}{2m})}$$

- 2d string theories, 3d non-critical M-theory

$$F_{3d}(\omega,\mu) = -\frac{1}{6\omega^2}\mu^3 + \frac{\Lambda}{4\omega}\mu^2 - \frac{1}{2\pi\omega_0}\mu^2 \log(1 - e^{-2\pi\mu/\omega}) + \frac{1}{2\pi^2}\mu Li_2(1 - e^{-2\pi\mu/\omega}) + \frac{\omega}{4\pi^3} Li_3(1 - e^{-2\pi\mu/\omega})$$

- 2d Yang-Mills theory

$$Z_{SU(2)} = e^{A\lambda/16}(\theta_3(e^{-A\lambda/16}) - 1)$$

- Klein-Gordon field in d-dimensional AdS

$$O^{(d)}(y_1, y_2) = (2\pi)^{-d/2}\left(\frac{(y_2 - y_1)^2}{\sqrt{m}}\right)^{\frac{d-2}{2}} K_{\frac{d-2}{2}}(\sqrt{m}(y_2 - y_1)^2)\,.$$

Is every QFT tame?

No! Can construct explicit counterexamples:

- infinite discrete symmetries $Z(g \cdot \lambda) = Z(\lambda)$
  - Need to be gauged or broken
  - Fits with no global symmetries conjecture [Banks,Dixon 88'][Banks,Seiberg 10']
- non-tame Lagrangian
  - Simple example : $V(\theta) = A\cos(\theta) + B\cos(\alpha\theta) \qquad \alpha$ irrational
  - Allows for infinite spirals $\rightarrow$ tension with distance conjecture [Grimm,Lanza,Li 22]
- Observables also need not be tame
  - Neutrino oscillations

Tameness of EFTs

Conjecture: All effective theories valid below a fixed finite energy cut-off scale Λ that can be coupled to QG are labelled by a tame parameter space and have scalar field spaces and Lagrangians that are tame in an o-minimal structure [Grimm 21']

Conjecture: $\mathbb{R}_{\mathrm{EFTd}}[\Lambda]$ are o-minimal structures, i.e. observables are also tame [Douglas,Grimm,LS - Part II]

What about string theory

- Perturbative string theory has infinitely many fields$\rightarrow$ not a tame theory
- The partition functions are expressible via θ functions $\rightarrow$ tame functions
- Any effective theory with finite cutoff Λ is tame.
- In 2d string theory one can understand what happens to the infinite discrete modes
- Goldstone modes of broken area-preserving diffeomorphisms of 3d theory
- Only a toy model!

Tameness of conformal field theories

Tameness of CFTs

Conjecture 1: All observables of a tame set $\mathcal{T}_{\mathrm{CFT}}$ are tame functions.
[Douglas,Grimm,LS - Part II]

Conjecture 2(a): The theory space $\mathcal{T}_{\mathrm{CFT}}$ in d=2 is tame if
- the central charge is bounded
- lowest operator dimension is bounded from below.

[Douglas,Grimm,LS - Part II]

Conjecture 2(b): The theory space $\mathcal{T}_{\mathrm{CFT}}$ in $d > 2$ is tame if
- an appropriate measure for the degrees of freedom is bounded
- theories differing by discrete gaugings are identified.

[Douglas,Grimm,LS - Part II]

Evidence for the conjectures - Observables

- Conformal symmetry fixes the form of the 2- and 3-point correlators to

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{(x_1-x_2)^{\Delta_i+\Delta_j}} \ ,$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{1,2,3}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{-\Delta_1+\Delta_2+\Delta_3}x_{13}^{\Delta_1-\Delta_2+\Delta_3}} \ ,$$

- Trivially tame in the positions and operator dimensions
- First non-trivial case is the 4-point correlator

Evidence for the conjectures - Observables

- Conformal symmetry fixes the dependence on the positions and weights in terms of Virasoro conformal blocks $W_{\mathcal{O}}$:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{\mathcal{O}\in\mathcal{O}_1\times\mathcal{O}_2} C_{1,2,\mathcal{O}}C_{3,4,\mathcal{O}}W_{\mathcal{O}} \ ,$$

$$W_{\mathcal{O}} = \frac{1}{C_{1,2,\mathcal{O}}C_{3,4,\mathcal{O}}} \sum_{\alpha\in\mathrm{descendants}} \langle 0|\,\mathcal{O}_1\mathcal{O}_2\,|\alpha\rangle\,\langle\alpha|\,\mathcal{O}_3\mathcal{O}_4\,|0\rangle$$

- In general very complicated functions, mostly unknown.
- In certain limits (large c, semi-classical, heavy-light exchange...) simplifies to hypergeometric function

Evidence for the conjectures - Observables

$$W^{2d}_{\mathcal{O},\text{classical}}(\Delta) \propto \ {}_2F_1\left(\frac{\Delta-\Delta_{12}}{2}, \frac{\Delta+\Delta_{34}}{2}; \Delta; z\right) \ ,$$

- Tame in the crossing ratio z
- Tameness in Δ more complicated
- Analysis shows that the differences in operator dimensions $\Delta_{i,j}$ need to be bounded!
- Fits nicely with the no parametric separation of scales conjecture [Lüst, Palti, Vafa 19']

Evidence for the conjectures - Space of theories - 2d

- The space of 2d CFTs is clearly not o-minimal
- Many infinite discrete sets exist, e.g. unitary minimal models

$$c = 1 - \frac{6}{(p+1)(p+2)} \xrightarrow{p\to\infty} 1 \qquad \Delta_1 = \frac{3}{4(p+1)(p+2)} \xrightarrow{p\to\infty} 0$$

- WZW models

$$c = 3 - \frac{6}{(p+2)} \xrightarrow{p\to\infty} 1 \qquad \Delta_1 = \frac{3}{4(p+2)} \xrightarrow{p\to\infty} 0$$

- For fixed lower bound on Δ_1 only finitely many theories

Evidence for the conjectures - Space of theories - 3d

- Again many families of theories parameterized by discrete choices of parameters
- 3d Chern-Simons theory: gauge group N and level k
  $\rightarrow$ naively a lattice $\mathbb{Z}^2$ of theories
- Dualities identify different choices, e.g level-rank duality
$$F(N,k) = F(k,N) = \frac{N}{2} log(k+N) + \dots$$
- For fixed upper bound of F only finitely many theories!

Applications of Tameness

- Used in proofs of many deep mathematical conjectures
  - Ax-Schanuel for Hodge Structures [Bakker, Tsimerman'17]
  - Griffiths'conjecture [Bakker, Brunebarbe, Tsimerman'18]
  - André-Oort conjecture [Pila, Shankar, Tsimerman'21]
  - Geometric André-Grothendieck Period Conjecture [Bakker, Tsimerman'22]
- Finiteness of of vacua [Bakker, Grimm, Schnell, Tsimerman'21]
  See Jeroen's talk on Thursday
- Allows to assign a complexity

Application: Complexity

Main idea: tameness=finite complexity

- Can assign a complexity to Pfaffian chains depending on the length (format) and degree of the chain
- Different types of complexities: Topological and computational
- Many physical systems (0d QFT correlators, wavefunctions in 1d QM, SU(2) SW theory...) are Pfaffian
- Just the starting point: generalizes to sharply $\sharp$o-minimal structures
- Stronger constraint than o-minimality

Summary

- Perturbative QFTs are tame if the Lagrangian is tame
- Non-perturbative tameness requires restrictions on the theories
  - CFT with bounded degrees of freedom and finite gap
    $\rightarrow$ leads to bounds on operator dimensions
  - EFT originating in QG
    $\rightarrow$Tameness as a swampland conjecture
- Dualities play an important role in the tameness of the theory space
- Sharply $\sharp$o-minimality/Pfaffian settings allow to define a complexity